THE THIRTEEN BOOKS
OF
EUCLID'S ELEMENTS

[illegible] ἰσ οὖν οὐκ ἐζήτη
[illegible] ἀξιον μεῖναι
χ[illegible] ἀλλὰ
[illegible] ἰσ αὐτῷ ἡ αἰχο
το[illegible] ὁμοί ἰσ ἐπεὶ τ' ἐπερ
[illegible] ἰσ οὐκ ἐπετέρα [illegible]
εἰ [illegible] οὐ [illegible] χαι [illegible]
τ ἰε

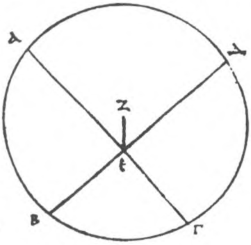
Δ

Δ ΤΑ Τ Γ

Δ Τ Γ Τ ΛΥ

Ε

ἐὰν ἐν κύκλῳ δύο εὐθεῖαι τέμνωσιν ἀλλήλας μὴ διὰ τοῦ κέντρου οὖσαι. οὐ τέμνουσιν ἀλλήλας δίχα· ἔστω κύκλος ὁ ΑΒΓΔ καὶ ἐν αὐτῷ δύο εὐθεῖαι αἱ ΑΓ ΒΔ τεμνέτωσαν ἀλλήλας κατὰ τὸ Ε μὴ διὰ τοῦ κέντρου οὖσαι· λέγω ὅτι οὐ τέμνουσιν ἀλλήλας δίχα· εἰ γὰρ δυνατόν. τεμνέτωσαν ἀλλήλας δίχα· ὥστε ἴσην εἶναι τὴν μὲν ΑΕ τῇ ΕΓ. τὴν δὲ ΒΕ τῇ ΕΔ καὶ εἰλήφθω τὸ κέντρον τοῦ ΑΒΓΔ κύκλου καὶ ἔστω τὸ Ζ καὶ ἐπεζεύχθω ἡ ΖΕ· ἐπεὶ οὖν εὐθεῖά τις διὰ τοῦ κέντρου ἡ ΖΕ εὐθεῖάν τινα τὴν ΑΓ δίχα τέμνει. καὶ πρὸς ὀρθὰς αὐτὴν τέμνει· ὀρθὴ ἄρα ἐστὶν ἡ ὑπὸ ΖΕΑ· πάλιν ἐπεὶ εὐθεῖά τις ἡ ΖΕ εὐθεῖαν τινα τὴν ΒΔ δίχα τέμνει. καὶ πρὸς ὀρθὰς αὐτὴν τέμνει· ὀρθὴ ἄρα ἡ ὑπὸ ΖΕΒ· ἐδείχθη δὲ καὶ ἡ ὑπὸ ΖΕΑ ὀρθή· ἴση ἄρα ἡ ὑπὸ ΖΕΑ τῇ ὑπὸ ΖΕΒ. ἡ ἐλάττων τῇ μείζονι. ὅπερ ἐστὶν ἀδύνατον· οὐκ ἄρα αἱ ΑΓ ΒΔ τέμνουσιν ἀλλήλας δίχα· ἐὰν ἄρα ἐν κύκλῳ δύο εὐθεῖαι τέμνωσιν ἀλλήλας καὶ τὰ ἑξῆς. ὅπερ ἔδει δεῖξαι:-



ἐὰν δύο κύκλοι τέμνωσιν ἀλλήλους. οὐκ ἔσται αὐτῶν τὸ αὐτὸ κέντρον· δύο γὰρ κύκλοι οἱ ΑΒΓ ΓΔΗ τεμνέτωσαν ἀλλήλους κατὰ τὰ ΒΓ σημεῖα· λέγω ὅτι οὐκ ἔσται αὐτῶν τὸ αὐτὸ κέντρον· εἰ γὰρ δυνατόν. ἔστω τὸ Ε καὶ ἐπεζεύχθω ἡ ΕΓ καὶ διήχθω ἡ ΕΖΗ ὡς ἔτυχεν· καὶ ἐπεὶ τὸ Ε σημεῖον κέντρον ἐστὶ τοῦ ΑΒΓ κύκλου. ἴση ἐστὶν ἡ ΕΓ τῇ ΕΖ· πάλιν ἐπεὶ τὸ Ε σημεῖον κέντρον ἐστὶ τοῦ ΓΔΗ κύκλου. ἴση ἐστὶν ἡ ΕΓ τῇ ΕΗ· ἐδείχθη δὲ ἡ ΕΓ καὶ τῇ ΕΖ ἴση. καὶ ἡ ΕΖ ἄρα τῇ ΕΗ

THE THIRTEEN BOOKS OF EUCLID'S ELEMENTS

TRANSLATED FROM THE TEXT OF HEIBERG

WITH INTRODUCTION AND COMMENTARY

BY

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PREFACE

"THERE never has been, and till we see it we never shall believe that there can be, a system of geometry worthy of the name, which has any material departures (we do not speak of *corrections* or *extensions* or *developments*) from the plan laid down by Euclid." De Morgan wrote thus in October 1848 (*Short supplementary remarks on the first six Books of Euclid's Elements* in the *Companion to the Almanac* for 1849); and I do not think that, if he had been living to-day, he would have seen reason to revise the opinion so deliberately pronounced sixty years ago. It is true that in the interval much valuable work has been done on the continent in the investigation of the first principles, including the formulation and classification of axioms or postulates which are necessary to make good the deficiencies of Euclid's own explicit postulates and axioms and to justify the further assumptions which he tacitly makes in certain propositions, content apparently to let their truth be inferred from observation of the figures as drawn; but, once the first principles are disposed of, the body of doctrine contained in the recent text-books of elementary geometry does not, and from the nature of the case cannot, show any substantial differences from that set forth in the *Elements*. In England it would seem that far less of scientific value has been done; the efforts of a multitude of writers have rather been directed towards producing alternatives for Euclid which shall be more suitable, that is to say, easier, for schoolboys. It is of course not surprising that, in these days of short cuts, there should have arisen a movement to get rid of Euclid and to substitute a "royal road to geometry"; the marvel is that a book which was not written for schoolboys but for grown men (as all internal evidence shows, and in particular the essentially theoretical character of the work and its aloofness from anything of the nature of "practical" geometry) should have held its own as a school-book for so long. And now that Euclid's proofs and arrangement are no longer required from candidates at examinations there has been a rush of competitors anxious to be first in the field with a new text-book on the more "practical" lines which now find so much favour. The natural desire of each teacher who writes such a text-book is to give prominence to some special nostrum which he has found successful with pupils. One result is, too often, a loss of a due sense of proportion; and, in any case, it is inevitable that there should be great diversity of treatment. It was with reference to such a danger that Lardner wrote in 1846: "Euclid once superseded, every teacher would esteem his own work the best, and every school would have its own class book. All that rigour and exactitude which have so long excited the admiration of men of science would be at an end. These very words would lose all definite meaning. Every school would have a different standard; matter of assumption in one being matter of demonstration in another; until, at length, GEOMETRY, in the ancient sense of the word, would be altogether frittered away or be only considered as a particular application of Arithmetic and Algebra." It is, perhaps, too early yet to prophesy what will be the ultimate outcome of the new order of things; but it would at least seem possible that history will repeat itself and that, when chaos has come again in geometrical teaching, there will be a return to Euclid more or less complete for the purpose of standardising it once more.

But the case for a new edition of Euclid is independent of any controversies as to how geometry shall be taught to schoolboys. Euclid's work will live long after all the text-books of the present day are superseded and forgotten. It is one of the noblest monuments of antiquity; no mathematician worthy of the name can afford not to know Euclid, the real Euclid as distinct from any revised or rewritten versions which will serve for schoolboys or engineers. And, to know Euclid, it is necessary to know his language, and, so far as it can be traced, the history of the "elements" which he collected in his immortal work.

This brings me to the *raison d'être* of the present edition. A new translation from the Greek was necessary for two reasons. First, though some time has elapsed since the appearance of Heiberg's definitive text and prolegomena, published between 1883 and 1888, there has not been, so far as I know, any attempt to make a faithful translation from it into English even of the Books which are commonly read. And, secondly, the other Books, VII. to X. and XIII., were not included by Simson and the editors who followed him, or apparently in any English translation since Williamson's (1781—8), so that they are now practically inaccessible to English readers in any form.

In the matter of notes, the edition of the first six Books in Greek and Latin with notes by Camerer and Hauber (Berlin, 1824—5) is a perfect mine of information. It would have been practically impossible to make the notes more exhaustive at the time when they were written. But the researches of the last thirty or forty years into the history of mathematics (I need only mention such names as those of Bretschneider, Hankel, Moritz Cantor, Hultsch, Paul Tannery, Zeuthen, Loria, and Heiberg) have put the whole subject upon a different plane. I have endeavoured in this edition to take account of all the main results of these researches up to the present date. Thus, so far as the geometrical Books are concerned, my notes are intended to form a sort of dictionary of the history of elementary geometry, arranged according to subjects; while the notes on the arithmetical Books VII.—IX. and on Book X. follow the same plan.

I desire to express here my thanks to my brother, Dr R. S. Heath, Vice-Principal of Birmingham University, for suggestions on the proof sheets and, in particular, for the reference to the parallelism between Euclid's definition of proportion and Dedekind's theory of irrationals, to Mr R. D. Hicks for advice on a number of difficult points of translation, to Professor A. A. Bevan for help in the transliteration of Arabic names, and to the Curators and Librarian of the Bodleian Library for permission to reproduce, as frontispiece, a page from the famous Bodleian MS. of the *Elements*. Lastly, my best acknowledgments are due to the Syndics of the Cambridge University Press for their ready acceptance of the work, and for the zealous and efficient coöperation of their staff which has much lightened the labour of seeing the book through the Press.

T. L. H.

November, 1908.

PREFACE TO THE SECOND EDITION

I LIKE to think that the exhaustion of the first edition of this work furnishes a new proof (if such were needed) that Euclid is far from being defunct or even dormant, and that, so long as mathematics is studied, mathematicians will find it necessary and worth while to come back again and again, for one purpose or another, to the twenty-two-centuries-old book which, notwithstanding its imperfections, remains the greatest elementary textbook in mathematics that the world is privileged to possess.

The present edition has been carefully revised throughout, and a number of passages (sometimes whole pages) have been rewritten, with a view to bringing it up to date. Some not inconsiderable additions have also been made, especially in the Excursuses to Volume I, which will, I hope, find interested readers.

Since the date of the first edition little has happened in the domain of geometrical teaching which needs to be chronicled. Two distinct movements however call for notice.

The first is a movement having for its object the mitigation of the difficulties (affecting in different ways students, teachers and examiners) which are found to arise from the multiplicity of the different textbooks and varying systems now in use for the teaching of elementary geometry. These difficulties have evoked a widespread desire among teachers for the establishment of an agreed sequence to be generally adopted in teaching the subject. One proposal to this end has already been made: but the chance of the acceptance of an agreed sequence has in the meantime been prejudiced by a second movement which has arisen in other quarters.

I refer to the movement in favour of reviving, in a modified form, the proposal made by Wallis in 1663 to replace Euclid's Parallel-Postulate by a Postulate of Similarity (as to which see pp. 210—11 of Volume I of this work). The form of Postulate now suggested is an assumption that "Given one triangle, there can be constructed, on any arbitrary base, another triangle equiangular with (or similar to) the given triangle." It may perhaps be held that this assumption has the advantage of not referring, in the statement of it, to the fact that a straight line is of unlimited length; but, on the other hand, as is well known, Saccheri showed (1733) that it involves more than is necessary to enable Euclid's Postulate to be proved. In any case it would seem certain that a scheme based upon the proposed Postulate, if made scientifically sound, must be more difficult than the procedure now generally followed. This being so, and having regard to the facts (1) that the difference between the suggested Postulate and that of Euclid is in effect so slight and (2) that the historic interest of Euclid's Postulate is so great, I am of opinion that the proposal is very much to be deprecated.

T. L. H.

December 1925.

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VOLUME I.

INTRODUCTION.

Facsimile of a page of the Bodleian ms. of the *Elements* . *Frontispiece*

This is a facsimile of a page (fol. 45 verso) of the famous Bodleian ms. of the *Elements*, D'Orville 301 (formerly x. 1 inf. 2, 30), written in the year 888. The scholium in the margin, not very difficult to decipher, though some letters are almost rubbed out, is one of the scholia Vaticana given by Heiberg (Vol. v. p. 263) as III. No. 15: *Διὰ τοῦ κέντρου οὐσῶν οὐκ ἦν ζητήσεως ἄξιον, εἰ δίχα τέμνουσιν ἀλλήλας· τὸ γὰρ κέντρον αὐτῶν ἡ διχοτομία. ὁμοίως καὶ ἡ εἰ τῆς ἑτέρας διὰ τοῦ κέντρου οὔσης ἡ ἑτέρα μὴ διὰ τοῦ κέντρου εἴη, ὅτι οὐ δίχα τέμνεται ἡ διὰ τοῦ κέντρου.* The *ἡ* before *εἰ* in the last sentence should be omitted. PFVat. read *ἡ* without *εἰ*. The marginal references lower down are of course to propositions quoted, (1) *διὰ τὸ α′ τοῦ γ′*, "by III. 1," and (2) *διὰ τὸ γ′ τοῦ αὐτοῦ*, "by 3 of the same."

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INTRODUCTION.

CHAPTER I.

EUCLID AND THE TRADITIONS ABOUT HIM.

As in the case of the other great mathematicians of Greece, so in Euclid's case, we have only the most meagre particulars of the life and personality of the man.

Most of what we have is contained in the passage of Proclus' summary relating to him, which is as follows[1]:

"Not much younger than these (sc. Hermotimus of Colophon and Philippus of Medma) is Euclid, who put together the Elements, collecting many of Eudoxus' theorems, perfecting many of Theaetetus', and also bringing to irrefragable demonstration the things which were only somewhat loosely proved by his predecessors. This man lived[2] in the time of the first Ptolemy. For Archimedes, who came immediately after the first (Ptolemy)[3], makes mention of Euclid: and, further, they say that Ptolemy once asked him if there was in geometry any shorter way than that of the elements, and he answered that there was no royal road to geometry[4]. He is then younger than the pupils of Plato but older than Eratosthenes and Archimedes; for the latter were contemporary with one another, as Eratosthenes somewhere says."

This passage shows that even Proclus had no direct knowledge of Euclid's birthplace or of the date of his birth or death. He proceeds by inference. Since Archimedes lived just after the first

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[1] Proclus, ed. Friedlein, p. 68, 6—20.

[2] The word γέγονε must apparently mean "flourished," as Heiberg understands it (*Litterargeschichtliche Studien über Euklid*, 1882, p. 26), not "was born," as Hankel took it: otherwise part of Proclus' argument would lose its cogency.

[3] So Heiberg understands ἐπιβαλὼν τῷ πρώτῳ (sc. Πτολεμαίῳ). Friedlein's text has καὶ between ἐπιβαλὼν and τῷ πρώτῳ; and it is right to remark that another reading is καὶ ἐν τῷ πρώτῳ (without ἐπιβαλών) which has been translated "in his first *book*," by which is understood *On the Sphere and Cylinder* I., where (1) in Prop. 2 are the words "let *BC* be made equal to *D by the second* (proposition) *of the first* of Euclid's (books)," and (2) in Prop. 6 the words "For these things are handed down in the Elements" (without the name of Euclid). Heiberg thinks the former passage is referred to, and that Proclus must therefore have had before him the words "by the second of the first of Euclid": a fair proof that they are genuine, though in themselves they would be somewhat suspicious.

[4] The same story is told in Stobaeus, *Ecl.* (II. p. 228, 30, ed. Wachsmuth) about Alexander and Menaechmus. Alexander is represented as having asked Menaechmus to teach him geometry concisely, but he replied: "O king, through the country there are royal roads and roads for common citizens, but in geometry there is one road for all."

Ptolemy, and Archimedes mentions Euclid, while there is an anecdote about *some* Ptolemy and Euclid, *therefore* Euclid lived in the time of the first Ptolemy.

We may infer then from Proclus that Euclid was intermediate between the first pupils of Plato and Archimedes. Now Plato died in 347/6, Archimedes lived 287–212, Eratosthenes *c.* 284–204 B.C. Thus Euclid must have flourished *c.* 300 B.C., which date agrees well with the fact that Ptolemy reigned from 306 to 283 B.C.

It is most probable that Euclid received his mathematical training in Athens from the pupils of Plato; for most of the geometers who could have taught him were of that school, and it was in Athens that the older writers of elements, and the other mathematicians on whose works Euclid's *Elements* depend, had lived and taught. He may himself have been a Platonist, but this does not follow from the statements of Proclus on the subject. Proclus says namely that he was of the school of Plato and in close touch with that philosophy[1]. But this was only an attempt of a New Platonist to connect Euclid with his philosophy, as is clear from the next words in the same sentence, "for which reason also he set before himself, as the end of the whole Elements, the construction of the so-called Platonic figures." It is evident that it was only an idea of Proclus' own to infer that Euclid was a Platonist because his *Elements* end with the investigation of the five regular solids, since a later passage shows him hard put to it to reconcile the view that the construction of the five regular solids was the end and aim of the *Elements* with the obvious fact that they were intended to supply a foundation for the study of geometry in general, "to make perfect the understanding of the learner in regard to the whole of geometry[2]." To get out of the difficulty he says[3] that, if one should ask him what was the aim (*σκοπός*) of the treatise, he would reply by making a distinction between Euclid's intentions (1) as regards the subjects with which his investigations are concerned, (2) as regards the learner, and would say as regards (1) that "the whole of the geometer's argument is concerned with the cosmic figures." This latter statement is obviously incorrect. It is true that Euclid's *Elements* end with the construction of the five regular solids; but the planimetrical portion has no direct relation to them, and the arithmetical no relation at all; the propositions about them are merely the conclusion of the stereometrical division of the work.

One thing is however certain, namely that Euclid taught, and founded a school, at Alexandria. This is clear from the remark of Pappus about Apollonius[4]: "he spent a very long time with the pupils of Euclid at Alexandria, and it was thus that he acquired such a scientific habit of thought."

It is in the same passage that Pappus makes a remark which might, to an unwary reader, seem to throw some light on the personality of Euclid. He is speaking about Apollonius' preface to the first book of his *Conics*, where he says that Euclid had not completely worked out the synthesis of the "three- and four-line locus," which in fact was not possible without some theorems first discovered by himself. Pappus says on this[1]: "Now Euclid—regarding Aristaeus as deserving credit for the discoveries he had already made in conics, and without anticipating him or wishing to construct anew the same system (such was his scrupulous fairness and his exemplary kindliness towards all who could advance mathematical science to however small an extent), being moreover in no wise contentious and, though exact, yet no braggart like the other [Apollonius]—wrote so much about the locus as was possible by means of the conics of Aristaeus, without claiming completeness for his demonstrations." It is however evident, when the passage is examined in its context, that Pappus is not following any tradition in giving this account of Euclid: he was offended by the terms of Apollonius' reference to Euclid, which seemed to him unjust, and he drew a fancy picture of Euclid in order to show Apollonius in a relatively unfavourable light.

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[1] Proclus, p. 68, 20, *καὶ τῇ προαιρέσει δὲ Πλατωνικός ἐστι καὶ τῇ φιλοσοφίᾳ ταύτῃ οἰκεῖος.*

[2] *ibid.* p. 71, 8.

[3] *ibid.* p. 70, 19 sqq.

[4] Pappus, VII. p. 678, 10—12, *συσχολάσας τοῖς ὑπὸ Εὐκλείδου μαθηταῖς ἐν Ἀλεξανδρείᾳ πλεῖστον χρόνον, ὅθεν ἔσχε καὶ τὴν τοιαύτην ἕξιν οὐκ ἀμαθῆ.*

Another story is told of Euclid which one would like to believe true. According to Stobaeus[2], "some one who had begun to read geometry with Euclid, when he had learnt the first theorem, asked Euclid, 'But what shall I get by learning these things?' Euclid called his slave and said 'Give him threepence, since he must make gain out of what he learns.'"

In the middle ages most translators and editors spoke of Euclid as Euclid *of Megara*. This description arose out of a confusion between our Euclid and the philosopher Euclid of Megara who lived about 400 B.C. The first trace of this confusion appears in Valerius Maximus (in the time of Tiberius) who says[3] that Plato, on being appealed to for a solution of the problem of doubling the cubical altar, sent the inquirers to "Euclid the geometer." There is no doubt about the reading, although an early commentator on Valerius Maximus wanted to correct "Eucliden" into "*Eudoxum*," and this correction is clearly right. But, if Valerius Maximus took Euclid the geometer for a contemporary of Plato, it could only be through confusing him with Euclid of Megara. The first specific reference to Euclid as Euclid of Megara belongs to the 14th century, occurring in the ὑπομνηματισμοί of Theodorus Metochita (d. 1332) who speaks of "Euclid of Megara, the Socratic philosopher, contemporary of Plato," as the author of treatises on plane and solid geometry, data, optics etc.: and a Paris MS. of the 14th century has "Euclidis philosophi Socratici liber elementorum." The misunderstanding was general in the period from Campanus' translation (Venice 1482) to those of Tartaglia (Venice 1565) and Candalla (Paris 1566). But one Constantinus Lascaris (d. about 1493) had already made the proper distinction by saying of our Euclid that "he was different from him of Megara of whom Laertius wrote, and who wrote dialogues"[1]; and to Commandinus belongs the credit of being the first translator[2] to put the matter beyond doubt: "Let us then free a number of people from the error by which they have been induced to believe that our Euclid is the same as the philosopher of Megara" etc.

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[1] Pappus, VII. pp. 676, 25—678, 6. Hultsch, it is true, brackets the whole passage pp. 676, 25—678, 15, but apparently on the ground of the diction only.

[2] Stobaeus, *l.c.*

[3] VIII. 12, ext. 1.

Another idea, that Euclid was born at Gela in Sicily, is due to the same confusion, being based on Diogenes Laertius' description[3] of the philosopher Euclid as being "of Megara, or, according to some, of Gela, as Alexander says in the Διαδοχαί."

In view of the poverty of Greek tradition on the subject even as early as the time of Proclus (410–485 A.D.), we must necessarily take *cum grano* the apparently circumstantial accounts of Euclid given by Arabian authors; and indeed the origin of their stories can be explained as the result (1) of the Arabian tendency to romance, and (2) of misunderstandings.

We read[4] that "Euclid, son of Naucrates, grandson of Zenarchus[5], called the author of geometry, a philosopher of somewhat ancient date, a Greek by nationality domiciled at Damascus, born at Tyre, most learned in the science of geometry, published a most excellent and most useful work entitled the foundation or elements of geometry, a subject in which no more general treatise existed before among the Greeks: nay, there was no one even of later date who did not walk in his footsteps and frankly profess his doctrine. Hence also Greek, Roman and Arabian geometers not a few, who undertook the task of illustrating this work, published commentaries, scholia, and notes upon it, and made an abridgment of the work itself. For this reason the Greek philosophers used to post up on the doors of their schools the well-known notice: 'Let no one come to our school, who has not first learned the elements of Euclid.'" The details at the beginning of this extract cannot be derived from Greek sources, for even Proclus did not know anything about Euclid's father, while it was not the Greek habit to record the names of grandfathers, as the Arabians commonly did. Damascus and Tyre were no doubt brought in to gratify a desire which the Arabians always showed to connect famous Greeks in some way or other with the East. Thus Naṣīraddīn, the translator of the *Elements*, who was of Ṭūs in Khurāsān, actually makes Euclid out to have been "Thusinus" also[6]. The readiness of the Arabians to run away with an idea is illustrated by the last words of the extract. Everyone knows the story of Plato's inscription over the porch of the Academy: "let no one unversed in geometry enter my doors"; the Arab turned geometry into *Euclid's* geometry, and told the story of Greek philosophers in general and "*their* Academies."

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[1] Letter to Fernandus Acuna, printed in Maurolycus, *Historia Siciliae*, fol. 21 r. (see Heiberg, *Euklid-Studien*, pp. 22—3, 25).

[2] Preface to translation (Pisauri, 1572).

[3] Diog. L. II. 106, p. 58 ed. Cobet.

[4] Casiri, *Bibliotheca Arabico-Hispana Escurialensis*, I. p. 339. Casiri's source is al-Qifṭī (d. 1248), the author of the *Ta'rīkh al-Ḥukamā*, a collection of biographies of philosophers, mathematicians, astronomers etc.

[5] The *Fihrist* says "son of Naucrates, the son of Berenice (?)" (see Suter's translation in *Abhandlungen zur Gesch. d. Math.* VI. Heft, 1892, p. 16).

[6] The same predilection made the Arabs describe Pythagoras as a pupil of the wise Salomo, Hipparchus as the exponent of Chaldaean philosophy or as the Chaldaean, Archimedes as an Egyptian etc. (Ḥājī Khalfa, *Lexicon Bibliographicum*, and Casiri).

Equally remarkable are the Arabian accounts of the relation of Euclid and Apollonius[1]. According to them the *Elements* were originally written, not by Euclid, but by a man whose name was Apollonius, a carpenter, who wrote the work in 15 books or sections[2]. In the course of time some of the work was lost and the rest became disarranged, so that one of the kings at Alexandria who desired to study geometry and to master this treatise in particular first questioned about it certain learned men who visited him and then sent for Euclid who was at that time famous as a geometer, and asked him to revise and complete the work and reduce it to order. Euclid then re-wrote it in 13 books which were thereafter known by his name. (According to another version Euclid composed the 13 books out of commentaries which he had published on two books of Apollonius on conics and out of introductory matter added to the doctrine of the five regular solids.) To the thirteen books were added two more books, the work of others (though some attribute these also to Euclid) which contain several things not mentioned by Apollonius. According to another version Hypsicles, a pupil of Euclid at Alexandria, offered to the king and published Books XIV. and XV., it being also stated that Hypsicles had "discovered" the books, by which it appears to be suggested that Hypsicles had edited them from materials left by Euclid.

We observe here the correct statement that Books XIV. and XV. were not written by Euclid, but along with it the incorrect information that Hypsicles, the author of Book XIV., wrote Book XV. also.

The whole of the fable about Apollonius having preceded Euclid and having written the *Elements* appears to have been evolved out of the preface to Book XIV. by Hypsicles, and in this way; the Book must in early times have been attributed to Euclid, and the inference based upon this assumption was left uncorrected afterwards when it was recognised that Hypsicles was the author. The preface is worth quoting:

"Basilides of Tyre, O Protarchus, when he came to Alexandria and met my father, spent the greater part of his sojourn with him on account of their common interest in mathematics. And once, when examining the treatise written by Apollonius about the comparison between the dodecahedron and the icosahedron inscribed in the same sphere, (showing) what ratio they have to one another, they thought that Apollonius had not expounded this matter properly, and accordingly they emended the exposition, as I was able to learn from my father. And I myself, later, fell in with another book published by Apollonius, containing a demonstration relating to the subject, and I was greatly interested in the investigation of the problem. The book published by Apollonius is accessible to all—for it has a large circulation, having apparently been carefully written out later—but I decided to send you the comments which seem to me to be necessary, for you will through your proficiency in mathematics in general and in geometry in particular form an expert judgment on what I am about to say, and you will lend a kindly ear to my disquisition for the sake of your friendship to my father and your goodwill to me."

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[1] The authorities for these statements quoted by Casiri and Ḥājī Khalfa are al-Kindī's tract *de instituto libri Euclidis* (al-Kindī died about 873) and a commentary by Qāḍīzāde ar-Rūmī (d. about 1440) on a book called *Ashkāl at-ta' sīs* (fundamental propositions) by Ashraf Shamsaddīn as-Samarqandī (*c.* 1276) consisting of elucidations of 35 propositions selected from the first books of Euclid. Naṣīraddīn likewise says that Euclid cut out two of 15 books of elements then existing and published the rest under his own name. According to Qāḍīzāde the king heard that there was a celebrated geometer named Euclid at *Tyre*: Naṣīr-addīn says that he sent for Euclid of Ṭūs.

[2] So says the *Fihrist*. Suter (*op. cit.* p. 49) thinks that the author of the *Fihrist* did not suppose Apollonius *of Perga* to be the writer of the *Elements*, as later Arabian authorities did, but that he distinguished another Apollonius whom he calls "a carpenter." Suter's argument is based on the fact that the *Fihrist's* article on Apollonius (of Perga) says nothing of the *Elements*; and that it gives the three great mathematicians, Euclid, Archimedes and Apollonius, in the correct chronological order.

The idea that Apollonius preceded Euclid must evidently have been derived from the passage just quoted. It explains other things besides. Basilides must have been confused with βασιλεύς, and we have a probable explanation of the "Alexandrian king," and of the "learned men who visited" Alexandria. It is possible also that in the "Tyrian" of Hypsicles' preface we have the origin of the notion that Euclid was born in Tyre. These inferences argue, no doubt, very defective knowledge of Greek: but we could expect no better from those who took the *Organon* of Aristotle to be "instrumentum musicum pneumaticum," and who explained the name of Euclid, which they variously pronounced as *Uclides* or *Icludes*, to be compounded of *Ucli* a key, and *Dis* a measure, or, as some say, geometry, so that *Uclides* is equivalent to the *key of geometry*!

Lastly the alternative version, given in brackets above, which says that Euclid made the *Elements* out of commentaries which he wrote on two books of Apollonius on conics and prolegomena added to the doctrine of the five solids, seems to have arisen, through a like confusion, out of a later passage[1] in Hypsicles' Book XIV.: "And this is expounded by Aristaeus in the book entitled 'Comparison of the five figures,' and by Apollonius in the second edition of his comparison of the dodecahedron with the icosahedron." The "doctrine of the five solids" in the Arabic must be the "Comparison of the five figures" in the passage of Hypsicles, for nowhere else have we any information about a work bearing this title, nor can the Arabians have had. The reference to the *two books* of Apollonius on *conics* will then be the result of mixing up the fact that Apollonius wrote a book on conics with the *second edition* of the other work mentioned by Hypsicles. We do not find elsewhere in Arabian authors any mention of a commentary by Euclid on Apollonius and Aristaeus: so that the story in the passage quoted is really no more than a variation of the fable that the *Elements* were the work of Apollonius.

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[1] Heiberg's Euclid, vol. V. p. 6.

CHAPTER II.

EUCLID'S OTHER WORKS.

IN giving a list of the Euclidean treatises other than the *Elements*, I shall be brief: for fuller accounts of them, or speculations with regard to them, reference should be made to the standard histories of mathematics[1].

I will take first the works which are mentioned by Greek authors.

1. The *Pseudaria*.

I mention this first because Proclus refers to it in the general remarks in praise of the *Elements* which he gives immediately after the mention of Euclid in his summary. He says[2]: "But, inasmuch as many things, while appearing to rest on truth and to follow from scientific principles, really tend to lead one astray from the principles and deceive the more superficial minds, he has handed down methods for the discriminative understanding of these things as well, by the use of which methods we shall be able to give beginners in this study practice in the discovery of paralogisms, and to avoid being misled. This treatise, by which he puts this machinery in our hands, he entitled (the book) of Pseudaria, enumerating in order their various kinds, exercising our intelligence in each case by theorems of all sorts, setting the true side by side with the false, and combining the refutation of error with practical illustration. This book then is by way of cathartic and exercise, while the Elements contain the irrefragable and complete guide to the actual scientific investigation of the subjects of geometry."

The book is considered to be irreparably lost. We may conclude however from the connexion of it with the *Elements* and the reference to its usefulness for beginners that it did not go outside the domain of elementary geometry[3].

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[1] See, for example, Loria, *Le scienze esatte nell' antica Grecia*, 1914, pp. 245—268; T. L. Heath, *History of Greek Mathematics*, 1921, I. pp. 421—446. Cf. Heiberg, *Litterargeschichtliche Studien über Euklid*, pp. 36—153; *Euclidis opera omnia*, ed. Heiberg and Menge, Vols. VI.—VIII.

[2] Proclus, p. 70, 1—18.

[3] Heiberg points out that Alexander Aphrodisiensis appears to allude to the work in his commentary on Aristotle's *Sophistici Elenchi* (fol. 25 *b*): "Not only those (ἔλεγχοι) which do not start from the principles of the science under which the problem is classed...but also those which do start from the proper principles of the science but in some respect admit a paralogism, e.g. the *Pseudographemata* of Euclid." Tannery (*Bull. des sciences math. et astr.* 2^e Série, VI., 1882, 1ère Partie, p. 147) conjectures that it may be from this treatise that the same commentator got his information about the quadratures of the circle by Antiphon and

2. The *Data*.

The *Data* (δεδομένα) are included by Pappus in the *Treasury of Analysis* (τόπος ἀναλυόμενος), and he describes their contents[1]. They are still concerned with elementary geometry, though forming part of the introduction to higher analysis. Their form is that of propositions proving that, if certain things in a figure are given (in magnitude, in species, etc.), something else is given. The subject-matter is much the same as that of the planimetrical books of the *Elements*, to which the *Data* are often supplementary. We shall see this later when we come to compare the propositions in the *Elements* which give us the means of solving the general quadratic equation with the corresponding propositions of the *Data* which give the solution. The *Data* may in fact be regarded as elementary exercises in analysis.

It is not necessary to go more closely into the contents, as we have the full Greek text and the commentary by Marinus newly edited by Menge and therefore easily accessible[2].

3. The book *On divisions* (*of figures*).

This work (περὶ διαιρέσεων βιβλίον) is mentioned by Proclus[3]. In one place he is speaking of the conception or definition (λόγος) of *figure*, and of the divisibility of a figure into others differing from it in kind; and he adds: "For the circle is divisible into parts unlike in definition or notion (ἀνόμοια τῷ λόγῳ), and so is each of the rectilineal figures; this is in fact the business of the writer of the Elements in his Divisions, where he divides given figures, in one case into like figures, and in another into unlike[4]." "Like" and "unlike" here mean, not "similar" and "dissimilar" in the technical sense, but "like" or "unlike *in definition* or *notion*" (λόγῳ): thus to divide a triangle into triangles would be to divide it into "like" figures, to divide a triangle into a triangle and a quadrilateral would be to divide it into "unlike" figures.

The treatise is lost in Greek but has been discovered in the Arabic. First John Dee discovered a treatise *De divisionibus* by one Muḥammad Bagdadinus[5] and handed over a copy of it (in Latin) in 1563 to Commandinus, who published it, in Dee's name and his own, in 1570[6]. Dee did not himself translate the tract from the Arabic; he

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Bryson, to say nothing of the lunules of Hippocrates. I think however that there is an objection to this theory so far as regards Bryson; for Alexander distinctly says that Bryson's quadrature did *not* start from the proper principles of geometry, but from some principles more general.

[1] Pappus, VII. p. 638.

[2] Vol. VI. in the Teubner edition of *Euclidis opera omnia* by Heiberg and Menge. A translation of the *Data* is also included in Simson's Euclid (though naturally his text left much to be desired).

[3] Proclus, p. 69, 4.

[4] *ibid.* 144, 22—26.

[5] Steinschneider places him in the 10th c. H. Suter (*Bibliotheca Mathematica*, IV$_3$, 1903, pp. 24, 27) identifies him with Abū (Bekr) Muḥ. b. ʿAbdalbāqī al-Bagdādī, Qāḍī (Judge) of Māristān (*circa* 1070–1141), to whom he also attributes the *Liber judei* (? judicis) *super decimum Euclidis* translated by Gherard of Cremona.

[6] *De superficierum divisionibus liber Machometo Bagdadino adscriptus, nunc primum Ioannis Dee Londinensis et Federici Commandini Urbinatis opera in lucem editus*, Pisauri, 1570, afterwards included in Gregory's Euclid (Oxford, 1703).

found it in Latin in a MS. which was then in his own possession but was about 20 years afterwards stolen or destroyed in an attack by a mob on his house at Mortlake[1]. Dee, in his preface addressed to Commandinus, says nothing of his having *translated* the book, but only remarks that the very illegible MS. had caused him much trouble and (in a later passage) speaks of "the actual, very ancient, copy from which I *wrote out*..." (in ipso unde descripsi vetustissimo exemplari). The Latin translation of this tract from the Arabic was probably made by Gherard of Cremona (1114–1187), among the list of whose numerous translations a "liber divisionum" occurs. The Arabic original cannot have been a direct translation from Euclid, and probably was not even a direct adaptation of it; it contains mistakes and unmathematical expressions, and moreover does not contain the propositions about the division of a circle alluded to by Proclus. Hence it can scarcely have contained more than a fragment of Euclid's work.

But Woepcke found in a MS. at Paris a treatise in Arabic on the division of figures, which he translated and published in 1851[2]. It is expressly attributed to Euclid in the MS. and corresponds to the description of it by Proclus. Generally speaking, the divisions are divisions into figures of the same kind as the original figures, e.g. of triangles into triangles; but there are also divisions into "unlike" figures, e.g. that of a triangle by a straight line parallel to the base. The missing propositions about the division of a circle are also here: "to divide into two equal parts a given figure bounded by an arc of a circle and two straight lines including a given angle" and "to draw in a given circle two parallel straight lines cutting off a certain part of the circle." Unfortunately the proofs are given of only four propositions (including the two last mentioned) out of 36, because the Arabic translator found them too easy and omitted them. To illustrate the character of the problems dealt with I need only take one more example: "To cut off a certain fraction from a (parallel-) trapezium by a straight line which passes through a given point lying inside or outside the trapezium but so that a straight line can be drawn through it cutting both the parallel sides of the trapezium." The genuineness of the treatise edited by Woepcke is attested by the facts that the four proofs which remain are elegant and depend on propositions in the *Elements*, and that there is a lemma with a true Greek ring: "to apply to a straight line a rectangle equal to the rectangle contained by *AB*, *AC and deficient by a square*." Moreover the treatise is no fragment, but finishes with the words "end of the treatise," and is a well-ordered and compact whole. Hence we may safely conclude that Woepcke's is not only Euclid's own work but the whole of it. A restoration of the work, with proofs, was attempted by Ofterdinger[3], who however does not give Woepcke's props. 30, 31, 34, 35, 36. We have now a satisfactory restoration, with ample notes and an introduction, by R. C. Archibald, who used for the purpose Woepcke's text and a section of Leonardo of Pisa's *Practica geometriae* (1220)[1].

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[1] R. C. Archibald, *Euclid's Book on the Division of Figures with a restoration based on Woepcke's text and on the Practica geometriae of Leonardo Pisano*, Cambridge, 1915, pp. 4—9.

[2] *Journal Asiatique*, 1851, p. 233 sqq.

[3] L. F. Ofterdinger, *Beiträge zur Wiederherstellung der Schrift des Euklides über die Theilung der Figuren*, Ulm, 1853.

4. The *Porisms*.

It is not possible to give in this place any account of the controversies about the contents and significance of the three lost books of Porisms, or of the important attempts by Robert Simson and Chasles to restore the work. These may be said to form a whole literature, references to which will be found most abundantly given by Heiberg and Loria, the former of whom has treated the subject from the philological point of view, most exhaustively, while the latter, founding himself generally on Heiberg, has added useful details, from the mathematical side, relating to the attempted restorations, etc.[2] It must suffice here to give an extract from the only original source of information about the nature and contents of the *Porisms*, namely Pappus[3]. In his general preface about the books composing the *Treasury of Analysis* (τόπος ἀναλυόμενος) he says:

"After the Tangencies (of Apollonius) come, in three books, the Porisms of Euclid, [in the view of many] a collection most ingeniously devised for the analysis of the more weighty problems, [and] although nature presents an unlimited number of such porisms[4], [they have added nothing to what was written originally by Euclid, except that some before my time have shown their want of taste by adding to a few (of the propositions) second proofs, each (proposition) admitting of a definite number of demonstrations, as we have shown, and Euclid having given one for each, namely that which is the most lucid. These porisms embody a theory subtle, natural, necessary, and of considerable generality, which is fascinating to those who can see and produce results].

"Now all the varieties of porisms belong, neither to theorems nor problems, but to a species occupying a sort of intermediate position [so that their enunciations can be formed like those of either theorems or problems], the result being that, of the great number of geometers, some regarded them as of the class of theorems, and others of problems, looking only to the form of the proposition. But that the ancients knew better the difference between these three things is clear from the definitions. For they said that a theorem is that which is proposed with a view to the demonstration of the very thing proposed, a problem that which is thrown out with a view to the construction of the very thing proposed, and a porism that which is proposed with a view to the producing of the very thing proposed. [But this definition of the porism was changed by the more recent writers who could not produce everything, but used these elements and proved only the fact that that which is sought really exists, but did not produce it[1] and were accordingly confuted by the definition and the whole doctrine. They based their definition on an incidental characteristic, thus: A porism is that which falls short of a locus-theorem in respect of its hypothesis[2]. Of this kind of porisms loci are a species, and they abound in the Treasury of Analysis; but this species has been collected, named and handed down separately from the porisms, because it is more widely diffused than the other species]. But it has further become characteristic of porisms that, owing to their complication, the enunciations are put in a contracted form, much being by usage left to be understood; so that many geometers understand them only in a partial way and are ignorant of the more essential features of their contents.

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[1] There is a remarkable similarity between the propositions of Woepcke's text and those of Leonardo, suggesting that Leonardo may have had before him a translation (perhaps by Gherard of Cremona) of the Arabic tract.

[2] Heiberg, *Euklid-Studien*, pp. 56—79, and Loria, *op. cit.*, pp. 253—265.

[3] Pappus, ed. Hultsch, VII. pp. 648—660. I put in square brackets the words bracketed by Hultsch.

[4] I adopt Heiberg's reading of a comma here instead of a full stop.

"[Now to comprehend a number of propositions in one enunciation is by no means easy in these porisms, because Euclid himself has not in fact given many of each species, but chosen, for examples, one or a few out of a great multitude[3]. But at the beginning of the first book he has given some propositions, to the number of ten, of one species, namely that more fruitful species consisting of loci.] Consequently, finding that these admitted of being comprehended in one enunciation, we have set it out thus:

> If, in a system of four straight lines[4] which cut each other two and two, three points on one straight line be given while the rest except one lie on different straight lines given in position, the remaining point also will lie on a straight line given in position[5].

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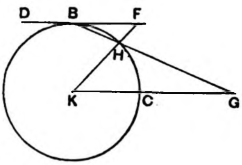
[1] Heiberg points out that Props. 5—9 of Archimedes' treatise *On Spirals* are porisms in this sense. To take Prop. 5 as an example, DBF is a tangent to a circle with centre K. It is then possible, says Archimedes, to draw a straight line KHF, meeting the circumference in H and the tangent in F, such that

$$FH : HK < (\text{arc } BH) : c,$$

where c is the circumference of *any* circle. To prove this he assumes the following construction. E being any straight line greater than c, he says: let KG be parallel to DF, "and let the line GH equal to E be placed *verging* to the point B." Archimedes must of course have known how to effect this construction, which requires conics. But that it is *possible* requires very little argument, for if we draw any straight line BHG meeting the circle in H and KG in G, it is obvious that as G moves away from C, HG becomes greater and greater and may be made as great as we please. The "later writers" would no doubt have contented themselves with this consideration without actually *constructing* HG.



[2] As Heiberg says, this translation is made certain by a preceding passage of Pappus (p. 648, 1—3) where he compares two enunciations, the latter of which "falls short of the former in *hypothesis* but goes beyond it in *requirement*." E.g. the first enunciation requiring us, given three circles, to draw a circle touching all three, the second may require us, given only *two* circles (one less datum), to draw a circle touching them and *of a given size* (an extra requirement).

[3] I translate Heiberg's reading with a full stop here followed by πρὸς ἀρχῇ δὲ ὅμως [πρὸς ἀρχὴν (δεδομένον) Hultsch] τοῦ πρώτου βιβλίου....

[4] The four straight lines are described in the text as (the sides) ὑπτίου ἢ παρυπτίου, i.e. sides of two sorts of quadrilaterals which Simson tries to explain (see p. 120 of the *Index Graecitatis* of Hultsch's edition of Pappus).

[5] In other words (Chasles, p. 23; Loria, p. 256), if a triangle be so deformed that each of its sides turns about one of three points in a straight line, and two of its vertices lie on two straight lines given in position, the third vertex will also lie on a straight line.

"This has only been enunciated of four straight lines, of which not more than two pass through the same point, but it is not known (to most people) that it is true of any assigned number of straight lines if enunciated thus:

> If any number of straight lines cut one another, not more than two (passing) through the same point, and all the points (of intersection situated) on one of them be given, and if each of those which are on another (of them) lie on a straight line given in position—

or still more generally thus:

> if any number of straight lines cut one another, not more than two (passing) through the same point, and all the points (of intersection situated) on one of them be given, while of the other points of intersection in multitude equal to a triangular number a number corresponding to the side of this triangular number lie respectively on straight lines given in position, provided that of these latter points no three are at the angular points of a triangle (*sc.* having for sides three of the given straight lines)—each of the remaining points will lie on a straight line given in position[1].

"It is probable that the writer of the Elements was not unaware of this but that he only set out the principle; and he seems, in the case of all the porisms, to have laid down the principles and the seed only [of many important things], the kinds of which should be distinguished according to the differences, not of their hypotheses, but of the results and the things sought. [All the hypotheses are different from one another because they are entirely special, but each of the results and things sought, being one and the same, follow from many different hypotheses.]

"We must then in the first book distinguish the following kinds of things sought:

"At the beginning of the book[2] is this proposition:

> I. '*If from two given points straight lines be drawn meeting on a straight line given in position, and one cut off from a straight line given in position (a segment measured) to a given point on it, the other will also cut off from another (straight line a segment) having to the first a given ratio.*'

"Following on this (we have to prove)

> II. that such and such a point lies on a straight line given in position;
>
> III. that the ratio of such and such a pair of straight lines is given;"
>
> etc. etc. (up to XXIX.).

"The three books of the porisms contain 38 lemmas; of the theorems themselves there are 171."

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[1] Loria (p. 256, *n.* 3) gives the meaning of this as follows, pointing out that Simson was the discoverer of it: "If a complete n-lateral be deformed so that its sides respectively turn about n points on a straight line, and $(n-1)$ of its $n(n-1)/2$ vertices move on as many straight lines, the other $(n-1)(n-2)/2$ of its vertices likewise move on as many straight lines: but it is necessary that it should be impossible to form with the $(n-1)$ vertices any triangle having for sides the sides of the polygon."

[2] Reading, with Heiberg, *τοῦ βιβλίου* [*τοῦ ζʹ* Hultsch].

Pappus further gives lemmas to the *Porisms* (pp. 866—918, ed. Hultsch).

With Pappus' account of Porisms must be compared the passages of Proclus on the same subject. Proclus distinguishes two senses in which the word *πόρισμα* is used. The first is that of *corollary* where something appears as an incidental result of a proposition, obtained without trouble or special seeking, a sort of bonus which the investigation has presented us with[1]. The other sense is that of Euclid's *Porisms*[2]. In this sense[3] "*porism* is the name given to things which are sought, but need some finding and are neither pure bringing into existence nor simple theoretic argument. For (to prove) that the angles at the base of isosceles triangles are equal is a matter of theoretic argument, and it is with reference to things existing that such knowledge is (obtained). But to bisect an angle, to construct a triangle, to cut off, or to place—all these things demand the making of something; and to find the centre of a given circle, or to find the greatest common measure of two given commensurable magnitudes, or the like, is in some sort between theorems and problems. For in these cases there is no bringing into existence of the things sought, but finding of them, nor is the procedure purely theoretic. For it is necessary to bring that which is sought into view and exhibit it to the eye. Such are the porisms which Euclid wrote, and arranged in three books of Porisms."

Proclus' definition thus agrees well enough with the first, "older," definition of Pappus. A porism occupies a place between a theorem and a problem: it deals with something already *existing*, as a theorem does, but has to *find* it (e.g. the centre of a circle), and, as a certain operation is therefore necessary, it partakes to that extent of the nature of a problem, which requires us to construct or produce something not previously existing. Thus, besides III. 1 of the *Elements* and X. 3, 4 mentioned by Proclus, the following propositions are real porisms: III. 25, VI. 11—13, VII. 33, 34, 36, 39, VIII. 2, 4, X. 10, XIII. 18. Similarly in Archimedes *On the Sphere and Cylinder* I. 2—6 might be called porisms.

The enunciation given by Pappus as comprehending ten of Euclid's propositions may not reproduce the *form* of Euclid's enunciations; but, comparing the result to be proved, that certain points lie on straight lines given in position, with the *class* indicated by II. above, where the question is of such and such a point lying on a straight line given in position, and with other classes, e.g. (V.) that such and such a line is given in position, (VI.) that such and such a line verges to a given point, (XXVII.) that there exists a given point such that straight lines drawn from it to such and such (circles) will contain a triangle given in species, we may conclude that a usual form of a porism was "to prove that it is possible to find a point with such and such a property" or "a straight line on which lie all the points satisfying given conditions" etc.

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[1] Proclus, pp. 212, 14; 301, 22.

[2] *ibid.* p. 212, 12. "The term porism is used of certain problems, like the *Porisms* written by Euclid."

[3] *ibid.* pp. 301, 25 sqq.

Simson defined a porism thus: "Porisma est propositio in qua proponitur demonstrare rem aliquam, vel plures datas esse, cui, vel quibus, ut et cuilibet ex rebus innumeris, non quidem datis, sed quae ad ea quae data sunt eandem habent relationem, convenire ostendendum est affectionem quandam communem in propositione descriptam[1]."

From the above it is easy to understand Pappus' statement that *loci* constitute a large class of porisms. A *locus* is well defined by Simson thus: "Locus est propositio in qua propositum est datam esse demonstrare, vel invenire lineam aut superficiem cuius quodlibet punctum, vel superficiem in qua quaelibet linea data lege descripta, communem quandam habet proprietatem in propositione descriptam." Heiberg cites an excellent instance of a *locus* which is a *porism*, namely the following proposition quoted by Eutocius[2] from the *Plane Loci* of Apollonius:

"Given two points in a plane, and a ratio between unequal straight lines, it is possible to draw, in the plane, a circle such that the straight lines drawn from the given points to meet on the circumference of the circle have (to one another) a ratio the same as the given ratio."

A difficult point, however, arises on the passage of Pappus, which says that a porism is "that which, in respect of its hypothesis, falls short of a locus-theorem" (*τοπικοῦ θεωρήματος*). Heiberg explains it by comparing the porism from Apollonius' *Plane Loci* just given with Pappus' enunciation of the same thing, to the effect that, if from two given points two straight lines be drawn meeting in a point, and these straight lines have to one another a given ratio, the point will lie on either a straight line or a circumference of a circle given in position. Heiberg observes that in this latter enunciation something is taken into the hypothesis which was not in the hypothesis of the enunciation of the porism, viz. "that the ratio of the straight lines is the same." I confess this does not seem to me satisfactory: for there is no real difference between the enunciations, and the supposed difference in hypothesis is very like playing with words. Chasles says: "*Ce qui constitue le porisme est ce qui manque à l'hypothèse d'un théorème local* (en d'autres termes, le porisme est inférieur, par l'hypothèse, au théorème local; c'est-à-dire que quand quelques parties d'une proposition locale n'ont pas dans l'énoncé la détermination qui leur est propre, cette proposition cesse d'être regardée comme un théorème et devient un porisme)." But the subject still seems to require further elucidation.

While there is so much that is obscure, it seems certain (1) that the *Porisms* were distinctly part of higher geometry and not of elementary geometry, (2) that they contained propositions belonging to the modern theory of transversals and to projective geometry. It should be remembered too that it was in the course of his researches on this subject that Chasles was led to the idea of *anharmonic ratios*.

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[1] This was thus expressed by Chasles: "Le porisme est une proposition dans laquelle on demande de démontrer qu'une chose ou plusieurs choses sont *données*, qui, ainsi que l'une quelconque d'une infinité d'autres choses non données, mais dont chacune est avec des choses données dans une même relation, ont une certaine propriété commune, décrite dans la proposition."

[2] Commentary on Apollonius' *Conics* (vol. II. p. 180, ed. Heiberg).

Lastly, allusion should be made to the theory of Zeuthen[1] on the subject of the porisms. He observes that the only porism of which Pappus gives the complete enunciation, "If from two given points straight lines be drawn meeting on a straight line given in position, and one cut off from a straight line given in position (a segment measured) towards a given point on it, the other will also cut off from another (straight line a segment) bearing to the first a given ratio," is also true if there be substituted for the first given straight line a conic regarded as the "locus with respect to four lines," and that this extended porism can be used for completing Apollonius' exposition of that locus. Zeuthen concludes that the *Porisms* were in part by-products of the theory of conics and in part auxiliary means for the study of conics, and that Euclid called them by the same name as that applied to corollaries because they were corollaries with respect to conics. But there appears to be no evidence to confirm this conjecture.

5. The *Surface-loci* (τόποι πρὸς ἐπιφανείᾳ).

The two books on this subject are mentioned by Pappus as part of the *Treasury of Analysis*[2]. As the other works in the list which were on plane subjects dealt only with straight lines, circles, and conic sections, it is *a priori* likely that among the loci in this treatise (loci which are surfaces) were included such loci as were cones, cylinders and spheres. Beyond this all is conjecture based on two lemmas given by Pappus in connexion with the treatise.

(1) The first of these lemmas[3] and the figure attached to it are not satisfactory as they stand, but a possible restoration is indicated by Tannery[4]. If the latter is right, it suggests that one of the loci contained all the points on the elliptical parallel sections of a cylinder and was therefore an oblique circular cylinder. Other assumptions with regard to the conditions to which the lines in the figure may be subject would suggest that other loci dealt with were cones regarded as containing all points on particular elliptical parallel sections of the cones[5].

(2) In the second lemma Pappus states and gives a complete proof of the focus-and-directrix property of a conic, viz. that *the locus of a point whose distance from a given point is in a given ratio to its distance from a fixed line is a conic section, which is an ellipse, a parabola or a hyperbola according as the given ratio is less than, equal to, or greater than unity*[6]. Two conjectures are possible as to the application of this theorem in Euclid's *Surface-loci*. (*a*) It may have been used to prove that the locus of a point whose distance from a given straight

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[1] *Die Lehre von den Kegelschnitten im Altertum*, chapter VIII.

[2] Pappus, VII. p. 636.

[3] *ibid.* VII. p. 1004.

[4] *Bulletin des sciences math. et astron.*, 2ᵉ Série, VI. 149.

[5] Further particulars will be found in *The Works of Archimedes*, pp. lxii—lxiv, and in Zeuthen, *Die Lehre von den Kegelschnitten*, p. 425 sqq.

[6] Pappus, VII. pp. 1006—1014, and Hultsch's Appendix, pp. 1270—3.

line is in a given ratio to its distance from a given plane is a certain cone. (*b*) It may have been used to prove that the locus of a point whose distance from a given point is in a given ratio to its distance from a given plane is the surface formed by the revolution of a conic about its major or conjugate axis[1]. Thus Chasles may have been correct in his conjecture that the *Surface-loci* dealt with surfaces of revolution of the second degree and sections of the same[2].

6. The *Conics*.

Pappus says of this lost work: "The four books of Euclid's Conics were completed by Apollonius, who added four more and gave us eight books of Conics[3]." It is probable that Euclid's work was lost even by Pappus' time, for he goes on to speak of "Aristaeus, who wrote the *still extant* five books of Solid Loci connected with the conics." Speaking of the relation of Euclid's work to that of Aristaeus on conics regarded as loci, Pappus says in a later passage (bracketed however by Hultsch) that Euclid, regarding Aristaeus as deserving credit for the discoveries he had already made in conics, did not (try to) anticipate him or construct anew the same system. We may no doubt conclude that the book by Aristaeus on solid loci preceded Euclid's on conics and was, at least in point of originality, more important. Though both treatises dealt with the same subject-matter, the object and the point of view were different; had they been the same, Euclid could scarcely have refrained, as Pappus says he did, from attempting to improve upon the earlier treatise. No doubt Euclid wrote on the general theory of conics as Apollonius did, but confined himself to those properties which were necessary for the analysis of the *Solid Loci* of Aristaeus. The *Conics* of Euclid were evidently superseded by the treatise of Apollonius.

As regards the contents of Euclid's *Conics*, the most important source of our information is Archimedes, who frequently refers to propositions in conics as well known and not needing proof, adding in three cases that they are proved in the "elements of conics" or in "the conics," which expressions must clearly refer to the works of Aristaeus and Euclid[4]

Euclid still used the old names for the conics (sections of a right-angled, acute-angled, or obtuse-angled cone), but he was aware that an ellipse could be obtained by cutting a cone in any manner by a plane not parallel to the base (assuming the section to lie wholly between the apex of the cone and its base) and also by cutting a cylinder. This is expressly stated in a passage from the *Phaenomena* of Euclid about to be mentioned[5].

7. The *Phaenomena*.

This is an astronomical work and is still extant. A much interpolated version appears in Gregory's Euclid. An earlier and better recension is however contained in the MS. Vindobonensis philos. Gr. 103, though the end of the treatise, from the middle of prop. 16 to the last (18), is missing. The book, now edited by Menge[1], consists of propositions in *spheric* geometry. Euclid based it on Autolycus' work *περὶ κινουμένης σφαίρας*, but also, evidently, on an earlier textbook of *Sphaerica* of exclusively mathematical content. It has been conjectured that the latter textbook may have been due to Eudoxus[2].

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[1] For further details see *The Works of Archimedes*, pp. lxiv, lxv, and Zeuthen, *l. c.*

[2] *Aperçu historique*, pp. 273—4.

[3] Pappus, VII. p. 672.

[4] For details of these propositions see my *Apollonius of Perga*, pp. xxxv, xxxvi.

[5] *Phaenomena*, ed. Menge, p. 6: "If a cone or a cylinder be cut by a plane not parallel to the base, the section is a section of an acute-angled cone, which is like a shield (θυρεός)."

8. The *Optics*.

This book needs no description, as it has been edited by Heiberg recently[3], both in its genuine form and in the recension by Theon. The *Catoptrica* published by Heiberg in the same volume is not genuine, and Heiberg suspects that in its present form it may be Theon's. It is not even certain that Euclid wrote *Catoptrica* at all, as Proclus may easily have had Theon's work before him and inadvertently assigned it to Euclid[4].

9. Besides the above-mentioned works, Euclid is said to have written the *Elements of Music*[5] (*αἱ κατὰ μουσικὴν στοιχειώσεις*). Two treatises are attributed to Euclid in our MSS. of the *Musici*, the *κατατομὴ κανόνος*, *Sectio canonis* (the theory of the intervals), and the *εἰσαγωγὴ ἁρμονική* (introduction to harmony)[6]. The first, resting on the Pythagorean theory of music, is mathematical, and the style and diction as well as the form of the propositions mostly agree with what we find in the *Elements*. Jan thought it genuine, especially as almost the whole of the treatise (except the preface) is quoted *in extenso*, and Euclid is twice mentioned by name, in the commentary on Ptolemy's *Harmonica* published by Wallis and attributed by him to Porphyry. Tannery was of the opposite opinion[7]. The latest editor, Menge, suggests that it may be a redaction by a less competent hand from the genuine Euclidean *Elements of Music*. The second treatise is not Euclid's, but was written by Cleonides, a pupil of Aristoxenus[8].

Lastly, it is worth while to give the Arabians' list of Euclid's works. I take this from Suter's translation of the list of philosophers and mathematicians in the *Fihrist*, the oldest authority of the kind that we possess[9]. "To the writings of Euclid belong further [in addition to the *Elements*]: the book of Phaenomena; the book of

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[1] *Euclidis opera omnia*, vol. VIII., 1916, pp. 2—156.

[2] Heiberg, *Euklid-Studien*, p. 46; Hultsch, *Autolycus*, p. xii; A. A. Björnbo, *Studien über Menelaos' Sphärik* (*Abhandlungen zur Geschichte der mathematischen Wissenschaften*, XIV. 1902), p. 56 sqq.

[3] *Euclidis opera omnia*, vol. VII. (1895).

[4] Heiberg, Euclid's *Optics*, *etc.* p. l.

[5] Proclus, p. 69, 3.

[6] Both treatises edited by Jan in *Musici Scriptores Graeci*, 1895, pp. 113—166, 167—207, and by Menge in *Euclidis opera omnia*, vol. VIII., 1916, pp. 157—183, 185—223.

[7] *Comptes rendus de l'Acad. des inscriptions et belles-lettres*, Paris, 1904, pp. 439—445. Cf. *Bibliotheca Mathematica*, VI$_3$, 1905–6, p. 225, note 1.

[8] Heiberg, *Euklid-Studien*, pp. 52—55; Jan, *Musici Scriptores Graeci*, pp. 169—174.

[9] H. Suter, *Das Mathematiker-Verzeichniss im Fihrist* in *Abhandlungen zur Geschichte der Mathematik*, VI., 1892, pp. 1—87 (see especially p. 17). Cf. Casiri, I. 339, 340, and Gartz, *De interpretibus et explanatoribus Euclidis Arabicis*, 1823, pp. 4, 5.

Given Magnitudes [*Data*]; the book of Tones, known under the name of Music, not genuine; the book of Division, emended by Thābit; the book of Utilisations or Applications [*Porisms*], not genuine; the book of the Canon; the book of the Heavy and Light; the book of Synthesis, not genuine; and the book of Analysis, not genuine."

It is to be observed that the Arabs already regarded the book of Tones (by which must be meant the *εἰσαγωγὴ ἁρμονική*) as spurious. The book of Division is evidently the book on *Divisions* (*of figures*). The next book is described by Casiri as "liber de utilitate suppositus." Suter gives reason for believing the *Porisms* to be meant[1], but does not apparently offer any explanation of why the work is supposed to be spurious. The book of the Canon is clearly the *κατατομὴ κανόνος*. The book on "the Heavy and Light" is apparently the tract *De levi et ponderoso*, included in the Basel Latin translation of 1537, and in Gregory's edition. The fragment, however, cannot safely be attributed to Euclid, for (1) we have nowhere any mention of his having written on mechanics, (2) it contains the notion of specific gravity in a form so clear that it could hardly be attributed to anyone earlier than Archimedes[2]. Suter thinks[3] that the works on Analysis and Synthesis (said to be spurious in the extract) may be further developments of the *Data* or *Porisms*, or may be the interpolated proofs of *Eucl.* XIII. 1—5, divided into *analysis* and *synthesis*, as to which see the notes on those propositions.

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[1] Suter, *op. cit.* pp. 49, 50. Wenrich translated the word as "utilia." Suter says that the nearest meaning of the Arabic word as of "porism" is *use*, *gain* (Nutzen, Gewinn), while a further meaning is explanation, observation, addition: a gain arising out of what has preceded (cf. Proclus' definition of the porism in the sense of a corollary).

[2] Heiberg, *Euklid-Studien*, pp. 9, 10.

[3] Suter, *op. cit.* p. 50.

CHAPTER III.

GREEK COMMENTATORS ON THE *ELEMENTS* OTHER THAN PROCLUS.

THAT there was no lack of commentaries on the *Elements* before the time of Proclus is evident from the terms in which Proclus refers to them; and he leaves us in equally little doubt as to the value which, in his opinion, the generality of them possessed. Thus he says in one place (at the end of his second prologue)[1]:

"Before making a beginning with the investigation of details, I warn those who may read me not to expect from me the things which have been dinned into our ears *ad nauseam* (*διατεθρύληται*) by those who have preceded me, viz. lemmas, cases, and so forth. For I am surfeited with these things and shall give little attention to them. But I shall direct my remarks principally to the points which require deeper study and contribute to the sum of philosophy, therein emulating the Pythagoreans who even had this common phrase for what I mean 'a figure and a platform, but not a figure and sixpence[2].'"

In another place[3] he says: "Let us now turn to the elucidation of the things proved by the writer of the Elements, selecting the more subtle of the comments made on them by the ancient writers, while cutting down their interminable diffuseness, giving the things which are more systematic and follow scientific methods, attaching more importance to the working-out of the real subject-matter than to the variety of cases and lemmas to which we see recent writers devoting themselves for the most part."

At the end of his commentary on Eucl. I. Proclus remarks[4] that the commentaries then in vogue were full of all sorts of confusion, and contained no account of *causes*, no dialectical discrimination, and no philosophic thought.

These passages and two others in which Proclus refers to "the commentators[5]" suggest that these commentators were numerous. He does not however give many names; and no doubt the only important commentaries were those of Heron, Porphyry, and Pappus.

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[1] Proclus, p. 84, 8.

[2] i.e. we reach a certain height, use the platform so attained as a base on which to build another stage, then use that as a base and so on.

[3] Proclus, p. 200, 10.

[4] *ibid.* p. 432, 15.

[5] *ibid.* p. 289, 11; p. 328, 16.

I. Heron.

Proclus alludes to Heron twice as Heron *mechanicus*[1], in another place[2] he associates him with Ctesibius, and in the three other passages[3] where Heron is mentioned there is no reason to doubt that the same person is meant, namely Heron of Alexandria. The date of Heron is still a vexed question. In the early stages of the controversy much was made of the supposed relation of Heron to Ctesibius. The best MS. of Heron's *Belopoeica* has the heading Ἥρωνος Κτησιβίου βελοποιϊκά, and an anonymous Byzantine writer of the tenth century, evidently basing himself on this title, speaks of Ctesibius as Heron's καθηγητής, "master" or "teacher." We know of two men of the name of Ctesibius. One was a barber who lived in the time of Ptolemy Euergetes II, i.e. Ptolemy VII, called Physcon (died 117 B.C.), and who is said to have made an improved water-organ[4]. The other was a mechanician mentioned by Athenaeus as having made an elegant drinking-horn in the time of Ptolemy Philadelphus (285–247 B.C.)[5]. Martin[6] took the Ctesibius in question to be the former and accordingly placed Heron at the beginning of the first century B.C., say 126–50 B.C. But Philo of Byzantium[7], who repeatedly mentions Ctesibius by name, says that the first mechanicians had the advantage of being under kings who loved fame and supported the arts. Hence our Ctesibius is more likely to have been the earlier Ctesibius who was contemporary with Ptolemy II Philadelphus.

But, whatever be the date of Ctesibius, we cannot safely conclude that Heron was his immediate pupil. The title "Heron's (edition of) Ctesibius's Belopoeica" does not, in fact, justify any inference as to the interval of time between the two works.

We now have better evidence for a *terminus post quem*. The *Metrica* of Heron, besides quoting Archimedes and Apollonius, twice refers to "the books about straight lines (chords) in a circle" (ἐν τοῖς περὶ τῶν ἐν κύκλῳ εὐθειῶν). Now we know of no work giving a Table of Chords earlier than that of Hipparchus. We get, therefore, at once, 150 B.C. or thereabouts as the *terminus post quem*. But, again, Heron's *Mechanica* quotes a definition of "centre of gravity" as given by "Posidonius, a Stoic": and, even if this Posidonius lived before Archimedes, as the context seems to imply, it is certain that another work of Heron's, the *Definitions*, owes something to Posidonius of Apamea or Rhodes, Cicero's teacher (135–51 B.C.). This brings Heron's date down to the end of the first century B.C., at least.

We have next to consider the relation, if any, between Heron and Vitruvius. In his *De Architectura*, brought out apparently in 14 B.C., Vitruvius quotes twelve authorities on *machinationes* including Archytas

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[1] Proclus, p. 305, 24; p. 346, 13.

[2] *ibid.* p. 41, 10.

[3] *ibid.* p. 196, 16; p. 323, 7; p. 429, 13.

[4] Athenaeus, *Deipno-Soph.* iv., c. 75, p. 174 *b—c*.

[5] *ibid.* xi., c. 97, p. 497 *b—c*.

[6] Martin, *Recherches sur la vie et les ouvrages d'Héron d'Alexandrie*, Paris, 1854, p. 27.

[7] Philo, *Mechan. Synt.*, p. 50, 38, ed. Schöne.

(second), Archimedes (third), Ctesibius (fourth) and Philo of Byzantium (sixth), but does not mention Heron. Nor is it possible to establish inter-dependence between Vitruvius and Heron; the differences between them seem on the whole more numerous and important than the resemblances (e.g. Vitruvius uses 3 as the value of π, while Heron always uses the Archimedean value $3\frac{1}{7}$). The inference is that Heron can hardly have written earlier than the first century A.D.

The most recent theory of Heron's date makes him later than Claudius Ptolemy the astronomer (100–178 A.D.). The arguments are mainly these. (1) Ptolemy claims as a discovery of his own a method of measuring the distance between two places (as an arc of a great circle on the earth's surface) in the case where the places are neither on the same meridian nor on the same parallel circle. Heron, in his *Dioptra*, speaks of this method as of a thing generally known to experts. (2) The dioptra described in Heron's work is a fine and accurate instrument, much better than anything Ptolemy had at his disposal. (3) Ptolemy, in his work Περὶ ῥοπῶν, asserted that water with water round it has no weight and that the diver, however deep he dives, does not feel the weight of the water above him. Heron, strangely enough, accepts as true what Ptolemy says of the diver, but is dissatisfied with the explanation given by "some," namely that it is because water is uniformly heavy—this seems to be equivalent to Ptolemy's dictum that water in water has no weight—and he essays a different explanation based on Archimedes. (4) It is suggested that the Dionysius to whom Heron dedicated his *Definitions* is a certain Dionysius who was *praefectus urbi* in 301 A.D.

On the other hand Heron was earlier than Pappus, who was writing under Diocletian (284–305 A.D.), for Pappus alludes to and draws upon the works of Heron. The net result, then, of the most recent research is to place Heron in the third century A.D. and perhaps little earlier than Pappus. Heiberg[1] accepts this conclusion, which may therefore, perhaps, be said to hold the field for the present[2].

That Heron wrote a systematic commentary on the *Elements* might be inferred from Proclus, but it is rendered quite certain by references to the commentary in Arabian writers, and particularly in an-Nairīzī's commentary on the first ten Books of the *Elements*. The *Fihrist* says, under Euclid, that "Heron wrote a commentary on this book [the *Elements*], endeavouring to solve its difficulties[3]"; and under Heron, "He wrote: the book of explanation of the obscurities in Euclid[4]...." An-Nairīzī's commentary quotes Heron by name very frequently, and often in such a way as to leave no doubt that the author had Heron's work actually before him. Thus the extracts are given in the first person, introduced by "Heron says" ("Dixit Yrinus" or "Heron"); and in other places we are told that Heron "says nothing," or "is not found to have said anything," on such and such a proposition. The commentary of an-Nairīzī is in part edited by Besthorn and Heiberg from a Leiden MS. of the translation of the *Elements* by al-Ḥajjāj with the commentary attached[1]. But this MS. only contains six Books, and several pages in the first Book, which contain the comments of Simplicius on the first twenty-two definitions of the first Book, are missing. Fortunately the commentary of an-Nairīzī has been discovered in a more complete form, in a Latin translation by Gherardus Cremonensis of the twelfth century, which contains the missing comments by Simplicius and an-Nairīzī's comments on the first ten Books. This valuable work has recently been edited by Curtze[2].

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[1] *Heronis Alexandrini opera*, vol. V. (Teubner, 1914), p. ix.

[2] Fuller details of the various arguments will be found in my *History of Greek Mathematics*, 1921, vol. II., pp. 298—306.

[3] *Das Mathematiker-Verzeichniss im Fihrist* (tr. Suter), p. 16.

[4] *ibid.* p. 22.

Thus from the three sources, Proclus, and the two versions of an-Nairīzī, which supplement one another, we are able to form a very good idea of the character of Heron's commentary. In some cases observations given by Proclus without the name of their author are seen from an-Nairīzī to be Heron's; in a few cases notes attributed by Proclus to Heron are found in an-Nairīzī without Heron's name; and, curiously enough, one alternative proof (of I. 25) given as Heron's by Proclus is introduced by the Arab with the remark that he has not been able to discover who is the author.

Speaking generally, the comments of Heron do not seem to have contained much that can be called important. We find

(1) A few general notes, e.g. that Heron would not admit more than three axioms.

(2) Distinctions of a number of particular *cases* of Euclid's propositions according as the figure is drawn in one way or in another.

Of this class are the different cases of I. 35, 36, III. 7, 8 (where the chords to be compared are drawn on *different* sides of the diameter instead of on the same side), III. 12 (which is not Euclid's, but Heron's own, adding the case of external contact to that of internal contact in III. 11), VI. 19 (where the triangle in which an additional line is drawn is taken to be the *smaller* of the two), VII. 19 (where he gives the particular case of *three* numbers in continued proportion, instead of four proportionals).

(3) Alternative proofs. Of these there should be mentioned (*a*) the proofs of II. 1—10 "without a figure," being simply the algebraic forms of proof, easy but uninstructive, which are so popular nowadays, the proof of III. 25 (placed after III. 30 and starting from the *arc* instead of the chord), III. 10 (proved by III. 9), III. 13 (a proof preceded by a lemma to the effect that a straight line cannot meet a circle in more than two points). Another class of alternative proof is

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[1] *Codex Leidensis* 399, 1. *Euclidis Elementa ex interpretatione al-Hadschdschadschii cum commentariis al-Narizii*. Five parts carrying the work to the end of Book IV. were issued in 1893, 1897, 1900, 1905 and 1910 respectively.

[2] *Anaritii in decem libros priores elementorum Euclidis commentarii ex interpretatione Gherardi Cremonensis...edidit* Maximilianus Curtze (Teubner, Leipzig, 1899).

(*b*) that which is intended to meet a particular *objection* (*ἔνστασις*) which had been or might be raised to Euclid's construction. Thus in certain cases he avoids *producing* a particular straight line, where Euclid produces it, in order to meet the objection of any one who should deny our right to assume that there is *any space available*[1]. Of this class are Heron's proofs of I. 11, I. 20, and his note on I. 16. Similarly on I. 48 he supposes the right-angled triangle which is constructed to be constructed on the *same* side of the common side as the given triangle is. A third class (*c*) is that which avoids *reductio ad absurdum*. Thus, instead of indirect proofs, Heron gives direct proofs of I. 19 (for which he requires, and gives, a preliminary lemma), and of I. 25.

(4) Heron supplies certain *converses* of Euclid's propositions, e.g. converses of II. 12, 13, VIII. 27.

(5) A few additions to, and extensions of, Euclid's propositions are also found. Some are unimportant, e.g. the construction of isosceles and scalene triangles in a note on I. 1, the construction of *two* tangents in III. 17, the remark that VII. 3 about finding the greatest common measure of three numbers can be applied to as many numbers as we please (as Euclid tacitly assumes in VII. 31). The most important extension is that of III. 20 to the case where the angle at the circumference is greater than a right angle, and the direct deduction from this extension of the result of III. 22. Interesting also are the notes on I. 37 (on I. 24 in Proclus), where Heron proves that two triangles with two sides of one equal to two sides of the other and with the included angles supplementary are equal, and compares the areas where the sum of the two included angles (one being supposed greater than the other) is less or greater than two right angles, and on I. 47, where there is a proof (depending on preliminary lemmas) of the fact that, in the figure of the proposition, the straight lines *AL*, *BK*, *CF* meet in a point. After IV. 16 there is a proof that, in a regular polygon with an even number of sides, the bisector of one angle also bisects its opposite, and an enunciation of the corresponding proposition for a regular polygon with an odd number of sides.

Van Pesch[2] gives reason for attributing to Heron certain other notes found in Proclus, viz. that they are designed to meet the same sort of points as Heron had in view in other notes undoubtedly written by him. These are (*a*) alternative proofs of I. 5, I. 17, and I. 32, which avoid the *producing* of certain straight lines, (*b*) an alternative proof of I. 9 avoiding the construction of the equilateral triangle on the side of *BC* opposite to *A*; (*c*) partial converses of I. 35—38, starting from the equality of the areas and the fact of the parallelograms or triangles being in the same parallels, and proving that the bases are the same or equal, may also be Heron's. Van Pesch further supposes that it was in Heron's commentary that the proof by Menelaus of I. 25 and the proof by Philo of I. 8 were given.

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[1] Cf. Proclus, 275, 7 εἰ δὲ λέγοι τις τόπον μὴ εἰδέναι..., 289, 18 λέγει οὖν τις ὅτι οὐκ ἔστι τόπος....

[2] *De Procli fontibus*, Lugduni-Batavorum, 1900.

The last reference to Heron made by an-Nairīzī occurs in the note on VIII. 27, so that the commentary of the former must at least have reached that point.

II. Porphyry.

The Porphyry here mentioned is of course the Neo-Platonist who lived about 232–304 A.D. Whether he really wrote a systematic commentary on the *Elements* is uncertain. The passages in Proclus which seem to make this probable are two in which he mentions him (1) as having demonstrated the necessity of the words "not on the same side" in the enunciation of I. 14[1], and (2) as having pointed out the necessity of understanding correctly the enunciation of I. 26, since, if the particular injunctions as to the sides of the triangles to be taken as equal are not regarded, the student may easily fall into error[2]. These passages, showing that Porphyry carefully analysed Euclid's *enunciations* in these cases, certainly suggest that his remarks were part of a systematic commentary. Further, the list of mathematicians in the *Fihrist* gives Porphyry as having written "a book on the Elements." It is true that Wenrich takes this book to have been a work by Porphyry mentioned by Suidas and Proclus (*Theolog. Platon.*), περὶ ἀρχῶν libri II.[3]

There is nothing of importance in the notes attributed to Porphyry by Proclus.

(1) Three alternative proofs of I. 20, which avoid *producing* a side of the triangle, are assigned to Heron and Porphyry without saying which belonged to which. If the first of the three was Heron's, I agree with van Pesch that it is more probable that the two others were both Porphyry's than that the second was Heron's and only the third Porphyry's. For they are similar in character, and the third uses a result obtained in the second[4].

(2) Porphyry gave an alternative proof of I. 18 to meet a childish objection which is supposed to require the part of AC equal to AB to be cut off from CA and not from AC.

Proclus gives a precisely similar alternative proof of I. 6 to meet a similar supposed objection; and it may well be that, though Proclus mentions no name, this proof was also Porphyry's, as van Pesch suggests[5].

Two other references to Porphyry found in Proclus cannot have anything to do with commentaries on the *Elements*. In the first a work called the Συμμικτά is quoted, while in the second a philosophical question is raised.

III. Pappus.

The references to Pappus in Proclus are not numerous; but we have other evidence that he wrote a commentary on the *Elements*. Thus a scholiast on the definitions of the *Data* uses the phrase "as

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[1] Proclus, pp. 297, 1—298, 10.

[2] *ibid.* p. 352, 13, 14 and the pages preceding.

[3] *Fihrist* (tr. Suter), p. 9, 10 and p. 45 (note 5).

[4] Van Pesch, *De Procli fontibus*, pp. 129, 130. Heiberg assigned them as above in his *Euklid-Studien* (p. 160), but seems to have changed his view later. (See Besthorn-Heiberg, *Codex Leidensis*, p. 93, note 2.)

[5] Van Pesch, *op. cit.* pp. 130—1.

Pappus says at the beginning of his (commentary) on the 10th (book) of Euclid[1]." Again in the *Fihrist* we are told that Pappus wrote a commentary to the tenth book of Euclid in two parts[2]. Fragments of this still survive in a MS. described by Woepcke[3], Paris. No. 952. 2 (supplément arabe de la Bibliothèque impériale), which contains a translation by Abū 'Uthmān (beginning of 10th century) of a Greek commentary on Book X. It is in two books, and there can now be no doubt that the author of the Greek commentary was Pappus[4]. Again Eutocius, in his note on Archimedes, *On the Sphere and Cylinder* I. 13, says that Pappus explained in his commentary on the *Elements* how to inscribe in a circle a polygon similar to a polygon inscribed in another circle; and this would presumably come in his commentary on Book XII., just as the problem is solved in the second scholium on Eucl. XII. 1. Thus Pappus' commentary on the *Elements* must have been pretty complete, an additional confirmation of this supposition being forthcoming in the reference of Marinus (a pupil and follower of Proclus) in his preface to the *Data* to "the commentaries of Pappus on the book[5]."

The actual references to Pappus in Proclus are as follows:

(1) On the Postulate (4) that all right angles are equal, Pappus is quoted as saying that the converse, viz. that all angles equal to a right angle are right, is not true[6], since the angle included between the arcs of two semicircles which are equal, and have their diameters at right angles and terminating at one point, is equal to a right angle, but is not a right angle.

(2) On the axioms Pappus is quoted as saying that, in addition to Euclid's axioms, others are on record as well (συναναγράφεσθαι) about unequals added to equals and equals added to unequals[7]; these, says Proclus, follow from the Euclidean axioms, while others given by Pappus are involved by the definitions, namely those which assert that "all parts of the plane and of the straight line coincide with one another," that "a point divides a straight line, a line a surface, and a surface a solid," and that "the infinite is (obtained) in magnitudes both by addition and diminution[8]."

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[1] Euclid's *Data*, ed. Menge, p. 262.

[2] *Fihrist* (tr. Suter), p. 22.

[3] *Mémoires présentés à l'académie des sciences*, 1856, XIV. pp. 658—719.

[4] Woepcke read the name of the author, in the title of the first book, as *B.los* (the dot representing a missing vowel). He quotes also from other MSS. (e.g. of the *Ta'rīkh al-Ḥukamā* and of the *Fihrist*) where he reads the name of the commentator as *B.lis*, *B.n.s* or *B.l.s*. Woepcke takes this author to be Valens, and thinks it possible that he may be the same as the astrologer Vettius Valens. This Heiberg (*Euklid-Studien*, pp. 169, 170) proves to be impossible, because, while one of the MSS. quoted by Woepcke says that "*B.n.s*, le *Roûmi*" (late-Greek) was later than Claudius Ptolemy and the *Fihrist* says "*B.l.s*, le *Roûmi*" wrote a commentary on Ptolemy's *Planisphaerium*, Vettius Valens seems to have lived under Hadrian, and must therefore have been an *elder* contemporary of Ptolemy. But Suter shows (*Fihrist*, p. 22 and p. 54, note 92) that *Banos* is only distinguished from *Babos* by the position of a certain dot, and *Balos* may also easily have arisen from an original *Babos* (there is no P in Arabic), so that Pappus must be the person meant. This is further confirmed by the fact that the *Fihrist* gives this author and Valens as the subjects of two separate paragraphs, attributing to the latter astrological works only.

[5] Heiberg, *Euklid-Studien*, p. 173; Euclid's *Data*, ed. Menge, pp. 256, lii.

[6] Proclus, pp. 189, 190.

[7] *ibid.* p. 197, 6—10.

[8] *ibid.* p. 198, 3—15.

(3) Pappus gave a pretty proof of I. 5. This proof has, I think, been wrongly understood; on this point see my note on the proposition.

(4) On I. 47 Proclus says[1]: "As the proof of the writer of the Elements is manifest, I think that it is not necessary to add anything further, but that what has been said is sufficient, since indeed those who have added more, like Heron and Pappus, were obliged to make use of what is proved in the sixth book, without attaining any important result." We shall see what Heron's addition consisted of; what Pappus may have added we do not know, unless it was something on the lines of his extension of I. 47 found in the *Synagoge* (IV. p. 176, ed. Hultsch).

We may fairly conclude, with van Pesch[2], that Pappus is drawn upon in various other passages of Proclus where he quotes no authority, but where the subject-matter reminds us of other notes expressly assigned to Pappus or of what we otherwise know to have been favourite questions with him. Thus:

1. We are reminded of the curvilineal angle which is equal to but not a right angle by the note on I. 32 to the effect that the converse (that a figure with its interior angles together equal to two right angles is a triangle) is not true unless we confine ourselves to rectilineal figures. This statement is supported by reference to a figure formed by four semicircles whose diameters form a square, and one of which is turned inwards while the others are turned outwards. The figure forms two angles "equal to" right angles in the sense described by Pappus on Post. 4, while the other curvilineal angles are not considered to be angles at all, and are left out in summing the internal angles. Similarly the allusions in the notes on I. 4, 23 to curvilineal angles of which certain moon-shaped angles (μηνοειδεῖς) are shown to be "equal to" rectilineal angles savour of Pappus.

2. On I. 9 Proclus says[3] that "Others, starting from the Archimedean spirals, divided any given rectilineal angle in any given ratio." We cannot but compare this with Pappus IV. p. 286, where the spiral is so used; hence this note, including remarks immediately preceding about the conchoid and the quadratrix, which were used for the same purpose, may very well be due to Pappus.

3. The subject of isoperimetric figures was a favourite one with Pappus, who wrote a recension of Zenodorus' treatise on the subject[4]. Now on I. 35 Proclus speaks[5] about the paradox of parallelograms having equal area (between the same parallels) though the two sides between the parallels may be of any length, adding that of parallelograms with equal perimeter the rectangle is greatest if the base be given, and the square greatest if the base be not given etc. He returns to the subject on I. 37 about triangles[6]. Compare[7] also his note on I. 4. These notes may have been taken from Pappus.

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[1] Proclus, p. 429, 9—15.

[2] Van Pesch, *De Procli fontibus*, p. 134 sqq.

[3] Proclus, p. 272, 10.

[4] Pappus, V. pp. 304—350; for Zenodorus' own treatise see Hultsch's Appendix, pp. 1189—1211.

[5] Proclus, pp. 396—8.

[6] *ibid.* pp. 403—4.

[7] *ibid.* pp. 236—7.

4. Again, on I. 21, Proclus remarks on the paradox that straight lines may be drawn from the base to a point within a triangle which are (1) together greater than the two sides, and (2) include a less angle, provided that the straight lines may be drawn from points in the base other than its extremities. The subject of straight lines satisfying condition (1) was treated at length, with reference to a variety of cases, by Pappus[1], after a collection of "paradoxes" by Erycinus, of whom nothing more is known. Proclus gives Pappus' first case, and adds a rather useless proof of the possibility of drawing straight lines satisfying condition (2) *alone*, adding that "the proposition stated has been proved by me without using the parallels of the commentators[2]." By "the commentators" Pappus is doubtless meant.

5. Lastly, the "four-sided triangle," called by Zenodorus the "hollow-angled,"[3] is mentioned in the notes on I. Def. 24—29 and I. 21. As Pappus wrote on Zenodorus' work in which the term occurred[4], Pappus may be responsible for these notes.

IV. Simplicius.

According to the *Fihrist*[5], Simplicius the Greek wrote "a commentary to the beginning of Euclid's book, which forms an introduction to geometry." And in fact this commentary on the definitions, postulates and axioms (including the postulate known as the Parallel-Axiom) is preserved in the Arabic commentary of an-Nairīzī[6]. On two subjects this commentary of Simplicius quotes a certain "Aganis," the first subject being the definition of an angle, and the second the definition of parallels and the parallel-postulate. Simplicius gives word for word, in a long passage placed by an-Nairīzī after I. 29, an attempt by "Aganis" to prove the parallel-postulate. It starts from a definition of parallels which agrees with Geminus' view of them as given by Proclus[7], and is closely connected with the definition given by Posidonius[8]. Hence it has been assumed that "Aganis" is none other than Geminus, and the historical importance of the commentary of Simplicius has been judged accordingly. But it has been recently shown by Tannery that the identification of "Aganis" with Geminus is practically impossible[9]. In the translation of Besthorn-Heiberg Aganis is called by Simplicius in one place "philosophus Aganis," in another "magister noster Aganis," in Gherard's version he is "socius Aganis" and "socius noster Aganis." These expressions seem to leave no doubt that Aganis was a contemporary and friend, if not master, of Simplicius; and it is impossible to suppose that Simplicius (fl. about 500 A.D.) could have used them of a man who lived four and a half centuries before his time. A phrase in Simplicius' word-for-word quotation from Aganis leads to the same conclusion. He speaks of people who objected "even in ancient times" (iam antiquitus) to the use by geometers of this postulate. This would not have been an appropriate phrase had Geminus been the writer. I do not think that this difficulty can be got over by Suter's suggestion[1] that the passages in question may have been taken out of *Heron's* commentary, and that an-Nairīzī may have forgotten to name the author; it seems clear that Simplicius is the person who described "Aganis." Hence we are driven to suppose that Aganis was not Geminus, but some unknown contemporary of Simplicius[2]. Considerable interest will however continue to attach to the comments of Simplicius so fortunately preserved.

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[1] Pappus, III. pp. 104—130.

[2] Proclus, p. 328, 15.

[3] Proclus, p. 165, 24; cf. pp. 328, 329.

[4] See Pappus, ed. Hultsch, pp. 1154, 1206.

[5] *Fihrist* (tr. Suter), p. 21.

[6] An-Nairīzī, ed. Besthorn-Heiberg, pp. 9—41, 119—133, ed. Curtze, pp. 1—37, 65—73. The *Codex Leidensis*, from which Besthorn and Heiberg's edition is taken, has unfortunately lost some leaves, so that there is a gap from Def. 1 to Def. 23 (parallels). The loss is, however, made good by Curtze's edition of the translation by Gherard of Cremona.

[7] Proclus, p. 177, 21.

[8] *ibid.* p. 176, 7.

[9] *Bibliotheca Mathematica*, 11$_3$, 1900, pp. 9—11.

Proclus tells us that one Aegaeas (? Aenaeas) of Hierapolis wrote an epitome of the *Elements*[3]; but we know nothing more of him or of it.

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[1] *Zeitschrift für Math. u. Physik*, XLIV., hist.-litt. Abth. p. 61.

[2] The above argument seems to me quite insuperable. The other arguments of Tannery do not, however, carry conviction to my mind. I do not follow the reasoning based on Aganis' definition of an angle. It appears to me a pure assumption that Geminus would have seen that Posidonius' definition of parallels was not admissible. Nor does it seem to me to count for much that Proclus, while telling us that Geminus held that the postulate ought to be proved and warned the unwary against hastily concluding that two straight lines approaching one another must necessarily meet (cf. a curve and its asymptote), gives no hint that Geminus did try to prove the postulate. It may well be that Proclus omitted Geminus' "proof" (if he wrote one) because he preferred Ptolemy's attempt which he gives (pp. 365—7).

[3] Proclus, p. 361, 21.

CHAPTER IV.

PROCLUS AND HIS SOURCES[1].

IT is well known that the commentary of Proclus on Eucl. Book I. is one of the two main sources of information as to the history of Greek geometry which we possess, the other being the *Collection* of Pappus. They are the more precious because the original works of the forerunners of Euclid, Archimedes and Apollonius are lost, having probably been discarded and forgotten almost immediately after the appearance of the masterpieces of that great trio.

Proclus himself lived 410–485 A.D., so that there had already passed a sufficient amount of time for the tradition relating to the pre-Euclidean geometers to become obscure and defective. In this connexion a passage is quoted from Simplicius[2] who, in his account of the quadrature of certain lunes by Hippocrates of Chios, while mentioning two authorities for his statements, Alexander Aphrodisiensis (about 220 A.D.) and Eudemus, says in one place[3], "As regards Hippocrates of Chios we must pay more attention to Eudemus, *since he was nearer the times*, being a pupil of Aristotle."

The importance therefore of a critical examination of Proclus' commentary with a view to determining from what original sources he drew need not be further emphasised.

Proclus received his early training in Alexandria, where Olympiodorus was his instructor in the works of Aristotle, and mathematics was taught him by one Heron[4] (of course a different Heron from the "*mechanicus* Hero" of whom we have already spoken). He afterwards went to Athens where he was imbued by Plutarch, and by Syrianus, with the Neo-Platonic philosophy, to which he then devoted heart and soul, becoming one of its most prominent exponents. He speaks everywhere with the highest respect of his masters, and was in turn regarded with extravagant veneration by his contemporaries, as we learn from Marinus his pupil and biographer. On the death of Syrianus he was put at the head of the Neo-Platonic school. He was a man of untiring industry, as is shown by the number of books which he wrote, including a large number of commentaries, mostly on the dialogues of Plato. He was an acute dialectician, and pre-eminent among his contemporaries in the range of his learning[1]; he was a competent mathematician; he was even a poet. At the same time he was a believer in all sorts of myths and mysteries and a devout worshipper of divinities both Greek and Oriental.

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[1] My task in this chapter is made easy by the appearance, in the nick of time, of the dissertation *De Procli fontibus* by J. G. van Pesch (Lugduni-Batavorum, Apud L. van Nifterik, MDCCCC). The chapters dealing directly with the subject show a thorough acquaintance on the part of the author with all the literature bearing on it; he covers the whole field and he exercises a sound and sober judgment in forming his conclusions. The same cannot always be said of his only predecessor in the same inquiry, Tannery (in *La Géométrie grecque*, 1887), who often robs his speculations of much of their value through his proneness to run away with an idea; he does so in this case, basing most of his conclusions on an arbitrary and unwarranted assumption as to the significance of the words οἱ περί τινα (e.g. Ἥρωνα, Ποσειδώνιον etc.) as used in Proclus.

[2] Simplicius on Aristotle's *Physics*, ed. Diels, pp. 54—69.

[3] *ibid.* p. 68, 32.

[4] Cf. Martin, *Recherches sur la vie et les ouvrages d'Héron d'Alexandrie*, pp. 240—2.

Though he was a competent mathematician, he was evidently much more a philosopher than a mathematician[2]. This is shown even in his commentary on Eucl. I., where, not only in the Prologues (especially the first), but also in the notes themselves, he seizes any opportunity for a philosophical digression. He says himself that he attaches most importance to "the things which require deeper study and contribute to the sum of philosophy[3]"; alternative proofs, *cases*, and the like (though he gives many) have no attraction for him; and, in particular, he attaches no value to the addition of Heron to I. 47[4], which is of considerable mathematical interest. Though he esteemed mathematics highly, it was only as a handmaid to philosophy. He quotes Plato's opinion to the effect that "mathematics, as making use of hypotheses, falls short of the non-hypothetical and perfect science[5]"..."Let us then not say that Plato excludes mathematics from the sciences, but that he declares it to be secondary to the one supreme science[6]." And again, while "mathematical science must be considered desirable in itself, though not with reference to the needs of daily life," "if it is necessary to refer the benefit arising from it to something else, we must connect that benefit with intellectual knowledge (*νοερὰν γνῶσιν*), to which it leads the way and is a propaedeutic, clearing the eye of the soul and taking away the impediments which the senses place in the way of the knowledge of universals (*τῶν ὅλων*)[7]."

We know that in the Neo-Platonic school the younger pupils learnt mathematics; and it is clear that Proclus taught this subject, and that this was the origin of the commentary. Many passages show him as a master speaking to scholars. Thus "we have illustrated and made plain all these things in the case of the first problem, but it is necessary that *my hearers* should make the same inquiry as regards the others as well[1]," and "I do not indicate these things as a merely incidental matter but as preparing *us* beforehand for the doctrine of the Timaeus[2]." Further, the pupils whom he was addressing were *beginners* in mathematics; for in one place he says that he omits "for the present" to speak of the discoveries of those who employed the curves of Nicomedes and Hippias for trisecting an angle, and of those who used the Archimedean spiral for dividing an angle in any given ratio, because these things would be too difficult for beginners (δυσθεωρήτους τοῖς εἰσαγομένοις)[3]. Again, if his pupils had not been beginners, it would not have been necessary for Proclus to explain what is meant by saying that sides subtend certain angles[4], the difference between *adjacent* and *vertical* angles[5] etc., or to exhort them, as he often does, to work out other particular cases for themselves, for practice (γυμνασίας ἕνεκα)[6].

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[1] Zeller calls him "Der Gelehrte, dem kein Feld damaligen Wissens verschlossen ist."

[2] Van Pesch observes that in his commentaries on the *Timaeus* (pp. 671—2) he speaks as no real mathematician could have spoken. In the passage referred to the question is whether the sun occupies a middle place among the planets. Proclus rejects the view of Hipparchus and Ptolemy because "*ὁ θεουργός*" (sc. the Chaldean, says Zeller) thinks otherwise, "whom it is not lawful to disbelieve." Martin says rather neatly, "Pour Proclus, les Éléments d'Euclide ont l'heureuse chance de n'être contredits ni par les Oracles chaldaïques, ni par les spéculations des pythagoriciens anciens et nouveaux......"

[3] Proclus, p. 84, 13.

[4] *ibid.* p. 429, 12.

[5] *ibid.* p. 31, 20.

[6] *ibid.* p. 32, 2.

[7] *ibid.* p. 27, 27 to 28, 7; cf. also p. 21, 25, pp. 46, 47.

The commentary seems then to have been founded on Proclus' lectures to beginners in mathematics. But there are signs that it was revised and re-edited for a larger public; thus he gives notice in one place[7] "to those who shall come upon" his work (τοῖς ἐντευξομένοις). There are also passages which could not have been understood by the beginners to whom he lectured, e.g. passages about the cylindrical helix[8], conchoids and cissoids[9]. These passages may have been added in the revised edition, or, as van Pesch conjectures, the explanations given in the lectures may have been much fuller and more comprehensible to beginners, and they may have been shortened on revision.

In his comments on the propositions of Euclid, Proclus generally proceeds in this way: first he gives explanations regarding Euclid's proofs, secondly he gives a few different cases, mainly for the sake of practice, and thirdly he addresses himself to refuting objections raised by cavillers to particular propositions. The latter class of note he deems necessary because of "sophistical cavils" and the attitude of the people who rejoiced in finding paralogisms and in causing annoyance to scientific men[10]. His commentary does not seem to have been written for the purpose of correcting or improving Euclid. For there are very few passages of mathematical content in which Proclus can be supposed to be propounding anything of his own; nearly all are taken from the works of others, mostly earlier commentators, so that, for the purpose of improving on or correcting Euclid, there was no need for his commentary at all. Indeed only in one place does he definitely bring forward anything of his own to get over a difficulty which he finds in Euclid[11]; this is where he tries to prove the parallel-postulate, after first giving Ptolemy's attempt and then pointing out objections to it. On the other hand, there are a number of passages in which he extols Euclid; thrice[1] also he supports Euclid against Apollonius where the latter had given proofs which he considered better than Euclid's (I. 10, 11, and 23).

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[1] Proclus, p. 210, 18.

[2] *ibid.* p. 384, 2.

[3] *ibid.* p. 272, 12.

[4] *ibid.* p. 238, 12.

[5] *ibid.* p. 298, 14.

[6] Cf. p. 224, 15 (on I. 2).

[7] *ibid.* p. 84, 9.

[8] *ibid.* p. 105.

[9] *ibid.* p. 113.

[10] *ibid.* p. 375, 8.

[11] *ibid.* pp. 368—373.

Allusion must be made to the debated question whether Proclus continued his commentaries beyond Book I. His intention to do so is clear from the following passages. Just after the words above quoted about the trisection etc. of an angle by means of certain curves he says, "For we may perhaps more appropriately examine these things on the third book, where the writer of the Elements bisects a given circumference[2]." Again, after saying that of all parallelograms which have the same perimeter the square is the greatest "and the rhomboid least of all," he adds: "But this we will prove in another place; for it is more appropriate to the (discussion of the) hypotheses of the second book[3]." Lastly, when alluding (on I. 45) to the squaring of the circle, and to Archimedes' proposition that any circle is equal to the right-angled triangle in which the perpendicular is equal to the radius of the circle and the base to its perimeter, he adds, "But of this elsewhere[4]"; this may imply an intention to treat of the subject on Eucl. XII., though Heiberg doubts it[5]. But it is clear that, at the time when the commentary on Book I. was written, Proclus had not yet begun to write on the other Books and was uncertain whether he would be able to do so: for at the end he says[6], "For my part, if I should be able to discuss the other books[7] in the same manner, I should give thanks to the gods; but, if other cares should draw me away, I beg those who are attracted by this subject to complete the exposition of the other books as well, following the same method, and addressing themselves throughout to the deeper and better defined questions involved" (*τὸ πραγματειῶδες πανταχοῦ καὶ εὐδιαίρετον μεταδιώκοντας*).

There is in fact no satisfactory evidence that Proclus did actually write any more commentaries than that on Book I.[8] The contrary view receives support from two facts pointed out by Heiberg, viz. (1) that the scholiast's copy of Proclus was not so much better than our

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[1] Proclus, p. 280, 9; p. 282, 20; pp. 335, 336.

[2] *ibid.* p. 272, 14.

[3] *ibid.* p. 398, 18.

[4] *ibid.* p. 423, 6.

[5] Heiberg, *Euklid-Studien*, p. 165, note.

[6] Proclus, p. 432, 9.

[7] The words in the Greek are: *εἰ μὲν δυνηθείημεν καὶ τοῖς λοιποῖς τὸν αὐτὸν τρόπον ἐξελθεῖν.* For *ἐξελθεῖν* Heiberg would read *ἐπεξελθεῖν*.

[8] True, a Vatican MS. has a collection of scholia on Books I. (extracts from the extant commentary of Proclus), II., V., VI., X. headed **Εἰς τὰ Εὐκλείδου στοιχεῖα προλαμβανόμενα ἐκ τῶν Πρόκλου σποράδην καὶ κατ' ἐπιτομήν.** Heiberg holds that this title itself suggests that the authorship of Proclus was limited to the scholia on Book I.; for *προλαμβανόμενα ἐκ τῶν Πρόκλου* suits extracts from Proclus' *prologues*, but hardly scholia to later Books. Again, a certain scholium (Heiberg in *Hermes* XXXVIII., 1903, p. 341, No. 17) purports to quote words from the end of "a scholium of Proclus" on X. 9. The words quoted are from the scholium X. No. 62, one of the Scholia Vaticana. But none of the other, older, sources connect Proclus' name with X. No. 62; it is probable therefore that a Byzantine, who had in his MS. of Euclid the collection of Schol. Vat. and knew that those on Book I. came from Proclus, himself attached Proclus' name to the others.

MSS. as to suggest that the scholiast had further commentaries of Proclus which have vanished for us[1]; (2) that there is no trace in the scholia of the notes which Proclus promised in the passages quoted above.

Coming now to the question of the sources of Proclus, we may say that everything goes to show that his commentary is a compilation, though a compilation "in the better sense" of the term[2]. He does not even give us to understand that we shall find in it much of his own; "let us," he says, "now turn to the exposition of the theorems proved by Euclid, selecting the more subtle of the comments made on them by the ancient writers, and cutting down their interminable diffuseness...[3]": not a word about anything of his own. At the same time, he seems to imply that he will not necessarily on each occasion quote the source of each extract from an earlier commentary; and, in fact, while he quotes the name of his authority in many places, especially where the subject is important, in many others, where it is equally certain that he is not giving anything of his own, he mentions no authority. Thus he quotes Heron by name six times; but we now know, from the commentary of an-Nairīzī, that a number of other passages, where he mentions no name, are taken from Heron, and among them the not unimportant addition of an alternative proof to I. 19. Hence we can by no means conclude that, where no authority is mentioned, Proclus is giving notes of his own. The presumption is generally the other way; and it is often possible to arrive at a conclusion, either that a particular note is not Proclus' own, or that it is definitely attributable to someone else, by applying the ordinary principles of criticism. Thus, where the note shows an unmistakable affinity to another which Proclus definitely attributes to some commentator by name, especially when both contain some peculiar and distinctive idea, we cannot have much doubt in assigning both to the same commentator[4]. Again, van Pesch finds a criterion in the *form* of a note, where the explanation is so condensed as to be only just intelligible; the note is that in which a converse of I. 32 is proved[5] the proposition namely that a rectilineal figure which has all its interior angles together equal to two right angles is a triangle.

It is not safe to attribute a passage to Proclus himself because he uses the first person in such expressions as "I say" or "I will prove"—for he was in the habit of putting into his own words the substance of notes borrowed from others—nor because, in speaking of an objection raised to a particular proposition, he uses such expressions as "perhaps someone may object" (*ἴσως δ' ἄν τινες ἐνσταῖεν...*): for sometimes other words in the same passage indicate that the objection had actually been taken by someone[1]. Speaking generally, we shall not be justified in concluding that Proclus is stating something new of his own unless he indicates this himself in express terms.

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[1] While one class of scholia (Schol. Vat.) have some better readings than our MSS. of Proclus have, and partly fill up the gaps at I. 36, 37 and I. 41—43, the other class (Schol. Vind.) derive from an inferior Proclus MS. which also had the same lacunae.

[2] Knoche, *Untersuchungen über des Proklus Diadochus Commentar zu Euklid's Elementen* (1862), p. 11.

[3] Proclus, p. 200, 10—13.

[4] Instances of the application of this criterion will be found in the discussion of Proclus' indebtedness to the commentaries of Heron, Porphyry and Pappus.

[5] Van Pesch attributes this converse and proof to Pappus, arguing from the fact that the proof is followed by a passage which, on comparison with Pappus' note on the postulate that all right angles are equal, he feels justified in assigning to Pappus. I doubt if the evidence is sufficient.

As regards the form of Proclus' references to others by name, van Pesch notes that he very seldom mentions the particular *work* from which he is borrowing. If we leave out of account the references to Plato's dialogues, there are only the following references to books: the *Bacchae* of Philolaus[2], the *Symmikta* of Porphyry[3], Archimedes *On the Sphere and Cylinder*[4], Apollonius *On the cochlias*[5], a book by Eudemus on *The Angle*[6], a whole book of Posidonius directed against Zeno of the Epicurean sect[7], Carpus' *Astronomy*[8], Eudemus' *History of Geometry*[9], and a tract by Ptolemy on the parallel-postulate[10].

Again, Proclus does not always indicate that he is quoting something at second-hand. He often does so, e.g. he quotes Heron as the authority for a statement about Philippus, Eudemus as attributing a certain theorem to Oenopides etc.; but he says on I. 12 that "Oenopides first investigated this problem, thinking it useful for astronomy" when he cannot have had Oenopides' work before him.

It has been said above that Proclus was in the habit of stating in his own words the substance of the things which he borrowed. We are prepared for this when we find him stating that he will select the best things from ancient commentaries and "cut short their interminable diffuseness," that he will "briefly describe" (*συντομως ἱστορῆσαι*) the other proofs of I. 20 given by Heron and Porphyry and also the proofs of I. 25 by Menelaus and Heron. But the best evidence is of course to be found in the passages where he quotes works still extant, e.g. those of Plato, Aristotle and Plotinus. Examination of these passages shows great divergences from the original; even where he purports to quote textually, using the expressions "Plato says," or "Plotinus says," he by no means quotes word for word[11]. In fact, he seems to have had a positive distaste for quoting textually from other works. He cannot conquer this even when quoting from Euclid; he says in his note on I. 22, "we will follow the words of the geometer" but fails, nevertheless, to reproduce the text of Euclid unchanged[12].

We now come to the sources themselves from which Proclus drew in writing his commentary. Three have already been disposed of, viz. Heron, Porphyry and Pappus, who had all written commentaries on the *Elements*[1]. We go on to

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[1] Van Pesch illustrates this by an objection refuted in the note on I. 9, p. 273, 11 sqq. After using the above expression to introduce the objection, Proclus uses further on (p. 273, 25) the term "they say" (*φασίν*).

[2] Proclus, p. 22, 15.

[3] *ibid.* p. 56, 25.

[4] *ibid.* p. 71, 18.

[5] *ibid.* p. 105, 5.

[6] *ibid.* p. 125, 8.

[7] *ibid.* p. 200, 2.

[8] *ibid.* p. 241, 19.

[9] *ibid.* p. 352, 15.

[10] *ibid.* p. 362, 15.

[11] See the passages referred to by van Pesch (p. 70). The most glaring case is a passage (p. 21, 19) where he quotes Plotinus, using the expression "Plotinus says......" Comparison with Plotinus, *Ennead.* I. 3, 3, shows that *very few* words are those of Plotinus himself; the rest represent Plotinus' views in Proclus' own language.

[12] Proclus, p. 330, 19 sqq

Eudemus, the pupil of Aristotle, who, among other works, wrote a history of arithmetic, a history of astronomy, and a history of geometry. The importance of the last mentioned work is attested by the frequent use made of it by ancient writers. That there was no other history of geometry written after the time of Eudemus seems to be proved by the remark of Proclus in the course of his famous summary: "Those who compiled histories bring the development of this science up to this point. *Not much younger than these is Euclid*[2]...." The loss of Eudemus' history is one of the gravest which fate has inflicted upon us, for it cannot be doubted that Eudemus had before him a number of the actual works of earlier geometers, which, as before observed, seem to have vanished completely when they were superseded by the treatises of Euclid, Archimedes and Apollonius. As it is, we have to be thankful for the fragments from Eudemus which such writers as Proclus have preserved to us.

I agree with van Pesch[3] that there is no sufficient reason for doubting that the work of Eudemus was accessible to Proclus at first hand. For the later writers Simplicius and Eutocius refer to it in terms such as leave no room for doubt that *they* had it before them. I have already quoted a passage from Simplicius' account of the lunes of Hippocrates to the effect that Eudemus must be considered the best authority since he lived nearer the times[4]. In the same place Simplicius says[5], "I will set out what Eudemus says word for word (κατὰ λέξιν λεγόμενα), adding only a little explanation in the shape of reference to Euclid's Elements *owing to the memorandum-like style of Eudemus* (διὰ τὸν ὑπομνηματικὸν τρόπον τοῦ Εὐδήμου) who sets out his explanations in the abbreviated form usual with ancient writers. Now in the second book of the history of geometry he writes as follows[5]." It is not possible to suppose that Simplicius would have written in this way about the style of Eudemus if he had merely been copying certain passages second-hand out of some other author and had not the original work itself to refer to. In like manner, Eutocius speaks of the paralogisms handed down in connexion with the attempts of Hippocrates and Antiphon to square the circle[6], "with which I imagine that those are accurately acquainted who have examined (ἐπεσκεμμένους) the geometrical history of Eudemus and know the Ceria Aristotelica." How could the contemporaries of Eutocius have *examined* the work of Eudemus unless it was still extant in his time?

The passages in which Proclus quotes Eudemus by name as his authority are as follows:

(1) On I. 26 he says that Eudemus in his history of geometry

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[1] See pp. 20 to 27 above.

[2] Proclus, p. 68, 4—7.

[3] *De Procli fontibus*, pp. 73—75.

[4] See above, p. 29.

[5] Simplicius, *loc. cit.*, ed. Diels, p. 60, 27.

[6] Archimedes, ed. Heiberg, vol. III. p. 228.

referred this theorem to Thales, inasmuch as it was necessary to Thales' method of ascertaining the distance of ships from the shore[1].

(2) Eudemus attributed to Thales the discovery of Eucl. I. 15[2], and

(3) to Oenopides the problem of I. 23[3].

(4) Eudemus referred the discovery of the theorem in I. 32 to the Pythagoreans, and gave their proof of it, which Proclus reproduces[4].

(5) On I. 44 Proclus tells us[5] that Eudemus says that "these things are ancient, being discoveries of the Pythagorean muse, the application (παραβολή) of areas, their exceeding (ὑπερβολή) and their falling short (ἔλλειψις)." The next words about the appropriation of these terms (parabola, hyperbola and ellipse) by later writers (i.e. Apollonius) to denote the conic sections are of course not due to Eudemus.

Coming now to notes where Eudemus is not named by Proclus, we may fairly conjecture, with van Pesch, that Eudemus was really the authority for the statements (1) that Thales first proved that a circle is bisected by its diameter[6] (though the proof by *reductio ad absurdum* which follows in Proclus cannot be attributed to Thales[7]), (2) that "Plato made over to Leodamas the analytical method, by means of which *it is recorded* (ἱστόρηται) that the latter too made many discoveries in geometry[8]," (3) that the theorem of I. 5 was due to Thales, and that for equal angles he used the more archaic expression "similar" angles[9], (4) that Oenopides first investigated the problem of I. 12, and that he called the perpendicular the *gnomonic* line (κατὰ γνώμονα)[10], (5) that the theorem that only three sorts of polygons can fill up the space round a point, viz. the equilateral triangle, the square and the regular hexagon, was Pythagorean[11]. Eudemus may also be the authority for Proclus' description of the two methods, referred to Plato and Pythagoras respectively, of forming right-angled triangles in whole numbers[12].

We cannot attribute to Eudemus the beginning of the note on I. 47 where Proclus says that "if we listen to those who like to recount ancient history, we may find some of them referring this theorem to Pythagoras and saying that he sacrificed an ox in honour of his discovery[13]." As such a sacrifice was contrary to the Pythagorean tenets, and Eudemus could not have been unaware of this, the story cannot rest on his authority. Moreover Proclus speaks as though he were not certain of the correctness of the tradition; indeed, so far as the story of the sacrifice is concerned, the same thing is told of Thales in connexion with his discovery that the angle in a semicircle is a right angle[1], and Plutarch is not certain whether the ox was sacrificed on the discovery of I. 47 or of the problem about application of areas[2]. Plutarch's doubt suggests that he knew of no evidence for the story beyond the vague allusion in the distich of Apollodorus "Logisticus" (the "calculator") cited by Diogenes Laertius also[3]; and Proclus may have had in mind this couplet with the passages of Plutarch.

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[1] Proclus, p. 352, 14—18.

[2] *ibid.* p. 299, 3.

[3] *ibid.* p. 333, 5.

[4] *ibid.* p. 379, 1—16.

[5] *ibid.* p. 419, 15—18.

[6] *ibid.* p. 157, 10, 11.

[7] Cantor (*Gesch. d. Math.* I$_3$, p. 221) points out the connexion between the *reductio ad absurdum* and the analytical method said to have been discovered by Plato. Proclus gives the proof by *reductio ad absurdum* to meet an imaginary critic who desires a *mathematical* proof; possibly Thales may have been satisfied with the argument in the same sentence which mentions Thales, "the cause of the bisection being the unswerving course of the straight line through the centre."

[8] Proclus, p. 211, 19—23.

[9] *ibid.* p. 250, 20.

[10] *ibid.* p. 283, 7—10.

[11] *ibid.* pp. 304, 11—305, 3.

[12] *ibid.* pp. 428, 7—429, 9.

[13] *ibid.* p. 426, 6—9.

We come now to the question of the famous historical summary given by Proclus[4]. No one appears to maintain that Eudemus is the author of even the early part of this summary in the form in which Proclus gives it. It is, as is well known, divided into two distinct parts, between which comes the remark, "Those who compiled histories[5] bring the development of this science up to this point. Not much younger than these is Euclid, who put together the Elements, collecting many of the theorems of Eudoxus, perfecting many others by Theaetetus, and bringing to irrefragable demonstration the things which had only been somewhat loosely proved by his predecessors." Since Euclid was later than Eudemus, it is impossible that Eudemus can have written this. Yet the style of the summary after this point does not show any such change from that of the former portion as to suggest different authorship. The author of the earlier portion recurs frequently to the question of the origin of the *elements* of geometry in a way in which no one would be likely to do who was not later than Euclid; and it must be the same hand which in the second portion connects Euclid's *Elements* with the work of Eudoxus and Theaetetus[6].

If then the summary is the work of one author, and that author not Eudemus, who is it likely to have been? Tannery answers that it is Geminus[7]; but I think, with van Pesch, that he has failed to show why it should be Geminus rather than another. And certainly the extracts which we have from Geminus' work suggest that the sort of topics which it dealt with was quite different; they seem rather to have been general questions of the *content* of mathematics, and even Tannery admits that historical details could only have come incidentally into the work[8].

Could the author have been Proclus himself? Circumstances which seem to suggest this possibility are (1) that, as already stated, the question of the origin of the *Elements* is kept prominent, (2) that there is no mention of Democritus, whom Eudemus would not be likely to have ignored, while a follower of Plato would be likely enough to do him the injustice, following the example of Plato who was an opponent of Democritus, never once mentions him, and is said to have wished to burn all his writings[1], and (3) the allusion at the beginning to the "inspired Aristotle" (ὁ δαιμονιος Ἀριστοτέλης)[2], though this may easily have been inserted by Proclus in a quotation made by him from someone else. On the other hand there are considerations which suggest that Proclus himself was *not* the writer. (1) The style of the whole passage is not such as to point to him as the author. (2) If he wrote it, it is hardly conceivable that he would have passed over in silence the discovery of the analytical method, the invention of Plato to which he attached so much importance[3].

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[1] Diogenes Laertius, I. 24, p. 6, ed. Cobet.

[2] Plutarch, *non posse suaviter vivi secundum Epicurum*, 11; *Symp.* VIII, 2.

[3] Diog. Laert. VIII. 12, p. 207, ed. Cobet:

Ἡνίκα Πυθαγόρης τὸ περικλεὲς εὕρετο γράμμα,
κεῖν' ἐφ' ὅτῳ κλεινὴν ἤγαγε βουθυσίην.

See on this subject Tannery, *La Géométrie grecque*, p. 105.

[4] Proclus, pp. 64—70.

[5] The plural is well explained by Tannery, *La Géométrie grecque*, pp. 73, 74. No doubt the author of the summary tried to supplement Eudemus by means of any other histories which threw light on the subject. Thus e.g. the allusion (p. 64, 21) to the Nile recalls Herodotus. Cf. the expression in Proclus, p. 64, 19, παρὰ τῶν πολλῶν ἱστόρηται.

[6] Tannery, *La Géométrie grecque*, p. 75.

[7] *ibid.* pp. 66—75.

[8] *ibid.* p. 19.

There is nothing improbable in the conjecture that Proclus quoted the summary from a compendium of Eudemus' history made by some later writer: but as yet the question has not been definitely settled. All that is certain is that the early part of the summary must have been made up from scattered notices found in the great work of Eudemus.

Proclus refers to another work of Eudemus besides the history, viz. a book on *The Angle* (βιβλίον περὶ γωνίας)[4]. Tannery assumes that this must have been part of the history, and uses this assumption to confirm his idea that the history was arranged according to *subjects*, not according to chronological order[5]. The phraseology of Proclus however unmistakably suggests a separate work; and that the history was *chronologically* arranged seems to be clearly indicated by the remark of Simplicius that Eudemus "also counted Hippocrates among the more ancient writers" (ἐν τοῖς παλαιοτέροις)[6].

The passage of Simplicius about the lunes of Hippocrates throws considerable light on the style of Eudemus' history. Eudemus wrote in a memorandum-like or summary manner (τὸν ὑπομνηματικὸν τρόπον τοῦ Εὐδήμου)[7] when reproducing what he found in the ancient writers; sometimes it is clear that he left out altogether proofs or constructions of things by no means easy[8].

Geminus.

The discussions about the date and birthplace of Geminus form a whole literature, as to which I must refer the reader to Manitius and Tittel[9]. Though the name looks like a Latin name (Gemĭnus), Manitius concluded that, since it appears as Γεμῖνος in all Greek MSS. and as Γεμεῖνος in some inscriptions, it is Greek and possibly formed from γεμ as Ἐργῖνος is from ἐργ and Ἀλεξῖνος from ἀλεξ (cf. also Ἰκτῖνος, Κρατῖνος). Tittel is equally positive that it is Gemĭnus and suggests that Γεμῖνος is due to a *false* analogy with Ἀλεξῖνος etc. and Γεμεῖνος wrongly formed on the model of Ἀντωνεῖνος, Ἀγριππεῖνα. Geminus, a Stoic philosopher, born probably in the island of Rhodes, was the author of a comprehensive work on the classification of mathematics, and also wrote, about 73–67 B.C., a not less comprehensive commentary on the meteorological textbook of his teacher Posidonius of Rhodes.

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[1] Diog. Laertius, IX. 40, p. 237, ed. Cobet.

[2] Proclus, p. 64, 8.

[3] Proclus, p. 211, 19 sqq.; the passage is quoted above, p. 36.

[4] *ibid.* p. 125, 8.

[5] Tannery, *La Géométrie grecque*, p. 26.

[6] Simplicius, ed. Diels, p. 69, 23.

[7] *ibid.* p: 60, 29.

[8] Cf. Simplicius, p. 63, 19 sqq.; p. 64, 25 sqq.; also Usener's note "de supplendis Hippocratis quas omisit Eudemus constructionibus" added to Diels' preface, pp. xxiii—xxvi.

[9] Manitius, *Gemini elementa astronomiae* (Teubner, 1898), pp. 237—252; Tittel, art. "Geminos" in Pauly-Wissowa's *Real-Encyclopädie der classischen Altertumswissenschaft*, vol. VII.. 1910.

It is the former work in which we are specially interested here. Though Proclus made great use of it, he does not mention its title, unless we may suppose that, in the passage (p. 177, 24) where, after quoting from Geminus a classification of lines which never meet, he says, "these remarks I have selected from the φιλοκαλία of Geminus," φιλοκαλία is a title or an alternative title. Pappus however quotes a work of Geminus "on the classification of the mathematics" (ἐν τῷ περὶ τῆς τῶν μαθημάτων τάξεως)[1], while Eutocius quotes from "the sixth book of the doctrine of the mathematics" (ἐν τῷ ἕκτῳ τῆς τῶν μαθημάτων θεωρίας)[2]. Tannery[3] pointed out that the former title corresponds well enough to the long extract[4] which Proclus gives in his first prologue, and also to the fragments contained in the *Anonymi variae collectiones* published by Hultsch at the end of his edition of Heron[5]; but it does not suit most of the other passages borrowed by Proclus. The correct title was therefore probably that given by Eutocius, *The Doctrine*, or *Theory*, *of the Mathematics*; and Pappus probably refers to one particular portion of the work, say the first Book. If the sixth Book treated of conics, as we may conclude from Eutocius, there must have been more Books to follow, because Proclus has preserved us details about higher curves, which must have come later. If again Geminus finished his work and wrote with the same fulness about the other branches of mathematics as he did about geometry, there must have been a considerable number of Books altogether. At all events it seems to have been designed to give a complete view of the whole science of mathematics, and in fact to be a sort of encyclopaedia of the subject.

I shall now indicate first the certain, and secondly the probable, obligations of Proclus to Geminus, in which task I have only to follow van Pesch, who has embodied the results of Tittel's similar inquiry also[6]. I shall only omit the passages as regards which a case for attributing them to Geminus does not seem to me to have been made out.

First come the following passages which must be attributed to Geminus, because Proclus mentions his name:

(1) (In the first prologue of Proclus[7]) on the division of mathe-

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[1] Pappus, ed. Hultsch, p. 1026, 9.

[2] Apollonius, ed. Heiberg, vol. II. p. 170.

[3] Tannery, *La Géométrie grecque*, pp. 18, 19.

[4] Proclus, pp. 38, 1—42, 8.

[5] Heron, ed. Hultsch, pp. 246, 16—249, 12.

[6] Van Pesch, *De Procli fontibus*, pp. 97—113. The dissertation of Tittel is entitled *De Gemini Stoici studiis mathematicis* (1895).

[7] Proclus, pp. 38, 1—42, 8, except the allusion in p. 41, 8—10, to Ctesibius and Heron and

matical sciences into arithmetic, geometry, mechanics, astronomy, optics, geodesy, canonic (science of musical harmony), and logistic (apparently arithmetical problems);

(2) (in the note on the definition of a straight line) on the classification of lines (including curves) as simple (straight or circular) and mixed, composite and incomposite, uniform (ὁμοιομερεῖς) and non-uniform (ἀνομοιομερεῖς), lines "about solids" and lines produced by cutting solids, including conic and spiric sections[1];

(3) (in the note on the definition of a plane surface) on similar distinctions extended to surfaces and solids[2];

(4) (in the note on the definition of parallels) on lines which *do not meet* (ἀσύμπτωτοι) but which are not on that account parallel, e.g. a curve and its asymptote, showing that the property of *not meeting* does not make lines parallel—a favourite observation of Geminus—and, incidentally, on *bounded* lines or those which *enclose a figure* and those which do not[3];

(5) (in the same note) the definition of parallels given by Posidonius[4];

(6) on the distinction between postulates and axioms, the futility of trying to prove axioms, as Apollonius tried to prove Axiom 1, and the equal incorrectness of assuming what really requires proof, "as Euclid did in the fourth postulate [equality of right angles] and in the fifth postulate [the parallel-postulate][5]";

(7) on Postulates 1, 2, 3, which Geminus makes depend on the idea of a straight line being described by the motion of a point[6];

(8) (in the note on Postulate 5) on the inadmissibility in geometry of an argument which is merely plausible, and the danger in this particular case owing to the existence of lines which do converge *ad infinitum* and yet never meet[7];

(9) (in the note on I. 1) on the subject-matter of geometry, theorems, problems and διορισμοί (conditions of possibility) for problems[8];

(10) (in the note on I. 5) on a generalisation of I. 5 by Geminus through the substitution for the rectilineal base of "one uniform line (curve)," by means of which he proved that the only "uniform lines"

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their pneumatic devices (θαυματοποιϊκή), as regards which Proclus' authority may be Pappus (VIII. p. 1024, 24—27) who uses very similar expressions. Heron, even if not later than Geminus, could hardly have been included in a historical work by him. Perhaps Geminus may have referred to Ctesibius only, and Proclus may have inserted "and Heron" himself.

[1] Proclus, pp. 103, 21—107, 10; pp. 111, 1—113, 3.

[2] *ibid.* pp. 117, 14—120, 12, where perhaps in the passage pp. 117, 22—118, 23 we may have Geminus' own words.

[3] *ibid.* pp. 176, 18—177, 25; perhaps also p. 175. The note ends with the words "These things too we have selected from Geminus' Φιλοκαλία for the elucidation of the matters in question." Tannery (p. 27) takes these words coming at the end of the commentary on the definitions as referring to the whole of the portion of the commentary dealing with the definitions. Van Pesch properly regards them as only applying to the note on *parallels*. This seems to me clear from the use of the word *too* (τοσαῦτα καί).

[4] Proclus, p. 176, 5—17.

[5] *ibid.* pp. 178—182, 4; pp. 183, 14—184, 10; cf. p. 188, 3—11.

[6] *ibid.* p. 185, 6—25.

[7] *ibid.* p. 192, 5—29.

[8] *ibid.* pp. 200, 21—202, 25.

(alike in all their parts) are a straight line, a circle, and a cylindrical helix[1];

(11) (in the note on I. 10) on the question whether a line is made up of indivisible parts (ἀμερῆ), as affecting the problem of bisecting a given straight line[2];

(12) (in the note on I. 35) on *topical*, or *locus*-theorems[3], where the illustration of the equal parallelograms described between a hyperbola and its asymptotes may also be due to Geminus[4].

Other passages which may fairly be attributed to Geminus, though his name is not mentioned, are the following:

(1) in the prologue, where there is the same allusion as in the passage (8) above to a remark of Aristotle that it is equally absurd to expect scientific proofs from a rhetorician and to accept mere plausibilities from a geometer[5];

(2) a passage in the prologue about the subject-matter, methods, and bases of geometry, the latter including axioms and postulates[6];

(3) another on the definition and nature of *elements*[7];

(4) a remark on the Stoic use of the term axiom for every simple statement (ἀπόφανσις ἁπλῆ)[8];

(5) another discussion on theorems and problems[9], in the middle of which however there are some sentences by Proclus himself[10].

(6) another passage, in connexion with Def. 3, on lines including or not including a figure (with which cf. part of the passage (4) above)[11];

(7) a classification of different sorts of angles according as they are contained by simple or mixed lines (or curves)[12];

(8) a similar classification of figures[13], and of plane figures[14];

(9) Posidonius' definition of a *figure*[15];

(10) a classification of triangles into seven kinds[16];

(11) a note distinguishing lines (or curves) producible indefinitely or not so producible, whether forming a figure or not forming a figure (like the "single-turn spiral")[17];

(12) passages distinguishing different sorts of problems[18], different sorts of theorems[19], and two sorts of converses (complete and partial)[20];

(13) the definition of the term "porism" as used in the title of Euclid's *Porisms*, as distinct from the other meaning of "corollary"[21];

(14) a note on the Epicurean objection to I. 20 as being obvious even to an ass[22];

(15) a passage on the properties of parallels, with allusions to

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[1] Proclus, p. 251, 2—11.

[2] *ibid.* pp. 277, 25—279, 11.

[3] *ibid.* pp. 394, 11—395, 2 and p. 395, 13—21.

[4] *ibid.* p. 395, 8—12.

[5] *ibid.* pp. 33, 21—34, 1.

[6] *ibid.* pp. 57, 9—58, 3.

[7] *ibid.* pp. 72, 3—75, 4.

[8] *ibid.* p. 77, 3—6.

[9] *ibid.* pp. 77, 7—78, 13, and 79, 3—81, 4.

[10] *ibid.* pp. 78, 13—79, 2.

[11] *ibid.* pp. 102, 22—103, 18.

[12] *ibid.* pp. 126, 7—127, 16.

[13] *ibid.* pp. 159, 12—160, 9.

[14] *ibid.* pp. 162, 27—164, 6.

[15] *ibid.* p. 143, 5—11.

[16] *ibid.* p. 168, 4—12.

[17] *ibid.* p. 187, 19—27.

[18] *ibid.* pp. 220, 7—222, 14; also p. 330, 6—9.

[19] *ibid.* pp. 244, 14—246, 12.

[20] *ibid.* pp. 252, 5—254, 20.

[21] *ibid.* pp. 301, 21—302, 13.

[22] *ibid.* pp. 322, 4—323, 3.

Apollonius' *Conics*, and the curves invented by Nicomedes, Hippias and Perseus[1];

(16) a passage on the parallel-postulate regarded as the converse of I. 17[2].

Of the authors to whom Proclus was indebted in a less degree the most important is **Apollonius of Perga.** Two passages allude to his *Conics*[3], one to a work on irrationals[4], and two to a treatise *On the cochlias* (apparently the cylindrical helix) by Apollonius[5]. But more important for our purpose are six references to Apollonius in connexion with elementary geometry.

(1) He appears as the author of an attempt to explain the idea of a line (possessing length but no breadth) by reference to daily experience, e.g. when we tell someone to measure, merely, the length of a road or of a wall[6]; and doubtless the similar passage showing how we may in like manner get a notion of a surface (without depth) is his also[7].

(2) He gave a new general definition of an angle[8].

(3) He tried to prove certain axioms[9], and Proclus gives his attempt to prove Axiom 1, word for word[10].

Proclus further quotes:

(4) Apollonius' solution of the problem in Eucl. I. 10, avoiding Euclid's use of I. 9[11],

(5) his solution of the problem in I. 11, differing only slightly from Euclid's[12], and

(6) his solution of the problem in I. 23[13].

Heiberg[14] conjectures that Apollonius departed from Euclid's method in these propositions because he objected to solving problems of a more general, by means of problems of a more particular, character. Proclus however considers all three solutions inferior to Euclid's; and his remarks on Apollonius' handling of these elementary matters generally suggest that he was nettled by criticisms of Euclid in the work containing the things which he quotes from Apollonius, just as we conclude that Pappus was offended by the remarks of Apollonius about Euclid's incomplete treatment of the "three- and four-line locus[15]." If this was the case, Proclus can hardly have got his information about these things at second-hand; and there seems to be no reason to doubt that he had the actual work of Apollonius before him. This work may have been the treatise mentioned by Marinus in the words "Apollonius in his general treatise" ('Απολλώνιος ἐν τῇ καθόλου πραγματείᾳ)[16]. If the notice in the *Fihrist*[17] stating, on the authority of Thābit b. Qurra, that

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[1] Proclus, pp. 355, 20—356, 16.

[2] *ibid.* p. 364, 9—12; pp. 364, 20—365, 4.

[3] *ibid.* p. 71, 19; p. 356, 8, 6.

[4] *ibid.* p. 74, 23, 24.

[5] *ibid.* pp. 105, 5, 6, 14, 15.

[6] *ibid.* p. 100, 5—19.

[7] *ibid.* p. 114, 20—25.

[8] *ibid.* p. 123, 15—19 (cf. p. 124, 17, p. 125, 17).

[9] *ibid.* p. 183, 13, 14.

[10] *ibid.* pp. 194, 25—195, 5.

[11] *ibid.* pp. 279, 16—280, 4.

[12] *ibid.* p. 282, 8—19.

[13] *ibid.* pp. 335, 16—336, 5.

[14] *Philologus*, vol. XLIII. p. 489.

[15] See above, pp. 2, 3.

[16] *Marinus in Euclidis Data*, ed. Menge, p. 234, 16.

[17] *Fihrist*, tr. Suter, p. 19.

Apollonius wrote a tract on the parallel-postulate be correct, it may have been included in the same work. We may conclude generally that, in it, Apollonius tried to remodel the beginnings of geometry, reducing the number of axioms, appealing, in his definitions of lines, surfaces etc., more to experience than to abstract reason, and substituting for certain proofs others of a more general character.

The probabilities are that, in quoting from the tract of **Ptolemy** in which he tried to prove the parallel-postulate, Proclus had the actual work before him. For, after an allusion to it as "a certain book[1]" he gives two long extracts[2], and at the beginning of the second indicates the title of the tract, "in the (book) about the meeting of straight lines produced from (angles) less than two right angles," as he has very rarely done in other cases.

Certain things from **Posidonius** are evidently quoted at second-hand, the authority being Geminus (e.g. the definitions of *figure* and *parallels*); but besides these we have quotations from a separate work which he wrote to controvert Zeno of Sidon, an Epicurean who had sought to destroy the whole of geometry[3]. We are told that Zeno had argued that, even if we admit the fundamental principles (*ἀρχαί*) of geometry, the deductions from them cannot be proved without the admission of something else as well, which has not been included in the said principles[4]. On I. 1 Proclus gives at some length the arguments of Zeno and the reply of Posidonius as regards this proposition[5]. In this case Zeno's "something else" which he considers to be assumed is the fact that two straight lines cannot have a common segment, and then, as regards the "proof" of it by means of the bisection of a circle by its diameter, he objects that it has been assumed that two *circumferences* (arcs) of circles cannot have a common part. Lastly, he makes up, for the purpose of attacking it, another supposed "proof" of the fact that two straight lines cannot have a common part. Proclus appears, more than once, to be quoting the actual words of Zeno and Posidonius; in particular, two expressions used by Posidonius about "the acrid Epicurean" (τὸν δριμὺν Ἐπικούρειον)[6] and his "misrepresentations" (Ποσειδώνιός φησι· τὸν Ζήνωνα συκοφαντεῖν)[7]. It is not necessary to suppose that Proclus had the original work of Zeno before him, because Zeno's arguments may easily have been got from Posidonius' reply; but he would appear to have quoted direct from the latter at all events.

The work of **Carpus** *mechanicus* (a treatise on astronomy) quoted from by Proclus[8] must have been accessible to him at first-hand, because a portion of the extract from it about the relation of theorems and problems[9] is reproduced word for word. Moreover, if he were not using the book itself, Proclus would hardly be in a position to question whether the introduction of the subject of theorems and problems was opportune in the place where it was found (*εἰ μὲν κατὰ καιρὸν ἦ μή, παρείσθω πρὸς τὸ παρόν*)[1].

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[1] Proclus, p. 191, 23.

[2] *ibid.* pp. 362, 14—363, 18; pp. 365, 7—367, 27.

[3] *ibid.* p. 200, 1—3.

[4] *ibid.* pp. 199, 11—200, 1.

[5] *ibid.* pp. 214, 18—215, 13; pp. 216, 10—218, 11.

[6] *ibid.* p. 216, 21.

[7] *ibid.* p. 218, 1.

[8] *ibid.* pp. 241, 19—243, 11.

[9] *ibid.* pp. 242, 22—243, 11.

It is of course evident that Proclus had before him the original works of Plato, Aristotle, Archimedes and Plotinus, as well as the *Συμμικτά* of Porphyry and the works of his master Syrianus (*ὁ ἡμέτερος καθηγεμών*)[2], from whom he quotes in his note on the definition of an angle. Tannery also points out that he must have had before him a group of works representing the Pythagorean tradition on its mystic, as distinct from its mathematical, side, from Philolaus downwards, and comprising the more or less apocryphal *ἱερὸς λόγος* of Pythagoras, the Oracles (*λόγια*), and Orphic verses[3].

Besides quotations from writers whom we can identify with more or less certainty, there are many other passages which are doubtless quoted from other commentators whose names we do not know. A list of such passages is given by van Pesch[4], and there is no need to cite them here.

Van Pesch also gives at the end of his work[5] a convenient list of the books which, as the result of his investigation, he deems to have been accessible to and directly used by Proclus. The list is worth giving here, on the same ground of convenience. It is as follows:

Eudemus: *history of geometry.*
Geminus: *the theory of the mathematical sciences.*
Heron: *commentary on the Elements of Euclid.*
Porphyry: ” ” ”
Pappus: ” ” ”
Apollonius of Perga: a work relating to elementary geometry.
Ptolemy: *on the parallel-postulate.*
Posidonius: a book controverting Zeno of Sidon.
Carpus: *astronomy.*
Syrianus: a discussion on the *angle.*
Pythagorean philosophical tradition.
Plato's works.
Aristotle's works.
Archimedes' works.
Plotinus: *Enneades.*

Lastly we come to the question what passages, if any, in the commentary of Proclus represent his own contributions to the subject. As we have seen, the *onus probandi* must be held to rest upon him who shall maintain that a particular note is original on the part of Proclus. Hence it is not enough that it should be impossible to point to another writer as the probable source of a note; we must have a *positive* reason for attributing it to Proclus. The criterion must therefore be found either (1) in the general terms in which Proclus points out the deficiencies in previous commentaries and indicates the respects in which his own will differ from them, or (2) in specific expressions used by him in introducing particular notes which may indicate that he is giving his own views. Besides indicating that he paid more attention than his predecessors to questions requiring deeper study (τὸ πραγματειῶδες) and "pursued clear distinctions" (τὸ εὐδιαίρετον μεταδιώκοντας)[1]—by which he appears to imply that his predecessors had confused the different departments of their commentaries, viz. lemmas, cases, and objections (ἐνστάσεις)[2]—Proclus complains that the earlier commentators had failed to indicate the ultimate grounds or *causes* of propositions[3]. Although it is from Geminus that he borrowed a passage maintaining that it is one of the proper functions of geometry to inquire into causes (τὴν αἰτίαν καὶ τὸ διὰ τί)[4], yet it is not likely that Geminus dealt with Euclid's propositions one by one; and consequently, when we find Proclus, on I. 8, 16, 17, 18, 32, and 47[5], endeavouring to explain *causes*, we have good reason to suppose that the explanations are his own.

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[1] Proclus, p. 241, 21, 22.
[2] *ibid.* p. 123, 19.
[3] Tannery, *La Géométrie grecque*, pp. 25, 26.
[4] Van Pesch, *De Procli fontibus*, p. 139.
[5] *ibid.* p. 155.

Again, his remarks on certain things which he quotes from Pappus can scarcely be due to anyone else, since Pappus is the latest of the commentators whose works he appears to have used. Under this head come

(1) his objections to certain new axioms introduced by Pappus[6],

(2) his conjecture as to how Pappus came to think of his alternative proof of I. 5[7],

(3) an addition to Pappus' remarks about the curvilineal angle which is equal to a right angle without being one[8].

The defence of Geminus against Carpus, who combated his view of theorems and problems, is also probably due to Proclus[9], as well as an observation on I. 38 to the effect that I. 35—38 are really comprehended in VI. 1 as particular cases[10].

Lastly, we can have no hesitation in attributing to Proclus himself (1) the criticism of Ptolemy's attempt to prove the parallel-postulate[11], and (2) the other attempted proof given in the same note[12] (on I. 29) and assuming as an axiom that "if from one point two straight lines forming an angle be produced *ad infinitum* the distance between them when so produced *ad infinitum* exceeds any finite magnitude (i.e. length)," an assumption which purports to be the equivalent of a statement in Aristotle[13]. It is introduced by words in which the writer appears to claim originality for his proof: "To him who desires to see this proved (κατασκευαζόμενον) *let it be said by us* (λεγέσθω παρ' ἡμῶν)" etc.[14] Moreover, Philoponus, in a note on Aristotle's *Anal. post.* I. 10, says that "the geometer (Euclid) assumes this as an axiom, but it wants a great deal of proof, insomuch that both Ptolemy and Proclus wrote a whole book upon it[15]."

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[1] Proclus, p. 84, 13, p. 432, 14, 15.

[2] Cf. *ibid.* p. 289, 11—15; p. 432, 15—17.

[3] *ibid.* p. 432, 17.

[4] *ibid.* p. 202, 9—25.

[5] See Proclus, p. 270, 5—24 (I. 8); pp. 309, 3—310, 8 (I. 16); pp. 310, 19—311, 23 (I. 17); pp. 316, 14—318, 2 (I. 18); p. 384, 13—21 (I. 32); pp. 426, 22—427, 8 (I. 47).

[6] Proclus, p. 198, 5—15.

[7] *ibid.* p. 250, 12—19.

[8] *ibid.* p. 190, 9—23.

[9] *ibid.* p. 243, 12—29.

[10] *ibid.* pp. 405, 6—406, 9.

[11] *ibid.* p. 368, 1—23.

[12] *ibid.* pp. 371, 11—373, 2.

[13] Aristotle, *de caelo*, I. 5 (271 b 28—30).

[14] Proclus, p. 371, 10.

[15] Berlin Aristotle, vol. IV. p. 214 a 9—12.

CHAPTER V.

THE TEXT[1].

It is well known that the title of Simson's edition of Euclid (first brought out in Latin and English in 1756) claims that, in it, "the errors by which Theon, or others, have long ago vitiated these books are corrected, and some of Euclid's demonstrations are restored"; and readers of Simson's notes are familiar with the phrases used, where anything in the text does not seem to him satisfactory, to the effect that the demonstration has been spoiled, or things have been interpolated or omitted, by Theon "or some other unskilful editor." Now most of the MSS. of the Greek text prove by their titles that they proceed from the recension of the *Elements* by Theon; they purport to be either "from the edition of Theon" (ἐκ τῆς Θέωνος ἐκδόσεως) or "from the lectures of Theon" (ἀπὸ συνουσιῶν τοῦ Θέωνος). This was Theon of Alexandria (4th c. A.D.) who also wrote a commentary on Ptolemy, in which there occurs a passage of the greatest importance in this connexion[2]: "But that sectors in equal circles are to one another as the angles on which they stand *has been proved by me in my edition of the Elements at the end of the sixth book.*" Thus Theon himself says that he edited the *Elements* and also that the second part of VI. 33, found in nearly all the MSS., is his addition.

This passage is the key to the whole question of Theon's changes in the text of Euclid; for, when Peyrard found in the Vatican the MS. 190 which contained neither the words from the titles of the other MSS. quoted above nor the interpolated second part of VI. 33, he was justified in concluding, as he did, that in the Vatican MS. we have an edition more ancient than Theon's. It is also clear that the copyist of P, or rather of its archetype, had before him the two recensions and systematically gave the preference to the earlier one; for at XIII. 6 in P the first hand has added a note in the margin: "This theorem is not given in most copies of the *new edition*, but is found in those of the old." Thus we are more fortunate than Simson, since our judgment of Theon's recension can be formed on the basis, not of mere conjecture, but of the documentary evidence afforded by a comparison of the Vatican MS. just mentioned with what we may conveniently call, after Heiberg, the Theonine MSS.

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[1] The material for the whole of this chapter is taken from Heiberg's edition of the *Elements*, introduction to vol. V., and from the same scholar's *Litterargeschichtliche Studien über Euklid*, p. 174 sqq. and *Paralipomena zu Euklid* in *Hermes*, XXXVIII., 1903.

[2] I. p. 201 ed. Halma = p. 50 ed. Basel.

The MSS. used for Heiberg's edition of the *Elements* are the following:

(1) P = Vatican MS. numbered 190, 4to, in two volumes (doubtless one originally); 10th c.

This is the MS. which Peyrard was able to use; it was sent from Rome to Paris for his use and bears the stamp of the Paris Imperial Library on the last page. It is well and carefully written. There are corrections some of which are by the original hand, but generally in paler ink, others, still pretty old, by several different hands, or by one hand with different ink in different places (P m. 2), and others again by the latest hand (P m. rec.). It contains, first, the *Elements* I.—XIII. with scholia, then Marinus' commentary on the *Data* (without the name of the author), followed by the *Data* itself and scholia, then the *Elements* XIV., XV. (so called), and lastly three books and a part of a fourth of a commentary by Theon *εἰς τοὺς προχείρους κανόνας Πτολεμαίου.*

The other MSS. are "Theonine."

(2) F = MS. XXVIII, 3, in the Laurentian Library at Florence, 4to; 10th c.

This MS. is written in a beautiful and scholarly hand and contains the *Elements* I.—XV., the *Optics* and the *Phaenomena*, but is not well preserved. Not only is the original writing renewed in many places, where it had become faint, by a later hand of the 16th c., but the same hand has filled certain smaller lacunae by gumming on to torn pages new pieces of parchment, and has replaced bodily certain portions of the MS., which had doubtless become illegible, by fresh leaves. The larger gaps so made good extend from Eucl. VII. 12 to IX. 15, and from XII. 3 to the end; so that, besides the conclusion of the *Elements*, the *Optics* and *Phaenomena* are also in the later hand, and we cannot even tell what in addition to the *Elements* I.—XIII. the original MS. contained. Heiberg denotes the later hand by ϕ and observes that, while in restoring words which had become faint and filling up minor lacunae the writer used no other MS., yet in the two larger restorations he used the Laurentian MS. XXVIII, 6, belonging to the 13th—14th c. The latter MS. (which Heiberg denotes by f) was copied from the Viennese MS. (V) to be described below.

(3) B = Bodleian MS., D'Orville X. 1 inf. 2, 30, 4to; A.D. 888.

This MS. contains the *Elements* I.—XV. with many scholia. Leaves 15—118 contain I. 14 (from about the middle of the proposition) to the end of Book VI., and leaves 123—387 (wrongly numbered 397) Books VII.—XV. in one and the same elegant hand (9th c.). The leaves preceding leaf 15 seem to have been lost at some time, leaves 6 to 14 (containing *Elem.* I. to the place in I. 14 above referred to) being carelessly written by a later hand on thick and common parchment (13th c.). On leaves 2 to 4 and 122 are certain notes in the hand of Arethas, who also wrote a two-line epigram on leaf 5, the greater part of the scholia in uncial letters, a few notes and corrections, and two sentences on the last leaf, the first of which states that the MS. was written by one Stephen *clericus* in the year of the world 6397

(= 888 A.D.), while the second records Arethas' own acquisition of it. Arethas lived from, say, 865 to 939 A.D. He was Archbishop of Caesarea and wrote a commentary on the Apocalypse. The portions of his library which survive are of the greatest interest to palaeography on account of his exact notes of dates, names of copyists, prices of parchment etc. It is to him also that we owe the famous Plato MS. from Patmos (Cod. Clarkianus) which was written for him in November 895[1].

(4) V = Viennese MS. Philos. Gr. No. 103; probably 12th c.

This MS. contains 292 leaves, Eucl. *Elements* I.—XV. occupying leaves 1 to 254, after which come the *Optics* (to leaf 271), the *Phaenomena* (mutilated at the end) from leaf 272 to leaf 282, and lastly scholia, on leaves 283 to 292, also imperfect at the end. The different material used for different parts and the varieties of handwriting make it necessary for Heiberg to discuss this MS. at some length[2]. The handwriting on leaves 1 to 183 (Book I. to the middle of X. 105) and on leaves 203 to 234 (from XI. 31, towards the end of the proposition, to XIII. 7, a few lines down) is the same; between leaves 184 and 202 there are two varieties of handwriting, that of leaves 184 to 189 and that of leaves 200 (verso) to 202 being the same. Leaf 235 begins in the same handwriting, changes first gradually into that of leaves 184 to 189 and then (verso) into a third more rapid cursive writing which is the same as that of the greater part of the scholia, and also as that of leaves 243 and 282, although, as these leaves are of different material, the look of the writing and of the ink seems altered. There are corrections both by the first and a second hand, and scholia by many hands. On the whole, in spite of the apparent diversity of handwriting in the MS., it is probable that the whole of it was written at about the same time, and it may (allowing for changes of material, ink etc.) even have been written by the same man. It is at least certain that, when the Laurentian MS. XXVIII, 6 was copied from it, the whole MS. was in the condition in which it is now, except as regards the later scholia and leaves 283 to 292 which are not in the Laurentian MS., that MS. coming to an end where the *Phaenomena* breaks off abruptly in V. Hence Heiberg attributes the whole MS. to the 12th c.

But it was apparently in two volumes originally, the first consisting of leaves 1 to 183; and it is certain that it was not all copied at the same time or from one and the same original. For leaves 184 to 202 were evidently copied from two MSS. different both from one another and from that from which the rest was copied. Leaves 184 to the middle of leaf 189 (recto) must have been copied from a MS. similar to P, as is proved by similarity of readings, though not from P itself. The rest, up to leaf 202, were copied from the Bologna MS. (b) to be mentioned below. It seems clear that the content of leaves 184 to 202 was supplied from other MSS. because there was a lacuna in the original from which the rest of V was copied.

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[1] See Pauly-Wissowa, *Real-Encyclopädie der class. Altertumswissenschaft*, vol. II., 1896, p. 675.

[2] Heiberg, vol. V. pp. xxix—xxxiii.

Heiberg sums up his conclusions thus. The copyist of V first copied leaves 1 to 183 from an original in which two *quaterniones* were missing (covering from the middle of Eucl. X. 105 to near the end of XI. 31). Noticing the lacuna he put aside one *quaternio* of the parchment used up to that point. Then he copied onwards from the end of the lacuna in the original to the end of the *Phaenomena*. After this he looked about him for another MS. from which to fill up the lacuna; finding one, he copied from it as far as the middle of leaf 189 (recto). Then, noticing that the MS. from which he was copying was of a different class, he had recourse to yet another MS. from which he copied up to leaf 202. At the same time, finding that the lacuna was longer than he had reckoned for, he had to use twelve more leaves of a different parchment in addition to the *quaternio* which he had put aside. The whole MS. at first formed two volumes (the first containing leaves 1 to 183 and the second leaves 184 to 282); then, after the last leaf had perished, the two volumes were made into one to which two more *quaterniones* were also added. A few leaves of the latter of these two have since perished.

(5) b = MS. numbered 18—19 in the Communal Library at Bologna, in two volumes, 4to; 11th c.

This MS. has scholia in the margin written both by the first hand and by two or three later hands; some are written by the latest hand, Theodorus Cabasilas (a descendant apparently of Nicolaus Cabasilas, 14th c.) who owned the MS. at one time. It contains (*a*) in 14 *quaterniones* the definitions and the enunciations (without proofs) of the *Elements* I.—XIII. and of the *Data*, (*b*) in the remainder of the volumes the *Proem to Geometry* (published among the *Variae Collectiones* in Hultsch's edition of Heron, pp. 252, 24 to 274, 14) followed by the *Elements* I.—XIII. (part of XIII. 18 to the end being missing), and then by part of the *Data* (from the last three words of the enunciation of Prop. 38 to the end of the penultimate clause in Prop. 87, ed. Menge). From XI. 36 inclusive to the end of XII. this MS. appears to represent an entirely different recension. Heiberg is compelled to give this portion of b separately in an appendix. He conjectures that it is due to a Byzantine mathematician who thought Euclid's proofs too long and tiresome and consequently contented himself with indicating the course followed[1]. At the same time this Byzantine must have had an excellent MS. before him, probably of the ante-Theonine variety of which the Vatican MS. 190 (P) is the sole representative.

(6) p = Paris MS. 2466, 4to; 12th c.

This manuscript is written in two hands, the finer hand occupying leaves 1 to 53 (recto), and a more careless hand leaves 53 (verso) to 64, which are of the same parchment as the earlier leaves, and leaves 65 to 239, which are of a thinner and rougher parchment showing traces of writing of the 8th—9th c. (a Greek version of the Old Testament). The MS. contains the *Elements* I.—XIII. and some scholia after Books XI., XII. and XIII.

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[1] *Zeitschrift für Math. u. Physik*, XXIX., hist.-litt. Abtheilung, p. 13.

(7) q = Paris MS. 2344, folio; 12th c.

It is written by one hand but includes scholia by many hands. On leaves 1 to 16 (recto) are scholia with the same title as that found by Wachsmuth in a Vatican MS. and relied upon by him to prove that Proclus continued his commentaries beyond Book I.[1] Leaves 17 to 357 contain the *Elements* I.—XIII. (except that there is a lacuna from the middle of VIII. 25 to the *ἔκθεσις* of IX. 14); before Books VII. and X. there are some leaves filled with scholia only, and leaves 358 to 366 contain nothing but scholia.

(8) Heiberg also used a palimpsest in the British Museum (Add. 17211). Five pages are of the 7th—8th c. and are contained (leaves 49—53) in the second volume of the Syrian MS. Brit. Mus. 687 of the 9th c.; half of leaf 50 has perished. The leaves contain various fragments from Book X. enumerated by Heiberg, Vol. III., p. v, and nearly the whole of XIII. 14.

Since his edition of the *Elements* was published, Heiberg has collected further material bearing on the history of the text[2]. Besides giving the results of further or new examination of MSS., he has collected the fresh evidence contained in an-Nairīzī's commentary, and particularly in the quotations from Heron's commentary given in it (often word for word), which enable us in several cases to trace differences between our text and the text as Heron had it, and to identify some interpolations which actually found their way into the text from Heron's commentary itself; and lastly he has dealt with some valuable fragments of ancient papyri which have recently come to light, and which are especially important in that the evidence drawn from them necessitates some modification in the views expressed in the preface to Vol. V. as to the nature of the changes made in Theon's recension, and in the principles laid down for differentiating between Theon's recension and the original text, on the basis of a comparison between P and the Theonine MSS. alone.

The fragments of ancient papyri referred to are the following.

1. *Papyrus Herculanensis* No. 1061[3].

This fragment quotes Def. 15 of Book I. in Greek, and *omits* the words *ἣ καλεῖται περιφέρεια*, "which is called the circumference," found in all our MSS., and the further addition *πρὸς τὴν τοῦ κύκλου περιφέρειαν* also found in practically all the MSS. Thus Heiberg's assumption that both expressions are interpolations is now confirmed by this oldest of all sources.

2. *The Oxyrhynchus Papyri* I. p. 58, No. XXIX. of the 3rd or 4th c.

This fragment contains the enunciation of Eucl. II. 5 (with figure, apparently without letters, immediately following, and not, as usual in our MSS., at the end of the proof) and before it the part of a word *περιεχομε* belonging to II. 4 (with room for *–νῳ ὀρθογωνίῳ· ὅπερ ἔδει*

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[1] [εἰς τ]ὰ τοῦ Εὐκλείδου στοιχεῖα προλαμβανόμενα ἐκ τῶν Πρόκλου σποράδην καὶ κατ' ἐπιτομήν. Cf. p. 32, note 8, above.

[2] Heiberg, *Paralipomena zu Euklid* in *Hermes*, XXXVIII., 1903, pp. 46—74, 161—201, 321—356.

[3] Described by Heiberg in *Oversigt over det kngl. danske Videnskabernes Selskabs Forhandlinger*, 1900, p. 161.

δεῖξαι and a stroke to mark the end), showing that the fragment *had not* the Porism which appears in all the Theonine MSS. and (in a later hand) in P, and thereby confirming Heiberg's assumption that the Porism was due to Theon.

3. A fragment in *Fayum towns and their papyri*, p. 96, No. IX. of 2nd or 3rd c.

This contains I. 39 and I. 41 following one another and almost complete, showing that I. 40 was wanting, whereas it is found in all the MSS. and is recognised by Proclus. Moreover the text of the beginning of I. 39 is better than ours, since it has no double *διορισμός* but omits the first ("I say that they are also in the same parallels") and has "*and*" instead of "*for* let AD be joined" in the next sentence. It is clear that I. 40 was interpolated by someone who thought there ought to be a proposition following I. 39 and related to it as I. 38 is related to I. 37 and I. 36 to I. 35, although Euclid nowhere uses I. 40, and therefore was not likely to include it. The same interpolator failed to realise that the words "let AD be joined" were part of the *ἔκθεσις* or *setting-out*, and took them for the *κατασκευή* or "construction" which generally follows the *διορισμός* or "particular statement" of the conclusion to be proved, and consequently thought it necessary to insert a *διορισμός* *before* the words.

The conclusions drawn by Heiberg from a consideration of particular readings in this papyrus along with those of our MSS. will be referred to below.

We now come to the principles which Heiberg followed, when preparing his edition, in differentiating the original text from the Theonine recension by means of a comparison of the readings of P and of the Theonine MSS. The rules which he gives are subject to a certain number of exceptions (mostly in cases where one MS. or the other shows readings due to copyists' errors), but in general they may be relied upon to give conclusive results.

The possible alternatives which the comparison of P with the Theonine MSS. may give in particular passages are as follows:

I. There may be *agreement* in three different degrees.

(1) P and *all* the Theonine MSS. may agree.

In this case the reading common to all, even if it is corrupt or interpolated, is more ancient than Theon, i.e. than the 4th c.

(2) P may agree with *some* (only) of the Theonine MSS.

In this case Heiberg considered that the latter give the true reading of Theon's recension, and the other Theonine MSS. have departed from it.

(3) P and *one* only of the Theonine MSS may agree.

In this case too Heiberg assumed that the *one* Theonine MS. which agrees with P gives the true Theonine reading, and that this rule even supplies a sort of measure of the quality and faithfulness of the Theonine MSS. Now none of them agrees *alone* with P in preserving the true reading so often as F. Hence F must be held to have preserved Theon's recension more faithfully than the other Theonine MSS.; and it would follow that in those portions where F fails us P must carry rather more weight even though it may differ from the Theonine MSS. BVpq. (Heiberg gives many examples in proof of this, as of his main rules generally, for which reference must be made to his *Prolegomena* in Vol. V.) The specially close relation of F and P is also illustrated by passages in which they have the same *errors*; the explanation of these common errors (where not due to accident) is found by Heiberg in the supposition that they existed, but were not noticed by Theon, in the original copy in which he made his changes.

Although however F is by far the best of the Theonine MSS., there are a considerable number of passages where one of the others (B, V, p or q) *alone* with P gives the genuine reading of Theon's recension.

As the result of the discovery of the papyrus fragment containing I. 39, 41, the principles above enunciated under (2) and (3) are found by Heiberg to require some qualification. For there is in some cases a remarkable agreement between the papyrus and the Theonine MSS. (some or all) as against P. This shows that Theon took more trouble to follow older MSS., and made fewer arbitrary changes of his own, than has hitherto been supposed. Next, when the papyrus agrees with some of the Theonine MSS. against P, it must now be held that these MSS. (and not, as formerly supposed, those which agree with P) give the true reading of Theon. If it were otherwise, the agreement between the papyrus and the Theonine MSS. would be accidental: but it happens too often for this. It is clear also that there must have been contamination between the two recensions; otherwise, whence could the Theonine MSS. which agree with P and not with the papyrus have got their readings? The influence of the P class on the Theonine F is especially marked.

II. There may be *disagreement* between P and all the Theonine MSS.

The following possibilities arise.

(1) The Theonine MSS. differ also among themselves.

In this case Heiberg considered that P nearly always has the true reading, and the Theonine MSS. have suffered interpolation in different ways after Theon's time.

(2) The Theonine MSS. all combine against P.

In this case the explanation was assumed by Heiberg to be one or other of the following.

(α) The common reading is due to an error which cannot be imputed to Theon (though it may have escaped him when putting together the archetype of his edition); such error may either have arisen accidentally in all alike, or (more frequently) may be referred to a common archetype of all the MSS.

(β) There may be an accidental error in P; e.g. something has dropped out of P in a good many places, generally through ὁμοιοτέλευτον

(γ) There may be words interpolated in P.

(δ) Lastly, *we may have in the Theonine* MSS. *a change made by Theon himself.*

(The discovery of the ancient papyrus showing readings agreeing with some, or with all, of the Theonine MSS. against P now makes it necessary to be very cautious in applying these criteria.)

It is of course the last class (δ) of changes which we have to investigate in order to get a proper idea of Theon's recension.

Heiberg first observes, as regards these, that we shall find that Theon, in editing the *Elements*, altered hardly anything without some reason, often inadequate according to our ideas, but still some reason which seemed to him sufficient. Hence, in cases of very slight differences where both the Theonine MSS. and P have readings good and probable in themselves, Heiberg is not prepared to put the differences down to Theon. In those passages where we cannot see the least reason why Theon, if he had the reading of P before him, should have altered it, Heiberg would not at once assume the superiority of P unless there was such a consistency in the differences as would indicate that they were due not to accident but to design. In the absence of such indications, he thinks that the ordinary principles of criticism should be followed and that proper weight should be attached to the *antiquity* of the sources. And it cannot be denied that the sources of the Theonine version are the more ancient. For not only is the British Museum palimpsest (L), which is intimately connected with the rest of our MSS., at least two centuries older than P, but the other Theonine MSS. are so nearly allied that they must be held to have had a common archetype intermediate between them and the actual edition of Theon; and, since they themselves are as old as, or older than P, their archetype must have been much older. Heiberg gives (pp. xlvi, xlvii) a list of passages where, for this reason, he has followed the Theonine MSS. in preference to P.

It has been mentioned above that the copyist of P or rather of its archetype wished to give an ancient recension. Therefore (apart from clerical errors and interpolations) the first hand in P may be relied upon as giving a genuine reading even where a correction by the first hand has been made *at the same time*. But in many places the first hand has made corrections *afterwards*; on these occasions he must have used new sources, e.g. when inserting the scholia to the first Book which P *alone* has, and in a number of passages he has made additions from Theonine MSS.

We cannot make out any "family tree" for the different Theonine MSS. Although they all proceeded from a common archetype later than the edition of Theon itself, they cannot have been copied one from the other; for, if they had been, how could it have come about that in one place or other each of them agrees *alone* with P in preserving the genuine reading? Moreover the great variety in their agreements and disagreements indicates that they have all diverged to about the same extent from their archetype. As we have seen that P contains corrections from the Theonine family, so they show corrections from P or other MSS. of the same family. Thus V has part of the lacuna in the MS. from which it was copied filled up from a MS. similar to P, and has corrections apparently derived from the same; the copyist, however, in correcting V, also used another MS. to which he alludes in the additions to IX. 19 and 30 (and also on X. 23 Por.): "in the book of the Ephesian (this) is not found." Who this Ephesian of the 12th c. was, we do not know.

We now come to the alterations made by Theon in his edition of the *Elements*. I shall indicate *classes* into which these alterations may be divided but without details (except in cases where they affect the *mathematical content* as distinct from form or language pure and simple)[1].

I. *Alterations made by Theon where he found, or thought he found, mistakes in the original.*

1. Real blots in the original which Theon saw and tried to remove.

(*a*) Euclid has a porism (corollary) to VI. 19, the enunciation of which speaks of similar and similarly described *figures* though the proposition itself refers only to triangles, and therefore the porism should have come after VI. 20. Theon substitutes *triangle* for *figure* and proves the more general porism after VI. 20.

(*b*) In IX. 19 there is a statement which is obviously incorrect. Theon saw this and altered the proof by reducing four alternatives to two, with the result that it fails to correspond to the enunciation even with Theon's substitution of "if" for "when" in the enunciation.

(*c*) Theon omits a porism to IX. 11, although it is necessary for the proof of the succeeding proposition, apparently because, owing to an error in the text (*κατὰ τὸν* corrected by Heiberg into *ἐπὶ τὸ*), he could not get out of it the right sense.

(*d*) I should also put into this category a case which Heiberg classifies among those in which Theon merely fancied that he found mistakes, viz. the porism to V. 7 stating that, if four magnitudes are proportional, they are proportional inversely. Theon puts this after V. 4 with a proof, which however has no necessary connexion with V. 4 but is obvious from the definition of proportion.

(*e*) I should also put under this head XI. 1, where Euclid's argument to prove that two straight lines cannot have a common segment is altered.

2. Passages which seemed to Theon to contain blots, and which he therefore set himself to correct, though more careful consideration would have shown that Euclid's words are right or at least may be excused and offer no difficulty to an intelligent reader. Under this head come:

(*a*) an alteration in III. 24.

(*b*) a perfectly unnecessary alteration, in VI. 14, of "equiangular parallelograms" into "parallelograms having one angle equal to one angle," where Theon followed the false analogy of VI. 15.

(*c*) an omission of words in V. 26, owing to his having been misled by a wrong figure.

(*d*) an alteration of the order of XI. Deff. 27, 28.

(*e*) the substitution of "parallelepipedal solid" for "cube" in XI.

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[1] Exhaustive details under all the different heads are given by Heiberg (Vol. v. pp. lii—lxxv).

38, because Theon observed, correctly enough, that it was true of the parallelepipedal solid in general as well as of the cube, but failed to give weight to the fact that Euclid must have given the particular case of the cube for the simple reason that that was all he wanted for use in XIII. 17.

(*f*) the substitution of the letter Φ for Ω (*V* for *Z* in my figure) because he saw that the perpendicular from K to BΦ would fall on Φ itself, so that Φ, Ω coincide. But, if the substitution is made, it should be *proved* that Φ, Ω coincide. Euclid can hardly have failed to notice the fact, but it may be that he deliberately ignored it as unnecessary for his purpose, because he did not want to lengthen his proposition by giving the proof.

II. *Emendations intended to improve the form or diction of Euclid.*

Some of these emendations of Theon affect passages of appreciable length. Heiberg notes about ten such passages; the longest is in Eucl. XII. 4 where a whole page of Heiberg's text is affected and Theon's version is put in the Appendix. The kind of alteration may be illustrated by that in IX. 15 where Euclid uses successively the propositions VII. 24, 25, quoting the enunciation of the former but not of the latter; Theon does exactly the reverse. In a few of the cases here quoted by Heiberg, Theon shortened the original somewhat.

But, as a rule, the emendations affect only a few words in each sentence. Sometimes they are considerable enough to alter the conformation of the sentence, sometimes they are trifling alterations "more magistellorum ineptorum" and unworthy of Theon. Generally speaking, they were prompted by a desire to change anything which was out of the common in expression or in form, in order to reduce the language to one and the same standard or norm. Thus Theon changed the order of words, substituted one word for another where the latter was used in a sense unusual with Euclid (e.g. ἐπειδήπερ, "since," for ὅτι in the sense of "because"), or one expression for another in like circumstances (e.g. where, finding "that which was enjoined would be done" in a *theorem*, VII. 31, and deeming the phrase more appropriate to a *problem*, he substituted for it "that which is sought would be manifest"; probably also and for similar reasons he made certain variations between the two expressions usual at the end of propositions ὅπερ ἔδει δεῖξαι and ὅπερ ἔδει ποιῆσαι, *quod erat demonstrandum* and *quod erat faciendum*). Sometimes his alterations show carelessness in the use of technical terms, as when he uses ἅπτεσθαι (to *meet*) for ἐφάπτεσθαι (to *touch*) although the ancients carefully distinguished the two words. The desire of keeping to a standard phraseology also led Theon to omit or add words in a number of cases, and also, sometimes, to change the lettering of figures.

But Theon seems, in editing the *Elements*, to have bestowed the most attention upon

III. *Additions designed to supplement or explain Euclid.*

First, he did not hesitate to interpolate whole propositions where he thought there was room or use for them. We have already mentioned the addition to VI. 33 of the second part relating to *sectors*, for which Theon himself takes credit in his commentary on Ptolemy. Again, he interpolated the proposition commonly known as VII. 22 (*ex aequo in proportione perturbata* for numbers, corresponding to V. 23), and perhaps also VII. 20, a particular case of VII. 19 as VI. 17 is of VI. 16. He added a second case to VI. 27, a porism to II. 4, a second porism to III. 16, and a lemma after X. 12; perhaps also the porism to V. 19 and the first porism to VI. 20. He also inserted alternative proofs here and there, e.g. in II. 4 (where the alternative differs little from the original) and in VII. 31; perhaps also in X. 1, 6, and 9.

Secondly, he sometimes repeats an argument where Euclid had said "For the same reason," adds specific references to points, straight lines etc. in the figures in order to exclude the possibility of mistake arising from Euclid's reference to them in general terms, or inserts words to make the meaning of Euclid more plain, e.g. *componendo* and *alternately*, where Euclid had left them out. Sometimes he thought to increase by his additions the mathematical precision of Euclid's language in enunciations or elsewhere, sometimes to make smoother and clearer things which Euclid had expressed with unusual brevity and harshness or carelessness, in reliance on the intelligence of his readers.

Thirdly, he supplied intermediate steps where Euclid's argument seemed too rapid and not easy enough to follow. The form of these additions varies; they are sometimes placed as a definite intermediate step with "therefore" or "so that," sometimes they are additions to the statement of premisses, sometimes phrases introduced by "since," "for" and the like, after the inference.

Lastly, there is a very large class of additions of a word, or one or two words, for the sake of clearness or consistency. Heiberg gives a number of examples of the addition of such nouns as "triangle," "square," "rectangle," "magnitude," "number," "point," "side," "circle," "straight line," "area" and the like, of adjectives such as "remaining," "right," "whole," "proportional," and of other parts of speech, even down to words like "is" (ἐστί) which is added 600 times, δή, ἄρα, μέν, γάρ, καί and the like.

IV. *Omissions by Theon.*

Heiberg remarks that, Theon's object having been, as above shown, to amplify and explain Euclid, we should not naturally have expected to find him doing much in the contrary process of compression, and it is only owing to the recurrence of a certain sort of omissions so frequently (especially in the first Books) as to exclude the hypothesis of their being all due to chance that we are bound to credit him with alterations making for greater brevity. We have seen, it is true, that he made omissions as well as additions for the purpose of reducing the language to a certain standard form. But there are also a good number of cases where in the enunciation of propositions, and in the *exposition* (the re-statement of them with reference to the figure), he has left out words because, apparently, he regarded Euclid's language as being *too* careful and precise.

Again, he is apparently responsible for the frequent omission of the words ὅπερ ἔδει δεῖξαι (or ποιῆσαι), Q.E.D. (or F.), at the end of propositions. This is often the case at the end of porisms, where, in omitting the words, Theon seems to have deliberately departed from Euclid's practice. The MS. P seems to show clearly that, where Euclid put a porism at the end of a proposition, he omitted the Q.E.D. at the end of the proposition but inserted it at the end of the porism, as if he regarded the latter as being actually a part of the proposition itself. As in the Theonine MSS. the Q.E.D. is generally omitted, the omission would seem to have been due to Theon. Sometimes in these cases the Q.E.D. is interpolated at the end of the proposition.

Heiberg summed up the discussion of Theon's edition by the remark that Theon evidently took no pains to discover and restore from MSS. the actual words which Euclid had written, but aimed much more at removing difficulties that might be felt by learners in studying the book. His edition is therefore not to be compared with the editions of the Alexandrine grammarians, but rather with the work done by Eutocius in editing Apollonius and with an interpolated recension of some of the works of Archimedes by a certain Byzantine, Theon occupying a position midway between these two editors, being superior to the latter in mathematical knowledge but behind Eutocius in industry (these views now require to be somewhat modified, as above stated). But however little Theon's object may be approved by those of us who would rather know the *ipsissima verba* of Euclid, there is no doubt that his work was approved by his pupils at Alexandria for whom it was written; and his edition was almost exclusively used by later Greeks, with the result that the more ancient text is only preserved to us in one MS.

As the result of the above investigation, we may feel satisfied that, where P and the Theonine MSS. agree, they give us (except in a few accidental instances) Euclid as he was read by the Greeks of the 4th c. But even at that time the text had been passed from hand to hand through more than six centuries, so that it is certain that it had already suffered changes, due partly to the fault of copyists and partly to the interpolations of mathematicians. Some errors of copyists escaped Theon and were corrected in some MSS. by later hands. Others appear in all our MSS. and, as they cannot have arisen accidentally in all, we must put them down to a common source more ancient than Theon. A somewhat serious instance is to be found in III. 8; and the use of ἁπτέσθω for ἐφαπτέσθω in the sense of "touch" may also be mentioned, the proper distinction between the words having been ignored as it was by Theon also. But there are a number of imperfections in the ante-Theonine text which it would be unsafe to put down to the errors of copyists, those namely where the good MSS. agree and it is not possible to see any motive that a copyist could have had for altering a correct reading. In these cases it is possible that the imperfections are due to a certain degree of carelessness on the part of Euclid himself; for it is not possible "Euclidem ab omni naevo vindicare," to use the words of Saccheri[1], and consequently Simson is not right in attributing to Theon and other editors all the things in Euclid to which mathematical objection can be taken. Thus, when Euclid speaks of "the ratio compounded of the sides" for "the ratio compounded of the *ratios of the* sides," there is no reason for doubting that Euclid himself is responsible for the more slip-shod expression. Again, in the Books XI.—XIII. relating to solid geometry there are blots neither few nor altogether unimportant which can only be attributed to Euclid himself[2]; and there is the less reason for hesitation in so attributing them because solid geometry was then being treated in a thoroughly systematic manner for the first time. Sometimes the *conclusion* (συμπέρασμα) of a proposition does not correspond exactly to the enunciation, often it is cut short with the words καὶ τὰ ἑξῆς "and the rest" (especially from Book X. onwards), and very often in Books VIII., IX. it is omitted. Where all the MSS. agree, there is no ground for hesitating to attribute the abbreviation or omission to Euclid; though, of course, where one or more MSS. have the longer form, it must be retained because this is one of the cases where a copyist has a temptation to abbreviate.

Where the true reading is preserved in one of the Theonine MSS. alone, Heiberg attributes the wrong reading to a mistake which arose before Theon's time, and the right reading of the single MS. to a successful correction.

We now come to the most important question of the *Interpolations introduced before Theon's time.*

I. Alternative proofs or additional cases.

It is not in itself probable that Euclid would have given two proofs of the same proposition; and the doubt as to the genuineness of the alternatives is increased when we consider the character of some of them and the way in which they are introduced. First of all, we have those of VI. 20 and XII. 17 introduced by "we shall prove this otherwise *more readily* (προχειρότερον)" or that of X. 90 "it is possible to prove *more shortly* (συντομώτερον)." Now it is impossible to suppose that Euclid would have given one proof as that definitely accepted by him and then added another with the express comment that the latter has certain advantages over the former. Had he considered the two proofs and come to this conclusion, he would have inserted the latter in the received text instead of the former. These alternative proofs must therefore have been interpolated. The same argument applies to alternatives introduced with the words "or even thus" (ἢ καὶ οὕτως), "or even otherwise" (ἢ καὶ ἄλλως). Under this head come the alternatives for the last portions of III. 7, 8; and Heiberg also compares the alternatives for parts of III. 31 (that the angle in a semicircle is a right angle) and XIII. 18, and the alternative proof of the lemma after X. 32. The alternatives to X. 105 and 106, again, are condemned by the place in which they occur, namely after an alternative proof to X. 115. The above alternatives being all admitted to be spurious, suspicion must necessarily attach to the few others which are in themselves unobjectionable. Heiberg instances the alternative proofs to III. 9, III. 10, VI. 30, VI. 31 and XI. 22, observing that it is quite comprehensible that any of these might have occurred to a teacher or editor and seemed to him, rightly or wrongly, to be better than the corresponding proofs in Euclid. Curiously enough, Simson adopted the alternatives to III. 9, 10 in preference to the genuine proofs. Since Heiberg's preface was written, his suspicion has been amply confirmed as regards III. 10 by the commentary of an-Nairīzī (ed. Curtze) which shows not only that this alternative is Heron's, but also that the substantive proposition III. 12 in Euclid is also Heron's, having been given by him to supplement III. 11 which must originally have been enunciated of circles "touching one another" simply, i.e. so as to include the case of external as well as internal contact, though the proof covered the case of internal contact only. "Euclid, in the 11th proposition," says Heron, "supposed two circles touching one another internally and wrote the proposition on this case, proving what it was required to prove in it. *But I will show how it is to be proved if the contact be external*[1]." This additional proposition of Heron's is by way of adding another *case*, which brings us to that class of interpolation. It was the practice of Euclid and the ancients to give only one case (generally the most difficult one) and to leave the others to be investigated by the reader for himself. One interpolation of a second case (VI. 27) is due, as we have seen, to Theon. The two extra cases of XI. 23 were manifestly interpolated before Theon's time, for the preliminary distinction of three cases, "(the centre) will either be within the triangle *LMN*, or on one of the sides, or outside. First let it be within," is a spurious addition (B and V only). Similarly an unnecessary case is interpolated in III. 11.

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[1] *Euclides ab omni naevo vindicatus*, Mediolani, 1733.

[2] Cf. especially the assumption, without proof or definition, of the criterion for *equal* solid angles, and the incomplete proof of XII. 17.

II. Lemmas.

Heiberg has unhesitatingly placed in his Appendix to Vol. III. certain lemmas interpolated either by Theon (on X. 13) or later writers (on X. 27, 29, 31, 32, 33, 34, where V only has the lemmas). But we are here concerned with the lemmas found in all the MSS., which however are, for different reasons, necessarily suspected. We will deal with the Book X. lemmas last.

(1) There is an *a priori* ground of objection to those lemmas which come *after* the propositions to which they relate and prove properties used in those propositions; for, if genuine, they would be a sign of faulty arrangement such as would not be likely in a systematic work so carefully ordered as the *Elements*. The lemma to VI. 22 is one of this class, and there is the further objection to it that in VI. 28 Euclid makes an assumption which would equally require a lemma though none is found. The lemma after XII. 4 is open to the further objections that certain altitudes are used but are not drawn in the

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[1] An-Nairīzī, ed. Curtze, p. 121.

figure (which is not in the manner of Euclid), and that a peculiar expression "parallelepipedal solids *described on* (*ἀναγραφόμενα ἀπό*) *prisms*" betrays a hand other than Euclid's. There is an objection on the score of language to the lemma after XIII. 2. The lemmas on XI. 23, XIII. 13, XIII. 18, besides coming after the propositions to which they relate, are not very necessary in themselves and, as regards the lemma to XIII. 13, it is to be noticed that the writer of a gloss in the proposition could not have had it, and the words "as will be proved afterwards" in the text are rightly suspected owing to differences between the MS. readings. The lemma to XII. 2 also, to which Simson raised objection, comes *after* the proposition; but, if it is rejected, the words "as was proved before" used in XII. 5 and 18, and referring to this lemma, must be struck out.

(2) Reasons of substance are fatal to the lemma before X. 60, which is really assumed in X. 44 and therefore should have appeared there if anywhere, and to the lemma on X. 20, which tries to prove what is already stated in X. Def. 4.

We now come to the remaining lemmas in Book X., eleven in number, which come *before* the propositions to which they relate and remove difficulties in the way of their demonstration. That before X. 42 introduces a set of propositions with the words "that the said irrational straight lines are uniquely divided ... we will prove after premising the following lemma," and it is not possible to suppose that these words are due to an interpolator; nor are there any objections to the lemmas before X. 14, 17, 22, 33, 54, except perhaps that they are rather easy. The lemma before X. 10 and X. 10 itself should probably be removed from the *Elements*; for X. 10 really uses the following proposition X. 11, which is moreover numbered 10 by the first hand in P, and the words in X. 10 referring to the lemma "for we learnt (how to do this)" betray the interpolator. Heiberg gives reason also for rejecting the lemmas before X. 19 and 24 with the words "in any of the aforesaid ways" (omitted in the Theonine MSS.) in the enunciations of X. 19, 24 and in the *exposition* of X. 20. Lastly, the lemmas before X. 29 may be genuine, though there is an addition to the second of them which is spurious.

Heiberg includes under this heading of interpolated lemmas two which purport to be substantive propositions, XI. 38 and XIII. 6. These must be rejected as spurious for reasons which will be found in detail in my notes on XI. 37 and XIII. 6 respectively. The latter proposition is only quoted once (in XIII. 17); probably the words quoting it (with γραμμή instead of εὐθεῖα) are themselves interpolated, and Euclid thought the fact stated a sufficiently obvious inference from XIII. 1.

III. Porisms (or corollaries).

Most of the porisms in the text are both genuine and necessary; but some are shown by differences in the MSS. not to be so, e.g. those to I. 15 (though Proclus has it), III. 31 and VI. 20 (Por. 2). Sometimes parts of porisms are interpolated. Such are the last few lines in the porisms to IV. 5, VI. 8; the latter addition is proved later by means of VI. 4, 8, so that the writer of these proofs could not have had the addition to VI. 8 Por. before him. Lastly, interpolators have added a sort of proof to some porisms, as though they were not quite obvious enough; but to add a demonstration is inconsistent with the idea of a porism, which, according to Proclus, is a by-product of a proposition appearing without our seeking it.

IV. Scholia.

Several interpolated scholia betray themselves by their wording, e.g. those given by Heiberg in the Appendix to Book X. and containing the words καλεῖ, ἐκάλεσε ("he calls" or "called"); these scholia were apparently written as marginal notes before Theon's time, and, being adopted as such by Theon, found their way into the text in P and some of the Theonine MSS. The same thing no doubt accounts for the interpolated analyses and syntheses to XIII. 1—5, as to which see my note on XIII. 1.

V. Interpolations in Book X.

First comes the proposition "*Let it be proposed to us* to show that in square figures the diameter is incommensurable in length with the side," which, with a scholium after it, ends the tenth Book. The form of the enunciation is suspicious enough and the proposition, the proof of which is indicated by Aristotle and perhaps was Pythagorean, is perfectly unnecessary when X. 9 has preceded. The scholium ends with remarks about commensurable and incommensurable *solids*, which are of course out of place before the Books on solids. The scholiast on Book X. alludes to this particular scholium as being due to "Theon and some others." But it is doubtless much more ancient, and may, as Heiberg conjectures have been the beginning of Apollonius' more advanced treatise on incommensurables. Not only is everything in Book X. after X. 115 interpolated, but Heiberg doubts the genuineness even of X. 112—115, on the ground that X. 111 rounds off the theory of incommensurables as we want it in the Books on solid geometry, while X. 112—115 are not really connected with what precedes, nor wanted for the later Books, but seem to form the starting-point of a new and more elaborate theory of irrationals.

VI. Other minor interpolations are found of the same character as those above attributed to Theon. First there are two places (XI. 35 and XI. 26) where, after "similarly we shall prove" and "for the same reason," an actual proof is nevertheless given. Clearly the proofs are interpolated; and there are other similar interpolations. There are also interpolations of intermediate steps in proofs, unnecessary explanations and so on, as to which I need not enter into details.

Lastly, following Heiberg's order, I come to

VII. Interpolated definitions, axioms etc.

Apart from VI. Def. 5 (which may have been interpolated by Theon although it is found written in the margin of P by the first hand), the definition of a segment of a circle in Book I. is interpolated, as is clear from the fact that it occurs in a more appropriate place in Book III. and Proclus omits it. VI. Def. 2 (reciprocal figures) is rightly condemned by Simson—perhaps it was taken from Heron—and

Heiberg would reject VII. Def. 10, as to which see my note on that definition. Lastly the double definition of a solid angle (XI. Def. 11) constitutes a difficulty. The use of the word ἐπιφάνεια suggests that the first definition may have been older than Euclid, and he may have quoted it from older *elements*, especially as his own definition which follows only includes solid angles contained by *planes*, whereas the other includes other sorts (cf. the words γραμμῶν, γραμμαῖς) which are also distinguished by Heron (Def. 22). If the first definition had come last, it could have been rejected without hesitation: but it is not so easy to reject the first part up to and including "otherwise" (ἄλλως). No difficulty need be felt about the definitions of "oblong," "rhombus," and "rhomboid," which are not actually used in the *Elements*; they were no doubt taken from earlier *elements* and given for the sake of completeness.

As regards the axioms or, as they are called in the text, *common notions* (κοιναὶ ἔννοιαι), it is to be observed that Proclus says[1] that Apollonius tried to prove "the axioms," and he gives Apollonius' attempt to prove Axiom 1. This shows at all events that Apollonius had *some* of the axioms now appearing in the text. But how could Apollonius have taken a controversial line against Euclid on the subject of axioms if these axioms had not been Euclid's to his knowledge? And, if they had been interpolated between Euclid's time and his own, how could Apollonius, living so comparatively short a time after Euclid, have been ignorant of the fact? Therefore *some* of the axioms are Euclid's (whether he called them *common notions*, or *axioms*, as is perhaps more likely since Proclus calls them axioms): and we need not hesitate to accept as genuine the first three discussed by Proclus, viz. (1) things equal to the same equal to one another, (2) if equals be added to equals, wholes equal, (3) if equals be subtracted from equals, remainders equal. The other two mentioned by Proclus (whole greater than part, and congruent figures equal) are more doubtful, since they are omitted by Heron, Martianus Capella, and others. The axiom that "two lines cannot enclose a space" is however clearly an interpolation due to the fact that I. 4 appeared to require it. The others about equals added to unequals, doubles of the same thing, and halves of the same thing are also interpolated; they are connected with other interpolations, and Proclus clearly used some source which did not contain them.

Euclid evidently limited his formal axioms to those which seemed to him most essential and of the widest application; for he not unfrequently assumes other things as axiomatic, e.g. in VII. 28 that, if a number measures two numbers, it measures their difference.

The differences of reading appearing in Proclus suggest the question of the comparative purity of the sources used by Proclus, Heron and others, and of our text. The omission of the definition of a segment in Book I. and of the old gloss "which is called the circumference" in I. Def. 15 (also omitted by Heron, Taurus, Sextus

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[1] Proclus, pp. 194, 10 sqq.

Empiricus and others) indicates that Proclus had better sources than we have; and Heiberg gives other cases where Proclus omits words which are in all our MSS. and where Proclus' reading should perhaps be preferred. But, except in these instances (where Proclus may have drawn from some ancient source such as one of the older commentaries), Proclus' MS. does not seem to have been among the best. Often it agrees with our worst MSS., sometimes it agrees with F where F alone has a certain reading in the text, so that (e.g. in I. 15 Por.) the common reading of Proclus and F must be rejected, thrice only does it agree with P alone, sometimes it agrees with P and some Theonine MSS., and once it agrees with the Theonine MSS. against P and other sources.

Of the other external sources, those which are older than Theon generally agree with our best MSS., e.g. Heron, allowing for the difference in the plan of his definitions and the somewhat free adaptation to his purpose of the Euclidean definitions in Books X., XI.

Heiberg concludes that the *Elements* were most spoiled by interpolations about the 3rd c., for Sextus Empiricus had a correct text, while Iamblichus had an interpolated one; but doubtless the purer text continued for a long time in circulation, as we conclude from the fact that our MSS. are free from interpolations already found in Iamblichus' MS.

CHAPTER VI.

THE SCHOLIA.

HEIBERG has collected scholia, to the number of about 1500, in Vol. V. of his edition of Euclid, and has also discussed and classified them in a separate short treatise, in which he added a few others[1].

These scholia cannot be regarded as doing much to facilitate the reading of the *Elements*. As a rule, they contain only such observations as any intelligent reader could make for himself. Among the few exceptions are XI. Nos. 33, 35 (where XI. 22, 23 are extended to solid angles formed by any number of plane angles), XII. No. 85 (where an assumption tacitly made by Euclid in XII. 17 is proved), IX. Nos. 28, 29 (where the scholiast has pointed out the error in the text of IX. 19).

Nor are they very rich in historical information; they cannot be compared in this respect with Proclus' commentary on Book I. or with those of Eutocius on Archimedes and Apollonius. But even under this head they contain some things of interest, e.g. II. No. 11 explaining that the gnomon was invented by geometers for the sake of brevity, and that its name was suggested by an incidental characteristic, namely that "from it the whole is known (*γνωρίζεται*), either of the whole area or of the remainder, when it (the *γνώμων*) is either placed round or taken away"; II. No. 13, also on the gnomon; IV. No. 2 stating that Book IV. was the discovery of the Pythagoreans; V. No. 1 attributing the content of Book V. to Eudoxus; X. No. 1 with its allusion to the discovery of incommensurability by the Pythagoreans and to Apollonius' work on irrationals; X. No. 62 definitely attributing X. 9 to Theaetetus; XIII. No. 1 about the "Platonic" figures, which attributes the cube, the pyramid, and the dodecahedron to the Pythagoreans, and the octahedron and icosahedron to Theaetetus.

Sometimes the scholia are useful in connexion with the settlement of the text, (1) directly, e.g. III. No. 16 on the interpolation of the word "within" (*ἐντός*) in the enunciation of III. 6, and X. No. 1 alluding to the discussion by "Theon and some others" of irrational "surfaces" and "solids," as well as "lines," from which we may conclude that the scholium at the end of Book X. is not genuine; (2) indirectly in that they sometimes throw light on the connexion of certain MSS.

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[1] Heiberg, *Om Scholierne til Euklids Elementer*, Kjøbenhavn, 1888. The tract is written in Danish, but, fortunately for those who do not read Danish easily, the author has appended (pp. 70—78) a résumé in French.

Lastly, they have their historical importance as enabling us to judge of the state of mathematical science at the times when they were written.

Before passing to the classification of the scholia, Heiberg remarks that we must separate from them a number of additions in the nature of scholia which are found in the text of our MSS. but which can, in one way or another, be proved to be spurious. As they are found both in P and in the Theonine MSS., they must have been in the MSS. anterior to Theon (4th c.). But they are, in great part, only found in the margin of P and the Theonine MSS.; in V they are half in the text and half in the margin. This can hardly be explained except on the supposition that these additions were originally (in the MSS. before Theon) in the margin, and that Theon kept them there in his edition, but that they afterwards found their way gradually into the text of P as well as of the Theonine MSS., or were omitted altogether, while particular MSS. have in certain places preserved the old arrangement. Of such spurious additions Heiberg enumerates the following: the axiom about equals subtracted from unequals, the last lines of the porism to VI. 8, second porisms to V. 19 and to VI. 20, the porism to III. 31, VI. Def. 5, various additions in Book X., the analyses and syntheses of XIII. 1—5, and the proposition XIII. 6.

The two first classes of scholia distinguished by Heiberg are denoted by the convenient abbreviations "Schol. Vat." and "Schol. Vind."

I. **Schol. Vat.**

It is first necessary to set out the letters by which Heiberg denotes certain collections of scholia.

P = Scholia in P written by the first hand.

B = Scholia in B by a hand of the same date as the MS. itself, generally that of Arethas.

F = Scholia in F by the first hand.

Vat. = Scholia of the Vatican MS. 204 of the 10th c., which has these scholia on leaves 198—205 (the end is missing) as an independent collection. It does not contain the text of the *Elements*.

V^c = Scholia found on leaves 283—292 of V and written in the same hand as that part of the MS. itself which begins at leaf 235.

Vat. 192 = a Vatican MS. of the 14th c. which contains, after (1) the *Elements* I.—XIII. (without scholia), (2) the *Data* with scholia, (3) Marinus on the *Data*, the Schol. Vat. as an independent collection and in their entirety, beginning with I. No. 88 and ending with XIII. No. 44.

The Schol. Vat., the most ancient and important collection of scholia, comprise those which are found in PBF Vat. and, from VII. 12 to IX. 15, in PB Vat. only, since in that portion of the *Elements* F was restored by a later hand without scholia; they also include I.

No. 88 which only happens to be erased in F, and IX. Nos. 28, 29 which may be left out because F. here has a different text. In F and Vat. the collection ends with Book X.; but it must also include Schol. PB of Books XI.—XIII., since these are found along with Schol. Vat. to Books I.—X. in several MSS. (of which Vat. 192 is one) as a separate collection. The Schol. Vat. to Books X.—XIII. are also found in the collection V^c (where, curiously enough, XIII. Nos. 43, 44 are at the beginning). The Schol. Vat. accordingly include Schol. PBV^c Vat. 192, and doubtless also those which are found in two of these sources. The total number of scholia classified by Heiberg as Schol. Vat. is 138.

As regards the contents of Schol. Vat. Heiberg has the following observations. The thirteen scholia to Book I. are extracts made from Proclus by a writer thoroughly conversant with the subject, and cleverly recast (with some additions). Their author does not seem to have had the two lacunae which our text of Proclus has (at the end of the note on I. 36 and the beginning of the next note, and at the beginning of the note on I. 43), for the scholia I. Nos. 125 and 137 seem to fill the gaps appropriately, at least in part. In some passages he had better readings than our MSS. have. The rest of Schol. Vat. (on Books II.—XIII.) are essentially of the same character as those on Book I., containing prolegomena, remarks on the object of the propositions, critical remarks on the text, converses, lemmas; they are, in general, exact and true to tradition. The reason of the resemblance between them and Proclus appears to be due to the fact that they have their origin in the commentary of Pappus, of which we know that Proclus also made use. In support of the view that Pappus is the source, Heiberg places some of the Schol. Vat. to Book X. side by side with passages from the commentary of Pappus in the Arabic translation discovered by Woepcke[1]; he also refers to the striking confirmation afforded by the fact that XII. No. 2 contains the solution of the problem of inscribing in a given circle a polygon similar to a polygon inscribed in another circle, which problem Eutocius says[2] that Pappus gave in his commentary on the *Elements*.

But, on the other hand, Schol. Vat. contain some things which cannot have come from Pappus, e.g. the allusion in X. No. 1 to Theon and irrational surfaces and solids, Theon being later than Pappus; III. No. 10 about *porisms* is more like Proclus' treatment of the subject than Pappus', though one expression recalls that of Pappus about *forming* (σχηματίζεσθαι) the enunciations of porisms like those of either theorems or problems.

The Schol. Vat. give us important indications as regards the text of the *Elements* as Pappus had it. In particular, they show that he could not have had in his text certain of the lemmas in Book X. For example, three of these are identical with what we find in Schol.

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[1] *Om Scholierne til Euklids Elementer*, pp. 11, 12: cf. *Euklid-Studien*, pp. 170, 171; Woepcke, *Mémoires présent. à l'Acad. des Sciences*, 1856, XIV. p. 658 sqq.

[2] Archimedes, ed. Heiberg, III. p. 28, 19—22.

Vat. (the lemma to x. 17 = Schol. x. No. 106, and the lemmas to x. 54, 60 come in Schol. x. No. 328); and it is not possible to suppose that these lemmas, if they were already in the text, would also be given as scholia. Of these three lemmas, that before x. 60 has already been condemned for other reasons; the other two, unobjectionable in themselves, must be rejected on the ground now stated. There were four others against which Heiberg found nothing to urge when writing his prolegomena to Vol. v., viz. the lemmas before x. 42, x. 14, x. 22 and x. 33. Of these, the lemma to x. 22 is not reconcilable with Schol. x. No. 161, which takes up the assumption in the text of Eucl. x. 22 as if no lemma had gone before. The lemma to x. 42, which, on account of the words introducing it (see p. 60 above), Heiberg at first hesitated to regard as an interpolation, is identical with Schol. x. No. 270. It is true that in Schol. x. No. 269 we find the words "this lemma has been proved before (*ἐν τοῖς ἔμπροσθεν*), but it shall also be proved now for convenience' sake (*τοῦ ἐτοίμου ἕνεκα*)," and it is possible to suppose that "before" may mean in Euclid's text *before* x. 42; but a proof in that place would surely have been as "convenient" as could be desired, and it is therefore more probable that the proof had been given by *Pappus* in some earlier place. (It may be added that the lemma to x. 14, which is identical with the lemma to XI. 23, condemned on other grounds, is for that reason open to suspicion.)

Heiberg's conclusion is that *all* the lemmas are spurious, and that most or all of them have found their way into the text from Pappus' commentary, though at a time anterior to Theon's edition, since they are found in all our MSS. This enables us to fix a date for these interpolations, namely the first half of the 4th c.

Of course Pappus had not in his text the interpolations which, from the fact of their appearing only in some of our MSS., are seen to be later than those above-mentioned. Such are the lemmas which are found in the text of V only after x. 29 and x. 31 respectively and are given in Heiberg's Appendix to Book x. (numbered 10 and 11). On the other hand it appears from Woepcke's tract[1] that Pappus already had x. 115 in his text: though it does not follow from this that the proposition is genuine but only that interpolations began very early.

Theon interpolated a proposition (or lemma) between x. 12 and x. 13 (No. 5 in Heiberg's Appendix). Schol. Vat. has the same thing (x. No. 125). The writer of the scholia therefore did not find this lemma in the text. Schol. Vat. IX. Nos. 28, 29 show that neither did he find in his text the alterations which Theon made in Eucl. IX. 19; the scholia in fact only agree with the text of P, not with Theon's. This suggests that Schol. Vat. were written for use with a MS. of the ante-Theonine recension such as P is. This probability is further confirmed by a certain independence which P shows in several places when compared with the Theonine MSS. Not only has P better readings in some passages, but more substantial divergences; and, in particular, the absence in P of three notes of a historical character which are added, wholly or partly from Proclus, in the Theonine MSS. attests an independent and more primitive point of view in P.

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[1] Woepcke, *op. cit.* p. 702.

In view of the distinctive character of P, it is possible that some of the scholia found in it in the first hand, but not in the other sources of Schol. Vat., also belong to that collection; and several circumstances confirm this. Schol. XIII. No. 45, found in P only, which relates to a passage in Eucl. XIII. 13, shows that certain words in the text, though older than Theon, are interpolated; and, as the scholium is itself older than Theon, is headed "*third* lemma," and follows a "second lemma" relating to a passage in the text immediately preceding, which "second lemma" belongs to Schol. Vat. and is taken from Pappus, the "third" in all probability came from Pappus also. The same is true of Schol. XII. No. 72 and XIII. No. 69, which are respectively identical with the propositions *vulgo* XI. 38 (Heiberg, App. to Book XI., No. 3) and XIII. 6; for both of these interpolations are older than Theon. Moreover most of the scholia which P in the first hand alone has are of the same character as Schol. Vat. Thus VII. No. 7 and XIII. No. 1 introducing Books VII. and XIII. respectively are of the same historical character as several of Schol. Vat.; that VII. No. 7 appears in the *text* of P at the beginning of Book VII. constitutes no difficulty. There are a number of *converses*, remarks on the relation of propositions to one another, explanations such as XII. No. 89 in which it is remarked that Φ, Ω in Euclid's figure to XII. 17 (Z, V in my figure) are really the same point but that this makes no difference in the proof. Two other Schol. P on XII. 17 are connected by their headings with XII. No. 72 mentioned above. XI. No. 10 (P) is only another form of XI. No. 11 (B); and B often, alone with P, has preserved Schol. Vat. On the whole Heiberg considers some 40 scholia found in P alone to belong to Schol. Vat.

The history of Schol. Vat. appears to have been, in its main outlines, the following. They were put together after 500 A.D., since they contain extracts from Proclus, to which we ought not to assign a date too near to that of Proclus' work itself; and they must at least be earlier than the latter half of the 9th c., in which B was written. As there must evidently have been several intermediate links between the archetype and B, we must assign them rather to the first half of the period between the two dates, and it is not improbable that they were a new product of the great development of mathematical studies at the end of the 6th c. (Isidorus of Miletus). The author extracted what he found of interest in the commentary of Proclus on Book I. and in that of Pappus on the rest of the work, and put these extracts in the margin of a MS. of the class of P. As there are no scholia to I. 1—22, the first leaves of the archetype or of one of the earliest copies must have been lost at an early date, and it was from that mutilated copy that partly P and partly a MS. of the Theonine class were taken, the scholia being put in the margin in both. Then the collection spread through the Theonine MSS., gradually losing some scholia which could not be read or understood, or which were accidentally or deliberately omitted. Next it was extracted from one of these MSS. and made into a separate work which has been preserved, in part, in its entirety (Vat. 192 etc.) and, in part, divided into sections, so that the scholia to Books X.—XIII. were detached (V^c). It had the same fate in the MSS. which kept the original arrangement (in the margin), and in consequence there are some MSS. where the scholia to the stereometric Books are missing, those Books having come to be less read in the period of decadence. It is from one of these MSS. that the collection was extracted as a separate work such as we find it in Vat. (10th c.).

II. The second great division of the scholia is **Schol. Vind.**

This title is taken from the Viennese MS. (V), and the letters used by Heiberg to indicate the sources here in question are as follows.

V^a = scholia in V written by the same hand that copied the MS. itself from fol. 235 onward.

q = scholia of the Paris MS. 2344 (q) written by the first hand.

l = scholia of the Florence MS. Laurent. XXVIII, 2 written in the 13th—14th c., mostly in the first hand, but partly in two later hands.

V^b = scholia in V written by the same hand as the first part (leaves 1—183) of the MS. itself; V^b wrote his scholia after V^a.

q^1 = scholia of the Paris MS. (q) found here and there in another hand of early date.

Schol. Vind. include scholia found in V^aq. l is nearly related to q; and in fact the three MSS. which, so far as Euclid's text is concerned, show no direct interdependence, are, as regards their scholia, derived from one original. Heiberg proves this by reference to the readings of the three in two passages (found in Schol. I. No. 109 and X. No. 39 respectively). The common source must have contained, besides the scholia found in the three MSS. V^aql, those also which are contained in two of them, for it is more unlikely that two of the three should contain common interpolations than that a particular scholium should drop out of one of them. Besides V^a and q, the scholia V^b and q^1 must equally be referred to Schol. Vind., since the greater part of their scholia are found in l. There is a lacuna in q from Eucl. VIII. 25 to IX. 14, so that for this portion of the *Elements* Schol. Vind. are represented by Vl only. Heiberg gives about 450 numbers in all as belonging to this collection.

Schol. Vind. did not all come from one source; this is shown by differences of substance, e.g. between X. Nos. 36 and 39, and by differences of time of writing: e.g. VI. No. 52 refers at the beginning to No. 55 with the words "as the scholium has it" and is therefore later than that scholium; X. No. 247 is also later than X. No. 246.

The scholia to Book I. are here also extracts from Proclus, but more copious and more verbatim than in Schol. Vat. The author has not always understood Proclus; and he had a text as bad as that of our MSS., with the same lacunae. The scholia to the other

Books are partly drawn (1) from Schol. Vat., the MSS. representing Schol. Vind. and Schol. Vat. in these cases showing nearly all possible combinations; but there is no certain trace in Schol. Vind. of the scholia peculiar to P. The author used a copy of Schol. Vat. in the form in which they were attached to the Theonine text; thus Schol. Vind. correspond to BF Vat., where these diverge from P, and especially closely to B. Besides Schol. Vat., the editors of Schol. Vind. used (2) other old collections of scholia of which we find traces in B and F; Schol. Vind. have also some scholia common with b. The scholia which Schol. Vind. have in common with BF come from two different sources, and were apparently afterwards introduced into the other MSS.; one result of this is that several scholia are reproduced twice.

But, besides the scholia derived from these sources, Schol. Vind. contain a large number of others of late date, characterised by incorrect language or by triviality of content (there are many examples in numbers, citations of propositions used, absurd *ἀπορίαι*, and the like). Unlike Schol. Vat., these scholia often quote words from Euclid as a heading (in one case a heading is inserted in Schol. Vind. where a scholium without the heading is quoted from Schol. Vat., see V. No. 14). The explanations given often presuppose very little knowledge on the part of the reader and frequently contain obscurities and gross errors.

Schol. Vind. were collected for use with a MS. of the Theonine class; this follows from the fact that they contain a note on the proposition *vulgo* VII. 22 interpolated by Theon (given in Heiberg's App. to Vol. II. p. 430). Since the scholium to VII. 39 given in V and p in the text after the title of Book VIII. quotes the proposition as VII. 39, it follows that this scholium must have been written before the interpolation of the two propositions *vulgo* VII. 20, 22; Schol. Vind. contain (VII. No. 80) the first sentence of it, but without the heading referring to VII. 39. Schol. VII. No. 97 quotes VII. 33 as VII. 34, so that the proposition *vulgo* VII. 22 may have stood in the scholiast's text but not the later interpolation *vulgo* VII. 20 (later because only found in B in the margin by the first hand). Of course the scholiast had also the interpolations earlier than Theon.

For the date of the collection we have a lower limit in the date (12th c.) of MSS. in which the scholia appear. That it was not much earlier than the 12th c. is indicated (1) by the poverty of its contents, (2) by the quality of the MS. of Proclus which was used in the compilation of it (the Munich MS. used by Friedlein with which the scholiast's excerpts are essentially in agreement belongs to the 11th—12th c.), (3) by the fact that Schol. Vind. appear only in MSS. of the 12th c. and no trace of them is found in our MSS. belonging to the 9th—10th c. in which Schol. Vat. are found. The collection may therefore probably be assigned to the 11th c. Perhaps it may be in part due to Psellus who lived towards the end of that century: for in a Florence MS. (Magliabecch. XI, 53 of the 15th c.) containing a mathematical compendium intended for use in the reading of Aristotle the scholia I. Nos. 40 and 49 appear with the name of Psellus attached.

Schol. Vind. are not found without the admixture of foreign elements in any of our three sources. In l there are only very few such in the first hand. In q there are several new scholia in the first hand, for the most part due to the copyist himself. The collection of scholia on Book X. in q (Heiberg's q^c) is also in the first hand; it is not original, and it may perhaps be due to Psellus (Maglb. has some definitions of Book X. with a heading "scholia of... Michael Psellus on the definitions of Euclid's 10th *Element*" and Schol. X. No. 9), whose name must have been attached to it in the common source of Maglb. and q; to a great extent it consists of extracts from Schol. Vind. taken from the same source as Vl. The scholia q^1 (in an ancient hand in q), confined to Book II., partly belong to Schol. Vind. and partly correspond to b^1 (Bologna MS.). q^a and q^b are in one hand (Theodorus Antiochita), the nearest to the first hand of q; they are doubtless due to an early possessor of the MS. of whom we know nothing more.

V^a has, besides Schol. Vind., a number of scholia which also appear in other MSS., one in BFb, some others in P, and some in v (Codex Vat. 1038, 13th c.); these scholia were taken from a source in which many abbreviations were used, as they were often misunderstood by V^a. Other scholia in V^a which are not found in the older sources—some appearing in V^a alone—are also not original, as is proved by mistakes or corruptions which they contain; some others may be due to the copyist himself.

V^b seldom has scholia common with the other older sources; for the most part they either appear in V^b alone or only in the later sources as v or F^2 (later scholia in F), some being original, others not. In Book X. V^b has three series of numerical examples, (1) with Greek numerals, (2) alternatives added later, also mostly with Greek numerals, (3) with Arabic numerals. The last class were probably the work of the copyist himself. These examples (cf. p. 74 below) show the facility with which the Byzantines made calculations at the date of the MS. (12th c.). They prove also that the use of the Arabic numerals (in the East-Arabian form) was thoroughly established in the 12th c.; they were actually known to the Byzantines a century earlier, since they appear, in the first hand, in an Escurial MS. of the 11th c.

Of collections in other hands in V distinguished by Heiberg (see preface to Vol. V.), V^1 has very few scholia which are found in other sources, the greater part being original; V^2, V^3 are the work of the copyist himself; V^4 are so in part only, and contain several scholia from Schol. Vat. and other sources. V^3 and V^4 are later than 13th—14th c., since they are not found in f (cod. Laurent. XXVIII, 6) which was copied from V and contains, besides V^a V^b, the greater part of V^1 and VI. No. 20 of V^2 (in the text).

In P there are, besides P^3 (a quite late hand, probably one of the old Scriptores Graeci at the Vatican), two late hands (P^2), one of which has some new and independent scholia, while the other has added the greater part of Schol. Vind., partly in the margin and partly on pieces of leaves stitched on.

Our sources for Schol. Vat. also contain other elements. In P there were introduced a certain number of extracts from Proclus, to supplement Schol. Vat. to Book I.; they are all written with a different ink from that used for the oldest part of the MS., and the text is inferior. There are additions in the other sources of Schol. Vat. (F and B) which point to a common source for FB and which are nearly all found in other MSS., and, in particular, in Schol. Vind., which also used the same source; that they are not assignable to Schol. Vat. results only from their not being found in Vat. Of other additions in F, some are peculiar to F and some common to it and b; but they are not original. F^2 (scholia in a later hand in F) contains three original scholia; the rest come from V. B contains, besides scholia common to it and F, b or other sources, several scholia which seem to have been put together by Arethas, who wrote at least a part of them with his own hand.

Heiberg has satisfied himself, by a closer study of b, that the scholia which he denotes by b, β and b^1 are by one hand; they are mostly to be found in other sources as well, though some are original. By the same hand (Theodorus Cabasilas, 15th c.) are also the scholia denoted by b^2, B^2, b^3 and B^3. These scholia come in great part from Schol. Vind., and in making these extracts Theodorus probably used one of our sources, l, mistakes in which often correspond to those of Theodorus. To one scholium is attached the name of Demetrius (who must be Demetrius Cydonius, a friend of Nicolaus Cabasilas, 14th c.); but it could not have been written by him, since it appears in B and Schol. Vind. Nor are all the scholia which bear the name of Theodorus due to Theodorus himself, though some are so.

As B^3 (a late hand in B) contains several of the original scholia of b^2, B^3 must have used b itself as his source, and, as all the scholia in B^3 are in b, the latter is also the source of the scholia in B^3 which are found in other MSS. B and b were therefore, in the 15th c., in the hands of the same person; this explains, too, the fact that b in a late hand has some scholia which can only come from B. We arrive then at the conclusion that Theodorus Cabasilas, in the 15th c., owned both the MSS. B and b, and that he transferred to B scholia which he had before written in b, either independently or after other sources, and inversely transferred some scholia from B to b. Further, B^2 are earlier than Theodorus Cabasilas, who certainly himself wrote B^3 as well as b^2 and b^3.

An author's name is also attached to the scholia VI. No. 6 and X. No. 223, which are attributed to Maximus Planudes (end of 13th c.) along with scholia on I. 31, X. 14 and X. 18 found in l in a quite late hand and published on pp. 46, 47 of Heiberg's dissertation. These seem to have been taken from lectures of Planudes on the *Elements* by a pupil who used l as his copy.

There are also in l two other Byzantine scholia, written by a late hand, and bearing the names Ioannes and Pediasimus respectively;

these must in like manner have been written by a pupil after lectures of Ioannes Pediasimus (first half of 14th c.), and this pupil must also have used l.

Before these scholia were edited by Heiberg, very few of them had been published in the original Greek. The Basel *editio princeps* has a few (V. No. 1, VI. Nos. 3, 4 and some in Book X.) which are taken, some from the Paris MS. (Paris. Gr. 2343) used by Grynaeus, others probably from the Venice MS. (Marc. 301) also used by him; one published by Heiberg, not in his edition of Euclid but in his paper on the scholia, may also be from Venet. 301, but appears also in Paris. Gr. 2342. The scholia in the Basel edition passed into the Oxford edition in the text, and were also given by August in the Appendix to his Vol. II.

Several specimens of the two series of scholia (Vat. and Vind.) were published by C. Wachsmuth (*Rhein. Mus.* XVIII. p. 132 sqq.) and by Knoche (*Untersuchungen über die neu aufgefundenen Scholien des Proklus*, Herford, 1865).

The scholia published in Latin were much more numerous. G. Valla (*De expetendis et fugiendis rebus*, 1501) reproduced apparently some 200 of the scholia included in Heiberg's edition. Several of these he obtained from two Modena MSS. which at one time were in his possession (Mutin. III B, 4 and II E, 9, both of the 15th c.); but he must have used another source as well, containing extracts from other series of scholia, notably Schol. Vind. with which he has some 87 scholia in common. He has also several that are new.

Commandinus included in his translation under the title "Scholia antiqua" the greater part of the Schol. Vat. which he certainly obtained from a MS. of the class of Vat. 192; on the whole he adhered closely to the Greek text. Besides these scholia Commandinus has the scholia and lemmas which he found in the Basel *editio princeps*, and also three other scholia not belonging to Schol. Vat., as well as one new scholium (to XII. 13) not included in Heiberg's edition, which are distinguished by different type and were doubtless taken from the Greek MS. used by him along with the Basel edition.

In Conrad Dasypodius' *Lexicon mathematicum* published in 1573 there is (on fol. 42—44) "Graecum scholion in definitiones Euclidis libri quinti elementorum appendicis loco propter pagellas vacantes annexum." This contains four scholia, and part of two others, published in Heiberg's edition, with some variations of readings, and with some new matter added (for which see pp. 64—6 of Heiberg's pamphlet). The source of these scholia is revealed to us by another work of Dasypodius, *Isaaci Monachi Scholia in Euclidis elementorum geometriae sex priores libros per C. Dasypodium in latinum sermonem translata et in lucem edita* (1579). This work contains, besides excerpts from Proclus on Book I. (in part closely related to Schol. Vind.), some 30 scholia included in Heiberg's edition, several new scholia, and the above-mentioned scholia to the definitions of Book V. published in Greek in 1573. After the scholia follow "Isaaci Monachi prolegomena in Euclidis Elementorum geometriae libros" (two definitions of geometry) and "Varia miscellanea ad geometriae cognitionem necessaria ab Isaaco Monacho collecta" (mostly the same as pp. 252, 24—272, 27 in the *Variae Collectiones* included in Hultsch's Heron); lastly, a note of Dasypodius to the reader says that these scholia were taken "ex clarissimi viri Joannis Sambuci antiquo codice manu propria Isaaci Monachi scripto." Isaak Monachus is doubtless Isaak Argyrus, 14th c.; and Dasypodius used a MS. in which, besides the passage in Hultsch's *Variae Collectiones*, there were a number of scholia marked in the margin with the name of Isaak (cf. those in b under the name of Theodorus Cabasilas). Whether the new scholia are original cannot be decided until they are published in Greek; but it is not improbable that they are at all events independent arrangements of older scholia. All but five of the others, and all but one of the Greek scholia to Book V., are taken from Schol. Vat.; three of the excepted ones are from Schol. Vind., and the other three seem to come from F (where some words of them are illegible, but can be supplied by means of Mut. III B, 4, which has these three scholia and generally shows a certain likeness to Isaak's scholia).

Dasypodius also published in 1564 the arithmetical commentary of Barlaam the monk (14th c.) on Eucl. Book II., which finds a place in Appendix IV. to the Scholia in Heiberg's edition.

Hultsch has some remarks on the origin of the scholia[1]. He observes that the scholia to Book I. contain a considerable portion of Geminus' commentary on the definitions and are specially valuable because they contain extracts from Geminus *only*, whereas Proclus, though drawing mainly upon him, quotes from others as well. On the postulates and axioms the scholia give more than is found in Proclus. Hultsch conjectures that the scholium on Book V., No. 3, attributing the discovery of the theorems to Eudoxus but their arrangement to Euclid, represents the tradition going back to Geminus, and that the scholium XIII., No. 1, has the same origin.

A word should be added about the numerical illustrations of Euclid's propositions in the scholia to Book X. They contain a large number of calculations with sexagesimal fractions[2]; the fractions go as far as *fourth-sixtieths* ($1/60^4$). Numbers expressed in these fractions are handled with skill and include some results of surprising accuracy[3]

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[1] Art. "Eukleides" in Pauly-Wissowa's *Real-Encyclopädie.*

[2] Hultsch has written upon these in *Bibliotheca Mathematica*, V_3, 1904, pp. 225—233.

[3] Thus $\sqrt{(27)}$ is given (allowing for a slight correction by means of the context) as 5 11′ 46″ 10‴, which gives for $\sqrt{3}$ the value 1 43′ 55″ 23‴, being the same value as that given by Hipparchus in his Table of Chords, and correct to the seventh decimal place. Similarly $\sqrt{8}$ is given as 2 49′ 42″ 20‴ 10⁗, which is equivalent to $\sqrt{2} = 1\cdot41421335$. Hultsch gives instances of the various operations, addition, subtraction, etc., carried out in these fractions, and shows how the extraction of the square root was effected. Cf. T. L. Heath, *History of Greek Mathematics*, I., pp. 59—63.

CHAPTER VII.

EUCLID IN ARABIA.

WE are told by Ḥājī Khalfa[1] that the Caliph al-Mansūr (754–775) sent a mission to the Byzantine Emperor as the result of which he obtained from him a copy of Euclid among other Greek books, and again that the Caliph al-Ma'mūn (813–833) obtained manuscripts of Euclid, among others, from the Byzantines. The version of the *Elements* by al-Ḥajjāj b. Yūsuf b. Matar is, if not the very first, at least one of the first books translated from the Greek into Arabic[2]. According to the *Fihrist*[3] it was translated by al-Ḥajjāj twice; the first translation was known as "Hārūni" ("for Hārūn"), the second bore the name "Ma'mūni" ("for al-Ma'mūn") and was the more trustworthy. Six Books of the second of these versions survive in a Leiden MS. (Codex Leidensis 399, 1) now in part published by Besthorn and Heiberg[4]. In the preface to this MS. it is stated that, in the reign of Hārūn ar-Rashīd (786–809), al-Ḥajjāj was commanded by Yaḥyā b. Khālid b. Barmak to translate the book into Arabic. Then, when al-Ma'mūn became Caliph, as he was devoted to learning, al-Ḥajjāj saw that he would secure the favour of al-Ma'mūn "if he illustrated and expounded this book and reduced it to smaller dimensions. He accordingly left out the superfluities, filled up the gaps, corrected or removed the errors, until he had gone through the book and reduced it, when corrected and explained, to smaller dimensions, as in this copy, but without altering the substance, for the use of men endowed with ability and devoted to learning, the earlier edition being left in the hands of readers."

The *Fihrist* goes on to say that the work was next translated by Isḥāq b. Ḥunain, and that this translation was improved by Thābit b. Qurra. This Abū Yaʿqūb Isḥāq b. Ḥunain b. Isḥāq al-ʿIbādī (d. 910) was the son of the most famous of Arabic translators, Ḥunain b. Isḥāq al-ʿIbādī (809–873), a Christian and physician to the Caliph al-Mutawakkil (847–861). There seems to be no doubt that Isḥāq, who must have known Greek as well as his father, made his translation direct from the Greek. The revision must apparently have been the subject of an arrangement between Isḥāq and Thābit, as the latter died in 901 or nine years before Isḥāq. Thābit undoubtedly consulted Greek MSS. for the purposes of his revision. This is expressly stated in a marginal note to a Hebrew version of the *Elements*, made from Isḥāq's, attributed to one of two scholars belonging to the same family, viz. either to Moses b. Tibbon (about 1244–1274) or to Jakob b. Machir (who died soon after 1306)[1]. Moreover Thābit observes, on the proposition which he gives as IX. 31, that he had not found this proposition and the one before it in the Greek but only in the Arabic; from which statement Klamroth draws two conclusions, (1) that the Arabs had already begun to interest themselves in the authenticity of the text and (2) that Thābit did not alter the numbers of the propositions in Isḥāq's translation[2]. The *Fihrist* also says that Yuḥannā al-Qass (i.e. "the Priest") had seen in the Greek copy in his possession the proposition in Book I. which Thābit took credit for, and that this was confirmed by Nazīf, the physician, to whom Yuhannā had shown it. This proposition may have been wanting in Isḥāq, and Thābit may have added it, but without claiming it as his own discovery[3]. As a fact, I. 45 is missing in the translation by al-Ḥajjāj.

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[1] *Lexicon bibliogr. et encyclop.* ed. Flügel, III. pp. 91, 92.

[2] Klamroth, *Zeitschrift der Deutschen Morgenländischen Gesellschaft*, XXXV. p. 303.

[3] *Fihrist* (tr. Suter), p. 16.

[4] *Codex Leidensis* 399, 1. *Euclidis Elementa ex interpretatione al-Hadschdschadschii cum commentariis al-Narizii*, Hauniae, part I. i. 1893, part I. ii. 1897, part II. i. 1900, part II. ii. 1905, part III. i. 1910.

The original version of Isḥāq *without* the improvements by Thābit has probably not survived any more than the first of the two versions by al-Ḥajjāj; the divergences between the MSS. are apparently due to the voluntary or involuntary changes of copyists, the former class varying according to the degree of mathematical knowledge possessed by the copyists and the extent to which they were influenced by considerations of practical utility for teaching purposes[4]. Two MSS. of the Isḥāq-Thābit version exist in the Bodleian Library (No. 279 belonging to the year 1238, and No. 280 written in 1260–1)[5]; Books I.—XIII. are in the Isḥāq-Thābit version, the non-Euclidean Books XIV., XV. in the translation of Qusṭā b. Lūqā al-Baʿlabakkī (d. about 912). The first of these MSS. (No. 279) is that (O) used by Klamroth for the purpose of his paper on the Arabian Euclid. The other MS. used by Klamroth is (K) Kjobenhavn LXXXI, undated but probably of the 13th c., containing Books V.—XV., Books V.—X. being in the Isḥāq-Thābit version, Books XI.—XIII. purporting to be in al-Ḥajjāj's translation, and Books XIV., XV. in the version of Qusṭā b. Lūqā. In not a few propositions K and O show not the slightest difference, and, even where the proofs show considerable differences, they are generally such that, by a careful comparison, it is possible to reconstruct the common archetype, so that it is fairly clear that we have in these cases, not two recensions of one translation, but arbitrarily altered and shortened copies of one and the same recension[1]. The Bodleian MS. No. 280 contains a preface, translated by Nicoll, which cannot be by Thābit himself because it mentions Avicenna (980–1037) and other later authors. The MS. was written at Marāġa in the year 1260–1 and has in the margin readings and emendations from the edition of Naṣīraddīn aṭ-Ṭūsī (shortly to be mentioned) who was living at Marāġa at the time. Is it possible that aṭ-Ṭūsī himself is the author of the preface[2]? Be this as it may, the preface is interesting because it throws light on the liberties which the Arabians allowed themselves to take with the text. After the observation that the book (in spite of the labours of many editors) is not free from errors, obscurities, redundancies, omissions etc., and is without certain definitions necessary for the proofs, it goes on to say that the man has not yet been found who could make it perfect, and next proceeds to explain (1) that Avicenna "cut out postulates and many definitions" and attempted to clear up difficult and obscure passages, (2) that Abū'l Wafā al-Būzjānī (939–997) "introduced unnecessary additions and left out many things of great importance and entirely necessary," inasmuch as he was too long in various places in Book VI. and too short in Book X. where he left out entirely the proofs of the *apotomae*, while he made an unsuccessful attempt to emend XII. 14, (3) that Abū Ja'far al-Khāzin (d. between 961 and 971) arranged the postulates excellently but "disturbed the number and order of the propositions, reduced several propositions to one" etc. Next the preface describes the editor's own claims[3] and then ends with the sentences, "But we have kept to the order of the books and propositions in the work itself (i.e. Euclid's) except in the twelfth and thirteenth books. For we have dealt in Book XIII. with the (solid) bodies and in Book XII. with the surfaces by themselves."

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[1] Steinschneider, *Zeitschrift für Math. u. Physik*, XXXI., hist.-litt. Abtheilung, pp. 85, 86, 99.

[2] Klamroth, p. 279.

[3] Steinschneider, p. 88.

[4] Klamroth, p. 306.

[5] These MSS. are described by Nicoll and Pusey, *Catalogus cod. mss. orient. bibl. Bodleianae*, pt. II. 1835 (pp. 257—262).

After Thābit the *Fihrist* mentions Abū 'Uthmān ad-Dimashqī as having translated some Books of the *Elements* including Book X. (It is Abū 'Uthmān's translation of Pappus' commentary on Book X. which Woepcke discovered at Paris.) The *Fihrist* adds also that "Nazīf the physician told me that he had seen the tenth Book of Euclid in Greek, that it had 40 propositions more than the version in common circulation which had 109 propositions, and that he had determined to translate it into Arabic."

But the third form of the Arabian Euclid actually accessible to us is the edition of Abū Ja'far Muḥ. b. Muḥ. b. al-Ḥasan Naṣīraddīn aṭ-Ṭūsī (whom we shall call aṭ-Ṭūsī for short), born at Ṭūs (in Khurāsān) in 1201 (d. 1274). This edition appeared in two forms, a larger and a smaller. The larger is said to survive in Florence only (Pal. 272 and 313, the latter MS. containing only six Books); this was published at Rome in 1594, and, remarkably enough, some copies of this edition are to be found with 12 and some with 13 Books, some with a Latin title and some without[1]. But the book was printed in Arabic, so that Kästner remarks that he will say as much about it as can be said about a book which one cannot read[2]. The shorter form, which however, in most MSS., is in 15 Books, survives at Berlin, Munich, Oxford, British Museum (974, 1334[3], 1335), Paris (2465, 2466), India Office, and Constantinople; it was printed at Constantinople in 1801, and the first six Books at Calcutta in 1824[4].

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[1] Klamroth, pp. 306—8.

[2] Steinschneider, p. 98. Heiberg has quoted the whole of this preface in the *Zeitschrift für Math. u. Physik*, XXIX., hist.-litt. Abth. p. 16.

[3] This seems to include a rearrangement of the contents of Books XIV., XV. added to the *Elements*.

Aṭ-Ṭūsī's work is however not a *translation* of Euclid's text, but a re-written Euclid based on the older Arabic translations. In this respect it seems to be like the Latin version of the *Elements* by Campanus (Campano), which was first published by Erhard Ratdolt at Venice in 1482 (the first printed edition of Euclid[5]). Campanus (13th c.) was a mathematician, and it is likely enough that he allowed himself the same liberty as aṭ-Ṭūsī in reproducing Euclid. Whatever may be the relation between Campanus' version and that of Athelhard of Bath (about 1120), and whether, as Curtze thinks[6], they both used one and the same Latin version of 10th—11th c., or whether Campanus used Athelhard's version in the same way as aṭ-Ṭūsī used those of his predecessors[7], it is certain that both versions came from an Arabian source, as is evident from the occurrence of Arabic words in them[8]. Campanus' version is not of much service for the purpose of forming a judgment on the relative authenticity of the Greek and Arabian tradition; but it sometimes preserves traces of the purer source, as when it omits Theon's addition to VI. 33[9]. A curious circumstance is that, while Campanus' version agrees with aṭ-Ṭūsī's in the number of the propositions in all the genuine Euclidean Books except V. and IX., it agrees with Athelhard's in having 34 propositions in Book V. (as against 25 in other versions), which confirms the view that the two are not independent, and also leads, as Klamroth says, to this dilemma: either the additions to Book V. are Athelhard's own, or he used an Arabian Euclid which is not known to us[10]. Heiberg also notes that Campanus' Books XIV., XV. show a certain agreement with the preface to the Thābit-Isḥāq version, in which the author claims to have (1) given a method of inscribing spheres in the five regular solids, (2) carried further the solution of the problem how to inscribe any one of the solids in any other and (3) noted the cases where this could not be done[1].

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[1] Suter, *Die Mathematiker und Astronomen der Araber*, p. 151. The Latin title is *Euclidis elementorum geometricorum libri tredecim. Ex traditione doctissimi Nasiridini Tusini nunc primum arabice impressi.* Romae in typographia Medicea MDXCIV. Cum licentia superiorum.

[2] Kästner, *Geschichte der Mathematik*, I. p. 367.

[3] Suter has a note that this MS. is very old, having been copied from the original in the author's lifetime.

[4] Suter, p. 151.

[5] Described by Kästner, *Geschichte der Mathematik*, I. pp. 289—299, and by Weissenborn, *Die Übersetzungen des Euklid durch Campano und Zamberti*, Halle a. S., 1882, pp. 1—7. See also *infra*, Chapter VIII, p. 97.

[6] Sonderabdruck des *Jahresberichtes über die Fortschritte der klassischen Alterthumswissenschaft vom Okt.* 1879—1882, Berlin, 1884.

[7] Klamroth, p. 271.

[8] Curtze, *op. cit.* p. 20; Heiberg, *Euklid-Studien*, p. 178.

[9] Heiberg's Euclid, vol. V. p. ci.

[10] Klamroth, pp. 273—4.

With a view to arriving at what may be called a common measure of the Arabian tradition, it is necessary to compare, in the first place, the numbers of propositions in the various Books. Ḥājī Khalfa says that al-Ḥajjāj's translation contained 468 propositions, and Thābit's 478; this is stated on the authority of aṭ-Ṭūsī, whose own edition contained 468[2]. The fact that Thābit's version had 478 propositions is confirmed by an index in the Bodleian MS. 279 (called O by Klamroth). A register at the beginning of the Codex Leidensis 399, 1 which gives Isḥaq's numbers (although the translation is that of al-Ḥajjāj) apparently makes the total 479 propositions (the number in Book XIV. being apparently 11, instead of the 10 of O[3]). I subjoin a table of relative numbers taken from Klamroth, to which I have added the corresponding numbers in August's and Heiberg's editions of the Greek text.

	The Arabian Euclid			The Greek Euclid		
Books	Isḥāq	aṭ-Ṭūsī	Campanus	Gregory	August	Heiberg
I	48	48	48	48	48	48
II	14	14	14	14	14	14
III	36	36	36	37	37	37
IV	16	16	16	16	16	16
V	25	25	34	25	25	25
VI	33	32	32	33	33	33
VII	39	39	39	41	41	39
VIII	27	25	25	27	27	27
IX	38	36	39	36	36	36
X	109	107	107	117	116	115
XI	41	41	41	40	40	39
XII	15	15	15	18	18	18
XIII	21	18	18	18	18	18
	462	452	464	470	469	465
[XIV	10	10	18	7		?
XV	6	6	13	10		
	478	468	495	487		?]

The numbers in the case of Heiberg include all propositions which he has printed in the text; they include therefore XIII. 6 and III. 12 now to be regarded as spurious, and X. 112—115 which he brackets as doubtful. He does not number the propositions in Books XIV., XV., but I conclude that the numbers in P reach at least 9 in XIV., and 9 in XV.

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[1] Heiberg, *Zeitschrift für Math. u. Physik*, XXIX., hist.-litt. Abtheilung, p. 21.

[2] Klamroth, p. 274; Steinschneider, *Zeitschrift für Math. u. Physik*, XXXI., hist.-litt. Abth. p. 98.

[3] Besthorn-Heiberg read "11?" as the number, Klamroth had read it as 21 (p. 273).

The *Fihrist* confirms the number 109 for Book X., from which Klamroth concludes that Isḥāq's version was considered as by far the most authoritative.

In the text of O, Book IV. consists of 17 propositions and Book XIV. of 12, differing in this respect from its own table of contents; IV. 15, 16 in O are really two proofs of the same proposition.

In al-Ḥajjāj's version Book I. consists of 47 propositions only, I. 45 being omitted. It has also one proposition fewer in Book III., the Heronic proposition III. 12 being no doubt omitted.

In speaking of particular propositions, I shall use Heiberg's numbering, except where otherwise stated.

The difference of 10 propositions between Thābit-Isḥāq and aṭ-Ṭūsī is accounted for thus:

(1) The three propositions VI. 12 and X. 28, 29 which both Isḥāq and the Greek text have are omitted in aṭ-Ṭūsī.

(2) Isḥāq divides each of the propositions XIII. 1—3 into two, making six instead of three in aṭ-Ṭūsī and in the Greek.

(3) Isḥāq has four propositions (numbered by him VIII. 24, 25, IX. 30, 31) which are neither in the Greek Euclid nor in aṭ-Ṭūsī.

Apart from the above differences al-Ḥajjāj (so far as we know), Isḥāq and aṭ-Ṭūsī agree; but their Euclid shows many differences from our Greek text. These differences we will classify as follows[1].

1. *Propositions.*

The Arabian Euclid omits VII. 20, 22 of Gregory's and August's editions (Heiberg, App. to Vol. II. pp. 428–32); VIII. 16, 17; X. 7, 8, 13, 16, 24, 112, 113, 114, besides a lemma *vulgo* X. 13, the proposition X. 117 of Gregory's edition, and the scholium at the end of the Book (see for these Heiberg's Appendix to Vol. III. pp. 382, 408—416); XI. 38 in Gregory and August (Heiberg, App. to Vol. IV. p. 354); XII. 6, 13, 14; (also all but the first third of Book XV.).

The Arabian Euclid makes III. 11, 12 into one proposition, and divides some propositions (X. 31, 32; XI. 31, 34; XIII. 1—3) into two each.

The order is also changed in the Arabic to the following extent. V. 12, 13 are interchanged and the order in Books VI., VII., IX.—XIII. is:

VI. 1—8, 13, 11, 12, 9, 10, 14—17, 19, 20, 18, 21, 22, 24, 26, 23, 25, 27—30, 32, 31, 33.

VII. 1—20, 22, 21, 23—28, 31, 32, 29, 30, 33—39.

IX. 1—13, 20, 14—19, 21—25, 27, 26, 28—36, with two new propositions coming before prop. 30.

X. 1—6, 9—12, 15, 14, 17—23, 26—28, 25, 29—30, 31, 32, 33—111, 115.

XI. 1—30, 31, 32, 34, 33, 35—39.

XII. 1—5, 7, 9, 8, 10, 12, 11, 15, 16—18.

XIII. 1—3, 5, 4, 6, 7, 12, 9, 10, 8, 11, 13, 15, 14, 16—18.

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[1] See Klamroth, pp. 275—6, 280, 282—4, 314—15, 326; Heiberg, vol. V. pp. xcvi, xcvii.

2. *Definitions.*

The Arabic omits the following definitions: IV. Deff. 3—7, VII. Def. 9 (or 10), XI. Deff. 5—7, 15, 17, 23, 25—28; but it has the spurious definitions VI. Deff. 2, 5, and those of *proportion* and *ordered proportion* in Book V. (Deff. 8, 19 August), and wrongly interchanges V. Deff. 11, 12 and also VI. Deff. 3, 4.

The order of the definitions is also different in Book VII. where, after Def. 11, the order is 12, 14, 13, 15, 16, 19, 20, 17, 18, 21, 22, 23, and in Book XI. where the order is 1, 2, 3, 4, 8, 10, 9, 13, 14, 16, 12, 21, 22, 18, 19, 20, 11, 24.

3. *Lemmas and porisms.*

All are omitted in the Arabic except the porisms to VI. 8, VIII. 2, X. 3; but there are slight additions here and there, not found in the Greek, e.g. in VIII. 14, 15 (in K).

4. *Alternative proofs.*

These are all omitted in the Arabic, except that in X. 105, 106 they are substituted for the genuine proofs; but one or two alternative proofs are peculiar to the Arabic (VI. 32 and VIII. 4, 6).

The analyses and syntheses to XIII. 1—5 are also omitted in the Arabic.

Klamroth is inclined, on a consideration of all these differences, to give preference to the Arabian tradition over the Greek (1) "on historical grounds," subject to the proviso that no Greek MS. as ancient as the 8th c. is found to contradict his conclusions, which are based generally (2) on the improbability that the Arabs would have *omitted* so much if they had found it in their Greek MSS., it being clear from the *Fihrist* that the Arabs had already shown an anxiety for a pure text, and that the old translators were subjected in this matter to the check of public criticism. Against the "historical grounds," Heiberg is able to bring a considerable amount of evidence[1]. First of all there is the British Museum palimpsest (L) of the 7th or the beginning of the 8th c. This has fragments of propositions in Book X. which are omitted in the Arabic; the numbering of one proposition, which agrees with the numbering in other Greek MS., is not comprehensible on the assumption that eight preceding propositions were omitted in it, as they are in the Arabic; and lastly, the readings in L are tolerably like those of our MSS., and surprisingly like those of B. It is also to be noted that, although P dates from the 10th c. only, it contains, according to all appearance, an ante-Theonine recension.

Moreover there is positive evidence against certain omissions by the Arabians. Aṭ-Ṭūsī omits VI. 12, but it is scarcely possible that, if Eutocius had not had it, he would have quoted VI. 23 by that number[2]. This quotation of VI. 23 by Eutocius also tells against Isḥāq who has the proposition as VI. 25. Again, Simplicius quotes VI. 10 by that number, whereas it is VI. 13 in Isḥāq; and Pappus quotes, by number, XIII. 2 (Isḥāq 3, 4), XIII. 4 (Isḥāq 8), XIII. 16 (Isḥāq 19).

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[1] Heiberg in *Zeitschrift für Math. u. Physik*, XXIX., hist.-litt. Abth. p. 3 sqq.

[2] Apollonius, ed. Heiberg, vol. II. p. 218, 3—5.

On the other hand the contraction of III. 11, 12 into one proposition in the Arabic tells in favour of the Arabic.

Further, the omission of certain porisms in the Arabic cannot be supported; for Pappus quotes the porism to XIII. 17[1], Proclus those to II. 4, III. 1, VII. 2[2], and Simplicius that to IV. 15.

Lastly, some propositions omitted in the Arabic are required in later propositions. Thus X. 13 is used in X. 18, 22, 23, 26 etc.; X. 17 is wanted in X. 18, 26, 36; XII. 6, 13 are required for XII. 11 and XII. 15 respectively.

It must also be remembered that some of the things which were properly omitted by the Arabians are omitted or marked as doubtful in Greek MSS. also, especially in P, and others are rightly suspected for other reasons (e.g. a number of alternative proofs, lemmas, and porisms, as well as the analyses and syntheses of XIII. 1—5). On the other hand, the Arabic has certain interpolations peculiar to our inferior MSS. (cf. the definition VI. Def. 2 and those of *proportion* and *ordered proportion*).

Heiberg comes to the general conclusion that, not only is the Arabic tradition not to be preferred offhand to that of the Greek MSS., but it must be regarded as inferior in authority. It is a question how far the differences shown in the Arabic are due to the use of Greek MSS. differing from those which have been most used as the basis of our text, and how far to the arbitrary changes made by the Arabians themselves. Changes of order and arbitrary omissions could not surprise us, in view of the preface above quoted from the Oxford MS. of Thābit-Isḥāq, with its allusion to the many important and necessary things left out by Abū 'l Wafā and to the author's own rearrangement of Books XII., XIII. But there is evidence of differences due to the use by the Arabs of other Greek MSS. Heiberg[3] is able to show considerable resemblances between the Arabic text and the Bologna MS. b in that part of the MS. where it diverges so remarkably from our other MSS. (see the short description of it above, p. 49); in illustration he gives a comparison of the proofs of XII. 7 in b and in the Arabic respectively, and points to the omission in both of the proposition given in Gregory's edition as XI. 38, and to a remarkable agreement between them as regards the order of the propositions of Book XII. As above stated, the remarkable divergence of b only affects Books XI. (at end) and XII.; and Book XIII. in b shows none of the transpositions and other peculiarities of the Arabic. There are many differences between b and the Arabic, especially in the definitions of Book XI., as well as in Book XIII. It is therefore a question whether the Arabians made arbitrary changes, or the Arabic form is the more ancient, and b has been altered through contact with other MSS. Heiberg points out that the Arabians must be alone responsible for their definition of a prism, which only covers a prism with a triangular base. This could not have been Euclid's own, for the word *prism* already has the wider meaning in Archimedes, and

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[1] Pappus, v. p. 436, 5.

[2] Proclus, pp. 303—4.

[3] *Zeitschrift für Math. u. Physik*, XXIX., hist.-litt. Abth. p. 6 sqq.

Euclid himself speaks of prisms with parallelograms and polygons as bases (XI. 39; XII. 10). Moreover, a Greek would not have been likely to leave out the definitions of the "Platonic" regular solids.

Heiberg considers that the Arabian translator had before him a MS. which was related to b, but diverged still further from the rest of our MSS. He does not think that there is evidence of the existence of a redaction of Books I.—X. similar to that of Books XI., XII. in b; for Klamroth observes that it is the Books on solid geometry (XI.—XIII.) which are more remarkable than the others for omissions and shorter proofs, and it is a noteworthy coincidence that it is just in these Books that we have a divergent text in b.

An advantage in the Arabic version is the omission of VII. Def. 10, although, as Iamblichus had it, it may have been deliberately omitted by the Arabic translator. Another advantage is the omission of the analyses and syntheses of XIII. 1—5; but again these may have been omitted purposely, as were evidently a number of porisms which are really necessary.

One or two remarks may be added about the Arabic versions as compared with one another. Al-Ḥajjāj's object seems to have been less to give a faithful reflection of the original than to write a useful and convenient mathematical text-book. One characteristic of it is the careful references to earlier propositions when their results are used. Such specific quotations of earlier propositions are rare in Euclid; but in al-Ḥajjāj we find not only such phrases as "by prop. so and so," "which was proved" or "which we showed how to do in prop. so and so," but also still longer phrases. Sometimes he repeats a construction, as in I. 44 where, instead of constructing "the parallelogram *BEFG* equal to the triangle *C* in the angle *EBG* which is equal to the angle *D*" and placing it in a certain position, he produces *AB* to *G*, making *BG* equal to half *DE* (the base of the triangle *CDE* in his figure), and on *GB* so constructs the parallelogram *BHKG* by I. 42 that it is equal to the triangle *CDE*, and its angle *GBH* is equal to the given angle.

Secondly, al-Ḥajjāj, in the arithmetical books, in the theory of proportion, in the applications of the Pythagorean I. 47, and generally where possible, illustrates the proofs by numerical examples. It is true, observes Klamroth, that these examples are not apparently separated from the commentary of an-Nairīzī, and might not therefore have been due to al-Ḥajjāj himself; but the marginal notes to the Hebrew translation in Munich MS. 36 show that these additions were in the copy of al-Ḥajjāj used by the translator, for they expressly give these proofs in numbers as variants taken from al-Ḥajjāj[1].

These characteristics, together with al-Ḥajjāj's freer formulation of the propositions and expansion of the proofs, constitute an intelligible reason why Isḥāq should have undertaken a fresh translation from the Greek. Klamroth calls Isḥāq's version a model of a good translation of a mathematical text; the introductory and transitional phrases are stereotyped and few in number, the technical terms are simply and consistently rendered, and the less formal expressions connect themselves as closely with the Greek as is consistent with intelligibility and the character of the Arabic language. Only in isolated cases does the formulation of definitions and enunciations differ to any considerable extent from the original. In general, his object seems to have been to get rid of difficulties and unevennesses in the Greek text by neat devices, while at the same time giving a faithful reproduction of it[1].

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[1] Klamroth, p. 310; Steinschneider, pp. 85—6.

There are curious points of contact between the versions of al-Ḥajjāj and Thābit-Isḥāq. For example, the definitions and enunciations of propositions are often word for word the same. Presumably this is owing to the fact that Isḥāq found these definitions and enunciations already established in the schools in his time, where they would no doubt be learnt by heart, and refrained from translating them afresh, merely adopting the older version with some changes[2]. Secondly, there is remarkable agreement between the Arabic versions as regards the figures, which show considerable variations from the figures of the Greek text, especially as regards the letters; this is also probably to be explained in the same way, all the later translators having most likely borrowed al-Ḥajjāj's adaptation of the Greek figures[3]. Lastly, it is remarkable that the version of Books XI.—XIII. in the Kjøbenhavn MS. (K), purporting to be by al-Ḥajjāj, is almost exactly the same as the Thābit-Isḥāq version of the same Books in O. Klamroth conjectures that Isḥāq may not have translated the Books on solid geometry at all, and that Thābit took them from al-Ḥajjāj, only making some changes in order to fit them to the translation of Isḥāq[4].

From the facts (1) that aṭ-Ṭūsī's edition had the same number of propositions (468) as al-Ḥajjāj's version, while Thābit-Isḥāq's had 478, and (2) that aṭ-Ṭūsī has the same careful references to earlier propositions, Klamroth concludes that aṭ-Ṭūsī deliberately preferred al-Ḥajjāj's version to that of Isḥāq[5]. Heiberg, however, points out (1) that aṭ-Ṭūsī left out VI. 12 which, if we may judge by Klamroth's silence, al-Ḥajjāj had, and (2) al-Ḥajjāj's version had one proposition less in Books I. and III. than aṭ-Ṭūsī has. Besides, in a passage quoted by Ḥājī Khalfa[6] from aṭ-Ṭūsī, the latter says that "he separated the things which, in the approved editions, were taken from the archetype from the things which had been added thereto," indicating that he had compiled his edition from *both* the earlier translations[7].

There were a large number of Arabian commentaries on, or reproductions of, the *Elements* or portions thereof, which will be found fully noticed by Steinschneider[1]. I shall mention here the commentators etc. referred to in the *Fihrist*, with a few others.

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[1] Klamroth, p. 290, illustrates Isḥāq's method by his way of distinguishing *ἐφαρμόζειν* (to be congruent with) and *ἐφαρμόζεσθαι* (to be applied to), the confusion of which by translators was animadverted on by Savile. Isḥāq avoided the confusion by using two entirely different words.

[2] Klamroth, pp. 310—1.

[3] *ibid.* p. 287.

[4] *ibid.* pp. 304—5.

[5] *ibid.* p. 274.

[6] Ḥājī Khalfa, I. p. 383.

[7] Heiberg, *Zeitschrift für Math. u. Physik*, XXIX., hist.-litt. Abth. pp. 2, 3.

1. Abū 'l 'Abbās al-Faḍl b. Ḥātim **an-Nairīzī** (born at Nairīz, died about 922) has already been mentioned[2]. His commentary survives, as regards Books I.—VI., in the Codex Leidensis 399, I, now edited, as to four Books, by Besthorn and Heiberg, and as regards Books I.—X. in the Latin translation made by Gherard of Cremona in the 12th c. and now published by Curtze from a Cracow MS.[3] Its importance lies mainly in the quotations from Heron and Simplicius.

2. Ahmad b. 'Umar **al-Karābīsī** (date uncertain, probably 9th—10th c.), "who was among the most distinguished geometers and arithmeticians[4]."

3. Al-'Abbās b. Sa'īd **al-Jauharī** (fl. 830) was one of the astronomical observers under al-Ma'mūn, but devoted himself mostly to geometry. He wrote a commentary to the whole of the *Elements*, from the beginning to the end; also the "Book of the propositions which he added to the first book of Euclid[5]."

4. Muḥ. b. 'Īsā Abū 'Abdallāh **al-Māhānī** (d. between 874 and 884) wrote, according to the *Fihrist*, (1) a commentary on Eucl. Book V., (2) "On proportion," (3) "On the 26 propositions of the first Book of Euclid which are proved without *reductio ad absurdum*[6]." The work "On proportion" survives and is probably identical with, or part of, the commentary on Book V.[7] He also wrote, what is not mentioned by the *Fihrist*, a commentary on Eucl. Book X., a fragment of which survives in a Paris MS.[8]

5. Abū Ja'far **al-Khāzin** (i.e. "the treasurer" or "librarian"), one of the first mathematicians and astronomers of his time, was born in Khurāsān and died between the years 961 and 971. The *Fihrist* speaks of him as having written a commentary on the whole of the *Elements*[9], but only the commentary on the beginning of Book X. survives (in Leiden, Berlin and Paris); therefore either the notes on the rest of the Books have perished, or the *Fihrist* is in error[10]. The latter would seem more probable, for, at the end of his commentary, al-Khāzin remarks that the rest had already been commented on by Sulaimān b. 'Uṣma (Leiden MS.)[11] or 'Oqba (Suter), to be mentioned below. Al-Khāzin's method is criticised unfavourably in the preface to the Oxford MS. quoted by Nicoll (see p. 77 above).

6. **Abū 'l Wafā al-Būzjānī** (940–997), one of the greatest Arabian mathematicians, wrote a commentary on the *Elements*, but

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[1] Steinschneider, *Zeitschrift für Math. u. Physik*, XXXI., hist.-litt. Abth. pp. 86 sqq.

[2] Steinschneider, p. 86, *Fihrist* (tr. Suter), pp. 16, 67; Suter, *Die Mathematiker und Astronomen der Araber* (1900), p. 45.

[3] *Supplementum ad Euclidis opera omnia*, ed. Heiberg and Menge, Leipzig, 1899.

[4] *Fihrist*, pp. 16, 38; Steinschneider, p. 87; Suter, p. 65.

[5] *Fihrist*, pp. 16, 25; Steinschneider, p. 88; Suter, p. 12.

[6] *Fihrist*, pp. 16, 25, 58.

[7] Suter, p. 26, note, quotes the Paris MS. 2467, 16° containing the work "on proportion" as the authority for this conjecture.

[8] MS. 2457, 39° (cf. Woepcke in *Mém. prés. à l'acad. des sciences*, XIV., 1856, p. 669).

[9] *Fihrist*, p. 17.

[10] Suter, p. 58, note b.

[11] Steinschneider, p. 89.

did not complete it[1]. His method is also unfavourably regarded in the same preface to the Oxford MS. 280. According to Ḥājī Khalfa, he also wrote a book on geometrical constructions, in thirteen chapters. Apparently a book answering to this description was compiled by a gifted pupil from lectures by Abū 'l Wafā, and a Paris MS. (Anc. fonds 169) contains a Persian translation of this work, not that of Abū 'l Wafā himself. An analysis of the work was given by Woepcke[2], and some particulars will be found in Cantor[3]. Abū 'l Wafā also wrote a commentary on Diophantus, as well as a separate "book of proofs to the propositions which Diophantus used in his book and to what he (Abū 'l Wafā) employed in his commentary[4]."

7. **Ibn Rāhawaihi al-Arjānī** also commented on Eucl. Book X.[5]

8. 'Alī b. Ahmad Abū 'l-Qāsim **al-Anṭākī** (d. 987) wrote a commentary on the whole book[6]; part of it seems to survive (from the 5th Book onwards) at Oxford (Catal. MSS. orient. II. 281)[7].

9. **Sind b. 'Alī** Abū 't-Ṭaiyib was a Jew who went over to Islam in the time of al-Ma'mūn, and was received among his astronomical observers, whose head he became[8] (about 830); he died after 864. He wrote a commentary on the whole of the *Elements*; "Abū 'Alī saw nine books of it, and a part of the tenth[9]." His book "On the Apotomae and the Medials," mentioned by the *Fihrist*, may be the same as, or part of, his commentary on Book X.

10. Abū Yūsuf Ya'qūb b. Muḥ. **ar-Rāzī** "wrote a commentary on Book X., and that an excellent one, at the instance of Ibn al-'Amīd[10]."

11. The *Fihrist* next mentions **al-Kindī** (Abū Yūsuf Ya'qūb b. Isḥāq b. as-Ṣabbāḥ al-Kindī, d. about 873), as the author (1) of a work "on the objects of Euclid's book," in which occurs the statement that the *Elements* were originally written by Apollonius, the carpenter (see above, p. 5 and note), (2) of a book "on the improvement of Euclid's work," and (3) of another "on the improvement of the 14th and 15th Books of Euclid." "He was the most distinguished man of his time, and stood alone in the knowledge of the old sciences collectively; he was called 'the philosopher of the Arabians'; his writings treat of the most different branches of knowledge, as logic, philosophy, geometry, calculation, arithmetic, music, astronomy and others[11]." Among the other geometrical works of al-Kindī mentioned by the *Fihrist*[12] are treatises on the closer investigation of the results of Archimedes concerning the measure of the diameter of a circle in terms of its circumference, on the construction of the figure of the two mean proportionals, on the approximate determination of the chords

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[1] *Fihrist*, p. 17.

[2] Woepcke, *Journal Asiatique*, Sér. V. T. V. pp. 218—256 and 309—359.

[3] *Gesch. d. Math.* vol. I_3, pp. 743—6.

[4] *Fihrist*, p. 39; Suter, p. 71.

[5] *Fihrist*, p. 17; Suter, p. 17.

[6] *Fihrist*, p. 17.

[7] Suter, p. 64.

[8] *Fihrist*, p. 17, 29; Suter, pp. 13, 14.

[9] *Fihrist*, p. 17.

[10] *Fihrist*, p. 17; Suter, p. 66.

[11] *Fihrist*, p. 17, 10—15.

[12] The mere catalogue of al-Kindī's works on the various branches of science takes up four octavo pages (11—15) of Suter's translation of the *Fihrist*.

of the circle, on the approximate determination of the chord (side) of the nonagon, on the division of triangles and quadrilaterals and constructions for that purpose, on the manner of construction of a circle which is equal to the surface of a given cylinder, on the division of the circle, in three chapters etc.

12. The physician **Naẓīf** b. Yumn (or Yaman) al-Qass ("the priest") is mentioned by the *Fihrist* as having seen a Greek copy of Eucl. Book X. which had 40 more propositions than that which was in general circulation (containing 109), and having determined to translate it into Arabic[1]. Fragments of such a translation exist at Paris, Nos. 18 and 34 of the MS. 2457 (952, 2 Suppl. Arab. in Woepcke's tract); No. 18 contains "additions to some propositions of the 10th Book, existing in the Greek language[2]." Naẓīf must have died about 990[3].

13. **Yūḥannā** b. Yūsuf b. al-Ḥārith b. al-Biṭrīq **al-Qass** (d. about 980) lectured on the *Elements* and other geometrical books, made translations from the Greek, and wrote a tract on the "proof" of the case of two straight lines both meeting a third and making with it, on one side, two angles together less than two right angles[4]. Nothing of his appears to survive, except that a tract "on rational and irrational magnitudes," No. 48 in the Paris MS. just mentioned, is attributed to him.

14. Abū Muḥ. **al-Ḥasan** b. 'Ubaidallāh b. Sulaimān b. **Wahb** (d. 901) was a geometer of distinction, who wrote works under the two distinct titles "A commentary on the difficult parts of the work of Euclid" and "The Book on Proportion[5]." Suter thinks that another reading is possible in the case of the second title, and that it may refer to the Euclidean work "on the divisions (of figures)[6]."

15. **Qusṭā b. Lūqā** al-Ba'labakkī (d. about 912), a physician, philosopher, astronomer, mathematician and translator, wrote "on the difficult passages of Euclid's book" and "on the solution of arithmetical problems from the third book of Euclid[7]"; also an "introduction to geometry," in the form of question and answer[8].

16. **Thābit b. Qurra** (826–901), besides translating some parts of Archimedes and Books V.—VII. of the *Conics* of Apollonius, and revising Isḥāq's translation of Euclid's *Elements*, also revised the translation of the *Data* by the same Isḥāq and the book *On divisions of figures* translated by an anonymous writer. We are told also that he wrote the following works: (1) On the Premisses (Axioms, Postulates etc.) of Euclid, (2) On the Propositions of Euclid, (3) On the propositions and questions which arise when two straight lines are cut by a third (or on the "proof" of Euclid's famous postulate). The last tract is extant in the MS. discovered by Woepcke (Paris 2457, 32°). He is also credited with "an excellent work" in the shape of an "Introduction to the Book of Euclid," a treatise on

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[1] *Fihrist*, pp. 16, 17.

[2] Woepcke, *Mém. prés. à l'acad. des sciences*, XIV. pp. 666, 668.

[3] Suter, p. 68.

[4] *Fihrist*, p. 38; Suter, p. 60.

[5] *Fihrist*, p. 26, and Suter's note, p. 60.

[6] Suter, p. 211, note 23.

[7] *Fihrist*, p. 43.

[8] *Fihrist*, p. 43; Suter, p. 41.

Geometry dedicated to Ismāʿīl b. Bulbul, a Compendium of Geometry, and a large number of other works for the titles of which reference may be made to Suter, who also gives particulars as to which are extant[1].

17. Abū Saʿīd **Sinān** b. Thābit b. Qurra, the son of the translator of Euclid, followed in his father's footsteps as geometer, astronomer and physician. He wrote an "improvement of the book of ...... on the Elements of Geometry, in which he made various additions to the original." It is natural to conjecture that *Euclid* is the name missing in this description (by Ibn abī Uṣaibiʿa); Casiri has the name Aqāṭon[2]. The latest editor of the *Ta'rīkh al-Ḥukamā*, however, makes the name to be Iflāton (= Plato), and he refers to the statement by the *Fihrist* and Ibn al-Qifṭī attributing to Plato a work on the Elements of Geometry translated by Qusṭā. It is just possible, therefore, that at the time of Qusṭā the Arabs were acquainted with a book on the Elements of Geometry translated from the Greek, which they attributed to Plato[3]. Sinān died in 943.

18. Abū Sahl Wījan (or Waijan) b. Rustam **al-Kūhī** (fl. 988), born at Kūh in Ṭabaristān, a distinguished geometer and astronomer, wrote, according to the *Fihrist*, a "Book of the Elements" after that of Euclid[4]; the 1st and 2nd Books survive at Cairo, and a part of the 3rd Book at Berlin (5922)[5]. He wrote also a number of other geometrical works: Additions to the 2nd Book of Archimedes on the Sphere and Cylinder (extant at Paris, at Leiden, and in the India Office), On the finding of the side of a heptagon in a circle (India Office and Cairo), On two mean proportionals (India Office), which last may be only a part of the Additions to Archimedes' On the Sphere and Cylinder, etc.

19. Abū Naṣr Muḥ. b. Muḥ. b. Ṭarkhān b. Uzlaġ **al-Fārābī** (870–950) wrote a commentary on the difficulties of the introductory matter to Books I. and V.[6] This appears to survive in the Hebrew translation which is, with probability, attributed to Moses b. Tibbon[7].

20. Abū ʿAlī al-Ḥasan b. al-Hasan b. **al-Haitham** (about 965–1039), known by the name Ibn al-Haitham or Abū ʿAlī al-Baṣrī, was a man of great powers and knowledge, and no one of his time approached him in the field of mathematical science. He wrote several works on Euclid the titles of which, as translated by Woepcke from Usaibiʿa, are as follows[8]:

1. Commentary and abridgment of the *Elements*.
2. Collection of the Elements of Geometry and Arithmetic, drawn from the treatises of Euclid and Apollonius.
3. Collection of the Elements of the Calculus deduced from the principles laid down by Euclid in his *Elements*.

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[1] Suter, pp. 34—8.

[2] *Fihrist* (ed. Suter), p. 59, note 132; Suter, p. 52, note b.

[3] See Suter in *Bibliotheca Mathematica*, IV_3, 1903–4, pp. 296—7, review of Julius Lippert's *Ibn al-Qiftī. Ta'rīch al-ḥukamā*, Leipzig, 1903.

[4] *Fihrist*, p. 40.

[5] Suter, p. 75.

[6] Suter, p. 55.

[7] Steinschneider, p. 92.

[8] Steinschneider, pp. 92—3.

4. Treatise on "measure" after the manner of Euclid's *Elements*.

5. Memoir on the solution of the difficulties in Book I.

6. Memoir for the solution of a doubt about Euclid, relative to Book V.

7. Memoir on the solution of a doubt about the stereometric portion.

8. Memoir on the solution of a doubt about Book XII.

9. Memoir on the division of the two magnitudes mentioned in X. 1 (the theorem of exhaustion).

10. Commentary on the definitions in the work of Euclid (where Steinschneider thinks that some more general expression should be substituted for "definitions").

The last-named work (which Suter calls a commentary on the *Postulates* of Euclid) survives in an Oxford MS. (Catal. MSS. orient. I. 908) and in Algiers (1446, 1°).

A Leiden MS. (966) contains his Commentary "on the difficult places" up to Book V. We do not know whether in this commentary, which the author intended to form, with the commentary on the Musādarāt, a sort of complete commentary, he had collected the separate memoirs on certain doubts and difficult passages mentioned in the above list.

A commentary on Book V. and following Books found in a Bodleian MS. (Catal. II. p. 262) with the title "Commentary on Euclid and solution of his difficulties" is attributed to b. Haitham; this might be a continuation of the Leiden MS.

The memoir on X. 1 appears to survive at St Petersburg, MS. de l'Institut des langues orient. 192, 5° (Rosen, Catal. p. 125).

21. **Ibn Sīnā**, known as **Avicenna** (980–1037), wrote a Compendium of Euclid, preserved in a Leiden MS. No. 1445, and forming the geometrical portion of an encyclopaedic work embracing Logic, Mathematics, Physics and Metaphysics[1].

22. Aḥmad b. al-Ḥusain **al-Ahwāzī** al-Kātib wrote a commentary on Book X., a fragment of which (some 10 pages) is to be found at Leiden (970), Berlin (5923) and Paris (2467, 18°)[2].

23. Naṣīraddīn **aṭ-Ṭūsī** (1201–1274) who, as we have seen, brought out a Euclid in two forms, wrote:

1. A treatise on the postulates of Euclid (Paris, 2467, 5°).
2. A treatise on the 5th postulate, perhaps only a part of the foregoing (Berlin, 5942, Paris, 2467, 6°).
3. Principles of Geometry taken from Euclid, perhaps identical with No. 1 above (Florence, Pal. 298).
4. 105 problems out of the *Elements* (Cairo). He also edited the *Data* (Berlin, Florence, Oxford etc.)[3].

24. Muḥ. b. Ashraf Shamsaddīn **as-Samarqandī** (fl. 1276) wrote "Fundamental Propositions, being elucidations of 35 selected proposi-

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[1] Steinschneider, p. 92; Suter, p. 89.

[2] Suter, p. 57.

[3] Suter, pp. 150—1.

tions of the first Books of Euclid," which are extant at Gotha (1496 and 1497), Oxford (Catal. I. 967, 2°), and Brit. Mus.[1].

25. Mūsā b. Muḥ. b. Maḥmūd, known as **Qāḍīzāde** ar-Rūmī (i.e. the son of the judge from Asia Minor), who died between 1436 and 1446, wrote a commentary on the "Fundamental Propositions" just mentioned, of which many MSS. are extant[2]. It contained biographical statements about Euclid alluded to above (p. 5, note).

26. Abū Dā'ūd Sulaimān b. **'Uqba**, a contemporary of al-Khāzin (see above, No. 5), wrote a commentary on the second half of Book X., which is, at least partly, extant at Leiden (974) under the title "On the binomials and apotomae found in the 10th Book of Euclid[3]."

27. The Codex Leidensis 399, 1 containing al-Ḥajjāj's translation of Books I.—VI. is said to contain glosses to it by Sa'īd b. Mas'ūd b. al-Qass, apparently identical with Abū Naṣr Ġars al-Na'ma, son of the physician Mas'ūd b. al-Qass al-Bağdādī, who lived in the time of the last Caliph al-Musta'ṣim (d. 1258)[4].

28. Abū **Muḥammad b. Abdalbāqī** al-Bağdādī al-Faraḍī (d. 1141, at the age of over 70 years) is stated in the *Ta'rīkh al-Ḥukamā* to have written an excellent commentary on Book X. of the *Elements*, in which he gave numerical examples of the propositions[5]. This is published in Curtze's edition of an-Nairīzī where it occupies pages 252—386[6].

29. Yaḥyā b. Muḥ. b. 'Abdān b. 'Abdalwāḥid, known by the name of Ibn al-Lubūdī (1210–1268), wrote a Compendium of Euclid, and a short presentation of the postulates[7].

30. Abū 'Abdallāh Muḥ. b. Mu'ādh al-Jayyānī wrote a commentary on Eucl. Book V. which survives at Algiers (1446, 3°)[8].

31. Abū Naṣr Manṣūr b. 'Alī b. 'Irāq wrote, at the instance of Muḥ. b. Aḥmad Abū 'r-Raiḥān al-Bīrūnī (973–1048), a tract "on a doubtful (difficult) passage in Eucl. Book XIII." (Berlin, 5925)[9].

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[1] Suter, p. 157.
[2] *ibid.* p. 175.
[3] *ibid.* p. 56.
[4] *ibid.* pp. 153—4, 227.
[5] Gartz, p. 14; Steinschneider, pp. 94—5.
[6] Suter in *Bibliotheca Mathematica*, IV_3, 1903, pp. 25, 295; Suter has also an article on its contents, *Bibliotheca Mathematica*, VII_3, 1906–7, pp. 234—251.
[7] Steinschneider, p. 94; Suter, p. 146.
[8] Suter, *Nachträge und Berichtigungen*, in *Abhandlungen zur Gesch. der math. Wissenschaften*, XIV., 1902, p. 170.
[9] Suter, p. 81, and *Nachträge*, p. 172.

CHAPTER VIII.

PRINCIPAL TRANSLATIONS AND EDITIONS OF THE ELEMENTS.

CICERO is the first Latin author to mention Euclid[1]; but it is not likely that in Cicero's time Euclid had been translated into Latin or was studied to any considerable extent by the Romans; for, as Cicero says in another place[2], while geometry was held in high honour among the Greeks, so that nothing was more brilliant than their mathematicians, the Romans limited its scope by having regard only to its utility for measurements and calculations. How very little theoretical geometry satisfied the Roman *agrimensores* is evidenced by the work of Balbus *de mensuris*[3], where some of the definitions of Eucl. Book I. are given. Again, the extracts from the *Elements* found in the fragment attributed to Censorinus (fl. 238 A.D)[4] are confined to the definitions, postulates, and common notions. But by degrees the *Elements* passed even among the Romans into the curriculum of a liberal education; for Martianus Capella speaks of the effect of the enunciation of the proposition "how to construct an equilateral triangle on a given straight line" among a company of philosophers, who, recognising the first proposition of the *Elements*, straightway break out into encomiums on Euclid[5]. But the *Elements* were then (*c.* 470 A.D.) doubtless read in Greek; for what Martianus Capella gives[6] was drawn from a Greek source, as is shown by the occurrence of Greek words and by the wrong translation of I. def. 1 ("punctum vero est cuius pars *nihil* est"). Martianus may, it is true, have quoted, not from Euclid himself, but from Heron or some other ancient source.

But it is clear from a certain palimpsest at Verona that some scholar had already attempted to translate the *Elements* into Latin. This palimpsest[7] has part of the "Moral reflections on the Book of Job" by Pope Gregory the Great written in a hand of the 9th c. above certain fragments which in the opinion of the best judges date from the 4th c. Among these are fragments of Vergil and of Livy, as well as a geometrical fragment which purports to be taken from the 14tn and 15th Books of Euclid. As a matter of fact it is from Books XII. and XIII. and is of the nature of a free rendering, or rather a new arrangement, of Euclid with the propositions in different order[1]. The MS. was evidently the translator's own copy, because some words are struck out and replaced by synonyms. We do not know whether the translator completed the translation of the whole, or in what relation his version stood to our other sources.

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[1] *De oratore* III. 132.

[2] *Tusc.* I. 5.

[3] *Gromatici veteres*, I. 97 sq. (ed. F. Blume, K. Lachmann and A. Rudorff. Berlin, 1848, 1852).

[4] Censorinus, ed. Hultsch, pp. 60—3.

[5] Martianus Capella, VI. 724.

[6] *ibid.* VI. 708 sq.

[7] Cf. Cantor, I_3, p. 565.

Magnus Aurelius Cassiodorus (b. about 475 A.D.) in the geometrical part of his encyclopaedia *De artibus ac disciplinis liberalium literarum* says that geometry was represented among the Greeks by Euclid, Apollonius, Archimedes, and others, "of whom Euclid was given us translated into the Latin language by the same great man Boethius"; also in his collection of letters[2] is a letter from Theodoric to Boethius containing the words, "for in your translations... Nicomachus the arithmetician, and Euclid the geometer, are heard in the Ausonian tongue." The so-called Geometry of Boethius which has come down to us by no means constitutes a translation of Euclid. The MSS. variously give five, four, three or two Books, but they represent only two distinct compilations, one normally in five Books and the other in two. Even the latter, which was edited by Friedlein, is not genuine[3], but appears to have been put together in the 11th c., from various sources. It begins with the definitions of Eucl. I., and in these are traces of perfectly correct readings which are not found even in the MSS. of the 10th c., but which can be traced in Proclus and other ancient sources; then come the Postulates (five only), the Axioms (three only), and after these some definitions of Eucl. II., III., IV. Next come the enunciations of Eucl. I., of ten propositions of Book II., and of some from Books III., IV., but always without proofs; there follows an extraordinary passage which indicates that the author will now give something of his own in elucidation of Euclid, though what follows is a literal translation of the proofs of Eucl. I. 1—3. This latter passage, although it affords a strong argument against the genuineness of this part of the work, shows that the Pseudoboethius had a Latin translation of Euclid from which he extracted the three propositions.

Curtze has reproduced, in the preface to his edition of the translation by Gherard of Cremona of an-Nairīzī's Arabic commentary on Euclid, some interesting fragments of a translation of Euclid taken from a Munich MS. of the 10th c. They are on two leaves used for the cover of the MS. (Bibliothecae Regiae Universitatis Monacensis 2° 757) and consist of portions of Eucl. I. 37, 38 and II. 8, translated literally word for word from the Greek text. The translator seems to have been an Italian (cf. the words "capitolo nono" used for the ninth prop. of Book II.) who knew very little Greek and had moreover little mathematical knowledge. For example, he translates the capital letters denoting points in figures as if they were numerals: thus τὰ ΑΒΓ,

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[1] The fragment was deciphered by W. Studemund, who communicated his results to Cantor.

[2] Cassiodorus, *Variae*, I. 45, p. 40, 12 ed. Mommsen.

[3] See especially Weissenborn in *Abhandlungen zur Gesch. d. Math.* II. p. 185 sq.; Heiberg in *Philologus*, XLIII. p. 507 sq.; Cantor, I_3, p. 580 sq.

ΔΕΖ is translated "que primo secundo et tertio quarto quinto et septimo," T becomes "tricentissimo" and so on. The Greek MS. which he used was evidently written in uncials, for ΔΕΖΘ becomes in one place "quod autem septimo nono," showing that he mistook ΔΕ for the particle δέ, and καὶ ὁ ΣΤU is rendered "sicut tricentissimo et quadringentissimo," showing that the letters must have been written KAIOCTU.

The date of the Englishman Athelhard (Æthelhard) is approximately fixed by some remarks in his work *Perdifficiles Quaestiones Naturales* which, on the ground of the personal allusions they contain, must be assigned to the first thirty years of the 12th c.[1] He wrote a number of philosophical works. Little is known about his life. He is said to have studied at Tours and Laon, and to have lectured at the latter school. He travelled to Spain, Greece, Asia Minor and Egypt, and acquired a knowledge of Arabic, which enabled him to translate from the Arabic into Latin, among other works, the *Elements* of Euclid. The date of this translation must be put at about 1120. MSS. purporting to contain Athelhard's version are extant in the British Museum (Harleian No. 5404 and others), Oxford (Trin. Coll. 47 and Ball. Coll. 257 of 12th c.), Nürnberg (Johannes Regiomontanus' copy) and Erfurt.

Among the very numerous works of Gherard of Cremona (1114—1187) are mentioned translations of "15 Books of Euclid" and of the *Data*[2]. Till recently this translation of the *Elements* was supposed to be lost; but Axel Anthon Björnbo has succeeded (1904) in discovering a translation from the Arabic which is different from the two others known to us (those by Athelhard and Campanus respectively), and which he, on grounds apparently convincing, holds to be Gherard's. Already in 1901 Björnbo had found Books X.—XV. of this translation in a MS. at Rome (Codex Reginensis lat. 1268 of 14th c.)[3]; but three years later he had traced three MSS. containing the whole of the same translation at Paris (Cod. Paris. 7216, 15th c.), Boulogne-sur-Mer (Cod. Bononiens. 196, 14th c.), and Bruges (Cod. Brugens. 521, 14th c.), and another at Oxford (Cod. Digby 174, end of 12th c.) containing a fragment, XI. 2 to XIV. The occurrence of Greek words in this translation such as *rombus*, *romboides* (where Athelhard keeps the Arabic terms), *ambligonius*, *orthogonius*, *gnomo*, *pyramis* etc., show that the translation is independent of Athelhard's. Gherard appears to have had before him an old translation of Euclid from the Greek which Athelhard also often followed, especially in his terminology, using it however in a very different manner. Again, there are some Arabic terms, e.g. *meguar* for *axis of rotation*, which Athelhard did not use, but which is found in almost all the translations that are with certainty attributed to Gherard of Cremona; there occurs also the expression "superficies equidistantium laterum et rectorum angulorum," found also in Gherard's translation of an-Nairīzī, where Athelhard says "parallelogrammum rectangulum." The translation is much clearer than Athelhard's: it is neither abbreviated nor "edited" as Athelhard's appears to have been; it is a word-for-word translation of an Arabic MS. containing a revised and critical edition of Thābit's version. It contains several notes quoted from Thābit himself (*Thebit dixit*), e.g. about alternative proofs etc. which Thābit found "in another Greek MS.," and is therefore a further testimony to Thābit's critical treatment of the text after Greek MSS. The new editor also added critical remarks of his own, e.g. on other proofs which he found in other Arabic versions, but not in the Greek: whence it is clear that he compared the Thābit version before him with other versions as carefully as Thābit collated the Greek MSS. Lastly, the new editor speaks of "Thebit qui transtulit hunc librum in arabicam linguam" and of "translatio Thebit," which may tend to confirm the statement of al-Qifṭī who credited Thābit with an independent translation, and not (as the *Fihrist* does) with a mere improvement of the version of Isḥāq b. Ḥunain.

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[1] Cantor, *Gesch. d. Math.* I$_3$, p. 906.

[2] Boncompagni, *Della vita e delle opere di Gherardo Cremonese*, Rome, 1851, p. 5.

[3] Described in an appendix to *Studien über Menelaos' Sphärik* (*Abhandlungen zur Geschichte der mathematischen Wissenschaften*, XIV., 1902).

[4] See *Bibliotheca Mathematica*, VI$_3$, 1905–6, pp. 242—8.

Gherard's translation of the Arabic commentary of an-Nairīzī on the first ten Books of the *Elements* was discovered by Maximilian Curtze in a MS. at Cracow and published as a supplementary volume to Heiberg and Menge's Euclid[1]: it will often be referred to in this work.

Next in chronological order comes Johannes Campanus (Campano) of Novara. He is mentioned by Roger Bacon (1214–1294) as a prominent mathematician of his time[2], and this indication of his date is confirmed by the fact that he was chaplain to Pope Urban IV, who was Pope from 1261 to 1281[3]. His most important achievement was his edition of the *Elements* including the two Books XIV. and XV. which are not Euclid's. The sources of Athelhard's and Campanus' translations, and the relation between them, have been the subject of much discussion, which does not seem to have led as yet to any definite conclusion. Cantor (II_1, p. 91) gives references[4] and some particulars. It appears that there is a MS. at Munich (Cod. lat. Mon. 13021) written by Sigboto in the 12th c. at Prüfning near Regensburg, and denoted by Curtze by the letter R, which contains the enunciations of part of Euclid. The Munich MSS. of Athelhard and Campanus' translations have many enunciations textually identical with those in R, so that the source of all three must, for these enunciations, have been the same; in others Athelhard and Campanus diverge completely from R, which in these places follows the Greek text and is therefore genuine and authoritative. In the 32nd definition occurs the word "elinuam," the Arabic term for "rhombus," and throughout the translation are a number of Arabic figures. But R was not translated from the Arabic, as is shown by (among other things) its close resemblance to the translation from Euclid given on pp. 377 sqq. of the *Gromatici Veteres* and to the so-called geometry of Boethius. The explanation of the Arabic figures and the word "elinuam" in Def. 32 appears to be that R was a late copy of an earlier original with corruptions introduced in many places; thus in Def. 32 a part of the text was completely lost and was supplied by some intelligent copyist who inserted the word "elinuam," which was known to him, and also the Arabic figures. Thus Athelhard certainly was not the first to translate Euclid into Latin; there must have been in existence before the 11th c. a Latin translation which was the common source of R, the passage in the *Gromatici*, and "Boethius." As in the two latter there occur the *proofs* as well as the enunciations of I. 1—3, it is possible that this translation originally contained the proofs also. Athelhard must have had before him this translation of the *enunciations*, as well as the Arabic source from which he obtained his proofs. That some sort of translation, or at least fragments of one, were available before Athelhard's time even in England is indicated by some old English verses[1]:

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[1] *Anaritii in decem libros priores Elementorum Euclidis Commentarii ex interpretatione Gherardi Cremonensis in codice Cracoviensi* 569 *servata* edidit Maximilianus Curtze, Leipzig (Teubner), 1899.

[2] Cantor, II_1, p. 88.

[3] Tiraboschi, *Storia della letteratura italiana*, IV. 145—160.

[4] H. Weissenborn in *Zeitschrift für Math. u. Physik*, XXV., Supplement, pp. 143—166, and in his monograph, *Die Übersetzungen des Euklid durch Campano und Zamberti* (1882); Max. Curtze in *Philologische Rundschau* (1881), I. pp. 943—950, and in *Jahresbericht über die Fortschritte der classischen Alterthumswissenschaft*, XL. (1884, III.) pp. 19—22; Heiberg in *Zeitschrift für Math. u. Physik*, XXXV., hist.-litt. Abth., pp. 48—58 and pp. 81—6.

"The clerk Euclide on this wyse hit fonde
Thys craft of gemetry yn Egypte londe
Yn Egypte he tawghte hyt ful wyde,
In dyvers londe on every syde.
Mony erys afterwarde y understonde
Yer that the craft com ynto thys londe.
Thys craft com into England, as y yow say,
Yn tyme of good kyng Adelstone's day,"

which would put the introduction of Euclid into England as far back as 924–940 A.D.

We now come to the relation between Athelhard and Campanus. That their translations were not independent, as Weissenborn would have us believe, is clear from the fact that in all MSS. and editions, apart from orthographical differences and such small differences as are bound to arise when MSS. are copied by persons with some knowledge of the subject-matter, the definitions, postulates, axioms, and the 364 enunciations are word for word identical in Athelhard and Campanus; and this is the case not only where both have the same text as R but where they diverge from it. Hence it would seem that Campanus used Athelhard's translation and only developed the proofs by means of another redaction of the Arabian Euclid. It is true that the difference between the proofs of the propositions in the two translations is considerable; Athelhard's are short and compressed, Campanus' clearer and more complete, following the Greek text more closely, though still at some distance. Further, the arrangement in the two is different; in Athelhard the proofs regularly precede the enunciations, Campanus follows the usual order. It is a question how far the differences in the proofs, and certain additions in each, are due to the two translators themselves or go back to Arabic originals. The latter supposition seems to Curtze and Cantor the more probable one. Curtze's general view of the relation of Campanus to Athelhard is to the effect that Athelhard's translation was gradually altered, from the form in which it appears in the two Erfurt MSS. described by Weissenborn, by successive copyists and commentators *who had Arabic originals before them*, until it took the form which Campanus gave it and in which it was published. In support of this view Curtze refers to Regiomontanus' copy of the Athelhard-Campanus translation. In Regiomontanus' own preface the title is given, and this attributes the translation to Athelhard; but, while this copy agrees almost exactly with Athelhard in Book I., yet, in places where Campanus is more lengthy, it has similar additions, and in the later Books, especially from Book III. onwards, agrees absolutely with Campanus; Regiomontanus, too, himself implies that, though the translation was Athelhard's, Campanus had revised it; for he has notes in the margin such as the following, "Campani est hec," "dubito an demonstret hic Campanus" etc.

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[1] Quoted by Halliwell in *Rara Mathematica* (p. 56 note) from MS. Bib. Reg. Mus. Brit. 17 A. I. f. 2^b—3.

We come now to the printed editions of the whole or of portions of the *Elements*. This is not the place for a complete bibliography, such as Riccardi has attempted in his valuable memoir issued in five parts between 1887 and 1893, which makes a large book in itself[1]. I shall confine myself to saying something of the most noteworthy translations and editions. It will be convenient to give first the Latin translations which preceded the publication of the *editio princeps* of the Greek text in 1533, next the most important editions of the Greek text itself, and after them the most important translations arranged according to date of first appearance and languages, first the Latin translations after 1533, then the Italian, German, French and English translations in order.

It may be added here that the first allusion, in the West, to the Greek text as still extant is found in Boccaccio's commentary on the *Divina Commedia* of Dante[2]. Next Johannes Regiomontanus, who intended to publish the *Elements* after the version of Campanus, but with the latter's mistakes corrected, saw in Italy (doubtless when staying with his friend Bessarion) some Greek MSS. and noticed how far they differed from the Latin version (see a letter of his written in the year 1471 to Christian Roder of Hamburg)[3].

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[1] *Saggio di una Bibliografia Euclidea*, memoria del Prof. Pietro Riccardi (Bologna, 1887, 1888, 1890, 1893).

[2] I. p. 404.

[3] Published in C. T. de Murr's *Memorabilia Bibliothecarum Norimbergensium*, Part I. p. 190 sqq.

I. LATIN TRANSLATIONS PRIOR TO 1533.

1482. In this year appeared the first printed edition of Euclid, which was also the first printed mathematical book of any importance. This was printed at Venice by Erhard Ratdolt and contained Campanus' translation[1]. Ratdolt belonged to a family of artists at Augsburg, where he was born about 1443. Having learnt the trade of printing at home, he went in 1475 to Venice, and founded there a famous printing house which he managed for 11 years, after which he returned to Augsburg and continued to print important books until 1516. He is said to have died in 1528. Kästner[2] gives a short description of this first edition of Euclid and quotes the dedication to Prince Mocenigo of Venice which occupies the page opposite to the first page of text. The book has a margin of 2½ inches, and in this margin are placed the figures of the propositions. Ratdolt says in his dedication that at that time, although books by ancient and modern authors were printed every day in Venice, little or nothing mathematical had appeared: a fact which he puts down to the difficulty involved by the figures, which no one had up to that time succeeded in printing. He adds that after much labour he had discovered a method by which figures could be produced as easily as letters[3]. Experts are in doubt as to the nature of Ratdolt's discovery. Was it a method of making figures up out of separate parts of figures, straight or curved lines, put together as letters are put together to make words? In a life of Joh. Gottlob Immanuel Breitkopf, a contemporary of Kästner's own, this member of the great house of Breitkopf is credited with this particular discovery. Experts in that same house expressed the opinion that Ratdolt's figures were woodcuts, while the letters denoting points in the figures were like the other letters in the text; yet it was with carved wooden blocks that printing began. If Ratdolt was the first to print geometrical figures, it was not long before an emulator arose; for in the very same year Mattheus Cordonis of Windischgrätz employed woodcut mathematical figures in printing Oresme's *De latitudinibus*[4]. How eagerly the opportunity of spreading geometrical knowledge was seized upon is proved by the number of editions which followed in the next few years. Even the year 1482 saw two forms of the book, though they only differ in the first sheet. Another edition came out in 1486 (*Ulmae, apud Io. Regerum*) and another in 1491 (*Vincentiae per*

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[1] Curtze (An-Nairîzî, p. xiii) reproduces the heading of the first page of the text as follows (there is no title-page): Preclariſſimū opus elementoꝝ Euclidis megarēſis v̄na cū cōmentis Campani pſpicaciſſimi in artē geometriā incipit felicit', after which the definitions begin at once. Other copies have the shorter heading: Preclarissimus liber elementorum Euclidis perspicacissimi: in artem Geometrie incipit quam foelicissime. At the end stands the following: ¶ Opus elementorū euclidis megarenſis in geometriā artē Jn id quoqꝫ Campani pſpicaciſſimi Cōmentationes finiūt. Erhardus ratdolt Augustensis impreſſor ſolertiſſimus. venetijs impreſſit . Anno ſalutis . M.cccc.lxxxij . Octauis . Caleñ . Juñ . Lector . Vale.

[2] Kästner, *Geschichte der Mathematik*, I. p. 289 sqq. See also Weissenborn, *Die Übersetzungen des Euklid durch Campano und Zamberti*, pp. 1—7.

[3] "Mea industria non sine maximo labore effeci vt qua facilitate litterarum elementa imprimuntur ea etiam geometrice figure conficerentur."

[4] Curtze in *Zeitschrift für Math. u. Physik*, XX., hist.-litt. Abth. p. 58.

Leonardum de Basilea et Gulielmum de Papia), but without the dedication to Mocenigo who had died in the meantime (1485). If Campanus added anything of his own, his additions are at all events not distinguished by any difference of type or otherwise; the enunciations are in large type, and the rest is printed continuously in smaller type. There are no superscriptions to particular passages such as *Euclides ex Campano*, *Campanus*, *Campani additio*, or *Campani annotatio*, which are found for the first time in the Paris edition of 1516 giving both Campanus' version and that of Zamberti (presently to be mentioned).

1501. G. Valla included in his encyclopaedic work *De expetendis et fugiendis rebus* published in this year at Venice (*in aedibus Aldi Romani*) a number of propositions with proofs and scholia translated from a Greek MS. which was once in his possession (cod. Mutin. III B, 4 of the 15th c.).

1505. In this year Bartolomeo Zamberti (Zambertus) brought out at Venice the first translation, from the Greek text, of the whole of the *Elements*. From the title[1], as well as from his prefaces to the *Catoptrica* and *Data*, with their allusions to previous translators "who take some things out of authors, omit some, and change some," or "to that most barbarous translator" who filled a volume purporting to be Euclid's "with extraordinary scarecrows, nightmares and phantasies," one object of Zamberti's translation is clear. His animus against Campanus appears also in a number of notes, e.g. when he condemns the terms "helmuain" and "helmuariphe" used by Campanus as barbarous, un-Latin etc., and when he is roused to wrath by Campanus' unfortunate mistranslation of V. Def. 5. He does not seem to have had the penetration to see that Campanus was translating from an Arabic, and not from a Greek, text. Zamberti tells us that he spent seven years over his translation of the thirteen Books of the *Elements*. As he seems to have been born in 1473, and the *Elements* were printed as early as 1500, though the complete work (including the *Phaenomena*, *Optica*, *Catoptrica*, *Data* etc.) has the date 1505 at the end, he must have translated Euclid before the age of 30. Heiberg has not been able to identify the MS. of the *Elements* which Zamberti used; but it is clear that it belonged to the worse class of MSS., since it contains most of the interpolations of the Theonine variety. Zamberti, as his title shows, attributed the *proofs* to Theon.

1509. As a counterblast to Zamberti, Luca Paciuolo brought out an edition of Euclid, apparently at the expense of Ratdolt, at Venice (*per Paganinum de Paganinis*), in which he set himself to vindicate Campanus. The title-page of this now very rare edition[2] begins thus: "The works of Euclid of Megara, a most acute philosopher and without question the chief of all mathematicians, translated by Campanus their *most faithful interpreter.*" It proceeds to say that the translation had been, through the fault of copyists, so spoiled and deformed that it could scarcely be recognised as Euclid. Luca Paciuolo accordingly has polished and emended it with the most critical judgment, has corrected 129 figures wrongly drawn and added others, besides supplying short explanations of difficult passages. It is added that Scipio Vegius of Milan, distinguished for his knowledge "*of both languages*" (i.e. of course Latin and Greek), as well as in medicine and the more sublime studies, had helped to make the edition more perfect. Though Zamberti is not once mentioned, this latter remark must have reference to Zamberti's statement that his translation was from the Greek text; and no doubt Zamberti is aimed at in the wish of Paciuolo's "that others too would seek to acquire knowledge instead of merely showing off, or that they would not try to make a market of the things of which they are ignorant, as it were (selling) smoke[1]." Weissenborn observes that, while there are many trivialities in Paciuolo's notes, they contain some useful and practical hints and explanations of terms, besides some new proofs which of course are not difficult if one takes the liberty, as Paciuolo does, of diverging from Euclid's order and assuming for the proof of a proposition results not arrived at till later. Two not inapt terms are used in this edition to describe the figures of III. 7, 8, the former of which is called the *goose's foot* (*pes anseris*), the second the *peacock's tail* (*cauda pavonis*). Paciuolo as the *castigator* of Campanus' translation, as he calls himself, failed to correct the mistranslation of V. Def. 5[2]. Before the fifth Book he inserted a discourse which he gave at Venice on the 15th August, 1508, in S. Bartholomew's Church, before a select audience of 500, as an introduction to his elucidation of that Book.

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[1] The title begins thus: "Euclidis megaresis philosophi platonicj mathematicarum disciplinarum Janitoris: Habent in hoc volumine quicunque ad mathematicam substantiam aspirant: elementorum libros xiij cum expositione Theonis insignis mathematici. quibus multa quae deerant ex lectione graeca sumpta addita sunt nec non plurima peruersa et praepostere: voluta in Campani interpretatione: ordinata digesta et castigata sunt etc." For a description of the book see Weissenborn, p. 12 sqq.

[2] See Weissenborn, p. 30 sqq.

1516. The first of the editions giving Campanus' and Zamberti's translations in conjunction was brought out at Paris (*in officina Henrici Stephani e regione scholae Decretorum*). The idea that only the enunciations were Euclid's, and that Campanus was the author of the proofs in his translation, while Theon was the author of the proofs in the Greek text, reappears in the title of this edition; and the enunciations of the added Books XIV., XV. are also attributed to Euclid, Hypsicles being credited with the proofs[3]. The date is not on the title-page nor at the end, but the letter of dedication to François Briconnet by Jacques Lefèvre is dated the day after the Epiphany, 1516. The figures are in the margin. The arrangement of the propositions is as follows: first the enunciation with the heading *Euclides ex Campano*, then the proof with the note *Campanus*, and after that, as *Campani additio*, any passage found in the edition of Campanus' translation but not in the Greek text; then follows the text of the enunciation translated from the Greek with the heading *Euclides ex Zamberto*, and lastly the proof headed *Theo ex Zamberto*. There are separate figures for the two proofs. This edition was reissued with few changes in 1537 and 1546 at Basel (*apud Iohannem Hervagium*), but with the addition of the *Phaenomena*, *Optica*, *Catoptrica* etc. For the edition of 1537 the Paris edition of 1516 was collated with "a Greek copy" (as the preface says) by Christian Herlin, professor of mathematical studies at Strassburg, who however seems to have done no more than correct one or two passages by the help of the Basel *editio princeps* (1533), and add the Greek word in cases where Zamberti's translation of it seemed unsuitable or inaccurate.

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[1] "Atque utinam et alii cognoscere vellent non ostentare aut ea quae nesciunt veluti fumum venditare non conarentur."

[2] Campanus' translation in Ratdolt's edition is as follows: "Quantitates quae dicuntur continuam habere proportionalitatem, sunt, quarum equè multiplicia aut equa sunt aut equè sibi sine interruptione addunt aut minuunt" (!), to which Campanus adds the note: "Continuè proportionalia sunt quorum omnia multiplicia equalia sunt continuè proportionalia. Sed noluit ipsam diffinitionem proponere sub hac forma, quia tunc diffiniret idem per idem, aperte (? a parte) tamen rei est istud cum sua diffinitione convertibile."

[3] "Euclidis Megarensis Geometricorum Elementorum Libri XV. Campani Galli transalpini in eosdem commentariorum libri XV. Theonis Alexandrini Bartholomaeo Zamberto Veneto interprete, in tredecim priores, commentationum libri XIII. Hypsiclis Alexandrini in duos posteriores, eodem Bartholomaeo Zamberto Veneto interprete, commentariorum libri II." On the last page (261) is a similar statement of content, but with the difference that the expression "ex Campani...deinde Theonis...et Hypsiclis...*traditionibus*." For description see Weissenborn, p. 56 sqq.

We now come to

II. EDITIONS OF THE GREEK TEXT.

1533 is the date of the *editio princeps*, the title-page of which reads as follows:

ΕΥΚΛΕΙΔΟΥ ΣΤΟΙΧΕΙΩΝ ΒΙΒΛ➤ ΙΕ̄➤
ΕΚ ΤΩΝ ΘΕΩΝΟΣ ΣΥΝΟΥΣΙΩΝ.
Εἰς τοῦ αὐτοῦ τὸ πρῶτον, ἐξηγημάτων Πρόκλου *βιβλ.* δ.
Adiecta praefatiuncula in qua de disciplinis
Mathematicis nonnihil.
BASILEAE APVD IOAN. HERVAGIVM ANNO
M.D.XXXIII. MENSE SEPTEMBRI.

The editor was Simon Grynaeus the elder (d. 1541), who, after working at Vienna and Ofen, Heidelberg and Tübingen, taught last of all at Basel, where theology was his main subject. His "praefatiuncula" is addressed to an Englishman, Cuthbert Tonstall (1474–1559), who, having studied first at Oxford, then at Cambridge, where he became Doctor of Laws, and afterwards at Padua, where in addition he learnt mathematics—mostly from the works of Regiomontanus and Paciuolo—wrote a book on arithmetic[1] as "a farewell to the sciences," and then, entering politics, became Bishop of London and member of the Privy Council, and afterwards (1530) Bishop of Durham. Grynaeus tells us that he used two MSS. of the text of the *Elements*, entrusted to friends of his, one at Venice by "Lazarus Bayfius" (Lazare de Baïf, then the ambassador of the King of France at Venice), the other at Paris by "Ioann. Rvellius" (Jean Ruel, a French doctor and a Greek scholar), while the commentaries of Proclus were put at the disposal of Grynaeus himself by "Ioann. Claymundus" at Oxford. Heiberg has been able to identify the two MSS. used for the text; they are (1) cod. Venetus Marcianus 301 and (2) cod. Paris. gr. 2343 of the 16th c., containing Books I.—XV., with some scholia which are embodied in the text. When Grynaeus notes in the margin the readings from "the other copy," this "other copy" is as a rule the Paris MS., though sometimes the reading of the Paris MS. is taken into the text and the "other copy" of the margin is the Venice MS. Besides these two MSS. Grynaeus consulted Zamberti, as is shown by a number of marginal notes referring to "Zampertus" or to "latinum exemplar" in certain propositions of Books IX.—XI. When it is considered that the two MSS. used by Grynaeus are among the worst, it is obvious how entirely unauthoritative is the text of the *editio princeps*. Yet it remained the source and foundation of later editions of the Greek text for a long period, the editions which followed being designed, not for the purpose of giving, from other MSS., a text more nearly representing what Euclid himself wrote, but of supplying a handy compendium to students at a moderate price.

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[1] *De arte supputandi libri quatuor.*

1536. Orontius Finaeus (Oronce Fine) published at Paris (*apud Simonem Colinaeum*) "demonstrations on the first six books of Euclid's elements of geometry," "in which the Greek text of Euclid himself is inserted in its proper places, with the Latin translation of Barth. Zamberti of Venice," which seems to imply that only the enunciations were given in Greek. The preface, from which Kästner quotes[1], says that the University of Paris at that time required, from all who aspired to the laurels of philosophy, a most solemn oath that they had attended lectures on the said first six Books. Other editions of Fine's work followed in 1544 and 1551.

1545. The *enunciations* of the fifteen Books were published in Greek, with an Italian translation by Angelo Caiani, at Rome (*apud Antonium Bladum Asulanum*). The translator claims to have corrected the books and "purged them of six hundred things which did not seem to savour of the almost divine genius and the perspicuity of Euclid[2]"

1549. Joachim Camerarius published the enunciations of the first six Books in Greek and Latin (Leipzig). The book, with preface, purports to be brought out by Rhaeticus (1514–1576), a pupil of Copernicus. Another edition with proofs of the propositions of the first three Books was published by Moritz Steinmetz in 1577 (Leipzig); a note by the printer attributes the preface to Camerarius himself.

1550. Ioan. Scheubel published at Basel (also *per Ioan. Hervagium*) the first six Books in Greek and Latin "together with true and appropriate proofs of the propositions, without the use of letters" (i.e. letters denoting points in the figures), the various straight lines and angles being described in words[3].

1557 (also 1558). Stephanus Gracilis published another edition (repeated 1573, 1578, 1598) of the enunciations (alone) of Books I.—XV.

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[1] Kästner, I. p. 260.

[2] Heiberg, vol. V. p. cvii.

[3] Kästner, I. p. 359.

in Greek and Latin at Paris (*apud Gulielmum Cavellat*). He remarks in the preface that for want of time he had changed scarcely anything in Books I.—VI., but in the remaining Books he had emended what seemed obscure or inelegant in the Latin translation, while he had adopted in its entirety the translation of Book X. by Pierre Mondoré (Petrus Montaureus), published separately at Paris in 1551. Gracilis also added a few "scholia."

1564. In this year Conrad Dasypodius (Rauchfuss), the inventor and maker of the clock in Strassburg cathedral, similar to the present one, which did duty from 1571 to 1789, edited (Strassburg, Chr. Mylius) (1) Book I. of the *Elements* in Greek and Latin with scholia, (2) Book II. in Greek and Latin with Barlaam's arithmetical version of Book II., and (3) the *enunciations* of the remaining Books III.—XIII. Book I. was reissued with "vocabula quaedam geometrica" of Heron, the enunciations of all the Books of the *Elements*, and the other works of Euclid, all in Greek and Latin. In the preface to (1) he says that it had been for twenty-six years the rule of his school that all who were promoted from the classes to public lectures should learn the first Book, and that he brought it out, because there were then no longer any copies to be had, and in order to prevent a good and fruitful regulation of his school from falling through. In the preface to the edition of 1571 he says that the first Book was generally taught in all gymnasia and that it was prescribed in his school for the first class. In the preface to (3) he tells us that he published the enunciations of Books III.—XIII. in order not to leave his work unfinished, but that, as it would be irksome to carry about the whole work of Euclid in extenso, he thought it would be more convenient to students of geometry to learn the *Elements* if they were compressed into a smaller book.

1620. Henry Briggs (of Briggs' logarithms) published the first six Books in Greek with a Latin translation after Commandinus, "corrected in many places" (London, G. Jones).

1703 is the date of the Oxford edition by David Gregory which, until the issue of Heiberg and Menge's edition, was still the only edition of the complete works of Euclid[1]. In the Latin translation attached to the Greek text Gregory says that he followed Commandinus in the main, but corrected numberless passages in it by means of the books in the Bodleian Library which belonged to Edward Bernard (1638–1696), formerly Savilian Professor of Astronomy, who had conceived the plan of publishing the complete works of the ancient mathematicians in fourteen volumes, of which the first was to contain Euclid's *Elements* I.—XV. As regards the Greek text, Gregory tells us that he consulted, as far as was necessary, not a few MSS. of the better sort, bequeathed by the great Savile to the University, as well as the corrections made by Savile in his own hand in the margin of the Basel edition. He had the help of John Hudson, Bodley's Librarian, who punctuated the Basel text before it went to the printer, compared the Latin version with the Greek throughout, especially in the *Elements* and *Data*, and, *where they differed* or *where he suspected the Greek text*, consulted the Greek MSS. and put their readings in the margin if they agreed with the Latin and, if they did not agree, affixed an asterisk in order that Gregory might judge which reading was geometrically preferable. Hence it is clear that no Greek MS., but the Basel edition, was the foundation of Gregory's text, and that Greek MSS. were only referred to in the special passages to which Hudson called attention.

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[1] ΕΥΚΛΕΙΔΟΥ ΤΑ ΣΩΖΟΜΕΝΑ. Euclidis quae supersunt omnia. Ex recensione Davidis Gregorii M.D. Astronomiae Professoris Saviliani et R.S.S. Oxoniae, e Theatro Sheldoniano, An. Dom. MDCCIII.

1814–1818. A most important step towards a good Greek text was taken by F. Peyrard, who published at Paris, between these years, in three volumes, the *Elements* and *Data* in Greek, Latin and French[1]. At the time (1808) when Napoleon was having valuable MSS. selected from Italian libraries and sent to Paris, Peyrard managed to get two ancient Vatican MSS. (190 and 1038) sent to Paris for his use (Vat. 204 was also at Paris at the time, but all three were restored to their owners in 1814). Peyrard noticed the excellence of Cod. Vat. 190, adopted many of its readings, and gave in an appendix a conspectus of these readings and those of Gregory's edition; he also noted here and there readings from Vat. 1038 and various Paris MSS. He therefore pointed the way towards a better text, but committed the error of correcting the Basel text instead of rejecting it altogether and starting afresh.

1824–1825. A most valuable edition of Books I.—VI. is that of J. G. Camerer (and C. F. Hauber) in two volumes published at Berlin[2]. The Greek text is based on Peyrard, although the Basel and Oxford editions were also used. There is a Latin translation and a collection of notes far more complete than any other I have seen and well nigh inexhaustible. There is no editor or commentator of any mark who is not quoted from; to show the variety of important authorities drawn upon by Camerer, I need only mention the following names: Proclus, Pappus, Tartaglia, Commandinus, Clavius, Peletier, Barrow, Borelli, Wallis, Tacquet, Austin, Simson, Playfair. No words of praise would be too warm for this veritable encyclopaedia of information.

1825. J. G. C. Neide edited, from Peyrard, the text of Books I.—VI., XI. and XII. (*Halis Saxoniae*).

1826–9. The last edition of the Greek text before Heiberg's is that of E. F. August, who followed the Vatican MS. more closely than Peyrard did, and consulted at all events the Viennese MS. Gr. 103 (Heiberg's V). August's edition (Berlin, 1826–9) contains Books I.—XIII.

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[1] *Euclidis quae supersunt. Les Œuvres d'Euclide, en Grec, en Latin et en Français d'après un manuscrit très-ancien, qui était resté inconnu jusqu'à nos jours.* Par F. Peyrard. Ouvrage approuvé par l'Institut de France (Paris, chez M. Patris).

[2] *Euclidis elementorum libri sex priores graece et latine commentario e scriptis veterum ac recentiorum mathematicorum et Pfleidereri maxime illustrati* (Berolini, sumptibus G. Reimeri). Tom. I. 1824; tom. II. 1825.

III. LATIN VERSIONS OR COMMENTARIES AFTER 1533.

1545. Petrus Ramus (Pierre de la Ramée, 1515–1572) is credited with a translation of Euclid which appeared in 1545 and again in 1549 at Paris[1]. Ramus, who was more rhetorician and logician than geometer, also published in his *Scholae mathematicae* (1559, Frankfurt; 1569, Basel) what amounts to a series of lectures on Euclid's *Elements*, in which he criticises Euclid's arrangement of his propositions, the definitions, postulates and axioms, all from the point of view of logic.

1557. Demonstrations to the geometrical Elements of Euclid, six Books, by Peletarius (Jacques Peletier). The second edition (1610) contained the same with the addition of the "Greek text of Euclid"; but only the *enunciations* of the propositions, as well as the definitions etc., are given in Greek (with a Latin translation), the rest is in Latin only. He has some acute observations, for instance about the "angle" of contact.

1559. Johannes Buteo, or Borrel (1492–1572), published in an appendix to his book *De quadratura circuli* some notes "on the errors of Campanus, Zambertus, Orontius, Peletarius, Pena, interpreters of Euclid." Buteo in these notes proved, by reasoned argument based on original authorities, that Euclid himself and not Theon was the author of the proofs of the propositions.

1566. Franciscus Flussates Candalla (François de Foix, Comte de Candale, 1502–1594) "restored" the fifteen Books, following, as he says, the terminology of Zamberti's translation from the Greek, but drawing, for his proofs, on both Campanus and Theon (i.e. Zamberti) except where mistakes in them made emendation necessary. Other editions followed in 1578, 1602, 1695 (in Dutch).

1572. The most important Latin translation is that of Commandinus (1509–1575) of Urbino, since it was the foundation of most translations which followed it up to the time of Peyrard, including that of Simson and therefore of those editions, numerous in England, which give Euclid "chiefly after the text of Simson." Simson's first (Latin) edition (1756) has "ex versione Latina Federici Commandini" on the title-page. Commandinus not only followed the original Greek more closely than his predecessors but added to his translation some ancient scholia as well as good notes of his own. The title of his work is

> *Euclidis elementorum libri* XV, *una cum scholiis antiquis. A Federico Commandino Urbinate nuper in latinum conversi, commentariisque quibusdam illustrati* (Pisauri, apud Camillum Francischinum).

He remarks in his preface that Orontius Finaeus had only edited six Books without reference to any Greek MS., that Peletarius had followed Campanus' version from the Arabic rather than the Greek text, and that Candalla had diverged too far from Euclid, having rejected as inelegant the proofs given in the Greek text and substituted faulty proofs of his own. Commandinus appears to have used, in addition to the Basel *editio princeps*, some Greek MS., so far not identified; he also extracted his "scholia antiqua" from a MS. of the class of Vat. 192 containing the scholia distinguished by Heiberg as "Schol. Vat." New editions of Commandinus' translation followed in 1575 (in Italian), 1619, 1749 (in English, by Keill and Stone), 1756 (Books I.—VI., XI., XII. in Latin and English, by Simson), 1763 (Keill). Besides these there were many editions of parts of the whole work, e.g. the first six Books.

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[1] Described by Boncompagni, *Bullettino*, II. p. 389.

1574. The first edition of the Latin version by Clavius[1] (Christoph Klau [?]. born at Bamberg 1537, died 1612) appeared in 1574, and new editions of it in 1589, 1591, 1603, 1607, 1612. It is not a translation, as Clavius himself states in the preface, but it contains a vast amount of notes collected from previous commentators and editors, as well as some good criticisms and elucidations of his own. Among other things, Clavius finally disposed of the error by which Euclid had been identified with Euclid of Megara. He speaks of the differences between Campanus who followed the Arabic tradition and the "commentaries of Theon," by which he appears to mean the Euclidean proofs as handed down by Theon; he complains of predecessors who have either only given the first six Books, or have rejected the ancient proofs and substituted worse proofs of their own, but makes an exception as regards Commandinus, "a geometer not of the common sort, who has lately restored Euclid, in a Latin translation, to his original brilliancy." Clavius, as already stated, did not give a translation of the *Elements* but rewrote the proofs, compressing them or adding to them, where he thought that he could make them clearer. Altogether his book is a most useful work.

1621. Henry Savile's lectures (*Praelectiones tresdecim in principium Elementorum Euclidis Oxoniae habitae* MDC.XX., Oxonii 1621), though they do not extend beyond I. 8, are valuable because they grapple with the difficulties connected with the preliminary matter, the definitions etc., and the tacit assumptions contained in the first propositions.

1654. André Tacquet's *Elementa geometriae planae et solidae* containing apparently the eight geometrical Books arranged for general use in schools. It came out in a large number of editions up to the end of the eighteenth century.

1655. Barrow's *Euclidis Elementorum Libri* XV *breviter demonstrati* is a book of the same kind. In the preface (to the edition of 1659) he says that he would not have written it but for the fact that Tacquet gave only eight Books of Euclid. He compressed the work into a very small compass (less than 400 small pages, in the edition of 1659, for the whole of the fifteen Books and the *Data*) by abbreviating the proofs and using a large quantity of symbols (which, he says, are generally Oughtred's). There were several editions up to 1732 (those of 1660 and 1732 and one or two others are in English).

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[1] *Euclidis elementorum libri* XV. *Accessit* XVI. *de solidorum regularium comparatione. Omnes perspicuis demonstrationibus, accuratisque scholiis illustrati. Auctore Christophoro Clavio* (Romae, apud Vincentium Accoltum), 2 vols.

1658. Giovanni Alfonso Borelli (1608–1679) published *Euclides restitutus*, on apparently similar lines, which went through three more editions (one in Italian, 1663).

1660. Claude François Milliet Dechales' eight geometrical Books of Euclid's *Elements* made easy. Dechales' versions of the *Elements* had great vogue, appearing in French, Italian and English as well as Latin. Riccardi enumerates over twenty editions.

1733. Saccheri's *Euclides ab omni naevo vindicatus sive conatus geometricus quo stabiliuntur prima ipsa geometriae principia* is important for his elaborate attempt to prove the parallel-postulate, forming an important stage in the history of the development of non-Euclidean geometry.

1756. Simson's first edition, in Latin and in English. The Latin title is

> *Euclidis elementorum libri priores sex, item undecimus et duodecimus, ex versione latina Federici Commandini; sublatis iis quibus olim libri hi a Theone, aliisve, vitiati sunt, et quibusdam Euclidis demonstrationibus restitutis. A Roberto Simson M.D.* Glasguae, in aedibus Academicis excudebant Robertus et Andreas Foulis, Academiae typographi.

1802. *Euclidis elementorum libri priores* XII *ex Commandini et Gregorii versionibus latinis. In usum juventutis Academicae*...by Samuel Horsley, Bishop of Rochester. (Oxford, Clarendon Press.)

IV. ITALIAN VERSIONS OR COMMENTARIES.

1543. Tartaglia's version, a second edition of which was published in 1565[1], and a third in 1585. It does not appear that he used any Greek text, for in the edition of 1565 he mentions as available only "the first translation by Campano," "the second made by Bartolomeo Zamberto Veneto who is still alive," "the editions of Paris or Germany in which they have included both the aforesaid translations," and "our own translation into the vulgar (tongue)."

1575. Commandinus' translation turned into Italian and revised by him.

1613. The first six Books "reduced to practice" by Pietro Antonio Cataldi, re-issued in 1620, and followed by Books VII.—IX. (1621) and Book X. (1625).

1663. Borelli's Latin translation turned into Italian by Domenico Magni.

1680. *Euclide restituto* by Vitale Giordano.

1690. Vincenzo Viviani's *Elementi piani e solidi di Euclide* (Book V. in 1674).

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[1] The title-page of the edition of 1565 is as follows: *Euclide Megarense philosopho, solo introduttore delle scientie mathematice, diligentemente rassettato, et alla integrità ridotto, per il degno professore di tal scientie Nicolo Tartalea Brisciano. secondo le due tradottioni. con una ampla espositione dello istesso tradottore di nuouo aggiunta. talmente chiara, che ogni mediocre ingegno, senza la notitia, ouer suffragio di alcun' altra scientia con facilità serà capace a poterlo intendere.* In Venetia, Appresso Curtio Troiano, 1565.

1731. *Elementi geometrici piani e solidi di Euclide* by Guido Grandi. No translation, but an abbreviated version, of which new editions followed one another up to 1806.

1749. Italian translation of Dechales with Ozanam's corrections and additions, re-issued 1785, 1797.

1752. Leonardo Ximenes (the first six Books). Fifth edition, 1819.

1818. Vincenzo Flauti's *Corso di geometria elementare e sublime* (4 vols.) contains (Vol. I.) the first six Books, with additions and a dissertation on Postulate 5, and (Vol. II.) Books XI., XII. Flauti also published the first six Books in 1827 and the *Elements of geometry of Euclid* in 1843 and 1854.

V. GERMAN.

1558. The arithmetical Books VII.—IX. by Scheubel[1] (cf. the edition of the first six Books, with enunciations in Greek and Latin, mentioned above, under date 1550).

1562. The version of the first six Books by Wilhelm Holtzmann (Xylander)[2]. This work has its interest as the first edition in German, but otherwise it is not of importance. Xylander tells us that it was written for practical people such as artists, goldsmiths, builders etc., and that, as the simple amateur is of course content to know facts, without knowing how to prove them, he has often left out the proofs altogether. He has indeed taken the greatest possible liberties with Euclid, and has not grappled with any of the theoretical difficulties, such as that of the theory of parallels.

1651. Heinrich Hoffmann's *Teutscher Euclides* (2nd edition 1653), not a translation.

1694. Ant. Ernst Burkh. v. Pirckenstein's *Teutsch Redender Euclides* (eight geometrical Books), "for generals, engineers etc." "proved in a new and quite easy manner." Other editions 1699, 1744.

1697. Samuel Reyher's *In teutscher Sprache vorgestellter Euclides* (six Books), "made easy, with symbols algebraical or derived from the newest art of solution."

1714. *Euclidis* XV *Bücher teutsch*, "treated in a special and brief manner, yet completely," by Chr. Schessler (another edition in 1729).

1773. The first six Books translated from the Greek for the use of schools by J. F. Lorenz. The first attempt to reproduce Euclid in German word for word.

1781. Books XI., XII. by Lorenz (supplementary to the preceding). Also *Euklid's Elemente fünfzehn Bücher* translated from the Greek by Lorenz (second edition 1798; editions of 1809, 1818, 1824 by Mollweide, of 1840 by Dippe). The edition of 1824, and I presume those before it, are shortened by the use of symbols and the compression of the enunciation and "setting-out" into one.

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[1] *Das sibend acht und neunt buch des hochberümbten Mathematici Euclidis Megarensis... durch Magistrum Johann Scheybl, der löblichen universitet zu Tübingen, des Euclidis und Arithmetic Ordinarien, auss dem latein ins teutsch gebracht....*

[2] *Die sechs erste Bücher Euclidis vom anfang oder grund der Geometrj...Auss Griechischer sprach in die Teütsch gebracht aigentlich erklärt...Demassen vormals in Teütscher sprach nie gesehen worden...Durch Wilhelm Holtzman genant Xylander von Augspurg.* Getruckht zu Basel.

1807. Books I.—VI., XI., XII. "newly translated from the Greek," by J. K. F. Hauff.

1828. The same Books by Joh. Jos. Ign. Hoffmann "as guide to instruction in elementary geometry," followed in 1832 by observations on the text by the same editor.

1833. *Die Geometrie des Euklid und das Wesen derselben* by E. S. Unger; also 1838, 1851.

1901. Max Simon, *Euclid und die sechs planimetrischen Bücher.*

VI. FRENCH.

1564–1566. Nine Books translated by Pierre Forcadel, a pupil and friend of P. de la Ramée.

1604. The first nine Books translated and annotated by Jean Errard de Bar-le-Duc; second edition, 1605.

1615. Denis Henrion's translation of the 15 Books (seven editions up to 1676).

1639. The first six Books "demonstrated by symbols, by a method very brief and intelligible," by Pierre Hérigone, mentioned by Barrow as the only editor who, before him, had used symbols for the exposition of Euclid.

1672. Eight Books "rendus plus faciles" by Claude François Milliet Dechales, who also brought out *Les élémens d'Euclide expliqués d'une manière nouvelle et très facile*, which appeared in many editions, 1672, 1677, 1683 etc. (from 1709 onwards revised by Ozanam), and was translated into Italian (1749 etc.) and English (by William Halifax, 1685).

1804. In this year, and therefore before his edition of the Greek text, F. Peyrard published the *Elements* literally translated into French. A second edition appeared in 1809 with the addition of the fifth Book. As this second edition contains Books I.—VI. XI., XII. and X. 1, it would appear that the first edition contained Books I.—IV., VI., XI., XII. Peyrard used for this translation the Oxford Greek text and Simson.

VII. DUTCH.

1606. Jan Pieterszoon Dou (six Books). There were many later editions. Kästner, in mentioning one of 1702, says that Dou explains in his preface that he used Xylander's translation, but, having afterwards obtained the French translation of the six Books by Errard de Bar-le-Duc (see above), the proofs in which sometimes pleased him more than those of the German edition, he made his Dutch version by the help of both.

1617. Frans van Schooten, "The Propositions of the Books of Euclid's Elements"; the fifteen Books in this version "enlarged" by Jakob van Leest in 1662.

1695. C. J. Vooght, fifteen Books complete, with Candalla's "16th."

1702. Hendrik Coets, six Books (also in Latin, 1692); several editions up to 1752. Apparently not a translation but an edition for school use.

1763. Pybo Steenstra, Books I.—VI., XI., XII., likewise an abbreviated version, several times reissued until 1825.

VIII. ENGLISH.

1570 saw the first and the most important translation, that of Sir Henry Billingsley. The title-page is as follows:

THE ELEMENTS
OF GEOMETRIE
of the most auncient Philosopher
EVCLIDE
of Megara

Faithfully (now first) translated into the Englishe toung, by H. Billingsley, *Citizen of London. Whereunto are annexed certaine Scholies, Annotations, and Inuentions, of the best Mathematiciens, both of time past, and in this our age.*

With a very fruitfull Preface by M. I. Dee, *specifying the chiefe Mathematicall Sciēces, what they are, and whereunto commodious: where, also, are disclosed certaine new Secrets Mathematicall and Mechanicall, vntill these our daies, greatly missed.*

Imprinted at London by *John Daye.*

The Preface by the translator, after a sentence observing that without the diligent study of Euclides Elementes it is impossible to attain unto the perfect knowledge of Geometry, proceeds thus. "Wherefore considering the want and lacke of such good authors hitherto in our Englishe tounge, lamenting also the negligence, and lacke of zeale to their countrey in those of our nation, to whom God hath geuen both knowledge and also abilitie to translate into our tounge, and to publishe abroad such good authors and bookes (the chiefe instrumentes of all learninges): seing moreouer that many good wittes both of gentlemen and of others of all degrees, much desirous and studious of these artes, and seeking for them as much as they can, sparing no paines, and yet frustrate of their intent, by no meanes attaining to that which they seeke: I haue for their sakes, with some charge and great trauaile, faithfully translated into our vulgare tōge, and set abroad in Print, this booke of Euclide. Whereunto I haue added easie and plaine declarations and examples by figures, of the definitions. In which booke also ye shall in due place finde manifolde additions, Scholies, Annotations, and Inuentions: which I haue gathered out of many of the most famous and chiefe Mathematiciēs, both of old time, and in our age: as by diligent reading it in course, ye shall well perceaue...."

It is truly a monumental work, consisting of 464 leaves, and therefore 928 pages, of folio size, excluding the lengthy preface by Dee. The notes certainly include all the most important that had ever been written, from those of the Greek commentators, Proclus and the others whom he quotes, down to those of Dee himself on the last books. Besides the fifteen Books, Billingsley included the "sixteenth" added by Candalla. The print and appearance of the book are worthy of its contents; and, in order that it may be understood how no pains were spared to represent everything in the clearest and most perfect form, I need only mention that the figures of the propositions in Book XI. are nearly all duplicated, one being the figure of Euclid, the other an arrangement of pieces of paper (triangular, rectangular etc.) pasted at the edges on to the page of the book so that the pieces can be turned up and made to show the real form of the solid figures represented.

Billingsley was admitted Lady Margaret Scholar of St John's College, Cambridge, in 1551, and he is also said to have studied at Oxford, but he did not take a degree at either University. He was afterwards apprenticed to a London haberdasher and rapidly became a wealthy merchant. Sheriff of London in 1584, he was elected Lord Mayor on 31st December, 1596, on the death, during his year of office, of Sir Thomas Skinner. From 1589 he was one of the Queen's four "customers," or farmers of customs, of the port of London. In 1591 he founded three scholarships at St John's College for poor students, and gave to the College for their maintenance two messuages and tenements in Tower Street and in Mark Lane, Allhallows, Barking. He died in 1606.

1651. *Elements of Geometry. The first* VI *Boocks: In a compendious form contracted and demonstrated* by Captain Thomas Rudd, with the mathematicall preface of John Dee (London).

1660. The first English edition of Barrow's Euclid (published in Latin in 1655), appeared in London. It contained "the whole fifteen books compendiously demonstrated"; several editions followed, in 1705, 1722, 1732, 1751.

1661. *Euclid's Elements of Geometry, with a supplement of divers Propositions and Corollaries. To which is added a Treatise of regular Solids by Campane and Flussat; likewise Euclid's Data and Marinus his Preface. Also a Treatise of the Divisions of Superficies, ascribed to Machomet Bagdedine, but published by Commandine at the request of J. Dee of London.* Published by care and industry of John Leeke and Geo. Serle, students in the Math. (London). According to Potts this was a second edition of Billingsley's translation.

1685. William Halifax's version of Dechales' "Elements of Euclid explained in a new but most easy method" (London and Oxford).

1705. *The English Euclide; being the first six Elements of Geometry, translated out of the Greek, with annotations and usefull supplements by* Edmund Scarburgh (Oxford). A noteworthy and useful edition.

1708. Books I.—VI., XI., XII., translated from Commandinus' Latin version by Dr John Keill, Savilian Professor of Astronomy at Oxford.

Keill complains in his preface of the omissions by such editors as Tacquet and Dechales of many necessary propositions (e.g. VI. 27—29), and of their substitution of proofs of their own for Euclid's. He praises Barrow's version on the whole, though objecting to the "algebraical" form of proof adopted in Book II., and to the excessive use of notes and symbols, which (he considers) make the proofs *too* short and thereby obscure; his edition was therefore intended to hit a proper mean between Barrow's excessive brevity and Clavius' prolixity.

Keill's translation was revised by Samuel Cunn and several times reissued. 1749 saw the eighth edition, 1772 the eleventh, and 1782 the twelfth.

1714. W. Whiston's English version (abridged) of *The Elements of Euclid with select theorems out of Archimedes by the learned Andr. Tacquet.*

1756. Simson's first English edition appeared in the same year as his Latin version under the title:

> *The Elements of Euclid, viz. the first six Books together with the eleventh and twelfth. In this Edition the Errors by which Theon or others have long ago vitiated these Books are corrected and some of Euclid's Demonstrations are restored.* By Robert Simson (Glasgow).

As above stated, the Latin edition, by its title, purports to be "ex versione latina Federici Commandini," but to the Latin edition, as well as to the English editions, are appended

> *Notes Critical and Geometrical; containing an Account of those things in which this Edition differs from the Greek text; and the Reasons of the Alterations which have been made. As also Observations on some of the Propositions.*

Simson says in the Preface to some editions (e.g. the tenth, of 1799) that "the translation is much amended by the friendly assistance of a learned gentleman."

Simson's version and his notes are so well known as not to need any further description. The book went through some thirty successive editions. The first five appear to have been dated 1756, 1762, 1767, 1772 and 1775 respectively; the tenth 1799, the thirteenth 1806, the twenty-third 1830, the twenty-fourth 1834, the twenty-sixth 1844. The *Data* "in like manner corrected" was added for the first time in the edition of 1762 (the first octavo edition).

1781, 1788. In these years respectively appeared the two volumes containing the complete translation of the whole thirteen Books by James Williamson, the last English translation which reproduced Euclid word for word. The title is

> *The Elements of Euclid, with Dissertations intended to assist and encourage a critical examination of these Elements, as the most effectual means of establishing a juster taste upon mathematical subjects than that which at present prevails.* By James Williamson.

In the first volume (Oxford, 1781) he is described as "M.A. Fellow of Hertford College," and in the second (London, printed by T. Spilsbury, 1788) as "B.D." simply. Books V., VI. with the Conclusion in the first volume are paged separately from the rest.

1781. *An examination of the first six Books of Euclid's Elements*, by William Austin (London).

1795. John Playfair's first edition, containing "the first six Books of Euclid with two Books on the Geometry of Solids." The book reached a fifth edition in 1819, an eighth in 1831, a ninth in 1836, and a tenth in 1846.

1826. Riccardi notes under this date *Euclid's Elements of Geometry containing the whole twelve Books translated into English, from the edition of Peyrard*, by George Phillips. The editor, who was President of Queens' College, Cambridge, 1857–1892, was born in 1804 and matriculated at Queens' in 1826, so that he must have published the book as an undergraduate.

1828. A very valuable edition of the first six Books is that of Dionysius Lardner, with commentary and geometrical exercises, to which he added, in place of Books XI., XII., a Treatise on Solid Geometry mostly based on Legendre. Lardner compresses the propositions by combining the enunciation and the setting-out, and he gives a vast number of riders and additional propositions in smaller print. The book had reached a ninth edition by 1846, and an eleventh by 1855. Among other things, Lardner gives an Appendix "on the theory of parallel lines," in which he gives a short history of the attempts to get over the difficulty of the parallel-postulate, down to that of Legendre.

1833. T. Perronet Thompson's *Geometry without axioms, or the first Book of Euclid's Elements with alterations and notes; and an intercalary book in which the straight line and plane are derived from properties of the sphere, with an appendix containing notices of methods proposed for getting over the difficulty in the twelfth axiom of Euclid.*

Thompson (1783–1869) was 7th wrangler 1802, midshipman 1803, Fellow of Queens' College, Cambridge, 1804, and afterwards general and politician. The book went through several editions, but, having been well translated into French by Van Tenac, is said to have received more recognition in France than at home.

1845. Robert Potts' first edition (and one of the best) entitled:

> *Euclid's Elements of Geometry chiefly from the text of Dr Simson with explanatory notes...to which is prefixed an introduction containing a brief outline of the History of Geometry. Designed for the use of the higher forms in Public Schools and students in the Universities* (Cambridge University Press, and London, John W. Parker), to which was added (1847) *An Appendix to the larger edition of Euclid's Elements of Geometry, containing additional notes on the Elements, a short tract on transversals, and hints for the solution of the problems etc.*

1862. Todhunter's edition.

The later English editions I will not attempt to enumerate; their name is legion and their object mostly that of adapting Euclid for school use, with all possible gradations of departure from his text and order.

IX. SPANISH.

1576. The first six Books translated into Spanish by Rodrigo Çamorano.

1637. The first six Books translated, with notes, by L. Carduchi.

1689. Books I.—VI., XI., XII., translated and explained by Jacob Knesa.

X. RUSSIAN.

1739. Ivan Astaroff (translation from Latin).
1789. Pr. Suvoroff and Yos. Nikitin (translation from Greek).
1880. Vachtchenko-Zakhartchenko.
(1817. A translation into Polish by Jo. Czecha.)

XI. SWEDISH.

1744. Mårten Strömer, the first six Books; second edition 1748. The third edition (1753) contained Books XI.—XII. as well; new editions continued to appear till 1884.

1836. H. Falk, the first six Books.
1844, 1845, 1859. P. R. Bråkenhjelm, Books I.—VI., XI., XII.
1850. F. A. A. Lundgren.
1850. H. A. Witt and M. E. Areskong, Books I.—VI., XI., XII.

XII. DANISH.

1745. Ernest Gottlieb Ziegenbalg.
1803. H. C. Linderup, Books I.—VI.

XIII. MODERN GREEK.

1820. Benjamin of Lesbos.

I should add a reference to certain editions which have appeared in recent years.

A Danish translation (*Euklid's Elementer* oversat af Thyra Eibe) was completed in 1912; Books I.—II. were published (with an Introduction by Zeuthen) in 1897, Books III.—IV. in 1900, Books V.—VI. in 1904, Books VII.—XIII. in 1912.

The Italians, whose great services to elementary geometry are more than once emphasised in this work, have lately shown a noteworthy disposition to make the *ipsissima verba* of Euclid once more the object of study. Giovanni Vacca has edited the text of Book I. (*Il primo libro degli Elementi.* Testo greco, versione italiana, introduzione e note, Firenze 1916.) Federigo Enriques has begun the publication of a complete Italian translation (*Gli Elementi d' Euclide e la critica antica e moderna*); Books I.—IV. appeared in 1925 (Alberto Stock, Roma).

An edition of Book I. by the present writer was published in 1918 (*Euclid in Greek, Book I., with Introduction and Notes*, Camb. Univ. Press).

CHAPTER IX.

§ 1. ON THE NATURE OF *ELEMENTS*.

It would not be easy to find a more lucid explanation of the terms *element* and *elementary*, and of the distinction between them, than is found in Proclus[1], who is doubtless, here as so often, quoting from Geminus. There are, says Proclus, in the whole of geometry certain leading theorems, bearing to those which follow the relation of a principle, all-pervading, and furnishing proofs of many properties. Such theorems are called by the name of *elements*; and their function may be compared to that of the letters of the alphabet in relation to language, letters being indeed called by the same name in Greek (*στοιχεῖα*).

The term *elementary*, on the other hand, has a wider application: it is applicable to things "which extend to greater multiplicity, and, though possessing simplicity and elegance, have no longer the same dignity as the *elements*, because their investigation is not of general use in the whole of the science, e.g. the proposition that in triangles the perpendiculars from the angles to the transverse sides meet in a point."

"Again, the term *element* is used in two senses, as Menaechmus says. For that which is the means of obtaining is an element of that which is obtained, as the first proposition in Euclid is of the second, and the fourth of the fifth. In this sense many things may even be said to be elements of each other, for they are obtained from one another. Thus from the fact that the exterior angles of rectilineal figures are (together) equal to four right angles we deduce the number of right angles equal to the internal angles (taken together)[2], and *vice versa*. Such an element is like a *lemma*. But the term *element* is otherwise used of that into which, being more simple, the composite is divided; and in this sense we can no longer say that everything is an element of everything, but only that things which are more of the nature of principles are elements of those which stand to them in the relation of results, as postulates are elements of theorems. It is according to this signification of the term *element* that the elements found in Euclid were compiled, being partly those of plane geometry, and partly those of stereometry. In like manner many writers have drawn up elementary treatises in arithmetic and astronomy.

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[1] Proclus, *Comm. on Eucl.* I., ed. Friedlein, pp. 72 sqq.

[2] *τὸ πλῆθος τῶν ἐντὸς ὀρθαῖς ἴσων.* If the text is right, we must apparently take it as "the number of the angles equal to right angles that there are inside," i.e. that are made up by the internal angles.

"Now it is difficult, in each science, both to select and arrange in due order the elements from which all the rest proceeds, and into which all the rest is resolved. And of those who have made the attempt some were able to put together more and some less; some used shorter proofs, some extended their investigation to an indefinite length; some avoided the method of *reductio ad absurdum*, some avoided *proportion*; some contrived preliminary steps directed against those who reject the principles; and, in a word, many different methods have been invented by various writers of elements.

"It is essential that such a treatise should be rid of everything superfluous (for this is an obstacle to the acquisition of knowledge); it should select everything that embraces the subject and brings it to a point (for this is of supreme service to science); it must have great regard at once to clearness and conciseness (for their opposites trouble our understanding); it must aim at the embracing of theorems in general terms (for the piecemeal division of instruction into the more partial makes knowledge difficult to grasp). In all these ways Euclid's system of elements will be found to be superior to the rest; for its utility avails towards the investigation of the primordial figures[1], its clearness and organic perfection are secured by the progression from the more simple to the more complex and by the foundation of the investigation upon common notions, while generality of demonstration is secured by the progression through the theorems which are primary and of the nature of principles to the things sought. As for the things which seem to be wanting, they are partly to be discovered by the same methods, like the construction of the scalene and isosceles (triangle), partly alien to the character of a selection of elements as introducing hopeless and boundless complexity, like the subject of *unordered irrationals* which Apollonius worked out at length[2], and partly developed from things handed down (in the elements) as causes, like the many species of angles and of lines. These things then have been omitted in Euclid, though they have received full discussion in other works; but the knowledge of them is derived from the simple (elements)."

Proclus, speaking apparently on his own behalf, in another place distinguishes two objects aimed at in Euclid's *Elements*. The first has reference to the *matter* of the investigation, and here, like a good Platonist, he takes the whole subject of geometry to be concerned with the "cosmic figures," the five regular solids, which in Book XIII.

[1] *τῶν ἀρχικῶν σχημάτων*, by which Proclus probably means the regular polyhedra (Tannery, p. 143 *n.*).

[2] We have no more than the most obscure indications of the character of this work in an Arabic MS. analysed by Woepcke, *Essai d'une restitution de travaux perdus d'Apollonius sur les quantités irrationelles d'après des indications tirées d'un manuscrit arabe* in *Mémoires présentés à l'académie des sciences*, XIV. 658—720, Paris, 1856. Cf. Cantor, *Gesch. d. Math.* I$_3$, pp. 348—9: details are also given in my notes to Book X.

are constructed, inscribed in a sphere and compared with one another. The second object is relative to the learner; and, from this standpoint, the elements may be described as "a means of perfecting the learner's understanding with reference to the whole of geometry. For, starting from these (elements), we shall be able to acquire knowledge of the other parts of this science as well, while without them it is impossible for us to get a grasp of so complex a subject, and knowledge of the rest is unattainable. As it is, the theorems which are most of the nature of principles, most simple, and most akin to the first hypotheses are here collected, in their appropriate order; and the proofs of all other propositions use these theorems as thoroughly well known, and start from them. Thus Archimedes in the books on the sphere and cylinder, Apollonius, and all other geometers, clearly use the theorems proved in this very treatise as constituting admitted principles[1]."

Aristotle too speaks of *elements* of geometry in the same sense. Thus: "in geometry it is well to be thoroughly versed in the elements[2]"; "in general the first of the elements are, given the definitions, e.g. of a straight line and of a circle, most easy to prove, although of course there are not many data that can be used to establish each of them because there are not many middle terms[3]"; "among geometrical propositions we call those 'elements' the proofs of which are contained in the proofs of all or most of such propositions[4]"; "(as in the case of bodies), so in like manner we speak of the elements of geometrical propositions and, generally, of demonstrations; for the demonstrations which come first and are contained in a variety of other demonstrations are called elements of those demonstrations... the term element is applied by analogy to that which, being one and small, is useful for many purposes[5]."

§ 2. *ELEMENTS* ANTERIOR TO EUCLID'S.

The early part of the famous summary of Proclus was no doubt drawn, at least indirectly, from the history of geometry by Eudemus; this is generally inferred from the remark, made just after the mention of Philippus of Medma, a disciple of Plato, that "those who have written histories bring the development of this science up to this point." We have therefore the best authority for the list of writers of *elements* given in the summary. Hippocrates of Chios (fl. in second half of 5th c.) is the first; then Leon, who also discovered *diorismi*, put together a more careful collection, the propositions proved in it being more numerous as well as more serviceable[6]. Leon was a little older than Eudoxus (about 408–355 B.C.) and a little younger than Plato (428/7–347/6 B.C.), but did not belong to the latter's school. The geometrical text-book of the Academy was written by Theudius of Magnesia, who, with Amyclas of Heraclea, Menaechmus the pupil of Eudoxus, Menaechmus' brother Dinostratus and Athenaeus of Cyzicus consorted together in the Academy and carried on their investigations in common. Theudius "put together the elements admirably, making many partial (or limited) propositions more general[1]." Eudemus mentions no text-book after that of Theudius, only adding that Hermotimus of Colophon "discovered many of the elements[2]." Theudius then must be taken to be the immediate precursor of Euclid, and no doubt Euclid made full use of Theudius as well as of the discoveries of Hermotimus and all other available material. Naturally it is not in Euclid's *Elements* that we can find much light upon the state of the subject when he took it up; but we have another source of information in Aristotle. Fortunately for the historian of mathematics, Aristotle was fond of mathematical illustrations; he refers to a considerable number of geometrical propositions, definitions etc., in a way which shows that his pupils must have had at hand some text-book where they could find the things he mentions; and this text-book must have been that of Theudius. Heiberg has made a most valuable collection of mathematical extracts from Aristotle[3], from which much is to be gathered as to the changes which Euclid made in the methods of his predecessors; and these passages, as well as others not included in Heiberg's selection, will often be referred to in the sequel.

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[1] Proclus, pp. 70, 19—71, 21.

[2] *Topics* VIII. 14, 163 b 23.

[3] *Topics* VIII. 3, 158 b 35.

[4] *Metaph.* 998 a 25.

[5] *Metaph.* 1014 a 35—b 5.

[6] Proclus, p. 66, 20 ὥστε τὸν Λέοντα καὶ τὰ στοιχεῖα συνθεῖναι τῷ τε πλήθει καὶ τῇ χρείᾳ τῶν δεικνυμένων ἐπιμελέστερον.

§ 3. FIRST PRINCIPLES: DEFINITIONS, POSTULATES, AND AXIOMS.

On no part of the subject does Aristotle give more valuable information than on that of the first principles as, doubtless, generally accepted at the time when he wrote. One long passage in the *Posterior Analytics* is particularly full and lucid, and is worth quoting *in extenso*. After laying it down that every demonstrative science starts from necessary principles[4], he proceeds[5]:

"By first principles in each genus I mean those the truth of which it is not possible to prove. What is *denoted* by the first (terms) and those derived from them is assumed; but, as regards their *existence*, this must be assumed for the principles but proved for the rest. Thus what a unit is, what the straight (line) is, or what a triangle is (must be assumed); and the existence of the unit and of magnitude must also be assumed, but the rest must be proved. Now of the premisses used in demonstrative sciences some are peculiar to each science and others common (to all), the latter being common by analogy, for of course they are actually useful in so far as they are applied to the subject-matter included under the particular science. Instances of first principles peculiar to a science are the assumptions that a line is of such and such a character, and similarly for the straight (line); whereas it is a common principle, for instance, that, if equals be subtracted from equals, the remainders are equal. But it is enough that each of the common principles is true so far as regards the particular genus (subject-matter); for (in geometry) the effect will be the same even if the common principle be assumed to be true, not of everything, but only of magnitudes, and, in arithmetic, of numbers.

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[1] Proclus, p. 67, 14 καὶ γὰρ τὰ στοιχεῖα καλῶς συνέταξεν καὶ πολλὰ τῶν μερικῶν [ὁρικῶν (?) Friedlein] καθολικώτερα ἐποίησεν.

[2] Proclus, p. 67, 22 τῶν στοιχείων πολλὰ ἀνεῦρε.

[3] *Mathematisches zu Aristoteles* in *Abhandlungen zur Gesch. d. math. Wissenschaften*, XVIII. Heft (1904), pp. 1—49.

[4] *Anal. post.* I. 6, 74 b 5.

[5] *ibid.* I. 10, 76 a 31—77 a 4.

"Now the things peculiar to the science, the existence of which must be assumed, are the things with reference to which the science investigates the essential attributes, e.g. arithmetic with reference to units, and geometry with reference to points and lines. With these things it is assumed that they exist and that they are of such and such a nature. But, with regard to their essential properties, what is assumed is only the meaning of each term employed: thus arithmetic assumes the answer to the question what is (meant by) 'odd' or 'even,' 'a square' or 'a cube,' and geometry to the question what is (meant by) 'the irrational' or 'deflection' or (the so-called) 'verging' (to a point); but that there are such things is proved by means of the common principles and of what has already been demonstrated. Similarly with astronomy. For every demonstrative science has to do with three things, (1) the things which are assumed to exist, namely the genus (subject-matter) in each case, the essential properties of which the science investigates, (2) the common axioms so-called, which are the primary source of demonstration, and (3) the properties with regard to which all that is assumed is the meaning of the respective terms used. There is, however, no reason why some sciences should not omit to speak of one or other of these things. Thus there need not be any supposition as to the existence of the genus, if it is manifest that it exists (for it is not equally clear that number exists and that cold and hot exist); and, with regard to the properties, there need be no assumption as to the meaning of terms if it is clear: just as in the common (axioms) there is no assumption as to what is the meaning of subtracting equals from equals, because it is well known. But none the less is it true that there are three things naturally distinct, the subject-matter of the proof, the things proved, and the (axioms) from which (the proof starts).

"Now that which is *per se* necessarily true, and must necessarily be thought so, is not a hypothesis nor yet a postulate. For demonstration has not to do with reasoning from outside but with the reason dwelling in the soul, just as is the case with the syllogism. It is always possible to raise objection to reasoning from outside, but to contradict the reason within us is not always possible. Now anything that the teacher assumes, though it is matter of proof, without proving it himself, is a hypothesis if the thing assumed is believed by the learner, and it is moreover a hypothesis, not absolutely, but relatively to the particular pupil; but, if the same thing is assumed when the learner either has no opinion on the subject or is of a contrary opinion, it is a postulate. This is the difference between a hypothesis and a postulate; for a postulate is that which is rather contrary than otherwise to the opinion of the learner, or whatever is assumed and used without being proved, although matter for demonstration. Now definitions are not hypotheses, for they do not assert the existence or non-existence of anything, while hypotheses are among propositions. Definitions only require to be understood: a definition is therefore not a hypothesis, unless indeed it be asserted that any audible speech is a hypothesis. A hypothesis is that from the truth of which, if assumed, a conclusion can be established. Nor are the geometer's hypotheses false, as some have said: I mean those who say that 'you should not make use of what is false, and yet the geometer falsely calls the line which he has drawn a foot long when it is not, or straight when it is not straight.' The geometer bases no conclusion on the particular line which he has drawn being that which he has described, but (he refers to) what is *illustrated* by the figures. Further, the postulate and every hypothesis are either universal or particular statements; definitions are neither" (because the subject is of equal extent with what is predicated of it).

Every demonstrative science, says Aristotle, must start from indemonstrable principles: otherwise, the steps of demonstration would be endless. Of these indemonstrable principles some are (*a*) common to all sciences, others are (*b*) particular, or peculiar to the particular science; (*a*) the common principles are the *axioms*, most commonly illustrated by the axiom that, if equals be subtracted from equals, the remainders are equal. Coming now to (*b*) the principles peculiar to the particular science which must be assumed, we have first the *genus* or subject-matter, the *existence* of which must be assumed, viz. magnitude in the case of geometry, the unit in the case of arithmetic. Under this we must assume *definitions* of manifestations or attributes of the genus, e.g. straight lines, triangles, deflection etc. The definition in itself says nothing as to the existence of the thing defined: it only requires to be understood. But in geometry, in addition to the *genus* and the *definitions*, we have to assume the *existence* of a few *primary* things which are defined, viz. points and lines only: the existence of everything else, e.g. the various figures made up of these, as triangles, squares, tangents, and their properties, e.g. incommensurability etc., has to be proved (as it is proved by construction and demonstration). In arithmetic we assume the *existence* of the *unit*: but, as regards the rest, only the *definitions*, e.g. those of odd, even, square, cube, are assumed, and *existence* has to be *proved*. We have then clearly distinguished, among the indemonstrable principles, *axioms* and *definitions*. A *postulate* is also distinguished from a *hypothesis*, the latter being made with the assent of the learner, the former without such assent or even in opposition to his opinion (though, strangely enough, immediately after saying this, Aristotle gives a wider meaning to "postulate" which would cover "hypothesis" as well, namely whatever is assumed, though it is matter for proof, and used without being proved). Heiberg remarks that there is no trace in Aristotle of Euclid's Postulates, and that "postulate" in Aristotle has a different meaning. He seems to base this on the alternative description of postulate, indistinguishable from a hypothesis; but, if we take the other description in which it is distinguished from a hypothesis as being an assumption of something which is a proper subject of demonstration without the assent or against the opinion of the learner, it seems to fit Euclid's Postulates fairly well, not only the first three (postulating three constructions), but eminently also the other two, that all right angles are equal, and that two straight lines meeting a third and making the internal angles on the same side of it less than two right angles will meet on that side. Aristotle's description also seems to me to suit the "postulates" with which Archimedes begins his book *On the equilibrium of planes*, namely that equal weights balance at equal distances, and that equal weights at unequal distances do not balance but that the weight at the longer distance will prevail.

Aristotle's distinction also between *hypothesis* and *definition*, and between *hypothesis* and *axiom*, is clear from the following passage: "Among immediate syllogistic principles, I call that a *thesis* which it is neither possible to prove nor essential for any one to hold who is to learn anything; but that which it is necessary for any one to hold who is to learn anything whatever is an *axiom*: for there are some principles of this kind, and that is the most usual name by which we speak of them. But, of *theses*, one kind is that which assumes one or other side of a predication, as, for instance, that something exists or does not exist, and this is a *hypothesis*; the other, which makes no such assumption, is a *definition*. For a definition is a thesis: thus the arithmetician posits (τίθεται) that a unit is that which is indivisible in respect of quantity; but this is not a hypothesis, since what is meant by a unit and the fact that a unit exists are different things[1]."

Aristotle uses as an alternative term for axioms "common (things)," τὰ κοινά, or "common opinions" (κοιναὶ δόξαι), as in the following passages. "That, when equals are taken from equals, the remainders are equal is (a) common (principle) in the case of all quantities, but mathematics takes a separate department (ἀπολαβοῦσα) and directs its investigation to some portion of its proper subject-matter, as e.g. lines or angles, numbers, or any of the other quantities[2]." "The common (principles), e.g. that one of two contradictories must be true, that equals taken from equals etc., and the like[3]...." "With regard to the principles of demonstration, it is questionable whether they belong to one science or to several. By principles of demonstration I mean the *common opinions* from which all demonstration proceeds, e.g. that one of two contradictories must be true, and that it is impossible for the same thing to be and not be[4]." Similarly "every demonstrative (science) investigates, with regard to some subject-matter, the essential attributes, starting from the *common opinions*[5]." We have then here, as Heiberg says, a sufficient explanation of Euclid's term for axioms, viz. *common notions* (κοιναὶ ἔννοιαι), and there is no reason to suppose it to be a substitution for the original term due to the Stoics: cf. Proclus' remark that, according to Aristotle and the geometers, axiom and common notion are the same thing[1].

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[1] *Anal. post.* I. 2, 72 a 14—24.

[2] *Metaph.* 1061 b 19—24.

[3] *Anal. post.* I. 11, 77 a 30.

[4] *Metaph.* 996 b 26—30.

[5] *Metaph.* 997 a 20—22.

Aristotle discusses the *indemonstrable* character of the axioms in the *Metaphysics*. Since "all the demonstrative sciences use the axioms[2]," the question arises, to what science does their discussion belong[3]? The answer is that, like that of Being (οὐσία), it is the province of the (first) philosopher[4]. It is impossible that there should be demonstration of everything, as there would be an infinite series of demonstrations: if the axioms were the subject of a demonstrative science, there would have to be here too, as in other demonstrative sciences, a *subject-genus*, its *attributes* and corresponding *axioms*[5]; thus there would be axioms behind axioms, and so on continually. The axiom is the most firmly established of all principles[6]. It is ignorance alone that could lead any one to try to prove the axioms[7]; the supposed proof would be a *petitio principii*[8]. If it is admitted that not everything can be proved, no one can point to any principle more truly indemonstrable[9]. If any one thought he could prove them, he could at once be refuted; if he did not attempt to say anything, it would be ridiculous to argue with him: he would be no better than a vegetable[10]. The first condition of the possibility of any argument whatever is that words should signify something both to the speaker and to the hearer: without this there can be no reasoning with any one. And, if any one admits that words can mean anything to both hearer and speaker, he admits that something can be true without demonstration. And so on[11].

It was necessary to give some sketch of Aristotle's view of the first principles, if only in connexion with Proclus' account, which is as follows. As in the case of other sciences, so "the compiler of elements in geometry must give separately the principles of the science, and after that the conclusions from those principles, not giving any account of the principles but only of their consequences. No science proves its own principles, or even discourses about them: they are treated as self-evident....Thus the first essential was to distinguish the principles from their consequences. Euclid carries out this plan practically in every book and, as a preliminary to the whole enquiry, sets out the common principles of this science. Then he divides the common principles themselves into *hypotheses*, *postulates*, and *axioms*. For all these are different from one another: an axiom, a postulate and a hypothesis are not the same thing, as the inspired Aristotle somewhere says. But, whenever that which is assumed and ranked as a principle is both known to the learner and convincing in itself, such a thing is an *axiom*, e.g. the statement that things which are equal to the same thing are also equal to one another. When, on the other hand, the pupil has not the notion of what is told him which carries conviction in itself, but nevertheless lays it down and assents to its being assumed, such an assumption is a *hypothesis*. Thus we do not preconceive by virtue of a common notion, and without being taught, that the circle is such and such a figure, but, when we are told so, we assent without demonstration. When again what is asserted is both unknown and assumed even without the assent of the learner, then, he says, we call this a *postulate*, e.g. that all right angles are equal. This view of a postulate is clearly implied by those who have made a special and systematic attempt to show, with regard to one of the postulates, that it cannot be assented to by any one straight off. According then to the teaching of Aristotle, an axiom, a postulate and a hypothesis are thus distinguished[1]."

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[1] Proclus, p. 194, 8.

[2] *Metaph.* 997 a 10.

[3] *ibid.* 996 b 26.

[4] *ibid.* 1005 a 21—b 11.

[5] *ibid.* 997 a 5—8.

[6] *ibid.* 1005 b 11—17.

[7] *ibid.* 1006 a 5.

[8] *ibid.* 1006 a 17.

[9] *ibid.* 1006 a 10.

[10] *ibid.* 1006 a 11—15.

[11] *ibid.* 1006 a 18 sqq.

We observe, first, that Proclus in this passage confuses *hypotheses* and *definitions*, although Aristotle had made the distinction quite plain. The confusion may be due to his having in his mind a passage of Plato from which he evidently got the phrase about "not giving an account of" the principles. The passage is[2]: "I think you know that those who treat of geometries and calculations (arithmetic) and such things take for granted (ὑποθέμενοι) odd and even, figures, angles of three kinds, and other things akin to these in each subject, implying that they know these things, and, though using them as hypotheses, do not even condescend to give any account of them either to themselves or to others, but begin from these things and then go through everything else in order, arriving ultimately, by recognised methods, at the conclusion which they started in search of." But the hypothesis is here the assumption, e.g. 'that *there may be such a thing* as length without breadth, henceforward called a line[3],' and so on, without any attempt to show that there is such a thing; it is mentioned in connexion with the distinction between Plato's 'superior' and 'inferior' intellectual method, the former of which uses successive hypotheses as stepping-stones by which it mounts upwards to the idea of Good.

We pass now to Proclus' account of the difference between *postulates* and *axioms*. He begins with the view of Geminus, according to which "they differ from one another in the same way as theorems are also distinguished from problems. For, as in theorems we propose to see and determine what follows on the premisses, while in problems we are told to find and do something, in like manner in the *axioms* such things are assumed as are manifest of themselves and easily apprehended by our untaught notions, while in the *postulates* we assume such things as are easy to find and effect (our understanding suffering no strain in their assumption), and we require no complication of machinery[4]."..."Both must have the characteristic of being simple and readily grasped, I mean both the postulate and the axiom; but the postulate bids us contrive and find some subject-matter (ὕλη) to exhibit a property simple and easily grasped, while the axiom bids us assert some essential attribute which is self-evident to the learner, just as is the fact that fire is hot, or any of the most obvious things[1]."

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[1] Proclus, pp. 75, 10—77, 2.

[2] *Republic*, VI. 510 C. Cf. Aristotle, *Nic. Eth.* 1151 a 17.

[3] H. Jackson, *Journal of Philology*, vol. X. p. 144.

[4] Proclus, pp. 178, 12—179, 8. In illustration Proclus contrasts the drawing of a straight line or a circle with the drawing of a "single-turn spiral" or of an equilateral triangle, the

Again, says Proclus, "some claim that all these things are alike postulates, in the same way as some maintain that all things that are sought are problems. For Archimedes begins his first book on *In-equilibrium*[2] with the remark 'I postulate that equal weights at equal distances are in equilibrium,' though one would rather call this an axiom. Others call them all axioms in the same way as some regard as theorems everything that requires demonstration[3]."

"Others again will say that postulates are peculiar to geometrical subject-matter, while axioms are common to all investigation which is concerned with quantity and magnitude. Thus it is the geometer who knows that all right angles are equal and how to produce in a straight line any limited straight line, whereas it is a common notion that things which are equal to the same thing are also equal to one another, and it is employed by the arithmetician and any scientific person who adapts the general statement to his own subject[4]."

The third view of the distinction between a postulate and an axiom is that of Aristotle above described[5].

The difficulties in the way of reconciling Euclid's classification of postulates and axioms with any one of the three alternative views are next dwelt upon. If we accept the first view according to which an axiom has reference to something known, and a postulate to something done, then the 4th postulate (that all right angles are equal) is not a postulate; neither is the 5th which states that, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the straight lines, if produced indefinitely, will meet on that side on which are the angles less than two right angles. On the second view, the assumption that two straight lines cannot enclose a space, "which even now," says Proclus, "some add as an axiom," and which is peculiar to the subject-matter of geometry, like the fact that all right angles are equal, is not an axiom. According to the third (Aristotelian) view, "everything which is confirmed (πιστοῦται) by a sort of demonstration

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spiral requiring more complex machinery and even the equilateral triangle needing a certain method. "For the geometrical intelligence will say that by conceiving a straight line fixed at one end but, as regards the other end, moving round the fixed end, and a point moving along the straight line from the fixed end, I have described the single-turn spiral; for the end of the straight line describing a circle, and the point moving on the straight line simultaneously, when they arrive and meet at the same point, complete such a spiral. And again, if I draw equal circles, join their common point to the centres of the circles and draw a straight line from one of the centres to the other, I shall have the equilateral triangle. These things then are far from being completed by means of a single act or of a moment's thought" (p. 180, 8—21).

[1] Proclus, p. 181, 4—11.

[2] It is necessary to coin a word to render ἀνισορροπιῶν, which is moreover in the plural. The title of the treatise as we have it is *Equilibria of planes or centres of gravity of planes* in Book I and *Equilibria of planes* in Book II.

[3] Proclus, p. 181, 16—23.

[4] *ibid.* p. 182, 6—14.

[5] Pp. 118, 119.

will be a postulate, and what is incapable of proof will be an axiom[1]." This last statement of Proclus is loose, as regards the axiom, because it omits Aristotle's requirement that the axiom should be a self-evident truth, and one that must be admitted by any one who is to learn anything at all, and, as regards the postulate, because Aristotle calls a postulate something assumed without proof though it is "matter of demonstration" (ἀποδεικτὸν ὄν), but says nothing of a *quasi*-demonstration of the postulates. On the whole I think it is from Aristotle that we get the best idea of what Euclid understood by a postulate and an axiom or common notion. Thus Aristotle's account of an axiom as a principle common to all sciences, which is self-evident, though incapable of proof, agrees sufficiently with the contents of Euclid's *common notions* as reduced to five in the most recent text (not omitting the fourth, that "things which coincide are equal to one another"). As regards the *postulates*, it must be borne in mind that Aristotle says elsewhere[2] that, "other things being equal, that proof is the better which proceeds from the fewer postulates or hypotheses or propositions." If then we say that a geometer must lay down as principles, first certain axioms or common notions, and then an *irreducible minimum* of postulates in the Aristotelian sense concerned only with the subject-matter of geometry, we are not far from describing what Euclid in fact does. As regards the postulates we may imagine him saying: "Besides the common notions there are a few other things which I must assume without proof, but which differ from the common notions in that they are not self-evident. The learner may or may not be disposed to agree to them; but he must accept them at the outset on the superior authority of his teacher, and must be left to convince himself of their truth in the course of the investigation which follows. In the first place certain simple constructions, the drawing and producing of a straight line, and the drawing of a circle, must be assumed to be possible, and with the constructions the existence of such things as straight lines and circles; and besides this we must lay down some postulate to form the basis of the theory of parallels." It is true that the admission of the 4th postulate that all right angles are equal still presents a difficulty to which we shall have to recur.

There is of course no foundation for the idea, which has found its way into many text-books, that "the object of the postulates is to declare that the only instruments the use of which is permitted in geometry are the *rule* and *compass*[3]."

§ 4. THEOREMS AND PROBLEMS.

"Again the deductions from the first principles," says Proclus, "are divided into **problems** and **theorems**, the former embracing the generation, division, subtraction or addition of figures, and generally the changes which are brought about in them, the latter exhibiting the essential attributes of each[1]."

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[1] Proclus, pp. 182, 21—183, 13.

[2] *Anal. post.* I. 25, 86 a 33—35.

[3] Cf. Lardner's Euclid: also Todhunter.

"Now, of the ancients, some, like Speusippus and Amphinomus, thought proper to call them all theorems, regarding the name of theorems as more appropriate than that of problems to theoretic sciences, especially as these deal with eternal objects. For there is no becoming in things eternal, so that neither could the problem have any place with them, since it promises the generation and making of what has not before existed, e.g. the construction of an equilateral triangle, or the describing of a square on a given straight line, or the placing of a straight line at a given point. Hence they say it is better to assert that all (propositions) are of the same kind, and that we regard the generation that takes place in them as referring not to actual *making* but to *knowledge*, when we treat things existing eternally as if they were subject to becoming: in other words, we may say that everything is treated by way of theorem and not by way of problem[2] (*πάντα θεωρηματικῶς ἀλλ' οὐ προβληματικῶς λαμβάνεσθαι*).

"Others on the contrary, like the mathematicians of the school of Menaechmus, thought it right to call them all problems, describing their purpose as twofold, namely in some cases to furnish (*πορίσασθαι*) the thing sought, in others to take a determinate object and see either what it is, or of what nature, or what is its property, or in what relations it stands to something else.

"In reality both assertions are correct. Speusippus is right because the problems of geometry are not like those of mechanics, the latter being matters of sense and exhibiting becoming and change of every sort. The school of Menaechmus are right also because the discoveries even of theorems do not arise without an issuing-forth into matter, by which I mean intelligible matter. Thus forms going out into matter and giving it shape may fairly be said to be like processes of becoming. For we say that the motion of our thought and the throwing-out of the forms in it is what produces the figures in the imagination and the conditions subsisting in them. It is in the imagination that constructions, divisions, placings, applications, additions and subtractions (take place), but everything in the mind is fixed and immune from becoming and from every sort of change[3]."

"Now those who distinguish the theorem from the problem say that every problem implies the possibility, not only of that which is predicated of its subject-matter, but also of its opposite, whereas every theorem implies the possibility of the thing predicated but not of its opposite as well. By the subject-matter I mean the genus which is the subject of inquiry, for example, a triangle or a square or a circle, and by the property predicated the essential attribute, as equality, section, position, and the like. When then any one enunciates thus, *To inscribe an equilateral triangle in a circle*, he states a *problem*; for it is also possible to inscribe in it a triangle which is not equilateral. Again, if we take the enunciation *On a given limited straight line to construct an equilateral triangle*, this is a *problem*; for it is possible also to construct one which is not equilateral. But, when any one enunciates that *In isosceles triangles the angles at the base are equal*, we must say that he enunciates a *theorem*; for it is not also possible that the angles at the base of isosceles triangles should be unequal. It follows that, if any one were to use the form of a problem and say *In a semicircle to describe a right angle*, he would be set down as no geometer. For every angle in a semicircle is right[1]."

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[1] Proclus, p. 77, 7—12.

[2] *ibid.* pp. 77, 15—78, 8.

[3] *ibid.* pp. 78, 8—79, 2.

"Zenodotus, who belonged to the succession of Oenopides, but was a disciple of Andron, distinguished the theorem from the problem by the fact that the theorem inquires what is the property predicated of the subject-matter in it, but the problem what is the cause of what effect (τίνος ὄντος τί ἐστιν). Hence too Posidonius defined the one (the problem) as a proposition in which it is inquired whether a thing exists or not (εἰ ἔστιν ἢ μή), the other (the theorem[2]) as a proposition in which it is inquired what (a thing) is or of what nature (τί ἐστιν ἢ ποῖόν τι); and he said that the theoretic proposition must be put in a declaratory form, e.g., *Any triangle has two sides (together) greater than the remaining side* and *In any isosceles triangle the angles at the base are equal*, but that we should state the problematic proposition as if inquiring whether it is possible to construct an equilateral triangle upon such and such a straight line. For there is a difference between inquiring absolutely and indeterminately (ἁπλῶς τε καὶ ἀορίστως) whether there exists a straight line from such and such a point at right angles to such and such a straight line and investigating which is the straight line at right angles[3]."

"That there is a certain difference between the problem and the theorem is clear from what has been said; and that the Elements of Euclid contain partly problems and partly theorems will be made manifest by the individual propositions, where Euclid himself adds at the end of what is proved in them, in some cases, 'that which it was required to do,' and in others, 'that which it was required to prove,' the latter expression being regarded as characteristic of theorems, in spite of the fact that, as we have said, demonstration is found in problems also. In problems, however, even the demonstration is for the purpose of (confirming) the construction: for we bring in the demonstration in order to show that what was enjoined has been done; whereas in theorems the demonstration is worthy of study for its own sake as being capable of putting before us the nature of the thing sought. And you will find that Euclid sometimes interweaves theorems with problems and employs them in turn, as in the first book, while at other times he makes one or other preponderate. For the fourth book consists wholly of problems, and the fifth of theorems[1]."

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[1] Proclus, pp. 79, 11—80, 5.

[2] In the text we have τὸ δὲ πρόβλημα answering to τὸ μὲν without substantive: πρόβλημα was obviously inserted in error.

[3] Proclus, pp. 80, 15—81, 4.

Again, in his note on Eucl. I. 4, Proclus says that Carpus, the writer on mechanics, raised the question of theorems and problems in his treatise on astronomy. Carpus, we are told, "says that the class of problems is in order prior to theorems. For the subjects, the properties of which are sought, are discovered by means of problems. Moreover in a problem the enunciation is simple and requires no skilled intelligence; it orders you plainly to do such and such a thing, *to construct an equilateral triangle*, or, *given two straight lines, to cut off from the greater* (*a straight line*) *equal to the lesser*, and what is there obscure or elaborate in these things? But the enunciation of a theorem is a matter of labour and requires much exactness and scientific judgment in order that it may not turn out to exceed or fall short of the truth; an example is found even in this proposition (I. 4), the first of the theorems. Again, in the case of problems, one general way has been discovered, that of *analysis*, by following which we can always hope to succeed; it is this method by which the more obscure problems are investigated. But, in the case of theorems, the method of setting about them is hard to get hold of since 'up to our time,' says Carpus, 'no one has been able to hand down a general method for their discovery. Hence, by reason of their easiness, the class of problems would naturally be more simple.' After these distinctions, he proceeds: 'Hence it is that in the Elements too problems precede theorems, and the Elements begin from them; the first theorem is fourth in order, not because the fifth[2] is proved from the problems, but because, even if it needs for its demonstration none of the propositions which precede it, it was necessary that they should be first because they are problems, while it is a theorem. In fact, in this theorem he uses the common notions exclusively, and in some sort takes the same triangle placed in different positions; the coincidence and the equality proved thereby depend entirely upon sensible and distinct apprehension. Nevertheless, though the demonstration of the first theorem is of this character, the problems properly preceded it, because in general problems are allotted the order of precedence[3].'"

Proclus himself explains the position of Prop. 4 after Props. 1—3 as due to the fact that a theorem about the essential properties of triangles ought not to be introduced before we know that such a thing as a triangle can be constructed, nor a theorem about the equality of sides or straight lines until we have shown, by constructing them, that there can be two straight lines which are equal to one another[4]. It is plausible enough to argue in this way that Props. 2 and 3 at all events should precede Prop. 4. And Prop. 1 is used in

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[1] Proclus, p. 81, 5—22.

[2] τὸ πέμπτον. This should apparently be the *fourth* because in the next words it is implied that none of the first three propositions are required in proving it.

[3] Proclus, pp. 241, 19—243, 11.

[4] *ibid.* pp. 233, 21—234, 6.

Prop. 2, and must therefore precede it. But Prop. 1 showing how to construct an *equilateral* triangle on a given base is not important, in relation to Prop. 4, as dealing with the "production of triangles" in general: for it is of no use to say, as Proclus does, that the construction of the equilateral triangle is "common to the three species (of triangles)[1]," as we are not in a position to know this at such an early stage. The *existence* of triangles in general was doubtless assumed as following from the existence of straight lines and points in one plane and from the possibility of drawing a straight line from one point to another.

Proclus does not however seem to reject definitely the view of Carpus, for he goes on[2]: "And perhaps problems are in order before theorems, and especially for those who need to ascend from the arts which are concerned with things of sense to theoretical investigation. But in dignity theorems are prior to problems....It is then foolish to blame Geminus for saying that the theorem is more perfect than the problem. For Carpus himself gave the priority to problems in respect of *order*, and Geminus to theorems in point of more perfect *dignity*," so that there was no real inconsistency between the two.

Problems were classified according to the number of their possible solutions. Amphinomus said that those which had a unique solution (μοναχῶς) were called "ordered" (the word has dropped out in Proclus, but it must be τεταγμένα, in contrast to the third kind, ἄτακτα); those which had a definite number of solutions "intermediate" (μέσα); and those with an infinite variety of solutions "unordered" (ἄτακτα)[3]. Proclus gives as an example of the last the problem *To divide a given straight line into three parts in continued proportion*[4]. This is the same thing as solving the equations $x+y+z=a$, $xz=y^2$. Proclus' remarks upon the problem show that it was solved, like all quadratic equations, by the method of "application of areas." The straight line a was first divided into any two parts, $(x+z)$ and y, subject to the sole limitation that $(x+z)$ must not be less than $2y$, which limitation is the διορισμός, or condition of possibility. Then an area was applied to $(x+z)$, or $(a-y)$, "*falling short by a square figure*" (ἐλλεῖπον εἴδει τετραγώνῳ) and equal to the square on y. This determines x and z separately in terms of a and y. For, if z be the side of the square by which the area (i.e. rectangle) "falls short," we have $\{(a-y)-z\}z=y^2$, whence $2z=(a-y)\pm\sqrt{\{(a-y)^2-4y^2\}}$. And y may be chosen arbitrarily, provided that it is not greater than $a/3$. Hence there are an infinite number of solutions. If $y=a/3$, then, as Proclus remarks, the three parts are equal.

Other distinctions between different kinds of problems are added by Proclus. The word "problem," he says, is used in several senses. In its widest sense it may mean anything "propounded" (προτεινόμενον), whether for the purpose of instruction (μαθήσεως) or construction (ποιήσεως). (In this sense, therefore, it would include a theorem.)

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[1] Proclus, p. 234, 21.

[2] *ibid.* p. 243, 12—25.

[3] *ibid.* p. 220, 7—12.

[4] *ibid.* pp. 220, 16—221, 6.

But its special sense in mathematics is that of something "propounded with a view to a theoretic construction[1]."

Again you may apply the term (in this restricted sense) even to something which is *impossible*, although it is more appropriately used of what is *possible* and neither asks too much nor contains too little in the shape of data. According as a problem has one or other of these defects respectively, it is called (1) a problem *in excess* (*πλεονάζον*) or (2) a *deficient* problem (*ἐλλιπὲς πρόβλημα*). The problem *in excess* (1) is of two kinds, (*a*) a problem in which the properties of the figure to be found are either *inconsistent* (*ἀσύμβατα*) or *non-existent* (*ἀνύπαρκτα*), in which case the problem is called impossible, or (*b*) a problem in which the enunciation is merely redundant: an example of this would be a problem requiring us to construct an equilateral triangle with its vertical angle equal to two-thirds of a right angle; such a problem is possible and is called "more than a problem" (*μεῖζον ἢ πρόβλημα*). The *deficient* problem (2) is similarly called "less than a problem" (*ἔλασσον ἢ πρόβλημα*), its characteristic being that something has to be added to the enunciation in order to convert it from indeterminateness (*ἀοριστία*) to order (*τάξις*) and scientific determinateness (*ὅρος ἐπιστημονικός*): such would be a problem bidding you "to construct an isosceles triangle," for the varieties of isosceles triangles are unlimited. Such "problems" are not problems in the proper sense (*κυρίως λεγόμενα προβλήματα*), but only equivocally[2].

§ 5. THE FORMAL DIVISIONS OF A PROPOSITION.

"Every problem," says Proclus[3], "and every theorem which is complete with all its parts perfect purports to contain in itself all of the following elements: **enunciation** (*πρότασις*), **setting-out** (*ἔκθεσις*), **definition** or **specification** (*διορισμός*), **construction** or **machinery** (*κατασκευή*), **proof** (*ἀπόδειξις*), **conclusion** (*συμπέρασμα*). Now of these the *enunciation* states what is given and what is that which is sought, the perfect *enunciation* consisting of both these parts. The *setting-out* marks off what is given, by itself, and adapts it beforehand for use in the investigation. The *definition* or *specification* states separately and makes clear what the particular thing is which is sought. The *construction* or *machinery* adds what is wanting to the datum for the purpose of finding what is sought. The *proof* draws the required inference by reasoning scientifically from acknowledged facts. The *conclusion* reverts again to the *enunciation*, confirming what has been demonstrated. These are all the parts of problems and theorems, but the most essential and those which are found in all are *enunciation*, *proof*, *conclusion*. For it is equally necessary to know beforehand what is sought, to prove this by means of the intermediate steps, and to state the proved fact as a conclusion; it is impossible to dispense with any of these three things. The remaining parts are often brought in, but are often left out as serving no purpose.

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[1] Proclus, p. 221, 7—11.

[2] *ibid.* pp. 221, 13—222, 14.

[3] *ibid.* pp. 203, 1—204, 13; 204, 23—205, 8.

Thus there is neither *setting-out* nor *definition* in the problem of constructing an isosceles triangle having each of the angles at the base double of the remaining angle, and in most theorems there is no *construction* because the *setting-out* suffices without any addition for proving the required property from the data. When then do we say that the *setting-out* is wanting? The answer is, when there is nothing *given* in the *enunciation*; for, though the enunciation is in general divided into what is given and what is sought, this is not always the case, but sometimes it states only what is sought, i.e. what must be known or found, as in the case of the problem just mentioned. That problem does not, in fact, state beforehand with what datum we are to construct the isosceles triangle having each of the equal angles double of the remaining angle, but (simply) that we are to find such a triangle.... When, then, the enunciation contains both (what is given and what is sought), in that case we find both *definition* and *setting-out*, but, whenever the datum is wanting, they too are wanting. For not only is the *setting-out* concerned with the datum, but so is the *definition* also, as, in the absence of the datum, the *definition* will be identical with the enunciation. In fact, what could you say in defining the object of the aforesaid problem except that it is required to find an isosceles triangle of the kind referred to? But that is what the *enunciation* stated. If then the *enunciation* does not include, on the one hand, what is given and, on the other, what is sought, there is no *setting-out* in virtue of there being no datum, and the *definition* is left out in order to avoid a mere repetition of the *enunciation*."

The constituent parts of an Euclidean proposition will be readily identified by means of the above description. As regards the *definition* or *specification* (διορισμός) it is to be observed that we have here only one of its uses. Here it means a closer definition or description of the object aimed at, by means of the concrete lines or figures set out in the ἔκθεσις instead of the general terms used in the enunciation; and its purpose is to rivet the attention better, as Proclus indicates in a later passage (τρόπον τινὰ προσεχείας ἐστὶν αἴτιος ὁ διορισμός)[1].

The other technical use of the word to signify the limitations to which the possible solutions of a problem are subject is also described by Proclus, who speaks of διορισμοί determining "whether what is sought is impossible or possible, and how far it is practicable and in how many ways[2]"; and the διορισμός in this sense appears in Euclid as well as in Archimedes and Apollonius. Thus we have in Eucl. I. 22 the *enunciation* "From three straight lines which are equal to three given straight lines to construct a triangle," followed immediately by the *limiting condition* (διορισμός). "Thus two of the straight lines taken together in any manner must be greater than the remaining one." Similarly in VI. 28 the *enunciation* "To a given straight line to apply a parallelogram equal to a given rectilineal figure and falling short by a parallelogrammic figure similar to a given one" is at once followed by the necessary condition of possibility: "Thus the given rectilineal figure must not be greater than that described on half the line and similar to the defect."

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[1] Proclus, p. 208, 21.

[2] *ibid.* p. 202, 3.

Tannery supposed that, in giving the other description of the διορισμός as quoted above, Proclus, or rather his guide, was using the term incorrectly. The διορισμὸς in the better known sense of the determination of limits or conditions of possibility was, we are told, invented by Leon. Pappus uses the word in this sense only. The other use of the term might, Tannery thought, be due to a confusion occasioned by the use of the same words (δεῖ δὴ) in introducing the parts of a proposition corresponding to the two meanings of the word διορισμὸς[1]. On the other hand it is to be observed that Eutocius distinguishes clearly between the two uses and implies that the difference was well known[2]. The διορισμὸς in the sense of condition of possibility follows immediately on the enunciation, is even part of it; the διορισμὸς in the other sense of course comes immediately after the *setting-out*.

Proclus has a useful observation respecting the *conclusion* of a proposition[3]. "The conclusion they are accustomed to make double in a certain way: I mean, by proving it in the given case and then drawing a general inference, passing, that is, from the partial conclusion to the general. For, inasmuch as they do not make use of the *individuality* of the subjects taken, but only draw an angle or a straight line with a view to placing the datum before our eyes, they consider that this same fact which is established in the case of the particular figure constitutes a conclusion true of every other figure of the same kind. They pass accordingly to the general in order that we may not conceive the conclusion to be partial. And they are justified in so passing, since they use for the demonstration the particular things set out, not *quâ* particulars, but *quâ* typical of the rest. For it is not in virtue of such and such a size attaching to the angle which is set out that I effect the bisection of it, but in virtue of its being rectilineal and nothing more. Such and such size is peculiar to the angle set out, but its quality of being rectilineal is common to all rectilineal angles. Suppose, for example, that the given angle is a right angle. If then I had employed in the proof the fact of its being right, I should not have been able to pass to every species of rectilineal angle; but, if I make no use of its being right, and only consider it as rectilineal, the argument will equally apply to rectilineal angles in general."

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[1] *La Géométrie grecque*, p. 149 note. Where δεῖ δὴ introduces the closer description of the problem we may translate, "it is then required" or "thus it is required" (to construct etc.): when it introduces the condition of possibility we may translate "thus it is necessary etc." Heiberg originally wrote δεῖ δὲ in the latter sense in I. 22 on the authority of Proclus and Eutocius, and against that of the MSS. Later, on the occasion of XI. 23, he observed that he should have followed the MSS. and written δεῖ δὴ which he found to be, after all, the right reading in Eutocius (Apollonius, ed. Heiberg, II. p. 178). δεῖ δὴ is also the expression used by Diophantus for introducing conditions of possibility.

[2] See the passage of Eutocius referred to in last note.

[3] Proclus, p. 207, 4—25.

§ 6. OTHER TECHNICAL TERMS.

1. Things said to be given.

Proclus attaches to his description of the formal divisions of a proposition an explanation of the different senses in which the word *given* or *datum* (δεδομένον) is used in geometry. "Everything that is given is given in one or other of the following ways, *in position*, *in ratio*, *in magnitude*, or *in species*. The point is given *in position* only, but a line and the rest may be given in all the senses[1]."

The illustrations which Proclus gives of the four senses in which a thing may be *given* are not altogether happy, and, as regards things which are given *in position*, *in magnitude*, and *in species*, it is best, I think, to follow the definitions given by Euclid himself in his book of *Data*. Euclid does not mention the fourth class, things given *in ratio*, nor apparently do any of the great geometers.

(1) *Given in position* really needs no definition; and, when Euclid says (*Data*, Def. 4) that "Points, lines and angles are said to be *given in position* which always occupy the same place," we are not really the wiser.

(2) *Given in magnitude* is defined thus (*Data*, Def. 1): "Areas, lines and angles are called *given in magnitude* to which we can find equals." Proclus' illustration is in this case the following: when, he says, two unequal straight lines are given from the greater of which we have to cut off a straight line equal to the lesser, the straight lines are obviously *given in magnitude*, "for greater and less, and finite and infinite are predications peculiar to magnitude." But he does not explain that part of the implication of the term is that a thing is given in magnitude *only*, and that, for example, its position is not given and is a matter of indifference

(3) *Given in species.* Euclid's definition (*Data*, Def. 3) is: "Rectilineal figures are said to be *given in species* in which the angles are severally given and the ratios of the sides to one another are given." And this is the recognised use of the term (cf. Pappus, *passim*) Proclus uses the term in a much wider sense for which I am not aware of any authority. Thus, he says, when we speak of (bisecting) a given rectilineal angle, the angle is given in species by the word *rectilineal*, which prevents our attempting, by the same method, to bisect a curvilineal angle! On Eucl. I. 9, to which he here refers, he says that an angle is given in species when e.g. we say that it is right or acute or obtuse or rectilineal or "mixed," but that the actual angle in the proposition is given in species only. As a matter of fact, we should say that the actual angle in the figure of the proposition is given *in magnitude* and not *in species*, part of the implication of *given in species* being that the actual magnitude of the thing *given in species* is indifferent; an angle cannot be *given in species* in this sense at all. The confusion in Proclus' mind is shown when, after saying that a right angle is given *in species*, he describes a third of a right angle as given *in magnitude*.

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[1] Proclus, p. 205, 13—15.

No better example of what is meant by *given in species*, in its proper sense, as limited to rectilineal figures, can be quoted than the given parallelogram in Eucl. VI. 28, to which the required parallelogram has to be made similar; the former parallelogram is in fact *given in species*, though its actual size, or scale, is indifferent.

(4) *Given in ratio* presumably means something which is given by means of its ratio to some other given thing. This we gather from Proclus' remark (in his note on I. 9) that an angle may be given in ratio "as when we say that it is double and treble of such and such an angle or, generally, greater and less." The term, however, appears to have no authority and to serve no purpose. Proclus may have derived it from such expressions as "in a given ratio" which are common enough.

2. Lemma.

"The term *lemma*," says Proclus[1], "is often used of any proposition which is assumed for the construction of something else: thus it is a common remark that a proof has been made out of such and such lemmas. But the special meaning of *lemma* in geometry is a proposition requiring confirmation. For when, in either construction or demonstration, we assume anything which has not been proved but requires argument, then, because we regard what has been assumed as doubtful in itself and therefore worthy of investigation, we call it a *lemma*[2], differing as it does from the postulate and the axiom in being matter of demonstration, whereas they are immediately taken for granted, without demonstration, for the purpose of confirming other things. Now in the discovery of lemmas the best aid is a mental aptitude for it. For we may see many who are quick at solutions and yet do not work by method; thus Cratistus in our time was able to obtain the required result from first principles, and those the fewest possible, but it was his natural gift which helped him to the discovery.

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[1] Proclus, pp. 211, 1—212, 4.

[2] It would appear, says Tannery (p. 151 *n.*), that Geminus understood a lemma as being simply λαμβανόμενον, something assumed (cf. the passage of Proclus, p. 73, 4, relating to Menaechmus' view of *elements*): hence we cannot consider ourselves authorised in attributing to Geminus the more technical definition of the term here given by Proclus, according to which it is only used of propositions not proved beforehand. This view of a lemma must be considered as relatively modern. It seems to have had its origin in an imperfection of method. In the course of a demonstration it was necessary to assume a proposition which required proof, but the proof of which would, if inserted in the particular place, break the thread of the demonstration: hence it was necessary either to prove it beforehand as a preliminary proposition or to postpone it to be proved afterwards (ὡς ἑξῆς δειχθήσεται). When, after the time of Geminus, the progress of original discovery in geometry was arrested, geometers occupied themselves with the study and elucidation of the works of the great mathematicians who had preceded them. This involved the investigation of propositions explicitly quoted or tacitly assumed in the great classical treatises; and naturally it was found that several such remained to be demonstrated, either because the authors had omitted them as being easy enough to be left to the reader himself to prove, or because books in which they were proved had been lost in the meantime. Hence arose a class of complementary or auxiliary propositions which were called *lemmas*. Thus Pappus gives in his Book VII a collection of lemmas in elucidation of the treatises of Euclid and Apollonius included in the so-called "Treasury of Analysis" (τόπος ἀναλυόμενος). When Proclus goes on to distinguish three methods of discovering lemmas, *analysis*, *division*, and *reductio ad absurdum*, he seems to imply that the principal business of contemporary geometers was the investigation of these auxiliary propositions.

Nevertheless certain methods have been handed down. The finest is the method which by means of *analysis* carries the thing sought up to an acknowledged principle, a method which Plato, as they say, communicated to Leodamas[1], and by which the latter, too, is said to have discovered many things in geometry. The second is the method of *division*[2], which divides into its parts the genus proposed for consideration and gives a starting-point for the demonstration by means of the elimination of the other elements in the construction of what is proposed, which method also Plato extolled as being of assistance to all sciences. The third is that by means of the *reductio ad absurdum*, which does not show what is sought directly, but refutes its opposite and discovers the truth incidentally."

3. Case.

"The *case*[3] (πτῶσις)," Proclus proceeds[4], "announces different ways of construction and alteration of positions due to the transposition of points or lines or planes or solids. And, in general, all its varieties are seen in the figure, and this is why it is called *case*, being a transposition in the construction."

4. Porism.

"The term *porism* is used also of certain problems such as the Porisms written by Euclid. But it is specially used when from what has been demonstrated some other theorem is revealed at the same time without our propounding it, which theorem has on this very account been called a *porism* (corollary) as being a sort of incidental gain arising from the scientific demonstration[5]." Cf. the note on I. 15.

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[1] This passage and another from Diogenes Laertius (III. 24, p. 74 ed. Cobet) to the effect that "He [Plato] explained (εἰσηγήσατο) to Leodamas of Thasos the method of inquiry by analysis" have been commonly understood as ascribing to Plato the *invention* of the method of analysis; but Tannery points out forcibly (pp. 112, 113) how difficult it is to explain in what Plato's discovery could have consisted if *analysis* be taken in the sense attributed to it in Pappus, where we can see no more than a series of successive *reductions* of a problem until it is finally reduced to a known problem. On the other hand, Proclus' words about carrying up the thing sought to "an acknowledged principle" suggest that what he had in mind was the process described at the end of Book VI of the *Republic* by which the dialectician (unlike the mathematician) uses hypotheses as stepping-stones up to a principle which is not hypothetical, and then is able to descend step by step verifying every one of the hypotheses by which he ascended. This description does not of course refer to mathematical analysis, but it may have given rise to the idea that analysis was Plato's discovery, since *analysis* and *synthesis* following each other are related in the same way as the upward and the downward progression in the dialectician's intellectual method. And it may be that Plato's achievement was to observe the importance, from the point of view of logical rigour, of the confirmatory synthesis following analysis, and to regularise in this way and elevate into a completely irrefragable method the partial and uncertain analysis upon which the works of his predecessors depended.

[2] Here again the successive bipartitions of genera into species such as we find in the *Sophist* and *Republic* have very little to say to geometry, and the very fact that they are here mentioned side by side with analysis suggests that Proclus confused the latter with the philosophical method of *Rep.* VI.

[3] Tannery rightly remarks (p. 152) that the subdivision of a theorem or problem into several cases is foreign to the really classic form; the ancients preferred, where necessary, to multiply enunciations. As, however, some omissions necessarily occurred, the writers of lemmas naturally added separate *cases*, which in some instances found their way into the text. A good example is Euclid I. 7, the second case of which, as it appears in our text-books, was interpolated. On the commentary of Proclus on this proposition Th. Taylor rightly remarks that "Euclid everywhere avoids a multitude of cases."

[4] Proclus, p. 212, 5—11.

[5] Tannery notes however that, so far from distinguishing his corollaries from the con-

5. **Objection.**

"The *objection* (ἔνστασις) obstructs the whole course of the argument by appearing as an obstacle (or crying 'halt,' ἀπαντῶσα) either to the construction or to the demonstration. There is this difference between the *objection* and the *case*, that, whereas he who propounds the case has to prove the proposition to be true of it, he who makes the objection does not need to prove anything: on the contrary it is necessary to destroy the objection and to show that its author is saying what is false[1]."

That is, in general the *objection* endeavours to make it appear that the demonstration is not true in every case; and it is then necessary to prove, in refutation of the objection, either that the supposed case is impossible, or that the demonstration *is* true even for that case. A good instance is afforded by Eucl. I. 7. The text-books give a second case which is not in the original text of Euclid. Proclus remarks on the proposition as given by Euclid that the objection may conceivably be raised that what Euclid declares to be impossible may after all be possible in the event of one pair of straight lines falling completely within the other pair. Proclus then refutes the objection by proving the impossibility in that case also. His proof then came to be given in the text-books as part of Euclid's proposition.

The *objection* is one of the technical terms in Aristotle's logic and its nature is explained in the *Prior Analytics*[2]. "An *objection* is a proposition contrary to a proposition.... Objections are of two sorts, general or partial.... For when it is maintained that an attribute belongs to every (member of a class), we object either that it belongs to none (of the class) or that there is some one (member of the class) to which it does not belong."

6. **Reduction.**

This is again an Aristotelian term, explained in the *Prior Analytics*[3]. It is well described by Proclus in the following passage:

"*Reduction* (ἀπαγωγή) is a transition from one problem or theorem to another, the solution or proof of which makes that which is propounded manifest also. For example, after the doubling of the cube had been investigated, they transformed the investigation into another upon which it follows, namely the finding of the two means; and from that time forward they inquired how between two given straight lines two mean proportionals could be discovered. And they say that the first to effect the reduction of difficult constructions was Hippocrates of Chios, who also squared a lune and discovered many other things in geometry, being second to none in ingenuity as regards constructions[4]."

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clusions of his propositions, Euclid inserts them before the closing words "(being) what it was required to do" or "to prove." In fact the porism-corollary is with Euclid rather a modified form of the regular conclusion than a separate proposition.

[1] Proclus, p. 212, 18—23.

[2] *Anal. prior.* II. 26, 69 a 37.

[3] *ibid.* II. 25, 69 a 20.

[4] Proclus, pp. 212, 24—213, 11. This passage has frequently been taken as crediting Hippocrates with the discovery of the method of geometrical reduction: cf. Taylor (Translation of Proclus, II. p. 26), Allman (p. 41 *n.*, 59), Gow (pp. 169, 170). As Tannery remarks (p. 110), if the particular reduction of the duplication problem to that of the two means is

7. **Reductio ad absurdum.**

This is variously called by Aristotle "*reductio ad absurdum*" (ἡ εἰς τὸ ἀδύνατον ἀπαγωγή)[1], "proof *per impossibile*" (ἡ διὰ τοῦ ἀδυνάτου δεῖξις or ἀπόδειξις)[2], "proof leading to the impossible" (ἡ εἰς τὸ ἀδύνατον ἄγουσα ἀπόδειξις)[3]. It is part of "proof (starting) from a hypothesis[4]" (ἐξ ὑποθέσεως). "All (syllogisms) which reach the conclusion *per impossibile* reason out a conclusion which is false, and they prove the original contention (by the method starting) from a hypothesis, when something impossible results from assuming the contradictory of the original contention, as, for example, when it is proved that the diagonal (of a square) is incommensurable because, if it be assumed commensurable, it will follow that odd (numbers) are equal to even (numbers)[5]." Or again, "proof (leading) to the impossible differs from the direct (δεικτικῆς) in that it assumes what it desires to destroy [namely the hypothesis of the falsity of the conclusion] and then reduces it to something admittedly false, whereas the direct proof starts from premisses admittedly true[6]."

Proclus has the following description of the *reductio ad absurdum*. "Proofs by *reductio ad absurdum* in every case reach a conclusion manifestly impossible, a conclusion the contradictory of which is admitted. In some cases the conclusions are found to conflict with the common notions, or the postulates, or the hypotheses (from which we started); in others they contradict propositions previously established[7]"..."Every *reductio ad absurdum* assumes what conflicts with the desired result, then, using that as a basis, proceeds until it arrives at an admitted absurdity, and, by thus destroying the hypothesis, establishes the result originally desired. For it is necessary to understand generally that all mathematical arguments either proceed from the first principles or lead back to them, as Porphyry somewhere says. And those which proceed from the first principles are again of two kinds, for they start either from common notions and the clearness of the self-evident alone, or from results previously proved; while those which lead back to the principles are either by way of assuming the principles or by way of destroying them. Those which assume the principles are called *analyses*, and the opposite of these are *syntheses*—for it is possible to start from the said principles and to proceed in the regular order to the desired conclusion, and this process is *synthesis*—while the arguments which would destroy the principles are

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the first noted in history, it is difficult to suppose that it was really the first; for Hippocrates must have found instances of it in the Pythagorean geometry. Bretschneider, I think, comes nearer the truth when he boldly (p. 99) translates: "This reduction *of the aforesaid construction* is said to have been first given by Hippocrates." The words are πρῶτον δέ φασι τῶν ἀπορουμένων διαγραμμάτων τὴν ἀπαγωγὴν ποιήσασθαι, which must, literally, be translated as in the text above; but, when Proclus speaks vaguely of "difficult constructions," he probably means to say simply that "this first recorded instance of a reduction of a difficult construction is attributed to Hippocrates."

[1] Aristotle, *Anal. prior.* I. 7, 29 b 5; I. 44, 50 a 30.

[2] *ibid.* I. 21, 39 b 32; I. 29, 45 a 35.

[3] *Anal. post.* I. 24, 85 a 16 etc.

[4] *Anal. prior.* I. 23, 40 b 25.

[5] *Anal. prior.* I. 23, 41 a 24.

[6] *ibid.* II. 14, 62 b 29.

[7] Proclus, p. 254, 22—27.

called *reductiones ad absurdum*. For it is the function of this method to upset something admitted as clear[1]."

8. **Analysis and Synthesis.**

It will be seen from the note on Eucl. XIII. 1 that the MSS. of the *Elements* contain definitions of *Analysis* and *Synthesis* followed by alternative proofs of XIII. 1—5 after that method. The definitions and alternative proofs are interpolated, but they have great historical interest because of the possibility that they represent an ancient method of dealing with these propositions, anterior to Euclid. The propositions give properties of a line cut "in extreme and mean ratio," and they are preliminary to the construction and comparison of the five regular solids. Now Pappus, in the section of his *Collection* dealing with the latter subject[2], says that he will give the comparisons between the five figures, the pyramid, cube, octahedron, dodecahedron and icosahedron, which have equal surfaces, "not by means of the so-called *analytical* inquiry, by which some of the ancients worked out the proofs, but by the synthetical method[3]...." The conjecture of Bretschneider that the matter interpolated in Eucl. XIII. is a survival of investigations due to Eudoxus has at first sight much to commend it[4]. In the first place, we are told by Proclus that Eudoxus "greatly added to the number of the theorems which Plato originated regarding *the section*, and employed in them the method of analysis[5]." It is obvious that "*the section*" was some particular section which by the time of Plato had assumed great importance; and the one section of which this can safely be said is that which was called the "golden section," namely, the division of a straight line in extreme and mean ratio which appears in Eucl. II. 11 and is therefore most probably Pythagorean. Secondly, as Cantor points out[6], Eudoxus was the founder of the theory of proportions in the form in which we find it in Euclid V., VI., and it was no doubt through meeting, in the course of his investigations, with proportions not expressible by whole numbers that he came to realise the necessity for a new theory of proportions which should be applicable to incommensurable as well as commensurable magnitudes. The "golden section" would furnish such a case. And it is even mentioned by Proclus in this connexion. He is explaining[7] that it is only in arithmetic that all quantities bear "rational" ratios (ῥητὸς λόγος) to one another, while in geometry there are "irrational" ones (ἄρρητος) as well. "Theorems about sections like those in Euclid's second Book are common to both [arithmetic and geometry] *except that in which the straight line is cut in extreme and mean ratio*[8]."

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[1] Proclus, p. 255, 8—26.

[2] Pappus, V. p. 410 sqq.

[3] *ibid.* pp. 410, 27—412, 2.

[4] Bretschneider, p. 168. See however Heiberg's recent suggestion (*Paralipomena zu Euklid* in *Hermes*, XXXVIII., 1903) that the author was Heron. The suggestion is based on a comparison with the remarks on analysis and synthesis quoted from Heron by an-Nairīzī (ed. Curtze, p. 89) at the beginning of his commentary on Eucl. Book II. On the whole, this suggestion commends itself to me more than that of Bretschneider.

[5] Proclus, p. 67, 6.

[6] Cantor, *Gesch. d. Math.* I$_3$, p. 241.

[7] Proclus, p. 60, 7—9.

[8] *ibid.* p. 60, 16—19.

The definitions of *Analysis* and *Synthesis* interpolated in Eucl. XIII. are as follows (I adopt the reading of B and V, the only intelligible one, for the second).

"**Analysis** is an assumption of that which is sought as if it were admitted < and the passage > through its consequences to something admitted (to be) true.

"**Synthesis** is an assumption of that which is admitted < and the passage > through its consequences to the finishing or attainment of what is sought."

The language is by no means clear and has, at the best, to be filled out.

Pappus has a fuller account[1]:

"The so-called *ἀναλυόμενος* ('Treasury of Analysis') is, to put it shortly, a special body of doctrine provided for the use of those who, after finishing the ordinary Elements, are desirous of acquiring the power of solving problems which may be set them involving (the construction of) lines, and it is useful for this alone. It is the work of three men, Euclid the author of the Elements, Apollonius of Perga, and Aristaeus the elder, and proceeds by way of analysis and synthesis.

"**Analysis** then takes that which is sought as if it were admitted and passes from it through its successive consequences to something which is admitted as the result of synthesis: for in analysis we assume that which is sought as if it were (already) done (*γεγονός*), and we inquire what it is from which this results, and again what is the antecedent cause of the latter, and so on, until by so retracing our steps we come upon something already known or belonging to the class of first principles, and such a method we call analysis as being solution backwards (*ἀνάπαλιν λύσιν*).

"But in **synthesis**, reversing the process, we take as already done that which was last arrived at in the analysis and, by arranging in their natural order as consequences what were before antecedents, and successively connecting them one with another, we arrive finally at the construction of what was sought; and this we call synthesis.

"Now analysis is of two kinds, the one directed to searching for the truth and called *theoretical*, the other directed to finding what we are told to find and called *problematical*. (1) In the *theoretical* kind we assume what is sought as if it were existent and true, after which we pass through its successive consequences, as if they too were true and established by virtue of our hypothesis, to something admitted: then (*a*), if that something admitted is true, that which is sought will also be true and the proof will correspond in the reverse order to the analysis, but (*b*), if we come upon something admittedly false, that which is sought will also be false. (2) In the *problematical* kind we assume that which is propounded as if it were known, after which we pass through its successive consequences, taking them as true, up to something admitted: if then (*a*) what is admitted is possible and obtainable, that is, what mathematicians call *given*, what was originally proposed will also be possible, and the proof will again correspond in reverse order to the analysis, but if (*b*) we come upon something admittedly impossible, the problem will also be impossible."

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[1] Pappus, VII. pp. 634—6.

The ancient Analysis has been made the subject of careful studies by several writers during the last half-century, the most complete being those of Hankel, Duhamel and Zeuthen; others by Ofterdinger and Cantor should also be mentioned[1].

The method is as follows. It is required, let us say, to prove that a certain proposition A is true. We assume as a hypothesis that A is true and, starting from this we find that, if A is true, a certain other proposition B is true; if B is true, then C; and so on until we arrive at a proposition K which is *admittedly* true. The object of the method is to enable us to infer, in the reverse order, that, since K is true, the proposition A originally assumed is true. Now Aristotle had already made it clear that false hypotheses might lead to a conclusion which is true. There is therefore a possibility of error unless a certain precaution is taken. While, for example, B may be a necessary consequence of A, it may happen that A is not a necessary consequence of B. Thus, in order that the reverse inference from the truth of K that A is true may be logically justified, it is necessary that each step in the chain of inferences should be unconditionally convertible. As a matter of fact, a very large number of theorems in elementary geometry are unconditionally convertible, so that in practice the difficulty in securing that the successive steps shall be convertible is not so great as might be supposed. But care is always necessary. For example, as Hankel says[2], a proposition may not be unconditionally convertible in the form in which it is generally quoted. Thus the proposition "The vertices of all triangles having a common base and constant vertical angle lie on a circle" cannot be converted into the proposition that "All triangles with common base and vertices lying on a circle have a constant vertical angle"; for this is only true if the further conditions are satisfied (1) that the circle passes through the extremities of the common base and (2) that only that part of the circle is taken as the locus of the vertices which lies on *one* side of the base. If these conditions are added, the proposition is unconditionally convertible. Or again, as Zeuthen remarks[3], K may be obtained by a series of inferences in which A or some other proposition in the series is only *apparently* used; this would be the case e.g. when the method of modern algebra is being employed and the expressions on each side of the sign of equality have been inadvertently multiplied by some composite magnitude which is in reality equal to zero.

Although the above extract from Pappus does not make it clear that each step in the chain of argument must be convertible in the case taken, he almost implies this in the second part of the definition of Analysis where, instead of speaking of the consequences B, C...

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[1] Hankel, *Zur Geschichte der Mathematik in Alterthum und Mittelalter*, 1874, pp. 137—150; Duhamel, *Des méthodes dans les sciences de raisonnement*, Part I., 3 ed., Paris, 1885, pp. 39—68; Zeuthen, *Geschichte der Mathematik im Altertum und Mittelalter*, 1896, pp. 92—104; Ofterdinger, *Beiträge zur Geschichte der griechischen Mathematik*, Ulm, 1860; Cantor, *Geschichte der Mathematik*, I_3, pp. 220—2.

[2] Hankel, p. 139.

[3] Zeuthen, p. 103.

successively following from A, he suddenly changes the expression and says that we inquire *what it is* (B) *from which* A *follows* (A being thus the consequence of B, instead of the reverse), and then what (viz. C) is the antecedent cause of B; and in practice the Greeks secured what was wanted by always insisting on the *analysis* being confirmed by subsequent *synthesis*, that is, they laboriously worked backwards the whole way from K to A, reversing the order of the analysis, which process would undoubtedly bring to light any flaw which had crept into the argument through the accidental neglect of the necessary precautions.

Reductio ad absurdum a variety of analysis.

In the process of analysis starting from the hypothesis that a proposition A is true and passing through B, C... as successive consequences we may arrive at a proposition K which, instead of being admittedly true, is either admittedly false or the contradictory of the original hypothesis A or of some one or more of the propositions B, C... intermediate between A and K. Now correct inference from a true proposition cannot lead to a false proposition; and in this case therefore we may at once conclude, without any inquiry whether the various steps in the argument are convertible or not, that the hypothesis A is false, for, if it were true, all the consequences correctly inferred from it would be true and no incompatibility could arise. This method of proving that a given hypothesis is *false* furnishes an indirect method of proving that a given hypothesis A is *true*, since we have only to take the *contradictory* of A and to prove that it is false. This is the method of *reductio ad absurdum*, which is therefore a variety of analysis. The contradictory of A, or not-A, will generally include more than one case and, in order to prove its falsity, each of the cases must be separately disposed of: e.g., if it is desired to prove that a certain part of a figure is *equal* to some other part, we take separately the hypotheses (1) that it is *greater*, (2) that it is *less*, and prove that each of these hypotheses leads to a conclusion either admittedly false or contradictory to the hypothesis itself or to some one of its consequences.

Analysis as applied to problems.

It is in relation to problems that the ancient analysis has the greatest significance, because it was the one general method which the Greeks used for solving all "the more abstruse problems" (*τὰ ἀσαφέστερα τῶν προβλημάτων*)[1].

We have, let us suppose, to construct a figure satisfying a certain set of conditions If we are to proceed at all methodically and not by mere guesswork, it is first necessary to "analyse" those conditions. To enable this to be done we must get them clearly in our minds, which is only possible by assuming all the conditions to be actually fulfilled, in other words, by supposing the problem solved. Then we have to transform those conditions, by all the means which practice in such cases has taught us to employ, into other conditions which are necessarily fulfilled if the original conditions are, and to continue this transformation until we at length arrive at conditions which we are in a position to satisfy[1]. In other words, we must arrive at some relation which enables us to *construct* a particular part of the figure which, it is true, has been hypothetically assumed and even drawn, but which nevertheless really requires to be *found* in order that the problem may be solved. From that moment the particular part of the figure becomes one of the *data*, and a fresh relation has to be found which enables a fresh part of the figure to be determined by means of the original data and the new one together. When this is done, the second new part of the figure also belongs to the data; and we proceed in this way until all the parts of the required figure are found[2]. The first part of the analysis down to the point of discovery of a relation which enables us to say that a certain new part of the figure not belonging to the original data is *given*, Hankel calls the *transformation*; the second part, in which it is proved that all the remaining parts of the figure are "given," he calls the *resolution*. Then follows the *synthesis*, which also consists of two parts, (1) the *construction*, in the order in which it has to be actually carried out, and in general following the course of the second part of the analysis, the *resolution*; (2) the *demonstration* that the figure obtained does satisfy all the given conditions, which follows the steps of the first part of the analysis, the *transformation*, but in the reverse order. The second part of the analysis, the *resolution*, would be much facilitated and shortened by the existence of a systematic collection of *Data* such as Euclid's book bearing that title, consisting of propositions proving that, if in a figure certain parts or relations are *given*, other parts or relations are also *given*. As regards the first part of the analysis, the *transformation*, the usual rule applies that every step in the chain must be unconditionally convertible; and any failure to observe this condition will be brought to light by the subsequent synthesis. The second part, the *resolution*, can be directly turned into the *construction* since that only is *given* which can be constructed by the means provided in the *Elements*.

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[1] Proclus, p. 242, 16, 17.

It would be difficult to find a better illustration of the above than the example chosen by Hankel from Pappus[3].

Given a circle ABC *and two points* D, E *external to it, to draw straight lines* DB, EB *from* D, E *to a point* B *on the circle such that, if* DB, EB *produced meet the circle again in* C, A, AC *shall be parallel to* DE.

Analysis.

Suppose the problem solved and the tangent at A drawn, meeting ED produced in F.

(Part I. *Transformation.*)

Then, since AC is parallel to DE, the angle at C is equal to the angle CDE.

But, since FA is a tangent, the angle at C is equal to the angle FAE.

Therefore the angle FAE is equal to the angle CDE, whence A, B, D, F are concyclic.

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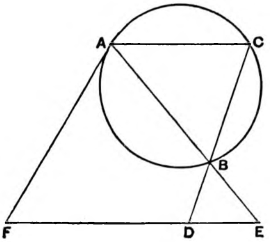
[1] Zeuthen, p. 93.

[2] Hankel, p. 141.

[3] Pappus, VII. pp. 830—2.

Therefore the rectangle *AE*, *EB* is equal to the rectangle *FE*, *ED*.



(Part II. *Resolution.*)

But the rectangle *AE*, *EB* is given, because it is equal to the square on the tangent from *E*.

Therefore the rectangle *FE*, *ED* is given;
and, since *ED* is given, *FE* is given (in length). [*Data*, 57.]

But *FE* is given in position also, so that *F* is also given. [*Data*, 27.]

Now *FA* is the tangent from a given point *F* to a circle *ABC* given in position;
therefore *FA* is given in position and magnitude. [*Data*, 90.]

And *F* is given; therefore *A* is given.

But *E* is also given; therefore the straight line *AE* is given in position. [*Data*, 26.]

And the circle *ABC* is given in position;
therefore the point *B* is also given. [*Data*, 25.]

But the points *D*, *E* are also given;
therefore the straight lines *DB*, *BE* are also given in position.

Synthesis.

(Part I. *Construction.*)

Suppose the circle *ABC* and the points *D*, *E* given.

Take a rectangle contained by *ED* and by a certain straight line *EF* equal to the square on the tangent to the circle from *E*.

From *F* draw *FA* touching the circle in *A*; join *ABE* and then *DB*, producing *DB* to meet the circle at *C*. Join *AC*.

I say then that *AC* is parallel to *DE*.

(Part II. *Demonstration.*)

Since, by hypothesis, the rectangle *FE*, *ED* is equal to the square on the tangent from *E*, which again is equal to the rectangle *AE*, *EB*, the rectangle *AE*, *EB* is equal to the rectangle *FE*, *ED*.

Therefore *A*, *B*, *D*, *F* are concyclic,
whence the angle *FAE* is equal to the angle *BDE*.

But the angle *FAE* is equal to the angle *ACB* in the alternate segment;
therefore the angle *ACB* is equal to the angle *BDE*.

Therefore *AC* is parallel to *DE*.

In cases where a διορισμός is necessary, i.e. where a solution is only possible under certain conditions, the analysis will enable those conditions to be ascertained. Sometimes the διορισμός is stated and proved at the end of the analysis, e.g. in Archimedes, *On the Sphere and Cylinder*, II. 7; sometimes it is stated in that place and the proof postponed till after the end of the synthesis, e.g. in the solution of the problem subsidiary to *On the Sphere and Cylinder*, II. 4, preserved in Eutocius' commentary on that proposition. The analysis should also enable us to determine the number of solutions of which the problem is susceptible.

§ 7. THE DEFINITIONS.

General. "Real" and "Nominal" Definitions.

It is necessary, says Aristotle, whenever any one treats of any whole subject, to divide the genus into its primary constituents, those which are indivisible in species respectively: e.g. number must be divided into triad and dyad; then an attempt must be made in this way to obtain definitions, e.g. of a straight line, of a circle, and of a right angle[1].

The word for definition is ὅρος. The original meaning of this word seems to have been "boundary," "landmark." Then we have it in Plato and Aristotle in the sense of standard or determining principle ("id quo alicuius rei natura constituitur vel definitur," *Index Aristotelicus*)[2]; and closely connected with this is the sense of *definition*. Aristotle uses both ὅρος and ὁρισμός for definition, the former occurring more frequently in the *Topics*, the latter in the *Metaphysics*.

Let us now first be clear as to what a definition does *not* do. There is nothing in connexion with definitions which Aristotle takes more pains to emphasise than that a definition asserts nothing as to the *existence* or *non-existence* of the thing defined. It is an answer to the question *what* a thing is (τί ἐστι), and does not say *that* it is (ὅτι ἔστι). The *existence* of the various things defined has to be *proved*, except in the case of a few primary things in each science, the existence of which is indemonstrable and must be *assumed* among the first principles of each science; e.g. points and lines in geometry must be *assumed* to exist, but the existence of everything else must be *proved*. This is stated clearly in the long passage quoted above under First Principles[3]. It is reasserted in such passages as the following. "The (answer to the question) *what is a man* and *the fact that a man exists* are different things[4]." "It is clear that, even according to the view of definitions now current, those who define things do not prove that they exist[5]." "We say that it is by *demonstration* that we must show that everything exists, except essence (εἰ μὴ οὐσία εἴη). But the *existence* of a thing is never essence; for the *existent* is not a genus. Therefore there must be demonstration that a thing exists. Thus, *what is meant by triangle* the geometer assumes, but that it exists he has to prove[6]." "Anterior knowledge of two sorts is necessary: for it is necessary to presuppose, with regard to some things, that they *exist*; in other cases it is necessary to understand *what* the thing described is, and in other cases it is necessary to do both. Thus, with the fact that one of two contradictories must be true, we must know that it exists (is true);

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[1] *Anal. post.* II. 13, 96 b 15.

[2] Cf. *De anima*, I. 2, 404 a 9, where "breathing" is spoken of as the ὅρος of "life," and the many passages in the *Politics* where the word is used to denote that which gives its *special character* to the several forms of government (virtue being the ὅρος of aristocracy, wealth of oligarchy, liberty of democracy, 1294 a 10); Plato, *Republic*, VIII. 551 C.

[3] *Anal. post.* I. 10, 76 a 31 sqq.

[4] *ibid.* II. 7, 92 b 10.

[5] *ibid.* 92 b 19.

[6] *ibid.* 92 b 12 sqq.

of the triangle we must know that it means such and such a thing; of the unit we must know both what it means and that it exists[1]." What is here so much insisted on is the very fact which Mill pointed out in his discussion of earlier views of Definitions, where he says that the so-called *real* definitions or definitions of *things* do not constitute a different kind of definition from *nominal* definitions, or definitions of *names*; the former is simply the latter *plus* something else, namely a covert assertion that the thing defined exists. "This covert assertion is not a definition but a postulate. The definition is a mere identical proposition which gives information only about the use of language, and from which no conclusion affecting matters of fact can possibly be drawn. The accompanying postulate, on the other hand, affirms a fact which may lead to consequences of every degree of importance. It affirms the actual or possible existence of Things possessing the combination of attributes set forth in the definition: and this, if true, may be foundation sufficient on which to build a whole fabric of scientific truth[2]." This statement really adds nothing to Aristotle's doctrine[3]: it has even the slight disadvantage, due to the use of the word "postulate" to describe "the covert assertion" in all cases, of not definitely pointing out that there are cases where existence has to be *proved* as distinct from those where it must be *assumed*. It is true that the existence of a definiend may have to be taken for granted provisionally until the time comes for proving it; but, so far as regards any case where existence must be proved sooner or later, the provisional assumption would be for Aristotle, not a *postulate*, but a *hypothesis*. In modern times, too, Mill's account of the true distinction between *real* and *nominal* definitions had been fully anticipated by Saccheri[4], the editor of *Euclides ab omni naevo vindicatus* (1733), famous in the history of non-Euclidean geometry. In his *Logica Demonstrativa* (to which he also refers in his Euclid) Saccheri lays down the clear distinction between what he calls *definitiones quid nominis* or *nominales*, and *definitiones quid rei* or *reales*, namely that the former are only intended to explain the meaning that is to be attached to a given term, whereas the latter, besides declaring the meaning of a word, affirm at the same time the existence of the thing defined or, in geometry, the possibility of constructing it. The *definitio quid nominis* becomes a *definitio quid rei* "by means of a *postulate*, or when we come to the question whether the thing *exists* and it is answered affirmatively[1]." *Definitiones quid nominis* are in themselves quite arbitrary, and neither require nor are capable of proof; they are merely provisional and are only intended to be turned as quickly as possible into *definitiones quid rei*, either (1) by means of a postulate in which it is asserted or conceded that what is defined exists or can be constructed, e.g. in the case of *straight lines* and *circles*, to which Euclid's first three postulates refer, or (2) by means of a demonstration reducing the construction of the figure defined to the successive carrying-out of a certain number of those elementary constructions, the possibility of which is *postulated*. Thus *definitiones quid rei* are in general obtained as the result of a series of demonstrations. Saccheri gives as an instance the construction of a square in Euclid I. 46. Suppose that it is objected that Euclid had no right to define a square, as he does at the beginning of the Book, when it was not certain that such a figure exists in nature; the objection, he says, could only have force if, before proving and making the construction, Euclid had assumed the aforesaid figure as given. That Euclid is not guilty of this error is clear from the fact that he never presupposes the existence of the square as defined until after I. 46.

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[1] *Anal. post.* I. 1, 71 a 11 sqq.

[2] Mill's *System of Logic*, Bk. I. ch. viii.

[3] It is true that it was in opposition to "the ideas of most of the *Aristotelian logicians*" (rather than of Aristotle himself) that Mill laid such stress on his point of view. Cf. his observation: "We have already made, and shall often have to repeat, the remark, that the philosophers who overthrew Realism by no means got rid of the consequences of Realism, but retained long afterwards, in their own philosophy, numerous propositions which could only have a rational meaning as part of a Realistic system. It had been handed down from Aristotle, and probably from earlier times, as an obvious truth, that the science of geometry is deduced from definitions. This, so long as a definition was considered to be a proposition 'unfolding the nature of the thing,' did well enough. But Hobbes followed and rejected utterly the notion that a definition declares the nature of the thing, or does anything but state the meaning of a name; yet he continued to affirm as broadly as any of his predecessors that the ἀρχαὶ, *principia*, or original premisses of mathematics, and even of all science, are definitions; producing the singular paradox that systems of scientific truth, nay, all truths whatever at which we arrive by reasoning, are deduced from the arbitrary conventions of mankind concerning the signification of words." Aristotle was guilty of no such paradox; on the contrary, he exposed it as plainly as did Mill.

[4] This has been fully brought out in two papers by G. Vailati, *La teoria Aristotelica della definizione* (*Rivista di Filosofia e scienze affini*, 1903), and *Di un' opera dimenticata del P. Gerolamo Saccheri* ("Logica Demonstrativa," 1697) (in *Rivista Filosofica*, 1903).

Confusion between the *nominal* and the *real* definition as thus described, i.e. the use of the former in demonstration before it has been turned into the latter by the necessary proof that the thing defined exists, is according to Saccheri one of the most fruitful sources of illusory demonstration, and the danger is greater in proportion to the "complexity" of the definition, i.e. the number and variety of the attributes belonging to the thing defined. For the greater is the possibility that there may be among the attributes some that are *incompatible*, i.e. the simultaneous presence of which in a given figure can be proved, by means of *other* postulates etc. forming part of the basis of the science, to be impossible.

The same thought is expressed by Leibniz also. "If," he says, "we give any definition, and it is not clear from it that the idea, which we ascribe to the thing, is possible, we cannot rely upon the demonstrations which we have derived from that definition, because, if that idea by chance involves a contradiction, it is possible that even contradictories may be true of it at one and the same time, and thus our demonstrations will be useless. Whence it is clear that definitions are not arbitrary. And this is a secret which is hardly sufficiently known[2]." Leibniz' favourite illustration was the "regular polyhedron with ten faces," the impossibility of which is not obvious at first sight.

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[1] "Definitio *quid nominis* nata est evadere definitio *quid rei* per *postulatum* vel dum venitur ad quaestionem *an est* et respondetur affirmative."

[2] *Opuscules et fragments inédits de Leibniz*, Paris, Alcan, 1903, p. 431. Quoted by Vailati.

It need hardly be added that, speaking generally, Euclid's definitions, and his use of them, agree with the doctrine of Aristotle that the definitions themselves say nothing as to the existence of the things defined, but that the existence of each of them must be proved or (in the case of the "principles") *assumed.* In geometry, says Aristotle, the existence of points and lines only must be assumed, the existence of the rest being proved. Accordingly Euclid's first three postulates declare the possibility of constructing straight lines and circles (the only "lines" except straight lines used in the *Elements*). Other things are defined and afterwards constructed and proved to exist: e.g. in Book I., Def. 20, it is explained what is meant by an equilateral triangle; then (I. 1) it is proposed to construct it, and, when constructed, it is proved to agree with the definition. When a square is defined (I. Def. 22), the question whether such a thing really exists is left open until, in I. 46, it is proposed to construct it and, when constructed, it is proved to satisfy the definition[1]. Similarly with the right angle (I. Def. 10, and I. 11) and parallels (I. Def. 23, and I. 27—29). The greatest care is taken to exclude mere presumption and imagination. The transition from the subjective definition of names to the objective definition of things is made, in geometry, by means of *constructions* (the first principles of which are postulated), as in other sciences it is made by means of experience[2].

Aristotle's requirements in a definition.

We now come to the positive characteristics by which, according to Aristotle, scientific definitions must be marked.

First, the different attributes in a definition, when taken separately, cover more than the notion defined, but the combination of them does not. Aristotle illustrates this by the "triad," into which enter the several notions of number, odd and prime, and the last "in both its two senses (*a*) of not being measured by any (other) number (*ὡς μὴ μετρεῖσθαι ἀριθμῷ*) and (*b*) of not being obtainable by adding numbers together" (*ὡς μὴ συγκεῖσθαι ἐξ ἀριθμῶν*), a unit not being a number. Of these attributes some are present in all other odd numbers as well, while the last [primeness in the second sense] belongs also to the dyad, but in nothing but the triad are they *all* present[3]." The fact can be equally well illustrated from geometry. Thus, e.g. into the definition of a square (Eucl. I., Def. 22) there enter the several notions of figure, four-sided, equilateral, and right-angled, each of which covers more than the notion into which *all* enter as attributes[4].

Secondly, a definition must be expressed in terms of things which are prior to, and better known than, the things defined[5]. This is clear, since the object of a definition is to give us knowledge of the thing defined, and it is by means of things prior and better known that we acquire fresh knowledge, as in the course of demonstrations. But the terms "prior" and "better known" are, as usual susceptible of two meanings; they may mean (1) *absolutely* or *logically* prior and better known, or (2) better known *relatively to us*. In the absolute sense, or from the standpoint of reason, a point is better known than a line, a line than a plane, and a plane than a solid, as also a unit is better known than number (for the unit is prior to, and the first principle of, any number). Similarly, in the absolute sense, a letter is prior to a syllable. But the case is sometimes different relatively to us; for example, a solid is more easily realised by the senses than a plane, a plane than a line, and a line than a point. Hence, while it is more scientific to begin with the *absolutely* prior, it may, perhaps, be permissible, in case the learner is not capable of following the scientific order, to explain things by means of what is more intelligible *to him*. "Among the definitions framed on this principle are those of the point, the line and the plane; all these explain what is prior by means of what is posterior, for the point is described as the extremity of a line, the line of a plane, the plane of a solid." But, if it is asserted that such definitions by means of things which are more intelligible relatively only to a particular individual are really definitions, it will follow that there may be many definitions of the same thing, one for each individual for whom a thing is being defined, and even different definitions for one and the same individual at different times, since at first sensible objects are more intelligible, while to a better trained mind they become less so. It follows therefore that a thing should be defined by means of the absolutely prior and not the relatively prior, in order that there may be one sole and immutable definition. This is further enforced by reference to the requirement that a good definition must state the *genus* and the *differentiae*, for these are among the things which are, in the absolute sense, better known than, and prior to, the species (τῶν ἁπλῶς γνωριμωτέρων καὶ προτέρων τοῦ εἴδους ἐστίν). For to destroy the genus and the differentia is to destroy the species, so that the former are *prior* to the species; they are also *better known*, for, when the species is known, the genus and the differentia must necessarily be known also, e.g. he who knows "man" must also know "animal" and "land-animal," but it does not follow, when the genus and differentia are known, that the species is known too, and hence the species is less known than they are[1]. It may be frankly admitted that the scientific definition will require superior mental powers for its apprehension; and the extent of its use must be a matter of discretion. So far Aristotle; and we have here the best possible explanation why Euclid supplemented his definition of a point by the statement in I. Def. 3 that *the extremities of a line are points* and his definition of a surface by I. Def. 6 to the effect that *the extremities of a surface are lines*. The supplementary explanations do in fact enable us to arrive at a better understanding of the formal definitions of a point and a line respectively, as is well explained by Simson in his note on Def. 1. Simson says, namely, that we must consider a solid, that is, a magnitude which has length, breadth and thickness, in order to understand aright the definitions of a point, a line and a surface. Consider, for instance, the boundary common to two solids which are contiguous or the boundary which divides one solid into two contiguous parts; this boundary is a surface. We can prove that it has no thickness by taking away either solid, when it remains the boundary of the other; for, if it had thickness, the thickness must either be a part of one solid or of the other, in which case to take away one or other solid would take away the thickness and therefore the boundary itself: which is impossible. Therefore the boundary or the surface has no thickness. In exactly the same way, regarding a line as the boundary of two contiguous surfaces, we prove that the line has no breadth; and, lastly, regarding a point as the common boundary or extremity of two lines, we prove that a point has no length, breadth or thickness.

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[1] Trendelenburg, *Elementa Logices Aristoteleae*, § 50.

[2] Trendelenburg, *Erläuterungen zu den Elementen der aristotelischen Logik*, 3 ed. p. 107. On construction as proof of existence in ancient geometry cf. H. G. Zeuthen, *Die geometrische Construction als "Existenzbeweis" in der antiken Geometrie* (in *Mathematische Annalen*, 47. Band).

[3] *Anal. post.* II. 13, 96 a 33—b 1.

[4] Trendelenburg, *Erläuterungen*, p. 108.

[5] *Topics* VI. 4, 141 a 26 sqq.

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[1] *Topics* VI. 4, 141 b 25—34.

Aristotle on unscientific definitions.

Aristotle distinguishes three kinds of definition which are unscientific because founded on what is *not* prior (μὴ ἐκ προτέρων). The *first* is a definition of a thing by means of its opposite, e.g. of "good" by means of "bad"; this is wrong because opposites are naturally evolved together, and the knowledge of opposites is not uncommonly regarded as one and the same, so that one of the two opposites cannot be better known than the other. It is true that, in some cases of opposites, it would appear that no other sort of definition is possible: e.g. it would seem impossible to define double apart from the half and, generally, this would be the case with things which in their very nature (καθ' αὑτά) are *relative* terms (πρός τι λέγεται), since one cannot be known without the other, so that in the notion of either the other must be comprised as well[1]. The *second* kind of definition which is based on what is not prior is that in which there is a complete circle through the unconscious use in the definition itself of the notion to be defined though not of the name[2]. Trendelenburg illustrates this by two current definitions, (1) that of magnitude as that which can be increased or diminished, which is bad because the positive and negative comparatives "more" and "less" presuppose the notion of the positive "great," (2) the famous Euclidean definition of a straight line as that which "lies evenly with the points on itself" (ἐξ ἴσου τοῖς ἐφ' ἑαυτῆς σημείοις κεῖται), where "lies evenly" can only be understood with the aid of the very notion of a straight line which is to be defined[3]. The *third* kind of vicious definition from that which is not prior is the definition of one of two coordinate species by means of its coordinate (ἀντιδιῃρημένον), e.g. a definition of "odd" as that which exceeds the even by a unit (the second alternative in Eucl. VII. Def. 7); for "odd" and "even" are coördinates, being *differentiae* of number[1]. This third kind is similar to the first. Thus, says Trendelenburg, it would be wrong to define a *square* as "a *rectangle* with equal sides."

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[1] *Topics* VI. 4, 142 a 22—31.

[2] *ibid.* 142 a 34—b 6.

[3] Trendelenburg, *Erläuterungen*, p. 115.

Aristotle's third requirement.

A third general observation of Aristotle which is specially relevant to geometrical definitions is that "to know *what* a thing is (τί ἐστιν) is the same as knowing *why* it is (διὰ τί ἐστιν)[2]." "*What* is an eclipse? A deprivation of light from the moon through the interposition of the earth. *Why* does an eclipse take place? Or *why* is the moon eclipsed? Because the light fails through the earth obstructing it. *What* is harmony? A ratio of numbers in high or low pitch. *Why* does the high-pitched harmonise with the low-pitched? Because the high and the low have a numerical ratio to one another[3]." "We seek the *cause* (τὸ διότι) when we are already in possession of the *fact* (τὸ ὅτι). Sometimes they both become evident at the same time, but at all events the cause cannot possibly be known [as a cause] before the fact is known[4]." "It is impossible to know *what* a thing is if we do not know *that* it is[5]" Trendelenburg paraphrases: "The definition of the notion does not fulfil its purpose until it is made *genetic*. It is the producing cause which first reveals the essence of the thing.... .The nominal definitions of geometry have only a provisional significance and are superseded as soon as they are made genetic by means of construction." E.g. the genetic definition of a parallelogram is evolved from Eucl. I. 31 (giving the construction for parallels) and I. 33 about the lines joining corresponding ends of two straight lines parallel and equal in length. Where existence is proved by *construction*, the cause and the fact appear *together*[6].

Again, "it is not enough that the defining statement should set forth the fact, as most definitions do; it should also contain and present the cause; whereas in practice what is stated in the definition is usually no more than a conclusion (συμπέρασμα). For example, what is quadrature? The construction of an equilateral right-angled figure equal to an oblong. But such a definition expresses merely the conclusion. Whereas, if you say that quadrature is the discovery of a mean proportional, then you state the reason[7]." This is better understood if we compare the statement elsewhere that "the cause is the middle term, and this is what is sought in all cases[8]," and the illustration of this by the case of the proposition that the angle in a semicircle is a right angle. Here the middle term which it is sought to establish by means of the figure is that the angle in the semi-circle is equal to *the half of two right angles*. We have then the syllogism: Whatever is half of two right angles is a right angle; the angle in a semi-circle is the half of two right angles; therefore (*conclusion*) the angle in a semi-circle is a right angle[9]. As with the demonstration, so it should be with the definition. A definition which is to show the *genesis* of the thing defined should contain the middle term or cause; otherwise it is a mere statement of a conclusion. Consider, for instance, the definition of "quadrature" as "making a square equal in area to a rectangle with unequal sides." This gives no hint as to whether a solution of the problem is possible or how it is solved: but, if you add that to find the mean proportional between two given straight lines gives another straight line such that the square on it is equal to the rectangle contained by the first two straight lines, you supply the necessary middle term or cause[1].

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[1] *Topics* VI. 4, 142 b 7—10.

[2] *Anal. post.* II. 2, 90 a 31.

[3] *Anal. post.* II. 2, 90 a 15—21.

[4] *ibid.* II. 8, 93 a 17.

[5] *ibid.* 93 a 20.

[6] Trendelenburg, *Erläuterungen*, p. 110.

[7] *De anima* II. 2, 413 a 13—20.

[8] *Anal. post.* II. 2, 90 a 6,

[9] *ibid.* II. 11, 94 a 28.

Technical terms not defined by Euclid.

It will be observed that what is here defined, "quadrature" or "squaring" (τετραγωνισμός), is not a geometrical figure, or an attribute of such a figure or a part of a figure, but a technical term used to describe a certain problem. Euclid does not define such things; but the fact that Aristotle alludes to this particular definition as well as to definitions of *deflection* (κεκλάσθαι) and of *verging* (νεύειν) seems to show that earlier text-books included among definitions explanations of a number of technical terms, and that Euclid deliberately omitted these explanations from his *Elements* as surplusage. Later the tendency was again in the opposite direction, as we see from the much expanded Definitions of Heron, which, for example, actually include a definition of a *deflected line* (κεκλασμένη γραμμή)[2]. Euclid uses the passive of κλᾶν occasionally[3], but evidently considered it unnecessary to explain such terms, which had come to bear a recognised meaning.

The mention too by Aristotle of a definition of *verging* (νεύειν) suggests that the problems indicated by this term were not excluded from elementary text-books before Euclid. The type of problem (νεῦσις) was that of placing a straight line across two lines, e.g. two straight lines, or a straight line and a circle, so that it shall *verge* to a given point (i.e. pass through it if produced) and at the same time the intercept on it made by the two given lines shall be of given length.

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[1] Other passages in Aristotle may be quoted to the like effect: e.g. *Anal. post.* I. 2, 71 b 9 "We consider that we know a particular thing in the absolute sense, as distinct from the sophistical and incidental sense, when we consider that we know the cause on account of which the thing is, in the sense of knowing that it is the cause of that thing and that it cannot be otherwise," *ibid.* I. 13, 79 a 2 "For here to know the *fact* is the function of those who are concerned with sensible things, to know the *cause* is the function of the mathematician; it is he who possesses the proofs of the causes, and often he does not know the fact." In view of such passages it is difficult to see how Proclus came to write (p. 202, 11) that Aristotle was the originator (Ἀριστοτέλους κατάρξαντος) of the idea of Amphinomus and others that geometry does not investigate the cause and the *why* (τὸ διὰ τί). To this Geminus replied that the investigation of the cause does, on the contrary, appear in geometry. "For how can it be maintained that it is not the business of the geometer to inquire for what reason, on the one hand, an infinite number of equilateral polygons are inscribed in a circle, but, on the other hand, it is not possible to inscribe in a sphere an infinite number of polyhedral figures, equilateral, equiangular, and made up of similar plane figures? Whose business is it to ask this question and find the answer to it if it is not that of the geometer? Now when geometers reason *per impossibile* they are content to discover the property, but when they argue by direct proof, if such proof be only partial (ἐπὶ μέρους), this does not suffice for showing the cause; if however it is general and applies to all like cases, the why (τὸ διὰ τί) is at once and concurrently made evident."

[2] Heron, Def. 12 (vol. IV. Heib. pp. 22–24).

[3] e.g. in III. 20 and in *Data* 89.

In general, the use of conics is required for the theoretical solution of these problems, or a mechanical contrivance for their practical solution[1]. Zeuthen, following Oppermann, gives reasons for supposing, not only that mechanical constructions were *practically* used by the older Greek geometers for solving these problems, but that they were *theoretically* recognised as a permissible means of solution when the solution could not be effected by means of the straight line and circle, and that it was only in later times that it was considered necessary to use conics in *every* case where that was possible[2]. Heiberg[3] suggests that the allusion of Aristotle to νεύσεις perhaps confirms this supposition, as Aristotle nowhere shows the slightest acquaintance with conics. I doubt whether this is a safe inference, since the problems of this type included in the elementary text-books might easily have been limited to those which could be solved by "plane" methods (i.e. by means of the straight line and circle). We know, e.g., from Pappus that Apollonius wrote two Books on *plane* νεύσεις[4]. But one thing is certain, namely that Euclid deliberately excluded this class of problem, doubtless as not being essential in a book of Elements.

Definitions not afterwards used.

Lastly, Euclid has definitions of some terms which he never afterwards uses, e.g. oblong (ἑτερόμηκες), rhombus, rhomboid. The "oblong" occurs in Aristotle; and it is certain that all these definitions are survivals from earlier books of Elements.

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[1] Cf. the chapter on νεύσεις in *The Works of Archimedes*, pp. c—cxxii.

[2] Zeuthen, *Die Lehre von den Kegelschnitten im Altertum*, ch. 12, p. 262.

[3] Heiberg, *Mathematisches zu Aristoteles*, p. 16.

[4] Pappus VII. pp. 670—2.

BOOK I.

DEFINITIONS.

1. A **point** is that which has no part.

2. A **line** is breadthless length.

3. The extremities of a line are points.

4. A **straight line** is a line which lies evenly with the points on itself.

5. A **surface** is that which has length and breadth only.

6. The extremities of a surface are lines.

7. A **plane surface** is a surface which lies evenly with the straight lines on itself.

8. A **plane angle** is the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line.

9. And when the lines containing the angle are straight, the angle is called **rectilineal.**

10. When a straight line set up on a straight line makes the adjacent angles equal to one another, each of the equal angles is **right,** and the straight line standing on the other is called a **perpendicular** to that on which it stands.

11. An **obtuse angle** is an angle greater than a right angle.

12. An **acute angle** is an angle less than a right angle.

13. A **boundary** is that which is an extremity of anything.

14. A **figure** is that which is contained by any boundary or boundaries.

15. A **circle** is a plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure are equal to one another;

16. And the point is called the **centre** of the circle.

17. A **diameter** of the circle is any straight line drawn through the centre and terminated in both directions by the circumference of the circle, and such a straight line also bisects the circle.

18. A **semicircle** is the figure contained by the diameter and the circumference cut off by it. And the centre of the semicircle is the same as that of the circle.

19. **Rectilineal figures** are those which are contained by straight lines, **trilateral** figures being those contained by three, **quadrilateral** those contained by four, and **multilateral** those contained by more than four straight lines.

20. Of trilateral figures, an **equilateral triangle** is that which has its three sides equal, an **isosceles triangle** that which has two of its sides alone equal, and a **scalene triangle** that which has its three sides unequal.

21. Further, of trilateral figures, a **right-angled triangle** is that which has a right angle, an **obtuse-angled triangle** that which has an obtuse angle, and an **acute-angled triangle** that which has its three angles acute.

22. Of quadrilateral figures, a **square** is that which is both equilateral and right-angled; an **oblong** that which is right-angled but not equilateral; a **rhombus** that which is equilateral but not right-angled; and a **rhomboid** that which has its opposite sides and angles equal to one another but is neither equilateral nor right-angled. And let quadrilaterals other than these be called **trapezia**.

23. **Parallel** straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction.

POSTULATES.

Let the following be postulated:

1. To draw a straight line from any point to any point.

2. To produce a finite straight line continuously in a straight line.

3. To describe a circle with any centre and distance.

4. That all right angles are equal to one another.

5. That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

COMMON NOTIONS.

1. Things which are equal to the same thing are also equal to one another.

2. If equals be added to equals, the wholes are equal.

3. If equals be subtracted from equals, the remainders are equal.

[7] 4. Things which coincide with one another are equal to one another.

[8] 5. The whole is greater than the part.

DEFINITION I.

Σημεῖόν ἐστιν, οὗ μέρος οὐθέν.

A point *is that which has no part.*

An exactly parallel use of μέρος (ἐστί) in the singular is found in Aristotle, *Metaph.* 1035 b 32 μέρος μὲν οὖν ἐστὶ καὶ τοῦ εἴδους, literally "There is a *part* even of the form"; Bonitz translates as if the plural were used, "Theile giebt es," and the meaning is simply "even the form is *divisible* (into parts)." Accordingly it would be quite justifiable to translate in this case "A point is that which is *indivisible into parts.*"

Martianus Capella (5th c. A.D.) alone or almost alone translated differently, "Punctum est cuius pars *nihil* est," "a point is that a part of which is *nothing.*" Notwithstanding that Max Simon (*Euclid und die sechs planimetrischen Bücher*, 1901) has adopted this translation (on grounds which I shall presently mention), I cannot think that it gives any sense. If a part of a point is *nothing*, Euclid might as well have said that a point is *itself* "nothing," which of course he does not do.

Pre-Euclidean definitions.

It would appear that this was not the definition given in earlier textbooks; for Aristotle (*Topics* VI. 4, 141 b 20), in speaking of "*the* definitions" of point, line, and surface, says that they *all* define the prior by means of the posterior, a point as an extremity of a line, a line of a surface, and a surface of a solid.

The first definition of a point of which we hear is that given by the Pythagoreans (cf. Proclus, p. 95, 21), who defined it as a "monad having position" or "with position added" (μονὰς προσλαβοῦσα θέσιν). It is frequently used by Aristotle, either in this exact form (cf. *De anima* I. 4, 409 a 6) or its equivalent: e.g. in *Metaph.* 1016 b 24 he says that that which is indivisible every way in respect of magnitude and *quâ* magnitude but has not position is a *monad*, while that which is similarly indivisible and has position is a *point.*

Plato appears to have objected to this definition. Aristotle says (*Metaph.*

992 a 20) that he objected "to this genus [that of points] as being a geometrical fiction (γεωμετρικὸν δόγμα), and called a point the beginning of a line (ἀρχὴ γραμμῆς), while again he frequently spoke of 'indivisible lines.'" To which Aristotle replies that even "indivisible lines" must have extremities, so that the same argument which proves the existence of *lines* can be used to prove that *points* exist. It would appear therefore that, when Aristotle objects to the definition of a point as the extremity of a line (πέρας γραμμῆς) as unscientific (*Topics* VI. 4, 141 b 21), he is aiming at Plato. Heiberg conjectures (*Mathematisches zu Aristoteles*, p. 8) that it was due to Plato's influence that the word for "point" generally used by Aristotle (στιγμή) was replaced by σημεῖον (the regular term used by Euclid, Archimedes and later writers), the latter term (= *nota*, a conventional mark) probably being considered more suitable than στιγμή (a *puncture*) which might appear to claim greater *reality* for a point.

Aristotle's conception of a point as that which is indivisible and has position is further illustrated by such observations as that a point is not a *body* (*De caelo* II. 13, 296 a 17) and has no *weight* (*ibid.* III. 1, 299 a 30); again, we can make no distinction between a point and the *place* (τόπος) where it is (*Physics* IV. 1, 209 a 11). He finds the usual difficulty in accounting for the transition from the indivisible, or infinitely small, to the finite or divisible magnitude. A point being *indivisible*, no accumulation of points, however far it may be carried, can give us anything divisible, whereas of course a line is a divisible magnitude. Hence he holds that points cannot make up anything continuous like a line, point cannot be continuous with point (οὐ γάρ ἐστιν ἐχόμενον σημεῖον σημείου ἢ στιγμὴ στιγμῆς, *De gen. et corr.* I. 2, 317 a 10), and a line is not *made up* of points (οὐ σύγκειται ἐκ στιγμῶν, *Physics* IV. 8, 215 b 19). A point, he says, is like the *now* in time: *now* is indivisible and is not a *part* of time, it is only the beginning or end, or a division, of time, and similarly a point may be an extremity, beginning or division of a line, but is not *part* of it or of magnitude (cf. *De caelo* III. 1, 300 a 14, *Physics* IV. 11, 220 a 1—21, VI. 1, 231 b 6 sqq.). It is only by *motion* that a point can generate a line (*De anima* I. 4, 409 a 4) and thus be the origin of magnitude.

Other ancient definitions.

According to an-Nairīzī (ed. Curtze, p. 3) one "Herundes" (not so far identified) defined a point as "the indivisible beginning of all magnitudes," and Posidonius as "an extremity which has no dimension, or an extremity of a line."

Criticisms by commentators.

Euclid's definition itself is of course practically the same as that which Aristotle's frequent allusions show to have been then current, except that it omits to say that the point must have position. Is it then sufficient, seeing that there are other things which are without parts or indivisible, e.g. the *now* in time, and the *unit* in number? Proclus answers (p. 93, 18) that the point is the only thing *in the subject-matter of geometry* that is indivisible. Relatively therefore to the particular science the definition is sufficient. Secondly, the definition has been over and over again criticised because it is purely negative. Proclus' answer to this is (p. 94, 10) that negative descriptions are appropriate to first principles, and he quotes Parmenides as having described his first and last cause by means of negations merely. Aristotle too admits that it may sometimes be necessary for one framing a definition to use negations, e.g. in defining privative terms such as "blind"; and he seems to accept as proper the negative element in the definition of a point, since he says (*De anima* III. 6, 430 b 20) that "the point and every division [e.g. in a length or in a period of time], and that which is indivisible in this sense, is exhibited as privation (δηλοῦται ὡς στέρησις)."

Simplicius (quoted by an-Nairīzī) says that "a point is the beginning of magnitudes and that from which they grow; it is also the only thing which, having position, is not divisible." He, like Aristotle, adds that it is by its *motion* that a point can generate a magnitude: the particular magnitude can only be "of one dimension," viz. a line, since the point does not "spread itself" (dimittat). Simplicius further observes that Euclid defined a point negatively because it was arrived at by detaching surface from body, line from surface, and finally point from line. "Since then body has three dimensions it follows that a point [arrived at after successively eliminating all three dimensions] has *none of the dimensions*, and has no part." This of course reappears in modern treatises (cf. Rausenberger, *Elementar-geometrie des Punktes, der Geraden und der Ebene*, 1887, p. 7).

An-Nairīzī adds an interesting observation. "If any one seeks to know the essence of a point, a thing more simple than a line, let him, in the sensible world, think of the centre of the universe and the *poles*." But there is nothing new under the sun: the same idea is mentioned, in an Aristotelian treatise, in controverting those who imagine that the poles have some influence in the motion of the sphere, "when the poles have no magnitude but are extremities and points" (*De motu animalium* 3, 699 a 21).

Modern views.

In the new geometry represented by the excellent treatises which start from new systems of postulates or axioms, the result of the profound study of the fundamental principles of geometry during recent years (I need only mention the names of Pasch, Veronese, Enriques and Hilbert), points come before lines, but the vain effort to define them *a priori* is not made; instead of this, the nearest material things in nature are mentioned as illustrations, with the remark that it is from them that we can get the abstract idea. Cf. the full statement as regards the notion of a point in Weber and Wellstein, *Encyclopädie der elementaren Mathematik*, II., 1905, p. 9. "This notion is evolved from the notion of the real or supposed *material* point by the process of limits, i.e. by an act of the mind which sets a term to a series of presentations in itself unlimited. Suppose a grain of sand or a mote in a sunbeam, which continually becomes smaller and smaller. In this way vanishes more and more the possibility of determining still smaller atoms in the grain of sand, and there is evolved, so we say, with growing certainty, the presentation of the point as a definite position in space which is one and is incapable of further division. But this view is untenable; we have, it is true, some idea how the grain of sand gets smaller and smaller, but only so long as it remains just visible; after that we are completely in the dark, and we cannot see or imagine the further diminution. That this procedure comes to an end is unthinkable; that nevertheless there exists a term beyond which it cannot go, we must believe or postulate without ever reaching it. . . . It is a pure act of *will*, not of the understanding." Max Simon observes similarly (*Euclid*, p. 25) "The notion 'point' belongs to the limit-notions (Grenzbegriffe), the necessary conclusions of continued, and in themselves unlimited, series of presentations." He adds, "The point is the limit of localisation; if this is more and more energetically continued, it leads to the limit-notion 'point,' better 'position,' which at the same time involves a change of notion. Content of *space* vanishes, relative *position* remains. 'Point' then, according to our interpretation of Euclid, is the extremest limit of that which we can still think of (not observe) as a *spatial* presentation, and if we go further than that, not only does extension cease but even relative *place*, and in this sense the 'part' is *nothing*." I confess I think that even the meaning which Simon intends to convey is better expressed by "it has *no* part" than by "the part is nothing," since to take a "part" of a thing in Euclid's sense of the result of a simple division, corresponding to an arithmetical fraction, would not be to change the *notion* from that of the thing divided to an entirely different one.

DEFINITION 2.

Γραμμὴ δὲ μῆκος ἀπλατές.

A line *is breadthless length.*

This definition may safely be attributed to the Platonic School, if not to Plato himself. Aristotle (*Topics* VI. 6, 143 b 11) speaks of it as open to objection because it "divides the genus by negation," length being necessarily either breadthless or possessed of breadth; it would seem however that the objection was only taken in order to score a point against the Platonists, since he says (*ibid.* 143 b 29) that the argument is "of service *only* against those who assert that the genus [sc. length] is one numerically, that is, those who assume *ideas*," e.g. the idea of length (αὐτὸ μῆκος) which they regard as a genus: for if the genus, being one and self-existent, could be divided into two species, one of which asserts what the other denies, it would be self-contradictory (Waitz).

Proclus (pp. 96, 21—97, 3) observes that, whereas the definition of a point is merely negative, the line introduces the first "dimension," and so its definition is to this extent positive, while it has also a negative element which denies to it the other "dimensions" (διαστάσεις). The negation of both breadth and depth is involved in the single expression "breadthless" (ἀπλατές), since everything that is without breadth is also destitute of depth, though the converse is of course not true.

Alternative definitions.

The alternative definition alluded to by Proclus, μέγεθος ἐφ' ἓν διαστατόν "magnitude in one dimension" or, better perhaps, "magnitude extended one way" (since διάστασις as used with reference to line, surface and solid scarcely corresponds to our use of "dimension" when we speak of "one," "two," or "three dimensions"), is attributed by an-Nairīzī to "Heromides," who must presumably be the same as "Herundes," to whom he attributes a certain definition of a point. It appears however in substance in Aristotle, though Aristotle does not use the adjective διαστατόν, nor does he apparently use διάστασις except of *body* as having *three* "dimensions" or "having dimension (or extension) *all* ways (πάντῃ)," the "dimensions" being in his view (1) up and down, (2) before and behind, and (3) right and left, and "up" being the principle or beginning of *length*, "right" of *breadth*, and "before" of *depth* (*De caelo* II. 2, 284 b 24). A line is, according to Aristotle, a magnitude "*divisible* in one way only" (μοναχῇ διαιρετόν), in contrast to a magnitude divisible in *two* ways (διχῇ διαιρετόν), or a surface, and a magnitude divisible "in all or in three ways" (πάντῃ καὶ τριχῇ διαιρετόν), or a body (*Metaph.* 1016 b 25—27); or it is a magnitude "*continuous* one way (or in one direction)," as compared with magnitudes continuous *two* ways or *three* ways, which curiously enough he describes as "breadth" and "depth" respectively (μέγεθος δὲ τὸ μὲν ἐφ᾽ ἓν συνεχὲς μῆκος, τὸ δ᾽ ἐπὶ δύο πλάτος, τὸ δ᾽ ἐπὶ τρία βάθος, *Metaph.* 1020 a 11), though he immediately adds that "length" means a line, "breadth" a surface, and "depth" a body.

Proclus gives another alternative definition as "*flux of a point*" (ῥύσις σημείου), i.e. the path of a point when moved. This idea is also alluded to in Aristotle (*De anima* I. 4, 409 a 4 above quoted): "they say that a line by its motion produces a surface, and a point by its motion a line." "This definition," says Proclus (p. 97, 8—13), "is a perfect one as showing the essence of the line: he who called it the flux of a point seems to define it from its genetic cause, and it is not every line that he sets before us, but only the immaterial line; for it is this that is produced by the point, which, though itself indivisible, is the cause of the existence of things divisible."

Proclus (p. 100, 5—19) adds the useful remark, which, he says, was current in the school of Apollonius, that we have the notion of a line when we ask for the length of a road or a wall measured merely as length; for in that case we mean something irrespective of breadth, viz. distance in one "dimension." Further we can obtain sensible perception of a line if we look at the division between the light and the dark when a shadow is thrown on the earth or the moon; for clearly the division is without breadth, but has length.

Species of "lines."

After defining the "line" Euclid only mentions *one* species of line, the straight line, although of course another species appears in the definition of a circle later. He doubtless omitted all *classification* of lines as unnecessary for his purpose, whereas, for example, Heron follows up his definition of a line by a division of lines into (1) those which are "straight" and (2) those which are not, and a further division of the latter into (*a*) "circular circumferences," (*b*) "spiral-shaped" (ἑλικοειδεῖς) lines and (*c*) "curved" (καμπύλαι) lines generally, and then explains the four terms. Aristotle tells us (*Metaph.* 986 a 25) that the Pythagoreans distinguished straight (εὐθύ) and curved (καμπύλον), and this distinction appears in Plato (cf. *Republic* X. 602 C) and in Aristotle (cf. "to a line belong the attributes straight or curved," *Anal. post.* I. 4, 73 b 19; "as in mathematics it is useful to know what is meant by the terms straight and curved," *De anima* I. 1, 402 b 19). But from the class of "curved" lines Plato and Aristotle separate off the περιφερής or "circular" as a distinct species often similarly contrasted with straight. Aristotle seems to recognise broken lines forming an angle as one line: thus "a line, if it be bent (κεκαμμένη), but yet continuous, is called one" (*Metaph.* 1016 a 2); "the straight line is more one than the bent line" (*ibid.* 1016 a 12). Cf. Heron, Def. 12, "A broken line (κεκλασμένη γραμμή) so-called is a line which, when produced, does not meet *itself*."

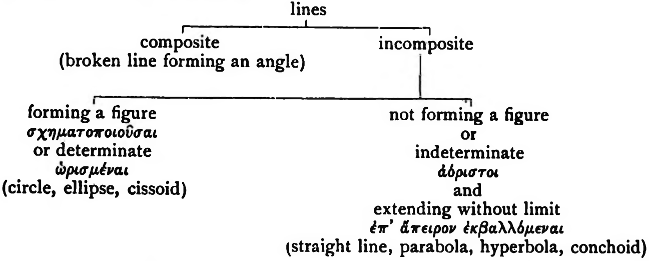
When Proclus says that both Plato and Aristotle divided lines into those which are "straight," "circular" (περιφερής) or "a mixture of the two," adding, as regards Plato, that he included in the last of these classes "those which are called helicoidal among plane (curves) and (curves) formed about solids, and such species of curved lines as arise from sections of solids" (p. 104, 1—5), he appears to be not quite exact. The reference as regards Plato seems to be to *Parmenides* 145 B: "At that rate it would seem that the one must have shape, either straight or round (στρογγύλου) or some combination of the two"; but this scarcely amounts to a formal classification of lines. As regards

Aristotle, Proclus seems to have in mind the passage (*De caelo* I. 2, 268 b 17) where it is stated that "all *motion* in space, which we call translation (φορά), is (in) a straight line, a circle, or a combination of the two; for the first two are the only simple (*motions*)."

For completeness it is desirable to add the substance of Proclus' account of the classification of lines, for which he quotes Geminus as his authority.

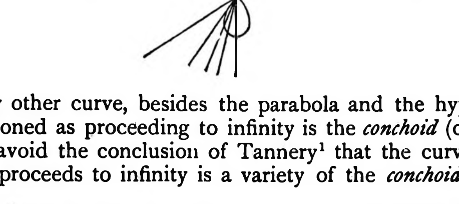
Geminus' first classification of lines.

This begins (p. 111, 1—9) with a division of lines into *composite* (σύνθετος) and incomposite (ἀσύνθετος). The only illustration given of the *composite* class is the "broken line which forms an angle" (ἡ κεκλασμένη καὶ γωνίαν ποιοῦσα); the subdivision of the *incomposite* class then follows (in the text as it stands the word "composite" is clearly an error for "incomposite"). The subdivisions of the incomposite class are repeated in a later passage (pp. 176, 27—177, 23) with some additional details. The following diagram reproduces the effect of both versions as far as possible (all the illustrations mentioned by Proclus being shown in brackets).


lines
composite
(broken line forming an angle)
incomposite
forming a figure
σχηματοποιοῦσαι
or determinate
ὡρισμέναι
(circle, ellipse, cissoid)
not forming a figure
or
indeterminate
ἀόριστοι
and
extending without limit
ἐπ' ἄπειρον ἐκβαλλόμεναι
(straight line, parabola, hyperbola, conchoid)

The additional details in the second version, which cannot easily be shown in the diagram, are as follows:

(1) Of the lines which extend without limit, some do not *form a figure* at all (viz. the straight line, the parabola and the hyperbola); but some first "come together and form a figure" (i.e. have a loop), "and, for the rest, extend without limit" (p. 177, 8).



As the only other curve, besides the parabola and the hyperbola, which has been mentioned as proceeding to infinity is the *conchoid* (of Nicomedes), we can hardly avoid the conclusion of Tannery[1] that the curve which has a loop and then proceeds to infinity is a variety of the *conchoid* itself. As is well known, the ordinary conchoid (which was used both for doubling the cube and for trisecting the angle) is obtained in this way. Suppose any number of rays passing through a fixed point (the *pole*) and intersecting a fixed straight line; and suppose that points are taken on the rays, beyond the fixed straight line, such that the portions of the rays intercepted between the fixed straight line and the point are equal to a constant *distance* (διάστημα), the locus of the points is a conchoid which has the fixed straight line for asymptote. If the "distance" a is measured from the intersection of the ray with the given straight line, not in the direction away from the pole, but towards the pole, we obtain three other curves according as a is less than, equal to, or greater than b, the distance of the pole from the fixed straight line, which is an asymptote in each case. The case in which $a > b$ gives a curve which forms a loop and then proceeds to infinity in the way Proclus describes. Now we know both from Eutocius (*Comm. on Archimedes*, ed. Heiberg, III. p. 98) and Proclus (p. 272, 3—7) that Nicomedes wrote on conchoid*s* (in the plural), and Pappus (IV. p. 244, 18) says that besides the "first" (used as above stated) there were "the second, the third and the fourth which are useful for other theorems."

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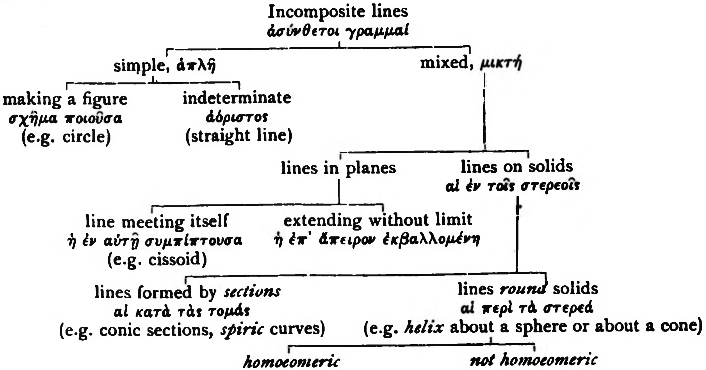
[1] *Notes pour l'histoire des lignes et surfaces courbes dans l'antiquité* in *Bulletin des sciences mathém. et astronom.* 2 sér. VIII. (1884), pp. 108—9 (*Mémoires scientifiques*, II. p. 23).

(2) Proclus next observes (p. 177, 9) that, of the lines which extend without limit, some are "*asymptotic*" (ἀσύμπτωτοι), namely "those which never meet, however they are produced," and some are "*symptotic*," namely "those which will meet sometime"; and, of the "asymptotic" class, some are in one plane, and others not. Lastly, of the "asymptotic" lines in one plane, some preserve always the same distance from one another, while others continually "lessen the distance, like the hyperbola with reference to the straight line, and the conchoid with reference to the straight line."

Geminus' second classification.

This (from Proclus, pp. 111, 9—20 and 112, 16—18) can be shown in a diagram thus:


Incomposite lines
ἀσύνθετοι γραμμαί
simple, ἁπλῆ
mixed, μικτή
making a figure
σχῆμα ποιοῦσα
(e.g. circle)
indeterminate
ἀόριστος
(straight line)
lines in planes
lines on solids
αἱ ἐν τοῖς στερεοῖς
line meeting itself
ἡ ἐν αὑτῇ συμπίπτουσα
(e.g. cissoid)
extending without limit
ἡ ἐπ' ἄπειρον ἐκβαλλομένη
lines formed by *sections*
αἱ κατὰ τὰς τομάς
(e.g. conic sections, *spiric* curves)
lines *round* solids
αἱ περὶ τὰ στερεά
(e.g. *helix* about a sphere or about a cone)
homoeomeric
(cylindrical helix)
not homoeomeric
(all others)

Notes on classes of "lines" and on particular curves.

We will now add the most interesting notes found in Proclus with reference to the above classifications or the particular curves mentioned.

1. Homoeomeric lines.

By this term (ὁμοιομερεῖς) are meant lines which are alike in all parts, so that in any one such curve any part can be made to coincide with any other part. Proclus observes that these lines are only three in number, two being "simple" and in a plane (the straight line and the circle), and the third "mixed," (subsisting) "about a solid," namely the cylindrical helix. The latter curve was also called the *cochlias* or *cochlion*, and its *homoeomeric* property was proved by Apollonius in his work περὶ τοῦ κοχλίου (Proclus, p. 105, 5). The fact that there are only three *homoeomeric* lines was proved by Geminus, "who proved, as a preliminary proposition, that, if from a point (ἀπό του σημείου, but on p. 251, 4 ἀφ' ἑνὸς σημείου) two straight lines be drawn to a homoeomeric line making equal angles with it, the straight lines are equal" (pp. 112, 1—113, 3, cf. p. 251, 2—19).

2. Mixed lines.

It might be supposed, says Proclus (p. 105, 11), that the cylindrical helix, being *homoeomeric*, like the straight line and the circle, must like them be *simple*. He replies that it is not simple, but *mixed*, because it is generated by *two unlike* motions. Two *like* motions, said Geminus, e.g. two motions at the same speed in the directions of two adjoining sides of a square, produce a *simple* line, namely a straight line (the diagonal); and again, if a straight line moves with its extremities upon the two sides of a right angle respectively, this same motion gives a *simple* curve (a circle) for the locus of the middle point of the straight line, and a *mixed* curve (an ellipse) for the locus of any other point on it (p. 106, 3—15).

Geminus also explained that the term "mixed," as applied to curves, and as applied to surfaces, respectively, is used in different senses. As applied to curves, "mixing" neither means simple "putting together" (σύνθεσις) nor "blending" (κρᾶσις). Thus the helix (or spiral) is a "mixed" line, but (1) it is not "mixed" in the sense of "putting together," as it would be if, say, part of it were straight and part circular, and (2) it is not mixed in the sense of "blending," because, if it is cut in any way, it does not present the appearance of any *simple* lines (of which it might be supposed to be compounded, as it were). The "mixing" in the case of lines is rather that in which the constituents are destroyed so far as their own character is concerned, and are replaced, as it were, by a *chemical* combination (ἔστιν ἐν αὐτῇ συνεφθαρμένα τὰ ἄκρα καὶ συγκεχυμένα). On the other hand "mixed" surfaces are mixed in the sense of a sort of "blending" (κατά τινα κρᾶσιν). For take a cone generated by a straight line passing through a fixed point and passing always through the circumference of a circle: if you cut this by a plane parallel to that of the circle, you obtain a circular section, and if you cut it by a plane through the vertex, you obtain a triangle, the "mixed" surface of the cone being thus cut into *simple* lines (pp. 117, 22—118, 23).

3. Spiric curves.

These curves, classed with conics as being sections of solids, were discovered by Perseus, according to an epigram of Perseus' own quoted by Proclus (p. 112, 1), which says that Perseus found "three lines upon (or, perhaps, in addition to) five sections" (τρεῖς γραμμὰς ἐπὶ πέντε τομαῖς). Proclus throws some light on these in the following passages:

"Of the spiric sections, one is interlaced, resembling the horse-fetter (ἵππου πέδῃ); another is widened out in the middle and contracts on each side (of the middle), a third is elongated and is narrower in the middle, broadening out on each side of it" (p. 112, 4—8).

"This is the case with the *spiric surface*; for it is conceived as generated by the revolution of a circle remaining at right angles [to a plane] and turning about a point which is not its centre [in other words, generated by the revolution of a circle about a straight line in its plane not passing through the centre]. Hence the *spire* takes three forms, for the centre [of rotation] is either on the circumference, or within it, or without it. And if the centre of rotation is on the circumference, we have the *continuous* spire (συνεχής), if within, the *interlaced* (ἐμπεπλεγμένη), and if without, the *open* (διεχής). And the spiric sections are three according to these three differences" (p. 119, 8—17).

"When the *hippopede*, which is one of the spiric curves, forms an angle with itself, this angle also is contained by mixed lines" (p. 127, 1—3).

"Perseus showed for spirics what was their property (σύμπτωμα)" (p. 356, 12).

Thus the spiric surface was what we call a *tore*, or (when open) an *anchor-ring*. Heron (Def. 97) says it was called alternatively *spire* (σπεῖρα) or *ring* (κρίκος); he calls the variety in which "the circle cuts itself," not "interlaced," but "crossing-itself" (ἐπαλλάττουσα).

Tannery[1] has discussed these passages, as also did Schiaparelli[2]. It is clear that Proclus' remark that the difference in the three curves which he mentions corresponds to the difference between the three surfaces is a slip, due perhaps to too hurried transcribing from Geminus: all three arise from plane sections of the *open* anchor-ring. If r is the radius of the revolving circle, a the distance of its centre from the axis of rotation, d the distance of the plane section (supposed to be parallel to the axis) from the axis, the three curves described in the first extract correspond to the following cases:

(1) $d = a - r$. In this case the curve is the *hippopede*, of which the lemniscate of Bernoulli is a particular case, namely that in which $a = 2r$.

The name *hippopede* was doubtless adopted for this one of Perseus' curves on the ground of its resemblance to the *hippopede* of Eudoxus, which seems to have been the curve of intersection of a sphere with a cylinder touching it internally.

(2) $a + r > d > a$. Here the curve is an oval.

(3) $a > d > a - r$. The curve is now narrowest in the middle.

Tannery explains the "three lines upon (in addition to) five sections" thus. He points out that with the *open tore* there are two other sections corresponding to

(4) $d = a$: transition from (2) to (3).

(5) $a - r > d > 0$, in which case the section consists of two symmetrical ovals.

He then shows that the sections of the *closed* or *continuous tore*, corresponding to $a = r$, give curves corresponding to (2), (3) and (4) only. Instead of (1) and (5) we have only a section consisting of two equal circles touching one another.

On the other hand, the *third* spire (the *interlaced* variety) gives three *new* forms, which make a group of three in addition to the first group of *five* sections.

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[1] *Pour l'histoire des lignes et surfaces courbes dans l'antiquité* in *Bulletin des sciences mathém. et astronom.* VIII. (1884), pp. 25—27 (*Mémoires scientifiques*, II. pp. 24—28).

[2] *Die homocentrischen Sphären des Eudoxus, des Kallippus und des Aristoteles* (*Abhandlungen zur Gesch. der Math.* I. Heft, 1877, pp. 149—152).

The difficulty which I see in this interpretation is the fact that, just after "three lines on five sections" are mentioned, Proclus describes three curves which were evidently the most important; but these three belong to three of the five sections of the open tore, and are not separate from them.

4. The cissoid.

This curve is assumed to be the same as that by means of which, according to Eutocius (*Comm. on Archimedes*, III. p. 66 sqq.), Diocles in his book περὶ πυρίων (*On burning-glasses*) solved the problem of doubling the cube. It is the locus of points which he found by the following construction. Let *AC*, *BD* be diameters at right angles in a circle with centre *O*.

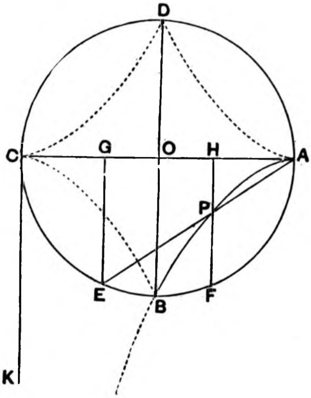
Let *E*, *F* be points on the quadrants *BC*, *BA* respectively such that the arcs *BE*, *BF* are equal.

Draw *EG*, *FH* perpendicular to *CA*. Join *AE*, and let *P* be its intersection with *FH*.

The cissoid is the locus of all the points *P* corresponding to different positions of *E* on the quadrant *BC* and of *F* at an equal distance from *B* along the arc *BA*.

A is the point on the curve corresponding to the position *C* for the point *E*, and *B* the point on the curve corresponding to the position of *E* in which it coincides with *B*.

It is easy to see that the curve extends in the direction *AB* beyond *B*, and that *CK* drawn perpendicular to *CA* is an asymptote. It may be regarded also as having a branch *AD* symmetrical with *AB*, and, beyond *D*, approaching *KC* produced as asymptote.



If *OA*, *OD* are coordinate axes, the equation of the curve is obviously

$$y^2(a+x)=(a-x)^3,$$

where a is the radius of the circle.

There is a cusp at *A*, and it agrees with this that Proclus should say (p. 126, 24) that "cissoidal lines converging to one point like the leaves of ivy—for this is the origin of their name—form an angle." He makes the slight correction (p. 128, 5) that it is not two *parts* of a curve, but *one* curve, which in this case makes an angle.

But what is surprising is that Proclus seems to have no idea of the curve passing outside the circle and having an asymptote, for he several times speaks of it as a *closed* curve (forming a figure and including an area): cf. p. 152, 7, "the plane (area) cut off by the cissoidal line has one bounding (line), but it has not in it a centre such that all (straight lines drawn to the curve) from it are equal." It would appear as if Proclus regarded the cissoid as formed by the *four* symmetrical cissoidal arcs shown in the figure.

Even more peculiar is Proclus' view of the

5. "Single-turn Spiral."

This is really the spiral of Archimedes traced by a point starting from the fixed extremity of a straight line and moving uniformly along it, while simultaneously the straight line itself moves uniformly in a plane about its fixed extremity. In Archimedes the spiral has of course any number of turns, the straight line making the same number of complete revolutions. Yet Proclus, while giving the same account of the generation of the spiral (p. 180, 8—12), regards the *single-turn spiral* as actually *stopping short* at the point reached after one complete revolution of the straight line: "it is necessary to know that extending without limit is not a property of all lines; for it neither belongs to the circle nor to the cissoid, nor in general to lines which form figures; nor even to those which do not form figures. For even the single-turn spiral does not extend without limit—*for it is constructed between two points*—nor does any of the other lines so generated do so" (p. 187, 19—25). It is curious that Pappus (VIII. p. 1110 sqq.) uses the same term μονόστροφος ἕλιξ to denote one turn, not of the spiral, but of the *cylindrical helix*.

DEFINITION 3.

Γραμμῆς δὲ πέρατα σημεῖα.
The extremities of a line are points.

It being unscientific, as Aristotle said, to define a point as the "extremity of a line" (πέρας γραμμῆς), thereby explaining the prior by the posterior, Euclid defined a point differently; then, as it was necessary to connect a point with a line, he introduced this explanation after the definitions of both had been given. This compromise is no doubt his own idea; the same thing occurs with reference to a surface and a line as its extremity in Def. 6, and with reference to a solid and a surface as its extremity in XI. Def. 2.

We miss a statement of the facts, equally requiring to be known, that a "division" (διαίρεσις) of a line, no less than its "beginning" or "end," is a point (this is brought out by Aristotle: cf. *Metaph.* 1060 b 15), and that the *intersection* of two lines is also a point. If these additional explanations had been given, Proclus would have been spared the difficulty which he finds in the fact that some of the lines used in Euclid (namely infinite straight lines on the one hand, and circles on the other) have no "extremities." So also the ellipse, which Proclus calls by the old name θυρεός ("shield"). In the case of the circle and ellipse we can, he observes (p. 103, 7), take a portion bounded by points, and the definition applies to that portion. His rather far-fetched distinction between two aspects of a circle or ellipse as a *line* and as a *closed figure* (thus, while you are *describing* a circle, you have two extremities at any moment, but they disappear when it is finished) is an unnecessarily elaborate attempt to establish the literal universality of the "definition," which is really no more than an explanation that, if a line *has* extremities, those extremities are points.

DEFINITION 4.

Εὐθεῖα γραμμή ἐστιν, ἥτις ἐξ ἴσου τοῖς ἐφ' ἑαυτῆς σημείοις κεῖται.
A straight line *is a line which lies evenly with the points on itself.*

The only definition of a straight line authenticated as pre-Euclidean is that of Plato, who defined it as "*that of which the middle covers the ends*" (relatively, that is, to an eye placed at either end and looking along the straight line). It appears in the *Parmenides* 137 E: "straight is whatever has its middle in front of (i.e. so placed as to obstruct the view of) both its ends" (εὐθύ γε οὗ ἂν τὸ μέσον ἀμφοῖν τοῖν ἐσχάτοιν ἐπίπροσθεν ᾖ). Aristotle quotes it in equivalent terms (*Topics* VI. 11, 148 b 27), οὗ τὸ μέσον ἐπιπροσθεῖ τοῖς πέρασιν; and, as he does not mention the name of its author, but states it in combination with the definition of a line as the extremity of a surface, we may assume that he used it as being well known. Proclus also quotes the definition as Plato's in almost identical terms, ἧς τὰ μέσα τοῖς ἄκροις ἐπιπροσθεῖ (p. 109, 21). This definition is ingenious, but implicitly appeals to the sense of sight and involves the postulate that the line of sight is straight. (Cf. the Aristotelian *Problems* 31, 20, 959 a 39, where the question is why we can better observe straightness in a row, say, of letters with one eye than with two.) As regards the straightness of "visual rays," ὄψεις, cf. Euclid's own *Optics*, Deff. 1, 2, assumed as *hypotheses*, in which he first speaks of the "straight lines" drawn from the eye, avoiding the word ὄψεις, and then says that the figure contained by the *visual rays* (ὄψεις) is a cone with its vertex in the eye.

As Aristotle mentions no definition of a straight line resembling Euclid's, but gives only Plato's definition and the other explaining it as the "extremity of a surface," the latter being evidently the current definition in contemporary textbooks, we may safely infer that Euclid's definition was a new departure of his own.

Proclus on Euclid's definition.

Coming now to the interpretation of Euclid's definition, εὐθεῖα γραμμή ἐστιν, ἥτις ἐξ ἴσου τοῖς ἐφ' ἑαυτῆς σημείοις κεῖται, we find any number of slightly different versions, but none that can be described as quite satisfactory; some authorities, e.g. Savile, have confessed that they could make nothing of it. It is natural to appeal to Proclus first; and we find that he does in fact give an interpretation which at first sight seems plausible. He says (p. 109, 8 sq.) that Euclid "shows by means of this that the straight line alone [of all lines] occupies a distance (κατέχειν διάστημα) equal to that between the points on it. For, as far as one of the points is distant from another, so great is the length (μέγεθος) of the straight line of which they are the extremities; and this is the meaning of lying ἐξ ἴσου to (or with) the points on it" [ἐξ ἴσου being thus, apparently, interpreted as "at" (or "over") "an equal distance"]. "But if you take two points on the circumference (of a circle) or any other line, the distance cut off between them along the line is greater than the interval separating them. And this is the case with every line except the straight line. Hence the ordinary remark, based on a common notion, that those who journey in a straight line only travel the necessary distance, while those who do not go straight travel more than the necessary distance." (Cf. Aristotle, *De caelo* I. 4, 271 a 13, "we always call the distance of anything the straight line" drawn to it.) Thus Proclus would interpret somewhat in this way: "a straight line is that which represents extension equal with (the distances separating) the points on it." This explanation seems to be an attempt to graft on to Euclid's definition the *assumption* (it is a λαμβανόμενον, not a definition) of Archimedes (*On the sphere and cylinder* I. *ad init.*) that "of all the lines which have the same extremities the straight line is least." For this purpose ἐξ ἴσου has apparently to be taken as meaning "at an equal distance," and again "lying at an equal distance" as equivalent to "extending over (or representing) an equal distance." This is difficult enough in itself, but is seen to be an impossible interpretation when applied to the similar definition of a plane by Euclid (Def. 7) as a surface "which lies evenly with the straight lines on itself." In that connexion Proclus tries to make the same words ἐξ ἴσου κεῖται mean "extends over an equal *area* with." He says namely (p. 117, 2) that, "if two straight lines are set out" on the plane, the plane surface "occupies a space equal to that between the straight lines." But two straight lines do not determine by themselves any space at all; it would be necessary to have a *closed* figure with its boundaries in the plane before we could arrive at the equivalent of the other assumption of Archimedes that "of surfaces which have the same extremities, if those extremities are in a plane, the plane is the least [in area]." This seems to be an impossible sense for ἐξ ἴσου even on the assumption that it means "at an equal distance" in the present definition. The necessity therefore of interpreting ἐξ ἴσου similarly in both definitions makes it impossible to regard it as referring to *distance* or *length* at all. It should be added that Simplicius gave the same explanations as Proclus (an-Nairīzī, p. 5).

The language and construction of the definition.

Let us now consider the actual wording and grammar of the phrase ἥτις ἐξ ἴσου τοῖς ἐφ' ἑαυτῆς σημείοις κεῖται. As regards the expression ἐξ ἴσου we note that Plato and Aristotle (whose use of it seems typical) commonly have it in the sense of "on a footing of equality": cf. οἱ ἐξ ἴσου in Plato's *Laws* 777 D, 919 D; Aristotle, *Politics* 1259 b 5 ἐξ ἴσου εἶναι βούλεται τὴν φύσιν, "tend to be on an equality in nature," *Eth. Nic.* VIII. 12, 1161 a 8 ἐνταῦθα πάντες ἐξ ἴσου, "there all are on a footing of equality." Slightly different are the uses in Aristotle, *Eth. Nic.* X. 8, 1178 a 25 τῶν μὲν γὰρ ἀναγκαίων χρεία καὶ ἐξ ἴσου ἔστω, "both need the necessaries of life *to the same extent*, let us say"; *Topics* IX. 15, 174 a 32 ἐξ ἴσου ποιοῦντα τὴν ἐρώτησιν, "asking the question indifferently" (i.e. without showing any expectation of one answer being given rather than another). The natural meaning would therefore appear to be "evenly placed" (or balanced), "in equal measure," "indifferently" or "without bias" one way or the other. Next, is the dative τοῖς ἐφ' ἑαυτῆς σημείοις constructed with ἐξ ἴσου or with κεῖται? In the first case the phrase must mean "that which lies *evenly with* (or in respect to) the points on it," in the second apparently "that which, in (or by) the points on it, lies (or is placed) evenly (or uniformly)." Max Simon takes the first construction to give the sense "die Gerade liegt in gleicher Weise wie ihre Punkte." If the last words mean "in the same way as (or in like manner as) its points," I cannot see that they tell us anything, although Simon attaches to the words the notion of *distance* (Abstand) like Proclus. The second construction he takes as giving "die Gerade liegt für (durch) ihre Punkte gleichmässig," "the straight line lies symmetrically for (or through) its points"; or, if κεῖται is taken as the passive of τίθημι, "die Gerade ist durch ihre Punkte gleichmässig gegeben worden," "the straight line is symmetrically determined by its points." He adds that the idea is here *direction*, and that both *direction* and *distance* (as between two different given points simply) would be to Euclid, as later to Bolzano (*Betrachtungen über einige Gegenstände der Elementargeometrie*, 1804, quoted by Schotten, *Inhalt und Methode des planimetrischen Unterrichts*, II. p. 16), primary irreducible notions.

While the language is thus seen to be hopelessly obscure, we can safely say that the sort of idea which Euclid wished to express was that of a line which presents the same shape at and relatively to all points on it, without any irregular or unsymmetrical feature distinguishing one part or side of it from another. Any such irregularity could, as Saccheri points out (Engel and Stäckel, *Die Theorie der Parallellinien von Euklid bis Gauss*, 1895, p. 109), be at once made perceptible by keeping the ends fixed and turning the line about them right round; if any two positions were distinguishable, e.g. one being to the left or right relatively to another, "it would not lie in a uniform manner between its points."

A conjecture as to its origin and meaning.

The question arises, what was the origin of Euclid's definition, or, how was it suggested to him? It seems to me that the basis of it was really Plato's definition of a straight line as "that line the middle of which covers the ends." Euclid was a Platonist, and what more natural than that he should have adopted Plato's definition in substance, while regarding it as essential to change the form of words in order to make it independent of any implied appeal to vision, which, as a physical fact, could not properly find a place in a purely geometrical definition? I believe therefore that Euclid's definition is simply an attempt (albeit unsuccessful, from the nature of the case) to express, in terms to which a geometer could not object as not being part of geometrical subject-matter, the same thing as the Platonic definition.

The truth is that Euclid was attempting the impossible. As Pfleiderer says (Scholia to Euclid), "It seems as though the notion of a *straight line*, owing to its simplicity, cannot be explained by any regular definition which does not introduce words already containing in themselves, by implication, the notion to be defined (such e.g. are direction, equality, uniformity or evenness of position, unswerving course), and as though it were impossible, if a person does not already know what the term *straight* here means, to teach it to him unless by putting before him in some way a picture or a drawing of it." This is accordingly done in such books as Veronese's *Elementi di geometria* (Part I., 1904, p. 10): "A stretched string, e.g. a plummet, a ray of light entering by a small hole into a dark room, are *rectilineal* objects. The image of them gives us the abstract idea of the limited line which is called a *rectilineal segment.*"

Other definitions.

We will conclude this note with some other famous definitions of a straight line. The following are given by Proclus (p. 110, 18—23).

1. *A line stretched to the utmost*, ἐπ' ἄκρον τεταμένη γραμμή. This appears in Heron also, with the words "towards the ends" (ἐπὶ τὰ πέρατα) added. (Heron, Def. 4).

2. *Part of it cannot be in the assumed plane while part is in one higher up* (ἐν μετεωροτέρῳ). This is a *proposition* in Euclid (XI. 1).

3. *All its parts fit on all (other parts) alike*, πάντα αὐτῆς τὰ μέρη πᾶσιν ὁμοίως ἐφαρμόζει. Heron has this too (Def. 4), but instead of "alike" he says παντοίως, "in all ways," which is better as indicating that the applied part may be applied one way or the *reverse* way, with the same result.

4. *That line which, when its ends remain fixed, itself remains fixed*, ἡ τῶν περάτων μενόντων καὶ αὐτὴ μένουσα. Heron's addition to this, "*when it is, as it were, turned round in the same plane*" (οἷον ἐν τῷ αὐτῷ ἐπιπέδῳ στρεφομένη), and his next variation, "and about the same ends having always the same position," show that the definition of a straight line as "that which does not change its position when it is turned about its extremities (or any two points in it) as poles" was no original discovery of Leibniz, or Saccheri, or Krafft, or Gauss, but goes back at least to the beginning of the Christian era. Gauss' form of this definition was: "The line in which lie all points that, during the revolution of a body (a part of space) about two fixed points, maintain their position unchanged is called a straight line." Schotten

(I. p. 315) maintains that the notion of a straight line and its property of being determined by two points are unconsciously assumed in this definition, which is therefore a logical "circle."

5. *That line which with one other of the same species cannot complete a figure*, ἡ μετὰ τῆς ὁμοειδοῦς μιᾶς σχῆμα μὴ ἀποτελοῦσα. This is an obvious ὕστερον-πρότερον, since it assumes the notion of a *figure*.

Lastly Leibniz' definition should be mentioned: *A straight line is one which divides a plane into two halves identical in all but position.* Apart from the fact that this definition introduces the plane, it does not seem to have any advantages over the definition last but one referred to.

Legendre uses the Archimedean property of a straight line as *the shortest distance between two points.* Van Swinden observes (*Elemente der Geometrie*, 1834, p. 4), that to take this as the definition involves *assuming* the proposition that any two sides of a triangle are greater than the third and *proving* that straight lines which have two points in common coincide throughout their length (cf. Legendre, *Éléments de Géométrie* I. 3, 8).

The above definitions all illustrate the observation of Unger (*Die Geometrie des Euklid*, 1833): "*Straight* is a simple notion, and hence all definitions of it must fail.... But if the proper idea of a straight line has once been grasped, it will be recognised in all the various definitions usually given of it; all the definitions must therefore be regarded as *explanations*, and among them that one is the best from which further inferences can immediately be drawn as to the essence of the straight line."

DEFINITION 5.

Ἐπιφάνεια δέ ἐστιν, ὃ μῆκος καὶ πλάτος μόνον ἔχει.

A surface *is that which has length and breadth only.*

The word ἐπιφάνεια was used by Euclid and later writers to denote *surface* in general, while they appropriated the word ἐπίπεδον for *plane* surface, thus making ἐπίπεδον a *species* of the *genus* ἐπιφάνεια. A solitary use of ἐπιφάνεια by Euclid when a plane is meant (XI. Def. 11) is probably due to the fact that the particular definition came from an earlier textbook. Proclus (p. 116, 17) remarks that the older philosophers, including Plato and Aristotle, used the words ἐπιφάνεια and ἐπίπεδον indifferently for any kind of surface. Aristotle does indeed use both words for a surface, with perhaps a tendency to use ἐπιφάνεια more than ἐπίπεδον for a surface not plane. Cf. *Categories* 6, 5 a 1 sq., where both words are used in one sentence: "You can find a common boundary at which the parts fit together, a point in the case of a line, and a line in the case of a surface (ἐπιφάνεια); for the parts of the surface (ἐπιπέδου) do fit together at some common boundary. Similarly also in the case of a body you can find a common boundary, a line or a surface (ἐπιφάνεια), at which the parts of the body fit together." Plato however does not use ἐπιφάνεια at all in the sense of surface, but only ἐπίπεδον for both *surface* and *plane surface.* There is reason therefore for doubting the correctness of the notice in Diogenes Laertius, III. 24, that Plato "was the first philosopher to name, among extremities, the *plane* surface" (ἐπίπεδος ἐπιφάνεια).

ἐπιφάνεια of course means literally the feature of a body which is *apparent* to the eye (ἐπιφανής), namely the surface.

Aristotle tells us (*De sensu* 3, 439 a 31) that the Pythagoreans called a surface χροιά, which seems to have meant *skin* as well as *colour.* Aristotle explains the term with reference to colour (χρῶμα) as a thing inseparable from the extremity (πέρας) of a body.

Alternative definitions.

The definitions of a surface correspond to those of a line. As in Aristotle a line is a magnitude "(extended) one way, or in one 'dimension'" (ἐφ' ἕν), "continuous one way" (ἐφ' ἓν συνεχές), or "divisible in one way" (μοναχῇ διαιρετόν), so a surface is a magnitude extended or continuous *two ways* (ἐπὶ δύο), or divisible *in two ways* (διχῇ). As in Euclid a surface has "length and breadth" only, so in Aristotle "breadth" is characteristic of the surface and is once used as synonymous with it (*Metaph.* 1020 a 12), and again "lengths are made up of long and short, *surfaces of broad and narrow*, and solids (ὄγκοι) of deep and shallow" (*Metaph.* 1085 a 10).

Aristotle mentions the common remark that *a line by its motion produces a surface* (*De anima* I. 4, 409 a 4). He also gives the *a posteriori* description of a surface as the "extremity of a solid" (*Topics* VI. 4, 141 b 22), and as "the section (τομή) or division (διαίρεσις) of a body" (*Metaph.* 1060 b 14).

Proclus remarks (p. 114, 20) that we get a notion of a surface when we measure areas and mark their boundaries in the sense of length and breadth; and we further get a sort of perception of it by looking at shadows, since these have no depth (for they do not penetrate the earth) but only have length and breadth.

Classification of surfaces.

Heron gives (Def. 74, p. 50, ed. Heiberg) two alternative divisions of surfaces into two classes, corresponding to Geminus' alternative divisions of lines, viz. into (1) *incomposite* and *composite* and (2) *simple* and *mixed*.

(1) *Incomposite* surfaces are "those which, when produced, fall into (or coalesce with) themselves" (ὅσαι ἐκβαλλόμεναι αὐταὶ καθ' ἑαυτῶν πίπτουσιν), i.e. are of continuous curvature, e.g. the sphere.

Composite surfaces are "those which, when produced, cut one another." Of composite surfaces, again, some are (*a*) made up of non-homogeneous (elements) (ἐξ ἀνομοιογενῶν) such as cones, cylinders and hemispheres, others (*b*) made up of homogeneous (elements), namely the rectilineal (or polyhedral) surfaces.

(2) Under the alternative division, *simple* surfaces are the plane and the spherical surfaces, but no others; the *mixed* class includes all other surfaces whatever and is therefore infinite in variety.

Heron specially mentions as belonging to the mixed class (*a*) the surface of cones, cylinders and the like, which are a mixture of plane and circular (μικταὶ ἐξ ἐπιπέδου καὶ περιφερείας) and (*b*) *spiric* surfaces, which are "a mixture of two circumferences" (by which he must mean a mixture of two circular elements, namely the generating circle and its circular motion about an axis in the same plane).

Proclus adds the remark that, curiously enough, *mixed* surfaces may arise by the revolution either of *simple* curves, e.g. in the case of the *spire*, or of *mixed* curves, e.g. the "right-angled conoid" from a parabola, "another conoid" from the hyperbola, the "oblong" (ἐπιμήκες, in Archimedes παραμᾶκες) and "flat" (ἐπιπλατύ) spheroids from an ellipse according as it revolves about the major or minor axis respectively (pp. 119, 6—120, 2). The *homoeomeric* surfaces, namely those any part of which will coincide with any other part, are *two* only (the plane and the spherical surface), not three as in the case of lines (p. 120, 7).

DEFINITION 6.

Ἐπιφανείας δὲ πέρατα γραμμαί.

The extremities of a surface are lines.

It being unscientific, as Aristotle says, to define a line as the extremity of a surface, Euclid avoids the error of defining the prior by means of the posterior in this way, and gives a different definition not open to this objection. Then, by way of compromise, and in order to show the connexion between a line and a surface, he adds the equivalent of the definition of a line previously current as an explanation.

As in the corresponding Def. 3 above, he omits to add what is made clear by Aristotle (*Metaph.* 1060 b 15) that a "division" (διαίρεσις) or "section" (τομή) of a solid or body is also a surface, or that the common boundary at which two parts of a solid fit together (*Categories* 6, 5 a 2) may be a surface.

Proclus discusses how the fact stated in Def. 6 can be said to be true of surfaces like that of the sphere "which is bounded (πεπέρασται), it is true, but not by lines." His explanation (p. 116, 8—14) is that, "if we take the surface (of a sphere), so far as it is extended two ways (διχῇ διαστατή), we shall find that it is bounded by lines as to length and breadth; and if we consider the spherical surface as possessing a form of its own and invested with a fresh quality, we must regard it as having fitted end on to beginning and made the two ends (or extremities) one, being thus one potentially only, and not in actuality."

DEFINITION 7.

Ἐπίπεδος ἐπιφάνειά ἐστιν, ἥτις ἐξ ἴσου ταῖς ἐφ' ἑαυτῆς εὐθείαις κεῖται.

A plane surface *is a surface which lies evenly with the straight lines on itself.*

The Greek follows exactly the definition of a straight line *mutatis mutandis*, i.e. with ταῖς...εὐθείαις for τοῖς...σημείοις. Proclus remarks that, in general, all the definitions of a straight line can be adapted to the plane surface by merely changing the *genus*. Thus, for instance, a plane surface is "a *surface* the middle of which covers the ends" (this being the adaptation of Plato's definition of a straight line). Whether Plato actually gave this as the definition of a plane surface or not, I believe that Euclid's definition of a plane surface as *lying evenly with the straight lines on itself* was intended simply to express the same idea without any implied appeal to vision (just as in the corresponding case of the definition of a straight line).

As already noted under Def. 4, Proclus tries to read into Euclid's definition the Archimedean assumption that "of surfaces which have the same extremities, if those extremities are in a plane, the plane is the least." But, as I have stated, his interpretation of the words seems impossible, although it is adopted by Simplicius also (see an-Nairīzī).

Ancient alternatives.

The other ancient definitions recorded are as follows.

1. *The surface which is stretched to the utmost* (ἐπ' ἄκρον τεταμένη): a definition which Proclus describes as equivalent to Euclid's definition (on Proclus' own view of that definition). Cf. Heron, Def. 9, "(a surface) which is right (and) stretched out" (ὀρθὴ οὖσα ἀποτεταμένη), words which he adds to Euclid's definition.

2. *The least surface among all those which have the same extremities.* Proclus is here (p. 117, 9) obviously quoting the Archimedean *assumption.*

3. *A surface all the parts of which have the property of fitting on* (*each other*) (Heron, Def. 9).

4. *A surface such that a straight line fits on all parts of it* (Proclus, p. 117, 8), or *such that the straight line fits on it all ways*, i.e. however placed (Proclus, p. 117, 20).

With this should be compared:

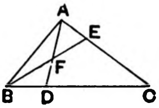
5. "(*A plane surface is*) *such that, if a straight line pass through two points on it, the line coincides wholly with it at every spot, all ways,*" i.e. however placed (one way or the reverse, no matter how), ἧς ἐπειδὰν δύο σημείων ἅψηται εὐθεῖα, καὶ ὅλη αὐτῇ κατὰ πάντα τόπον παντοίως ἐφαρμόζεται (Heron, Def. 9). This appears, with the words κατὰ πάντα τόπον παντοίως omitted, in Theon of Smyrna (p. 112, 5, ed. Hiller), so that it goes back at least as far as the 1st c. A.D. It is of course the same as the definition commonly attributed to Robert Simson, and very widely adopted as a substitute for Euclid's.

This same definition appears also in an-Nairīzī (ed. Curtze, p. 10) who, after quoting Simplicius' explanation (on the same lines as Proclus') of the meaning of Euclid's definition, goes on to say that "others defined the plane surface as that in which it is possible to draw a straight line from any point to any other."

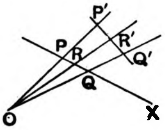
Difficulties in ordinary definitions.

Gauss observed in a letter to Bessel that the definition of a plane surface as *a surface such that, if any two points in it be taken, the straight line joining them lies wholly in the surface* (which, for short, we will call "Simson's" definition) contains more than is necessary, in that a plane can be obtained by simply projecting a straight line lying in it from a point outside the line but also lying on the plane; in fact the definition includes a theorem, or postulate, as well. The same is true of Euclid's definition of a plane as the surface which "lies evenly with (*all*) the straight lines on itself," because it is sufficient for a definition of a plane if the surface "lies evenly" with those lines only which pass through a fixed point on it and each of the several points of a straight line also lying in it but not passing through the point. But from Euclid's point of view it is immaterial whether a definition contains more than the necessary minimum *provided* that the *existence* of a thing possessing all the attributes contained in the definition is afterwards proved. This however is not done in regard to the plane. No proposition about the nature of a plane as such appears before Book XI., although its existence is presupposed in all the geometrical Books I.—IV. and VI.; nor in Book XI. is there any attempt to prove, e.g. by construction, the existence of a surface conforming to the definition. The explanation may be that the existence of the plane as defined was deliberately assumed from the beginning like that of points and lines, the existence of which, according to Aristotle, must be assumed as principles unproved, while the existence of everything else must be proved; and it may well be that Aristotle would have included plane surfaces with points and lines in this statement had it not been that he generally took his illustrations from *plane* geometry (excluding solid).

But, whatever definition of a plane is taken, the evolution of its essential properties is extraordinarily difficult. Crelle, who wrote an elaborate article *Zur Theorie der Ebene* (read in the Academie der Wissenschaften in 1834) of which account must be taken in any full history of the subject, observes that, since the plane is the field, as it were, of almost all the rest of geometry, while a proper conception of it is necessary to enable Eucl. I. 1 to be understood, it might have been expected that the theory of the plane would have been the subject of at least the same amount of attention as, say, that of parallels. This however was far from being the case, perhaps because the subject of parallels (which, for the rest, presuppose the notion of a plane) is *much easier* than that of the plane. The nature of the difficulties as regards the plane have also been pointed out recently by Mr Frankland (*The First Book of Euclid's Elements*, Cambridge, 1905): it would appear that, whatever definition is taken, whether the simplest (as containing the minimum necessary to determine a plane) or the more complex, e.g. Simson's, some postulate has to be assumed in addition before the fundamental properties, or the truth of the other definitions, can be established. Crelle notes the same thing as regards Simson's definition, containing *more* than is necessary. Suppose a plane in which lies the triangle ABC. Let AD join the vertex A to any point D on BC, and BE the vertex B to any point E on CA. Then, according to the definition, AD lies wholly in the plane of the triangle; so does BE. But, if both AD and BE are to lie wholly in the one plane, AD, BE must intersect, say at F: if they did not, there would be two planes in question, not one. But the fact that the lines intersect and that, say, AD does not pass above or below BE, is by no means self-evident.



Mr Frankland points out the similar difficulty as regards the simpler definition of a plane as the surface generated by a straight line passing always through a fixed point and always intersecting a fixed straight line. Let OPP', OQQ' drawn from O intersect the straight line X at P, Q respectively. Let R be any third point on X: then it needs to be proved that OR intersects $P'Q'$ in some point, say R'. Without some postulate, however, it is not easy to see how to prove this, or even to prove that $P'Q'$ intersects X.

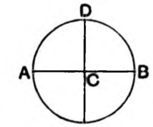


Crelle's essay. Definitions by Fourier, Deahna, Becker.

Crelle takes as the standard of a good definition that it shall be, not only as simple as possible, but also the best adapted for deducing, with the aid of the simplest possible principles, further properties belonging to the thing defined. He was much attracted by a very lucid definition, due, he says, to Fourier, according to which *a plane is formed by the aggregate of all the straight lines which, passing through one point on a straight line in space, are perpendicular to that straight line.* (This is really no more than an adaptation from Euclid's proposition XI. 5, to the effect that, if one of four concurrent straight lines be at right angles to each of the other three, those three are in one plane, which proposition is also used in Aristotle, *Meteorologica* III. 3, 373 a 13.) But Crelle confesses that he had not been able to deduce the necessary properties from this and had had to substitute the definition, already mentioned, of a plane as *the surface containing, throughout their whole length, all the straight lines passing through a fixed point and also intersecting a straight line in space*; and he only claims to have proved, after a long series of propositions, that the "Fourier"- or "perpendicular"-surface and the *plane* of the other definition just given are identical, after which the properties of the "Fourier"-surface can be used along with those of the plane. The advantage of the Fourier definition is that it leads easily, by means of the two propositions that triangles are equal in all respects (1) when two sides and the included angle are respectively equal and (2) when all three sides are respectively equal, to the property expressed in Simson's definition. But Crelle uses to establish these two congruence-theorems a number of propositions about *equal angles*, *supplementary* angles, *right* angles, *greater* and *less* angles; and it is difficult to question the soundness of Schotten's criticism that these notions in themselves really presuppose that of a plane. The difficulty due to Fourier's use of the word "perpendicular," if that were all, could no doubt be got over. Thus Deahna in a dissertation (Marburg, 1837) constructed a plane as follows. Presupposing the notions of a straight line and a sphere, he observes that, if a sphere revolve about a diameter, all the points of its surface which move describe closed curves (circles). Each of these circles, during the revolution, moves along itself, and one of them divides the surface of the sphere into two congruent parts. The aggregate then of the lines joining the centre to the points of this circle forms the *plane*. Again, J. K. Becker (*Die Elemente der Geometrie*, 1877) pointed out that the revolution of a right angle about one side of it produces a conical surface which differs from all other conical surfaces generated by the revolution of other angles in the fact that *the particular cone coincides with the cone vertically opposite to it*: this characteristic might therefore be taken in order to get rid of the use of the *right angle*.

W. Bolyai and Lobachewsky.

Very similar to Deahna's equivalent for Fourier's definition is the device of W. Bolyai and Lobachewsky (described by Frischauf, *Elemente der absoluten Geometrie*, 1876). They worked upon a fundamental idea first suggested, apparently, by Leibniz. Briefly stated, their way of evolving a *plane* and a *straight line* was as follows. Conceive an infinite number of pairs of concentric spheres described about two fixed points in space, O, O', as centres, and with equal radii, gradually increasing: these pairs of equal spherical surfaces intersect respectively in homogeneous curves (circles), and the "Inbegriff" or aggregate of these curves of intersection forms a *plane*. If A be a point on one of these circles (k say), suppose points M, M' to start simultaneously from A and to move in opposite directions *at the same speed* till they meet at B, say: B then is "opposite" to A, and A, B divide the circumference into two equal halves. If the points A, B be held fast and the whole system be turned about them until O takes the place of O', and O' of O, the circle k will occupy the same position as before (though turned a different way). Two opposite points, P, Q say, of each of the other circles will remain stationary during the motion as well as A, B: the "Inbegriff" or aggregate of all such points which remain stationary forms a *straight line*. It is next observed that the *plane* as defined can be generated by the revolution of the straight line about OO', and this suggests the following construction for a plane. Let a circle as one of the curves of intersection of the pairs of spherical surfaces be divided as before into two equal halves at A, B. Let the arc ADB be similarly bisected at D, and let C be the middle point of AB. This determines a straight line CD which is then *defined* as "perpendicular" to AB. The revolution of CD about AB generates a *plane*. The property stated in Simson's definition is then proved by means of the congruence-theorems proved in Eucl. I. 8 and I. 4. The first is taken as proved, practically by considerations of symmetry and homogeneity. If two spherical surfaces, not necessarily equal, with centres O, O' intersect, A and its "opposite" point B are taken as

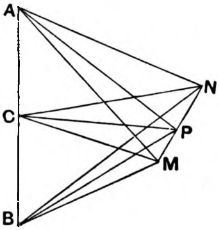


before on the curve of intersection (a circle) and, relatively to OO', the point A is taken to be convertible with B or any other point on the homogeneous curve. The second (that of Eucl. I. 4) is established by simple application. Rausenberger objects to these proofs on the grounds that the first *assumes* that the two spherical surfaces intersect in one single curve, not in several, and that the second compares *angles*: a comparison which, he says, is possible only in a *plane*, so that a plane is really presupposed. Perhaps as regards the particular comparison of angles Rausenberger is hypercritical; but it is difficult to regard the supposed proof of the theorem of Eucl. I. 8 as sufficiently rigorous (quite apart from the use of the uniform *motion* of points for the purpose of bisecting lines).

Simson's property is proved from the two congruence-theorems thus. Suppose that AB is "perpendicular" (as defined by Bolyai) to two generators CM, CN of a plane, or suppose CM, CN respectively to make with AB two angles congruent with one another. It is enough to prove that, if P be any point on the straight line MN, then CP, just as much as CM, CN respectively, makes with AB two angles congruent with one another and is therefore a generator. We prove successively the congruence of the following pairs of triangles:

ACM, BCM
ACN, BCN
AMN, BMN
AMP, BMP
ACP, BCP,

whence the angles ACP, BCP are congruent.



Other views.

Enriques and Amaldi (*Elementi di geometria*, Bologna, 1905), Veronese (in his *Elementi*) and Hilbert all assume as a *postulate* the property stated in Simson's definition. But G. Ingrami (*Elementi di geometria*, Bologna, 1904) proves it in the course of a remarkable series of closely argued proposition based upon a much less comprehensive postulate. He evolves the theory of the plane from that of a triangle, beginning with a triangle as a mere *three-side* (trilatero), i.e. a *frame*, as it were. His postulate relates to the *three-side* and is to the effect that each "(rectilineal) segment" joining a vertex to a point of the opposite side meets every segment similarly joining each of the other two vertices to the points of the sides opposite to them respectively, and, conversely, if a point be taken on a segment joining a vertex to a point of the opposite side, and if a straight line be drawn from another vertex to the point on the segment so taken, it will if produced meet the opposite side. A *triangle* is then defined as the figure formed by the aggregate of all the segments joining the respective vertices of a *three-side* to points on the opposite sides. After a series of propositions, Ingrami evolves a plane as *the figure formed by the "half straight-lines" which project from an internal point of the triangle the points of the perimeter*, and then, after two more theorems, proves that a plane is determined by any three of its points which are not in a straight line, and that *a straight line which has two points in a plane has all its points in it.*

The argument by which Bolyai and Lobachewsky evolved the plane is of course equivalent to the definition of a plane as *the locus of all points equidistant from two fixed points in space.*

Leibniz in a letter to Giordano defined a plane as *that surface which divides space into two congruent parts*. Adverting to Giordano's criticism that you could conceive of surfaces and lines which divided space or a plane into two congruent parts without being *plane* or *straight* respectively, Beez (*Über Euklidische und Nicht-Euklidische Geometrie*, 1888) pointed out that what was wanted to complete the definition was the further condition that the two congruent spaces could be *slid along each other* without the surfaces ceasing to coincide, and claimed priority for his completion of the definition in this way. But the idea of *all the parts* of a plane fitting exactly on *all other parts* is ancient, appearing, as we have seen, in Heron, Def. 9.

DEFINITIONS 8, 9.

8. Ἐπίπεδος δὲ γωνία ἐστὶν ἡ ἐν ἐπιπέδῳ δύο γραμμῶν ἁπτομένων ἀλλήλων καὶ μὴ ἐπ' εὐθείας κειμένων πρὸς ἀλλήλας τῶν γραμμῶν κλίσις.

9. Ὅταν δὲ αἱ περιέχουσαι τὴν γωνίαν γραμμαὶ εὐθεῖαι ὦσιν, εὐθύγραμμος καλεῖται ἡ γωνία.

8. *A* plane angle *is the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line.*

9. *And when the lines containing the angle are straight. the angle is called* rectilineal.

The phrase "not in a *straight line*" is strange, seeing that the definition purports to apply to angles formed by *curves* as well as straight lines. We should rather have expected *continuous* (συνεχής) with one another; and Heron takes this to be the meaning, since he at once adds an explanation as to what is meant by lines not being *continuous* (οὐ συνεχεῖς). It looks as though Euclid really intended to define a *rectilineal* angle, but on second thoughts, as a concession to the then common recognition of curvilineal angles, altered "straight lines" into "lines" and separated the definition into two.

I think all our evidence suggests that Euclid's definition of an angle as *inclination* (κλίσις) was a new departure. The word does not occur in Aristotle; and we should gather from him that the idea generally associated with an angle in his time was rather *deflection* or *breaking* of lines (κλάσις): cf. his common use of κεκλάσθαι and other parts of the verb κλᾶν, and also his reference to *one bent line* forming an angle (τὴν κεκαμμένην καὶ ἔχουσαν γωνίαν, *Metaph.* 1016 a 13)

Proclus has a long and elaborate note on this definition, much of which (pp. 121, 12—126, 6) is apparently taken direct from a work by his master Syrianus (ὁ ἡμέτερος καθηγεμών). Two criticisms contained in the note need occasion no difficulty. One of these asks how, if an angle be an inclination, one inclination can produce two angles. The other (p. 128, 2) is to the effect that the definition seems to exclude an angle formed by one and the same curve with itself, e.g. the complete *cissoid* [at what we call the "cusp"] or the curve known as the *hippopede* (horse-fetter) [shaped like a lemniscate]. But such an "angle" as this belongs to higher geometry, which Euclid may well be excused for leaving out of account in any case.

Other ancient definitions: Apollonius, Plutarch, Carpus.

Proclus' note records other definitions of great interest. Apollonius defined an angle as *a contracting of a surface or a solid at one point under a broken line or surface* (συναγωγὴ ἐπιφανείας ἢ στερεοῦ πρὸς ἑνὶ σημείῳ ὑπὸ κεκλασμένῃ γραμμῇ ἢ ἐπιφανείᾳ), where again an angle is supposed to be formed by *one* broken line or surface. Still more interesting, perhaps, is the definition by "those who say that *the first distance under the point* (τὸ πρῶτον *διάστημα ὑπὸ τὸ σημεῖον*) *is the angle*. Among these is Plutarch, who insists that Apollonius meant the same thing; for, he says, there must be *some* first distance under the breaking (or deflection) of the including lines or surfaces, though, the distance under the point being continuous, it is impossible to obtain the actual *first*, since every distance is divisible without limit" (ἐπ' ἄπειρον). There is some vagueness in the use of the word "distance" (διάστημα); thus it was objected that "if we anyhow separate off the *first*" (*distance* being apparently the word understood) "and draw a straight line *through it*, we get a *triangle* and not one angle." In spite of the objection, I cannot but see in the idea of Plutarch and the others the germ of a valuable conception in infinitesimals, an attempt (though partial and imperfect) to get at the *rate of divergence* between the lines at their point of meeting as a measure of the angle between them.

A third view of an angle was that of Carpus of Antioch, who said "that the angle was a *quantity* (ποσόν), namely a *distance* (διάστημα) between the lines or surfaces containing it. This means that it would be a distance (or divergence) in one *sense* (ἐφ' ἓν διεστώς), although the angle is not on that account a straight line. For it is not everything *extended in one sense* (τὸ ἐφ' ἓν διαστατόν) that is a line." This very phrase "extended one way" being held to define a *line*, it is natural that Carpus' idea should have been described as the greatest possible paradox (πάντων παραδοξότατον). The difficulty seems to have been caused by the want of a different technical term to express a new idea; for Carpus seems undoubtedly to have been anticipating the more modern idea of an angle as representing *divergence* rather than distance, and to have meant by ἐφ' ἓν *in one sense* (*rotationally*) as distinct from *one way* or *in one dimension* (linearly).

To what category does an angle belong?

There was much debate among philosophers as to the particular category (according to the Aristotelian scheme) in which an angle should be placed; is it, namely, a *quantum* (ποσόν), *quale* (ποιόν) or *relation* (πρός τι)?

1. Those who put it in the category of *quantity* argued from the fact that a plane angle is divided by a line and a solid angle by a surface. Since, then, it is a surface which is divided by a line, and a solid which is divided by a surface, they felt obliged to conclude that an angle *is* a surface or a solid, and therefore a magnitude. But homogeneous finite magnitudes, e.g. plane angles, must bear a ratio to one another, or one must be capable of being multiplied until it exceeds the other. This is, however, not the case with a rectilineal angle and the *horn-like* angle (κερατοειδής), by which latter is meant the "angle" between a circle and a tangent to it, since (Eucl. III. 16) the latter "angle" is less than any rectilineal angle whatever. The objection, it will be observed, assumes that the two sorts of angles *are* homogeneous. Plutarch and Carpus are classed among those who, in one way or other, placed an angle among *magnitudes*; and, as above noted, Plutarch claimed Apollonius as a supporter of his view, although the word *contraction* (of a surface or solid) used by the latter does not in itself suggest magnitude much more than Euclid's *inclination*. It was this last consideration which doubtless led "Aganis," the "friend" (socius) apparently of Simplicius, to substitute for Apollonius' wording "*a quantity which has dimensions and the extremities of which arrive at one point*" (an-Nairīzī, p. 13).

2. Eudemus the Peripatetic, who wrote a whole work on the angle, maintained that it belonged to the category of *quality*. Aristotle had given as his fourth variety of *quality* "figure and the shape subsisting in each thing, and,

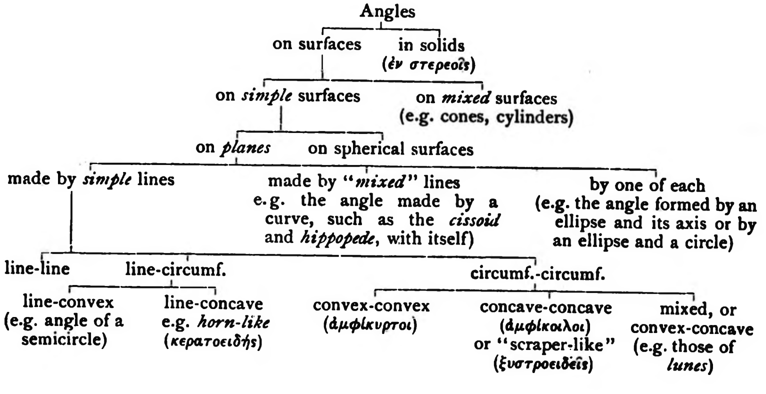
besides these, straightness, curvature, and the like" (*Categories* 8, 10 a 11). He says that each individual thing is spoken of as *quale* in respect of its form, and he instances a triangle and a square, using them again later on (*ibid.* 11 a 5) to show that it is not all qualities which are susceptible of *more* and *less*; again, in *Physics* I. 5, 188 a 25 *angle*, *straight*, *circular* are called kinds of *figure*. Aristotle would no doubt have regarded *deflection* (κεκλάσθαι) as belonging to the same category with straightness and curvature (καμπυλότης). At all events, Eudemus took up an angle as having its origin in the *breaking* or *deflection* (κλάσις) of lines: deflection, he argued, was quality if straightness was, and that which has its origin in quality is itself quality. Objectors to this view argued thus. If an angle be a quality (ποιότης) like heat or cold, how can it be bisected, say? It can in fact be divided; and, if things of which divisibility is an essential attribute are varieties of *quantum* and not qualities, an angle cannot be a quality. Further, the *more* and the *less* are the appropriate attributes of quality, not the equal and the unequal; if therefore an angle were a quality, we should have to say of angles, not that one is greater and another smaller, but that one is more an angle and another less an angle, and that two angles are not unequal but *dissimilar* (ἀνόμοιοι). As a matter of fact, we are told by Simplicius, 538, 21, on Arist. *De caelo* that those who brought the angle under the category of *quale* did call equal angles *similar* angles; and Aristotle himself speaks of *similar* angles in this sense in *De caelo* 296 b 20, 311 b 34.

3. Euclid and all who called an angle an inclination are held by Syrianus to have classed it as a *relation* (πρός τι). Yet Euclid certainly regarded angles as magnitudes; this is clear both from the earliest propositions dealing specifically with angles, e.g. I. 9, 13, and also (though in another way) from his describing an angle in the very next definition and always as *contained* (περιεχομένη) by the two lines forming it (Simon, *Euclid*, p. 28).

Proclus (i.e. in this case Syrianus) adds that the truth lies between these three views. The angle partakes in fact of all those categories: it needs the *quantity* involved in magnitude, thereby becoming susceptible of equality, inequality and the like; it needs the *quality* given it by its *form*, and lastly the *relation* subsisting between the lines or planes bounding it.

Ancient classification of "angles."

An elaborate classification of angles given by Proclus (pp. 126, 7—127, 16) may safely be attributed to Geminus. In order to show it by a diagram it


Angles
on surfaces
in solids
(ἐν στερεοῖς)
on *simple* surfaces
on *mixed* surfaces
(e.g. cones, cylinders)
on *planes*
on spherical surfaces
made by *simple* lines
made by "*mixed*" lines
e.g. the angle made by a curve, such as the *cissoid* and *hippopede*, with itself)
by one of each
(e.g. the angle formed by an ellipse and its axis or by an ellipse and a circle)
line-line
line-circumf.
circumf.-circumf.
line-convex
(e.g. angle of a semicircle)
line-concave
e.g. *horn-like*
(κερατοειδής)
convex-convex
(ἀμφίκυρτοι)
concave-concave
(ἀμφίκοιλοι)
or "scraper-like"
(ξυστροειδεῖς)
mixed, or
convex-concave
(e.g. those of *lunes*)

will be necessary to make a convention about terms. Angles are to be understood under each class, "line-circumference" means an angle contained by a straight line and an arc of a circle, "line-convex" an angle contained by a straight line and a circular arc with convexity *outwards*, and so on in every case.

Definitions of angle classified.

As for the point, straight line, and plane, so for the *angle*, Schotten gives a valuable summary, classification and criticism of the different modern views up to date (*Inhalt und Methode des planimetrischen Unterrichts*, II., 1893, pp. 94—183); and for later developments represented by Veronese reference may be made to the third article (by Amaldi) in *Questioni riguardanti le matematiche elementari*, I. (Bologna, 1912).

With one or two exceptions, says Schotten, the definitions of an angle may be classed in three groups representing generally the following views:

1. *The angle is the difference of direction between two straight lines.* (With this group may be compared Euclid's definition of an angle as an inclination.)

2. *The angle is the quantity or amount (or the measure) of the rotation necessary to bring one of its sides from its own position to that of the other side without its moving out of the plane containing both.*

3. *The angle is the portion of a plane included between two straight lines in the plane which meet in a point (or two rays issuing from the point).*

It is remarkable however that nearly all of the text-books which give definitions different from those in group 2 add to them something pointing to a connexion between an angle and rotation: a striking indication that the essential nature of an angle is closely connected with rotation, and that a good definition must take account of that connexion.

The definitions in the first group must be admitted to be tautologous, or *circular*, inasmuch as they really presuppose some conception of an angle. *Direction* (*as between two given points*) may no doubt be regarded as a primary notion; and it may be defined as "the immediate relation of two points which the *ray* enables us to realise" (Schotten). But "a *direction* is no intensive magnitude, and therefore two directions cannot have any quantitative difference" (Bürklen). Nor is direction susceptible of differences such as those between qualities, e.g. colours. Direction is a *singular* entity: there cannot be different *sorts* or *degrees* of direction. If we speak of "a *different* direction," we use the word equivocally; what we mean is simply "another" direction. The fact is that these definitions of an angle as a difference of direction unconsciously appeal to something outside the notion of direction altogether, to some conception equivalent to that of the angle itself.

Recent Italian views.

The second group of definitions are (says Amaldi) based on the idea of the rotation of a straight line or ray in a plane about a point: an idea which, logically formulated, may lead to a convenient method of introducing the angle. But it must be made independent of *metric* conceptions, or of the conception of *congruence*, so as to bring out *first* the notion of an angle, and *afterwards* the notion of *equal* angles.

The third group of definitions satisfy the condition of not including metric conceptions; but they do not entirely correspond to our intuitive conception of an angle, to which we attribute the character of an entity in *one* dimension (as Veronese says) with respect to the *ray* as element, or an entity in *two* dimensions with reference to *points* as elements, which may be called an *angular sector*. The defect is however easily remedied by considering the angle as "the aggregate of the rays issuing from the vertex and comprised in the angular sector."

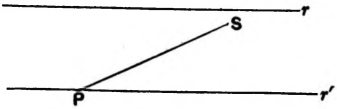
Proceeding to consider the principal methods of arriving at the logical formulation of the first superficial properties of the *plane* from which a definition of the angle may emerge, Amaldi distinguishes two points of view (1) the *genetic*, (2) the *actual*.

(1) From the first point of view we consider the *cluster of straight lines* or *rays* (the aggregate of all the straight lines in a plane passing through a point, or of all the rays with their extremities in that point) as generated by the movement of a straight line or ray in the plane, about a point. This leads to the *postulation* of a *closed order*, or *circular disposition*, of the straight lines or rays in a cluster. Next comes the connexion subsisting between the disposition of any two clusters whatever in one, plane, and so on.

(2) Starting from the point of view of the *actual*, we lay the foundation of the definition of an angle in *the division of the plane into two parts* (half-planes) *by the straight line*. Next, two straight lines (a, b) in the plane, intersecting at a point O, divide the plane into four *regions* which are called *angular sectors* (convex); and finally the *angle* (ab) or (ba) may be defined as *the aggregate of the rays issuing from* O *and belonging to the angular sector which has* a *and* b *for sides*.

Veronese's procedure (in his *Elementi*) is as follows. He begins with the first properties of the plane introduced by the following definition.

The figure given by all the straight lines joining the points of a straight line r to a point P outside it and by the parallel to r through P is called a *cluster of straight lines*, a *cluster of rays*, or a *plane*, according as we consider the *element* of the figure itself to be the *straight line*, the *ray* terminated at P, or a *point*.

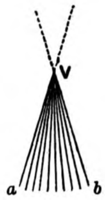


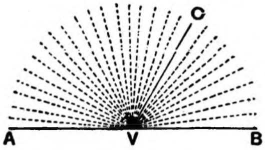
[It will be observed that this method of producing a plane involves using the *parallel* to r. This presents no difficulty to Veronese because he has previously defined parallels, without reference to the plane, by means of *reflex* or *opposite* figures, with respect to a point O: "two straight lines are called *parallel*, if one of them contains two points opposite to (or the reflex of) two points of the other with respect to the middle point of a common transversal (of the two lines)." He proves by means of a postulate that the parallel r' does belong to the plane Pr. Ingrami avoids the use of the parallel by defining a *plane* as "the figure formed by the half straight lines which project from an internal point of a triangle (i.e. a point on a line joining any vertex of a *three-side* to a point of the opposite side) the points of its perimeter," and then defining a *cluster* of rays as "the aggregate of the half straight lines in a plane starting from a given point of the plane and passing through the points of the perimeter of a triangle containing the point."]

Veronese goes on to the definition of an angle. "*We call an* angle *a part of a cluster of rays, bounded by two rays* (*as the segment is a part of a straight line bounded by two points*).

"*An angle of the cluster, the bounding rays of which are opposite, is called a* flat angle."

Then, after a postulate corresponding to postulates which he lays down for a *rectilineal segment* and for a *straight line*, Veronese proves that *all flat angles* are equal to one another.





Hence he concludes that "the cluster of rays is a homogeneous linear system in which the element is the *ray* instead of the *point*. The cluster being a homogeneous linear system, all the propositions deduced from [Veronese's] Post. 1 for the straight line apply to it, e.g. that relative to the sum and difference of the segments: it is only necessary to substitute the ray for the point, and the angle for the segment."

DEFINITIONS 10, 11, 12.

10. Ὅταν δὲ εὐθεῖα ἐπ᾽ εὐθεῖαν σταθεῖσα τὰς ἐφεξῆς γωνίας ἴσας ἀλλήλαις ποιῇ, ὀρθὴ ἑκατέρα τῶν ἴσων γωνιῶν ἐστι, καὶ ἡ ἐφεστηκυῖα εὐθεῖα κάθετος καλεῖται, ἐφ᾽ ἣν ἐφέστηκεν.

11. Ἀμβλεῖα γωνία ἐστὶν ἡ μείζων ὀρθῆς.

12. Ὀξεῖα δὲ ἡ ἐλάσσων ὀρθῆς.

10. *When a straight line set up on a straight line makes the adjacent angles equal to one another, each of the equal angles is* right, *and the straight line standing on the other is called a* perpendicular *to that on which it stands.*

11. *An* obtuse angle *is an angle greater than a right angle.*

12. *An* acute angle *is an angle less than a right angle.*

ἐφεξῆς is the regular term for *adjacent* angles, meaning literally "(next) in order." I do not find the term used in Aristotle of *angles*, but he explains its meaning in such passages as *Physics* VI. 1, 231 b 8: "those things are (next) in order which have nothing of the same kind (συγγενές) between them."

κάθετος, *perpendicular*, means literally *let fall*: the full expression is *perpendicular straight line*, as we see from the enunciation of Eucl. I. 11, and the notion is that of a straight line let fall *upon the surface of the earth*, a *plumb-line*. Proclus (p. 283, 9) tells us that in ancient times the perpendicular was called *gnomon-wise* (κατὰ γνώμονα), because the gnomon (an upright stick) was set up at right angles to the horizon.

The three kinds of angles are among the things which according to the Platonic Socrates (*Republic* VI. 510 C) the geometer assumes and argues from, declining to give any account of them because they are obvious. Aristotle discusses the *priority* of the right angle in comparison with the acute (*Metaph.* 1084 b 7): in one way the right angle is prior, i.e. *in being defined* (ὅτι ὥρισται) and by its *notion* (τῷ λόγῳ), in another way the acute is prior, i.e. as being a *part*, and because the right angle is divided into acute angles; the acute angle is prior as *matter*, the right angle in respect of *form*; cf. also *Metaph.* 1035 b 6, "the notion of the right angle is not divided into that of an acute angle, but the reverse; for, when defining an acute angle, you make use of the right angle." Proclus (p. 133, 15) observes that it is by the *perpendicular* that we measure the heights of figures, and that it is by reference to the right angle that we distinguish the other rectilineal angles, which are otherwise undistinguished the one from the other.

The Aristotelian *Problems* (16, 4, 913 b 36) contain an expression perhaps worth quoting. The question discussed is why things which fall on the ground and rebound make "similar" angles with the surface on both sides of the point of impact; and it is observed that "the right angle is the *limit* (ὅρος) of the opposite angles," where however "opposite" seems to mean, not "supplementary" (or acute and obtuse), but the equal angles made with the surface on opposite sides of the perpendicular.

Proclus, after his manner, remarks that the statement that an angle less than a right angle is acute is not true without qualification, for (1) the *horn-like* angle (between the circumference of a circle and a tangent) is less than a right angle, since it is less than an *acute* angle, but is not an acute angle, while (2) the "angle of a semicircle" (between the arc and a diameter) is also less than a right angle, but is not an acute angle.

The *existence* of the right angle is of course proved in I. 11.

DEFINITION 13.

Ὅρος ἐστίν, ὅ τινός ἐστι πέρας.

A boundary *is that which is an extremity of anything.*

Aristotle also uses the words ὅρος and πέρας as synonymous. Cf. *De gen. animal.* II. 6, 745 a 6, 9, where in the expression "limit of magnitude" first one and then the other word is used.

Proclus (p. 136, 8) remarks that the word boundary is appropriate to the origin of geometry, which began from the measurement of areas of ground and involved the marking of boundaries.

DEFINITION 14.

Σχῆμά ἐστι τὸ ὑπό τινος ἤ τινων ὅρων περιεχόμενον.

A figure *is that which is contained by any boundary or boundaries.*

Plato in the *Meno* observes that *roundness* (στρογγυλότης) or the *round* is a "figure," and that *the straight* and many other things are so too; he then inquires what there is common to all of them, in virtue of which we apply the term "figure" to them. His answer is (76 A): "with reference to every figure I say that *that in which the solid terminates* (τοῦτο, εἰς ὅ τὸ στερεὸν περαίνει) *is a figure*, or, to put it briefly, *a figure is an extremity of a solid.*" The first observation is similar to Aristotle's in the *Physics* I. 5, 188 a 25, where *angle*, *straight*, and *circular* are mentioned as genera of figure. In the *Categories* 8, 10 a 11, "figure" is placed with straightness and curvedness in the category of quality. Here however "figure" appears to mean *shape* (μορφή) rather than "figure" in our sense. Coming nearer to "figure" in our sense, Aristotle admits that figure is "*a sort of* magnitude" (*De anima* III. 1, 425 a 18), and he distinguishes *plane figures* of two kinds, in language not unlike Euclid's, as *contained* by straight and circular lines respectively: "every plane figure is either rectilineal or formed by circular lines (περιφερόγραμμον), and the rectilineal figure is contained by several lines, the circular by one line" (*De caelo* II. 4, 286 b 13). He is careful to explain that a plane is not a figure, nor a figure a plane, but that a plane figure constitutes one notion and is a *species* of the *genus* figure (*Anal. post.* II. 3, 90 b 37). Aristotle does not attempt to define figure in general, in fact he says it would be useless: "From this it is clear that there is one definition of soul in the same way as there is one definition of *figure*; for in the one case there is no figure except the triangle, quadrilateral, and so on, nor is there any soul other than those above mentioned. A definition might be constructed which should apply to all figures but not specially to any particular figure, and similarly with the species of soul referred to. [But such a general definition would serve no purpose.] Hence it is absurd here as elsewhere to seek a general definition which will not be properly a definition of anything in existence and will not be applicable to the particular irreducible species before us, to the neglect of the definition which is so applicable" (*De anima* II. 3, 414 b 20—28).

Comparing Euclid's definition with the above, we observe that by introducing *boundary* (ὅρος) he at once excludes the *straight* which Aristotle classed as figure; he doubtless excluded *angle* also, as we may judge by (1) Heron's statement that "neither one nor two straight lines can complete a figure," (2) the alternative definition of a straight line as "that which cannot with another line of the same species form a figure," (3) Geminus' distinction between the line which *forms a figure* (σχηματοποιοῦσα) and the line which *extends indefinitely* (ἐπ' ἄπειρον ἐκβαλλομένη), which latter term includes a hyperbola and a parabola. Instead of calling figure an *extremity* as Plato did in the expression "extremity (or limit) of a solid," Euclid describes a figure as *that which has* a boundary or boundaries. And lastly, in spite of Aristotle's objection, he does attempt a general definition to cover all kinds of figure, solid and plane. It appears certain therefore that Euclid's definition is entirely his own.

Another view of a figure, recalling that of Plato in *Meno* 76 A, is attributed by Proclus (p. 143, 8) to Posidonius. The latter regarded the *figure* as the *confining extremity* or *limit* (πέρας συγκλεῖον), "separating the notion of figure from *quantity* (or magnitude) and making it the cause of *definition*, *limitation*, and *inclusion* (τοῦ ὡρίσθαι καὶ πεπεράσθαι καὶ τῆς περιοχῆς)...Posidonius thus seems to have in view only the boundary placed round from outside, Euclid the whole content, so that Euclid will speak of the circle as a figure in respect of its whole plane (surface) and of its inclusion (from) without, whereas Posidonius (makes it a figure) in respect of its circumference...Posidonius wished to explain the notion of figure as itself *limiting* and *confining* magnitude."

Proclus observes that a logical and refining critic might object to Euclid's definition as defining the genus from the species, since that which is enclosed by one boundary and that which is enclosed by several are both species of figure. The best answer to this seems to be supplied by the passage of Aristotle's *De anima* quoted above.

DEFINITIONS 15, 16.

15. Κύκλος ἐστὶ σχῆμα ἐπίπεδον ὑπὸ μιᾶς γραμμῆς περιεχόμενον [ἣ καλεῖται περιφέρεια], πρὸς ἣν ἀφ' ἑνὸς σημείου τῶν ἐντὸς τοῦ σχήματος κειμένων πᾶσαι αἱ προσπίπτουσαι εὐθεῖαι [πρὸς τὴν τοῦ κύκλου περιφέρειαν] ἴσαι ἀλλήλαις εἰσίν.

16. Κέντρον δὲ τοῦ κύκλου τὸ σημεῖον καλεῖται.

15. *A* circle *is a plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure are equal to one another;*

16. *And the point is called the* centre *of the circle.*

The words *ἣ καλεῖται περιφέρεια*, "which is called the circumference," and *πρὸς τὴν τοῦ κύκλου περιφέρειαν*, "to the circumference of the circle," are bracketed by Heiberg because, although the MSS. have them, they are omitted in other ancient sources, viz. Proclus, Taurus, Sextus Empiricus and Boethius, and Heron also omits the second gloss. The recently discovered papyrus Herculanensis No. 1061 also quotes the definition without the words in question, confirming Heiberg's rejection of them (see Heiberg in *Hermes* XXXVIII., 1903, p. 47). The words were doubtless added in view of the occurrence of the word "circumference" in Deff. 17, 18 immediately following, without any explanation. But no explanation was needed. Though the word *περιφέρεια* does not occur in Plato, Aristotle uses it several times (1) in the general sense of *contour* without any special mathematical signification, (2) mathematically, with reference to the rainbow and the circumference, as well as an arc, of a circle. Hence Euclid was perfectly justified in employing the word in Deff. 17, 18 and elsewhere, but leaving it undefined as being a word universally understood and not involving in itself any mathematical conception. It may be added that an-Nairīzī had not the bracketed words in his text; for he comments on and tries to explain Euclid's omission to define the circumference.

The definition itself contained nothing new in substance. Plato (*Parmenides* 137 E) says: "*Round* is, I take it, that the extremes of which are every way equally distant from the middle" (*στρογγύλον γέ πού ἐστι τοῦτο, οὗ ἂν τὰ ἔσχατα πανταχῇ ἀπὸ τοῦ μέσου ἴσον ἀπέχῃ*). In Aristotle we find the following expressions: "the circular (*περιφερόγραμμον*) plane figure bounded by one line" (*De caelo* II. 4, 286 b 13—16); "the plane equal (i.e. extending equally all ways) from the middle" (*ἐπίπεδον τὸ ἐκ τοῦ μέσου ἴσον*), meaning a circle (*Rhetoric* III. 6, 1407 b 27); he also contrasts with the circle "any other figure which has not the lines from the middle equal, as for example an egg-shaped figure" (*De caelo* II. 4, 287 a 19). The word "centre" (*κέντρον*) was also regularly used: cf. Proclus' quotation from the "oracles" (*λόγια*), "the centre from which all (lines extending) as far as the rim are equal."

The definition as it stands has no *genetic* character. It says nothing as to the existence or non-existence of the thing defined or as to the method of constructing it. It simply explains what is meant by the word "circle," and is a provisional definition which cannot be used until the existence of circles is proved or assumed. Generally, in such a case, existence is proved by actual construction; but here the possibility of constructing the circle as defined, and consequently its existence, are *postulated* (Postulate 3). A *genetic* definition might state that a circle is the figure described when a straight line, always remaining in one plane, moves about one extremity as a fixed point until it returns to its first position (so Heron, Def. 27).

Simplicius indeed, who points out that the distance between the feet of a pair of compasses is a straight line from the centre to the circumference, will have it that Euclid intended by this definition to show how to construct a circle by the revolution of a straight line about one end as centre; and an-Nairīzī points to this as the explanation (1) of Euclid's definition of a circle as a *plane figure*, meaning the whole surface bounded by the circumference, and not the circumference itself, and (2) of his omission to mention the "circumference," since with this construction the circumference is not drawn separately as a *line*. But it is not necessary to suppose that Euclid himself did more than follow the traditional view; for the same conception of the circle as a *plane figure* appears, as we have seen, in Aristotle. While, however, Euclid is generally careful to say the "*circumference* of a circle" when he means the circumference, or an arc, only, there are cases where "circle" means "circumference of a circle," e.g. in III. 10: "A circle does not cut a circle in more points than two."

Heron, Proclus and Simplicius are all careful to point out that the centre is not the only point which is equidistant from all points of the circumference. The centre is the only point *in the plane of the circle* ("lying within the figure," as Euclid says) of which this is true; any point not in the same plane which is equidistant from all points of the circumference is a *pole*. If you set up a "gnomon" (an upright stick) at the centre of a circle (i.e. a line through the centre perpendicular to the plane of the circle), its upper extremity is a pole (Proclus, p. 153, 3); the perpendicular is the locus of all such poles.

DEFINITION 17.

Διάμετρος δὲ τοῦ κύκλου ἐστὶν εὐθεῖά τις διὰ τοῦ κέντρου ἠγμένη καὶ περατουμένη ἐφ' ἑκάτερα τὰ μέρη ὑπὸ τῆς τοῦ κύκλου περιφερείας, ἥτις καὶ δίχα τέμνει τὸν κύκλον.

A diameter *of the circle is any straight line drawn through the centre and terminated in both directions by the circumference of the circle, and such a straight line also bisects the circle.*

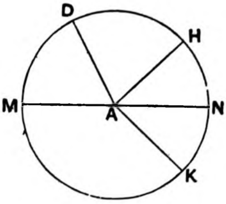
The last words, literally "which (straight line) also bisects the circle," are omitted by Simson and the editors who followed him. But they are necessary even though they do not "belong to the definition" but only express a property of the diameter as defined. For, without this explanation, Euclid would not have been justified in describing as a *semi*-circle a portion of a circle bounded by a diameter and the circumference cut off by it.

Simplicius observes that the *diameter* is so called because it passes *through* the whole surface of a circle as if *measuring* it, and also because it divides the circle into two equal parts. He might however have added that, in general, it is a line passing through a figure where it is *widest*, as well as dividing it equally: thus in Aristotle τὰ κατὰ διάμετρον κείμενα, "things diametrically situated" in space, are at their maximum distance apart. *Diameter* was the regular word in Euclid and elsewhere for the diameter of a *square*, and also of a parallelogram; *diagonal* (διαγώνιος) was a later term, defined by Heron (Def. 67) as the straight line drawn from an angle to an angle.

Proclus (p. 157, 10) says that Thales was the first to prove that a circle is bisected by its diameter; but we are not told how he proved it. Proclus gives as the *reason* of the property "the undeviating course of the straight line through the centre" (a simple appeal to symmetry), but adds that, if it is desired to prove it mathematically, it is only necessary to imagine the diameter drawn and one part of the circle applied to the other; it is then clear that they must coincide, for, if they did not, and one fell inside or outside the other, the straight lines from the centre to the circumference would not all be equal: which is absurd.

Saccheri's proof is worth quoting. It depends on three "Lemmas" immediately preceding, (1) that two straight lines cannot enclose a space, (2) that two straight lines cannot have one and the same segment common, (3) that, if two straight lines meet at a point, they do not touch, but cut one another, at it.

"Let *MDHNKM* be a circle, *A* its centre, *MN* a diameter. Suppose the portion *MNKM* of the circle turned about the fixed points *M*, *N*, so that it ultimately comes near to or coincides with the remaining portion *MNHDM*.



"Then (i) the whole diameter *MAN*, with all its points, clearly remains in the same position, since otherwise two straight lines would enclose a space (contrary to the first Lemma).

"(ii) Clearly no point *K* of the circumference *NKM* falls within or outside the surface enclosed by the diameter *MAN* and the other part, *NHDM*, of the circumference, since otherwise, contrary to the nature of the circle, a radius as *AK* would be less or greater than another radius as *AH*.

"(iii) Any radius *MA* can clearly be rectilineally produced only along a single other radius *AN*, since otherwise (contrary to the second Lemma) two lines assumed straight, e.g. *MAN*, *MAH*, would have one and the same common segment.

"(iv) All diameters of the circle obviously cut one another in the centre (Lemma 3 preceding), and they bisect one another there, by the general properties of the circle.

"From all this it is manifest that the diameter *MAN* divides its circle and the circumference of it just exactly into two equal parts, and the same may be generally asserted for every diameter whatsoever of the same circle; which was to be proved."

Simson observes that the property is easily deduced from III. 31 and 24; for it follows from III. 31 that the two parts of the circle are "similar segments" of a circle (segments containing equal angles, III. Def. 11), and from III. 24 that they are equal to one another.

DEFINITION 18.

Ἡμικύκλιον δέ ἐστι τὸ περιεχόμενον σχῆμα ὑπό τε τῆς διαμέτρου καὶ τῆς ἀπολαμβανομένης ὑπ' αὐτῆς περιφερείας. κέντρον δὲ τοῦ ἡμικυκλίου τὸ αὐτό, ὃ καὶ τοῦ κύκλου ἐστίν.

A semicircle *is the figure contained by the diameter and the circumference cut off by it. And the centre of the semicircle is the same as that of the circle.*

The last words, "And the centre of the semicircle is the same as that of the circle," are added from Proclus to the definition as it appears in the MSS. Scarburgh remarks that a semicircle has no centre, properly speaking, and thinks that the words are not Euclid's, but only a note by Proclus. I am however inclined to think that they are genuine, if only because of the very futility of an observation added by Proclus. He explains, namely, that the semicircle is the only plane figure that has its centre on its perimeter (!), "so that you may conclude that the centre has three positions, since it may be within the figure, as in the case of a circle, or on the perimeter, as with the semicircle, or outside, as with some conic lines (the single-branch hyperbola presumably)"!

Proclus and Simplicius point out that, in the order adopted by Euclid for these definitions of figures, the first figure taken is that bounded by *one* line (the circle), then follows that bounded by *two* lines (the semicircle), then the triangle, bounded by *three* lines, and so on. Proclus, as usual, distinguishes different kinds of figures bounded by two lines (pp. 159, 14—160, 9). Thus they may be formed

(1) by circumference and circumference, e.g. (*a*) those forming angles, as a *lune* (τὸ μηνοειδές) and the figure included by two arcs with convexities outward, and (*b*) the *angle-less* (ἀγώνιον), as the figure included between two concentric circles (the *coronal*);

(2) by circumference and straight line, e.g. the semicircle or segments of circles (ἁψῖδες is a name given to those less than a semicircle);

(3) by "mixed" line and "mixed" line, e.g. two ellipses cutting one another;

(4) by "mixed" line and circumference, e.g. intersecting ellipse and circle;

(5) by "mixed" line and straight line, e.g. half an ellipse.

Following Def. 18 in the MSS. is a definition of a *segment of a circle* which was obviously interpolated from III. Def. 6. Proclus, Martianus Capella and Boethius do not give it in this place, and it is therefore properly omitted.

DEFINITIONS 19, 20, 21.

19. Σχήματα εὐθύγραμμά ἐστι τὰ ὑπὸ εὐθειῶν περιεχόμενα, τρίπλευρα μὲν τὰ ὑπὸ τριῶν, τετράπλευρα δὲ τὰ ὑπὸ τεσσάρων, πολύπλευρα δὲ τὰ ὑπὸ πλειόνων ἢ τεσσάρων εὐθειῶν περιεχόμενα.

20. Τῶν δὲ τριπλεύρων σχημάτων ἰσόπλευρον μὲν τρίγωνόν ἐστι τὸ τὰς τρεῖς ἴσας ἔχον πλευράς, ἰσοσκελὲς δὲ τὸ τὰς δύο μόνας ἴσας ἔχον πλευράς, σκαληνὸν δὲ τὸ τὰς τρεῖς ἀνίσους ἔχον πλευράς.

21. Ἔτι δὲ τῶν τριπλεύρων σχημάτων ὀρθογώνιον μὲν τρίγωνόν ἐστι τὸ ἔχον ὀρθὴν γωνίαν, ἀμβλυγώνιον δὲ τὸ ἔχον ἀμβλεῖαν γωνίαν, ὀξυγώνιον δὲ τὸ τὰς τρεῖς ὀξείας ἔχον γωνίας.

19. Rectilineal figures *are those which are contained by straight lines*, trilateral *figures being those contained by three*, quadrilateral *those contained by four, and* multilateral *those contained by more than four straight lines.*

20. *Of trilateral figures, an* equilateral triangle *is that which has its three sides equal, an* isosceles triangle *that which has two of its sides alone equal, and a* scalene triangle *that which has its three sides unequal.*

21. *Further, of trilateral figures, a* right-angled triangle *is that which has a right angle, an* obtuse-angled triangle *that which has an obtuse angle, and an* acute-angled triangle *that which has its three angles acute.*

19.

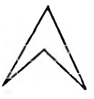
The latter part of this definition, distinguishing *three-sided, four-sided* and *many-sided* figures, is probably due to Euclid himself, since the words τρίπλευρον, τετράπλευρον and πολύπλευρον do not appear in Plato or Aristotle (only in one passage of the *Mechanics* and of the *Problems* respectively does even τετράπλευρον, *quadrilateral*, occur). By his use of τετράπλευρον, quadrilateral, Euclid seems practically to have put an end to any ambiguity in the use by mathematicians of the word τετράγωνον, literally "four-angled (figure)," and to have got it restricted to the *square*. Cf. note on Def. 22.

20.

Isosceles (ἰσοσκελής, with equal legs) is used by Plato as well as Aristotle. *Scalene* (σκαληνός, with the variant σκαληνής) is used by Aristotle of a triangle with no two sides equal: cf. also Tim. Locr. 98 B. Plato, *Euthyphro* 12 D, applies the term "scalene" to an *odd* number in contrast to "isosceles" used of an even number. Proclus (p. 168, 24) seems to connect it with σκάζω, *to limp*; others make it akin to σκολιός, *crooked*, *aslant*. Apollonius uses the same word "scalene" of an *oblique* circular cone.

Triangles are classified, first with reference to their sides, and then with reference to their angles. Proclus points out that seven distinct species of triangles emerge: (1) the *equilateral* triangle, (2) three species of *isosceles* triangles, the right-angled, the obtuse-angled and the acute-angled, (3) the same three varieties of *scalene* triangles.

Proclus gives an odd reason for the dual classification according to sides and angles, namely that Euclid was mindful of the fact that it is not every *triangle* that is *trilateral* also. He explains this statement by reference (p. 165, 22) to a figure which some called *barb-like* (ἀκιδοειδής) while Zenodorus called it *hollow-angled* (κοιλογώνιος). Proclus mentions it again in his note on I. 22 (p. 328, 21 sqq.) as one of the paradoxes of geometry, observing that it is seen in the figure of that proposition. This "triangle" is merely a *quadrilateral* with a re-entrant angle; and the idea that it has only three angles is due to the non-recognition of the fourth angle (which is greater than two right angles) as being an angle at all. Since Proclus speaks of the *four-sided triangle* as "one of the paradoxes in geometry," it is perhaps not safe to assume that the misconception underlying the expression existed in the mind of Proclus alone; but there does not seem to be any evidence that Zenodorus called the figure in question a triangle (cf. Pappus, ed. Hultsch, pp. 1154, 1206).



DEFINITION 22.

Τῶν δὲ τετραπλεύρων σχημάτων τετράγωνον μέν ἐστιν, ὃ ἰσόπλευρόν τέ ἐστι καὶ ὀρθογώνιον, ἑτερόμηκες δέ, ὃ ὀρθογώνιον μέν, οὐκ ἰσόπλευρον δέ, ῥόμβος δέ, ὃ ἰσόπλευρον μέν, οὐκ ὀρθογώνιον δέ, ῥομβοειδὲς δὲ τὸ τὰς ἀπεναντίον πλευράς τε καὶ γωνίας ἴσας ἀλλήλαις ἔχον, ὃ οὔτε ἰσόπλευρόν ἐστιν οὔτε ὀρθογώνιον· τὰ δὲ παρὰ ταῦτα τετράπλευρα τραπέζια καλείσθω.

Of quadrilateral figures, a square *is that which is both equilateral and right-angled; an* oblong *that which is right-angled but not equilateral; a* rhombus *that which is equilateral but not right-angled; and a* rhomboid *that which has its opposite sides and angles equal to one another but is neither equilateral nor right-angled. And let quadrilaterals other than these be called* trapezia.

τετράγωνον was already a *square* with the Pythagoreans (cf. Aristotle, *Metaph.* 986 a 26), and it is so most commonly in Aristotle; but in *De anima* II. 3, 414 b 31 it seems to be a quadrilateral, and in *Metaph.* 1054 b 2, "equal and equiangular τετράγωνα," it cannot be anything else but quadrilateral if "equiangular" is to have any sense. Though, by introducing τετράπλευρον for any quadrilateral, Euclid enabled ambiguity to be avoided, there seem to be traces of the older vague use of τετράγωνον in much later writers. Thus Heron (Def. 100) speaks of a cube as "contained by six equilateral and *equiangular* τετράγωνα" and Proclus (p. 166, 10) adds to his remark about the "four-sided triangle" that "you might have τετράγωνα with more than the four sides," where τετράγωνα can hardly mean squares.

ἑτερόμηκες, *oblong* (with sides of *different length*), is also a Pythagorean term.

The word *right-angled* (ὀρθογώνιον) as here applied to quadrilaterals must mean *rectangular* (i.e., practically, having all its angles right angles); for, although it is tempting to take the word in the same sense for a square as for a triangle (i.e. "having *one* right angle"), this will not do in the case of the oblong, which, unless it were stated that *three* of its angles are right angles, would not be sufficiently defined.

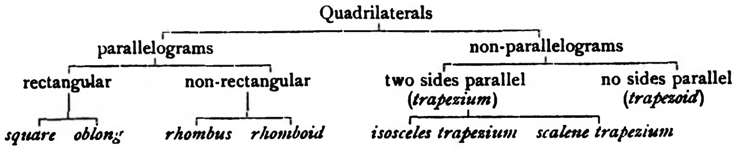
If it be objected, as it was by Todhunter for example, that the definition of a square assumes more than is necessary, since it is sufficient that, being equilateral, it should have one right angle, the answer is that, as in other cases, the superfluity does not matter from Euclid's point of view; on the contrary, the more of the essential attributes of a thing that could be included in its definition the better, provided that the existence of the thing defined and its possession of all those attributes is proved before the definition is actually used; and Euclid does this in the case of the square by construction in I. 46, making no use of the definition before that proposition.

The word *rhombus* (ῥόμβος) is apparently derived from ῥέμβω, to *turn round and round*, and meant among other things a *spinning-top*. Archimedes uses the term *solid rhombus* to denote a solid figure made up of two right cones with a common circular base and vertices turned in opposite directions. We can of course easily imagine this solid generated by *spinning*; and, if the cones were equal, the section through the common axis would be a *plane* rhombus, which would also be the *apparent* form of the spinning solid to the eye. The difficulty in the way of supposing the plane figure to have been named after the solid figure is that in Archimedes the cones forming the solid are not necessarily equal. It is however possible that the solid to which the name was originally given was made up of two equal cones, that the plane rhombus then received its name from that solid, and that Archimedes, in taking up the old name again, extended its signification (cf. J. H. T. Müller, *Beiträge zur Terminologie der griechischen Mathematiker*, 1860, p. 20). Proclus, while he speaks of a rhombus as being like a shaken, i.e. deformed, square, and of a rhomboid as an oblong that has been moved, tries to explain the rhombus by reference to the appearance of a *spinning* square (*τετράγωνον ῥομβούμενον*).

It is true that the definition of a rhomboid says more than is necessary in describing it as having its opposite sides *and angles* equal to one another. The answer to the objection is the same as the answer to the similar objection to the definition of a square.

Euclid makes no use in the *Elements* of the *oblong*, the *rhombus* and the *rhomboid*. The explanation of his inclusion of definitions of these figures is no doubt that they were taken from earlier text-books. From the words "*let* quadrilaterals other than these *be called* trapezia" we may perhaps infer that *trapezium* was a new name or a new application of an old name.

As Euclid has not yet defined parallel lines and does not anywhere define a *parallelogram*, he is not in a position to make the more elaborate classification of quadrilaterals attributed by Proclus to Posidonius and appearing also in Heron's Definitions. It may be shown by the following diagram, distinguishing seven species of quadrilaterals.



It will be observed that, while Euclid in the above definition classes as *trapezia* all quadrilaterals other than squares, oblongs, rhombi, and rhomboids, the word is in this classification restricted to quadrilaterals having two sides (only) parallel, and *trapezoid* is used to denote the rest. Euclid appears to have used *trapezium* in the restricted sense of a quadrilateral with two sides parallel in his book περὶ διαιρέσεων (on divisions of figures). Archimedes uses it in the same sense, but in one place describes it more precisely as a trapezium with its two sides parallel.

DEFINITION 23.

Παράλληλοί εἰσιν εὐθεῖαι, αἵτινες ἐν τῷ αὐτῷ ἐπιπέδῳ οὖσαι καὶ ἐκβαλλόμεναι εἰς ἄπειρον ἐφ' ἑκάτερα τὰ μέρη ἐπὶ μηδέτερα συμπίπτουσιν ἀλλήλαις.

Parallel *straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction.*

Παράλληλος (alongside one another) written in one word does not appear in Plato; but with Aristotle it was already a familiar term.

εἰς ἄπειρον cannot be translated "to infinity" because these words might seem to suggest a *region* or *place* infinitely distant, whereas εἰς ἄπειρον, which seems to be used indifferently with ἐπ' ἄπειρον, is adverbial, meaning "without limit," i.e. "indefinitely." Thus the expression is used of a magnitude being "infinitely divisible," or of a series of terms extending without limit.

In both directions, ἐφ' ἑκάτερα τὰ μέρη, literally "towards both the parts" where "parts" must be used in the sense of "regions" (cf Thuc. II. 96).

It is clear that with Aristotle the general notion of parallels was that of straight lines *which do not meet*, as in Euclid: thus Aristotle discusses the question whether to think that parallels do meet should be called a geometrical or an ungeometrical error (*Anal. post.* I. 12, 77 b 22), and (more interesting still in relation to Euclid) he observes that there is nothing surprising in different hypotheses leading to the same error, as one might conclude that parallels meet by starting from the assumption, either (*a*) that the interior (angle) is greater than the exterior, or (*b*) that the angles of a triangle make up more than two right angles (*Anal. prior.* II. 17, 66 a 11).

Another definition is attributed by Proclus to Posidonius, who said that "*parallel lines are those which,* (*being*) *in one plane, neither converge nor diverge, but have all the perpendiculars equal which are drawn from the points of one line to the other*, while such (straight lines) as make the perpendiculars less and less continually do converge to one another; for the perpendicular is enough to define (ὁρίζειν δύναται) the heights of areas and the distances between lines. For this reason, when the perpendiculars are equal, the distances between the straight lines are equal, but when they become greater and less, the interval is lessened, and the straight lines converge to one another in the direction in which the less perpendiculars are" (Proclus, p. 176, 6—17).

Posidonius' definition, with the explanation as to distances between straight lines, their convergence and divergence, amounts to the definition quoted by Simplicius (an-Nairīzī, p. 25, ed. Curtze) which described straight lines as parallel *if, when they are produced indefinitely both ways, the distance between them, or the perpendicular drawn from either of them to the other, is always equal and not different.* To the objection that it should be *proved* that the distance between two parallel lines is the perpendicular to them Simplicius replies that the definition will do equally well if all mention of the *perpendicular* be omitted and it be merely stated that the *distance* remains equal, although "for *proving* the matter in question it is necessary to say that one straight line is perpendicular to both" (an-Nairīzī, ed. Besthorn-Heiberg, p. 9). He then quotes the definition of "the philosopher Aganis": "*Parallel straight lines are straight lines, situated in the same plane, the distance between which, if they are produced indefinitely in both directions at the same time, is everywhere the same.*" (This definition forms the basis of the attempt of "Aganis" to prove the Postulate of Parallels.) On the definition Simplicius remarks that the words "situated in the same plane" are perhaps unnecessary, since, if the distance between the lines is everywhere the same, and one does not incline at all towards the other, they must for that reason be in the same plane. He adds that the "distance" referred to in the definition is the shortest line which joins things disjoined. Thus, between point and point, the distance is the straight line joining them; between a point and a straight line or between a point and a plane it is the perpendicular drawn from the point to the line or plane; "as regards the distance between two lines, that distance is, if the lines are parallel, one and the same, equal to itself at all places on the lines, it is the *shortest* distance and, at all places on the lines, perpendicular to both" (*ibid.* p. 10).

The same idea occurs in a quotation by Proclus (p. 177, 11) from Geminus. As part of a classification of lines which do not meet he observes: "Of lines which do not meet, some are in one plane with one another, others not. Of those which meet and are in one plane, *some are always the same distance from one another*, others lessen the distance continually, as the hyperbola (approaches) the straight line, and the conchoid the straight line (i.e. the asymptote in each case). For these, while the distance is being continually lessened, are continually (in the position of) not meeting, though they converge to one another; they never converge entirely, and this is the most paradoxical theorem in geometry, since it shows that the convergence of some lines is non-convergent. But of lines which are always an equal distance apart, those which are straight and never make the (distance) between them smaller, and which are in one plane, are parallel."

Thus the *equidistance*-theory of parallels (to which we shall return) is very fully represented in antiquity. I seem also to see traces in Greek writers of a conception equivalent to the vicious *direction*-theory which has been adopted in so many modern text-books. Aristotle has an interesting, though obscure, allusion in *Anal. prior.* II. 16, 65 a 4 to a *petitio principii* committed by "those who think that they draw parallels" (or "establish the theory of parallels," which is a possible translation of τὰς παραλλήλους γράφειν): "for they unconsciously assume such things as it is not possible to demonstrate if parallels do not exist." It is clear from this that there was a vicious circle in the then current theory of parallels; something which depended for its truth on the properties of parallels was assumed in the actual proof of those properties, e.g. that the three angles of a triangle make up two right angles. This is not the case in Euclid, and the passage makes it clear that it was Euclid himself who got rid of the *petitio principii* in earlier text-books by formulating and premising before I. 29 the famous Postulate 5, which must ever be regarded as among the most epoch-making achievements in the domain of geometry. But one of the commentators on Aristotle, Philoponus, has a note on the above passage purporting to give the specific character of the *petitio principii* alluded to; and it is here that a *direction*-theory of parallels may be hinted at, whether Philoponus is or is not right in supposing that this was what Aristotle had in mind. Philoponus says: "The same thing is done by those who draw parallels, namely begging the original question; for they will have it that it is possible to draw parallel straight lines from the meridian circle, and they assume a point, so to say, falling on the plane of that circle and thus they draw the straight lines. And what was sought is thereby assumed; for he who does not admit the genesis of the parallels will not admit the point referred to either." What is meant is, I think, somewhat as follows. Given a straight line and a point through which a parallel to it is to be drawn, we are to suppose the given straight line placed in the plane of the meridian. Then we are told to draw through the given point another straight line in the plane of the meridian (strictly speaking it should be drawn in a plane parallel to the plane of the meridian, but the idea is that, compared with the size of the meridian circle, the distance between the point and the straight line is negligible); and this, as I read Philoponus, is supposed to be equivalent to assuming a very distant point in the meridian plane and joining the given point to it. But obviously no ruler would stretch to such a point, and the objector would say that we cannot really direct a straight line to the assumed distant point except by drawing it, without more ado, *parallel* to the given straight line. And herein is the *petitio principii.* I am confirmed in seeing in Philoponus an allusion to a *direction*-theory by a remark of Schotten on a similar reference to the meridian plane supposed to be used by advocates of that theory. Schotten is arguing that direction is not in itself a conception such that you can predicate *one* direction of *two* different lines. "If any one should reply that nevertheless many lines can be conceived *which all have the direction from north to south,*" he replies that this represents only a nominal, not a real, identity of direction.

Coming now to modern times, we may classify under three groups practically all the different definitions that have been given of parallels (Schotten, *op. cit.* II. p. 188 sqq.).

(1) *Parallel straight lines have no point common,* under which general conception the following varieties of statement may be included:

(*a*) *they do not cut one another,*

(*b*) *they meet at infinity,* or

(*c*) *they have a common point at infinity.*

(2) *Parallel straight lines have the same, or like, direction or directions,* under which class of definitions must be included all those which introduce transversals and say that the parallels *make equal angles with a transversal.*

(3) *Parallel straight lines have the distance between them constant*; with which group we may connect the attempt to explain a parallel as *the geometrical locus of all points which are equidistant from a straight line.*

But the three points of view have a good deal in common; some of them lead easily to the others. Thus the idea of the lines having no point common led to the notion of their having a common point at infinity, through the influence of modern geometry seeking to embrace different cases under one conception; and then again the idea of the lines having a common point at infinity might suggest their having the same direction. The "non-secant" idea would also naturally lead to that of equidistance (3), since our observation shows that it is things which come nearer to one another that tend to meet, and hence, if lines are not to meet, the obvious thing is to see that they shall not come nearer, i.e. shall remain the same distance apart.

We will now take the three groups in order.

(1) The first observation of Schotten is that the varieties of this group which regard parallels as (*a*) meeting at infinity or (*b*) having a common point at infinity (first mentioned apparently by Kepler, 1604, as a "façon de parler" and then used by Desargues, 1639) are at least unsuitable definitions for elementary text-books. How do we know that the lines cut or meet at infinity? We are not entitled to assume either that they do or that they do not, because "infinity" is outside our field of observation and we cannot verify either. As Gauss says (letter to Schumacher), "Finite man cannot claim to be able to regard the infinite as something to be grasped by means of ordinary methods of observation." Steiner, in speaking of the rays passing through a point and successive points of a straight line, observes that as the point of intersection gets further away the ray moves continually in one and the same direction ("nach einer und derselben Richtung hin"); only in one position, that in which it is parallel to the straight line, "there is *no real cutting*" between the ray and the straight line; what we have to say is that the ray is "*directed towards the infinitely distant point on the straight line.*" It is true that higher geometry has to assume that the lines do meet at infinity: whether such lines exist in nature or not does not matter (just as we deal with "straight lines" although there is no such thing as a straight line). But if two lines do not cut at any finite distance, may not the same thing be true at infinity also? Are lines conceivable which would not cut even at infinity but always remain at the same distance from one another even there? Take the case of a line of railway. Must the two rails meet at infinity so that a train could not stand on them there (whether we could *see* it or not makes no difference)? It seems best therefore to leave to higher geometry the conception of infinitely distant points on a line and of two straight lines meeting at infinity, like *imaginary* points of intersection, and, for the purposes of elementary geometry, to rely on the plain distinction between "parallel" and "cutting" which average human intelligence can readily grasp. This is the method adopted by Euclid in his definition, which of course belongs to the group (1) of definitions regarding parallels as non-secant.

It is significant, I think, that such authorities as Ingrami (*Elementi di geometria*, 1904) and Enriques and Amaldi (*Elementi di geometria*, 1905), after all the discussion of principles that has taken place of late years, give definitions of parallels equivalent to Euclid's: "those straight lines in a plane which have not any point in common are called parallels." Hilbert adopts the same point of view. Veronese, it is true, takes a different line. In his great work *Fondamenti di geometria*, 1891, he had taken a ray to be parallel to another when a point at infinity on the second is situated on the first; but he appears to have come to the conclusion that this definition was unsuitable for his *Elementi*. He avoids however giving the Euclidean definition of parallels as "straight lines in a plane which, though produced indefinitely, never meet," because "no one has ever seen two straight lines of this sort," and because the postulate generally used in connexion with this definition is not evident in the way that, in the field of our experience, it is evident that only one straight line can pass through two points. Hence he gives a different definition, for which he claims the advantage that it is independent of the plane. It is based on a definition of figures "opposite to one another with respect to a point" (or *reflex* figures). "Two figures are opposite to one another with respect to a point O, e.g. the figures ABC ... and $A'B'C'$ ..., if to every point of the one there corresponds one sole point of the other, and if the segments

OA, *OB*, *OC*, ... joining the points of one figure to *O* are respectively equal and opposite to the segments *OA'*, *OB'*, *OC'*, ... joining to *O* the corresponding points of the second": then, a *transversal* of two straight lines being any segment having as its extremities one point of one line and one point of the other, "*two straight lines are called parallel if one of them contains two points opposite to two points of the other with respect to the middle point of a common transversal.*" It is true, as Veronese says, that the parallels so defined and the parallels of Euclid are in substance the same; but it can hardly be said that the definition gives as good an idea of the essential nature of parallels as does Euclid's. Veronese has to *prove*, of course, that his parallels have no point in common, and his "Postulate of Parallels" can hardly be called more evident than Euclid's: "If two straight lines are parallel, they are figures opposite to one another with respect to the middle points of all their transversal segments."

(2) The *direction*-theory.

The fallacy of this theory has nowhere been more completely exposed than by C. L. Dodgson (*Euclid and his modern Rivals*, 1879). According to Killing (*Einführung in die Grundlagen der Geometrie*, I. p. 5) it would appear to have originated with no less a person than Leibniz. In the text-books which employ this method the notion of *direction* appears to be regarded as a primary, not a derivative notion, since no definition is given. But we ought at least to know how the same direction or like directions can be recognised when two different straight lines are in question. But no answer to this question is forthcoming. The fact is that the whole idea as applied to non-coincident straight lines is derived from knowledge of the properties of *parallels*; it is a case of explaining a thing by itself. The idea of parallels being in the *same* direction perhaps arose from the conception of an angle as a *difference* of direction (the hollowness of which has already been exposed); sameness of direction for parallels follows from the same "difference of direction" which both exhibit relatively to a third line. But this is not enough. As Gauss said (*Werke*, IV. p. 365), "If it [identity of direction] is recognised by the equality of the angles formed with *one* third straight line, we do not yet know without an antecedent proof whether this same equality will also be found in the angles formed with a *fourth* straight line" (and any number of other transversals); and in order to make this theory of parallels valid, so far from getting rid of axioms such as Euclid's, you would have to assume as an axiom what is much less axiomatic, namely that "straight lines which make equal corresponding angles with a certain transversal do so with *any* transversal" (Dodgson, p. 101).

(3) In modern times the conception of parallels as *equidistant* straight lines was practically adopted by Clavius (the editor of Euclid, born at Bamberg, 1537) and (according to Saccheri) by Borelli (*Euclides restitutus*, 1658) although they do not seem to have *defined* parallels in this way. Saccheri points out that, before such a definition can be used, it has to be *proved* that "the geometrical locus of points equidistant from a straight line is a straight line." To do him justice, Clavius saw this and tried to prove it: he makes out that the locus is a straight line according to the definition of Euclid, because "it lies evenly with respect to all the points on it"; but there is a confusion here, because such "evenness" as the locus has is with respect to the straight line from which its points are equidistant, and there is nothing to show that it possesses this property with respect to itself. In fact the theorem cannot be proved without a postulate.

POSTULATE 1.

Ἠιτήσθω ἀπὸ παντὸς σημείου ἐπὶ πᾶν σημεῖον εὐθεῖαν γραμμὴν ἀγαγεῖν.

Let the following be postulated: to draw a straight line from any point to any point.

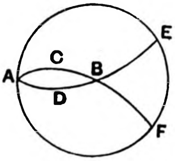
From any point to any point. In general statements of this kind the Greeks did not say, as we do, "*any* point," "*any* triangle" etc., but "*every* point," "*every* triangle" and the like. Thus the words are here literally "from every point to every point." Similarly the first words of Postulate 3 are "with *every* centre and distance," and the enunciation, e.g., of 1. 18 is "In *every* triangle the greater side subtends the greater angle."

It will be remembered that, according to Aristotle, the geometer must in general assume *what* a thing is, or its definition, but must prove *that* it is, i.e. the *existence* of the thing corresponding to the definition: only in the case of the two most primary things, points and lines, does he assume, without proof, both the definition and the existence of the thing defined. Euclid has indeed no separate assumption affirming the existence of *points* such as we find nowadays in text-books like those of Veronese, Ingrami, Enriques, "there exist distinct points" or "there exist an infinite number of points." But, as regards the only lines dealt with in the *Elements*, straight lines and circles, existence is asserted in Postulates 1 and 3 respectively. Postulate 1 however does much more than (1) postulate the existence of straight lines. It is (2) an answer to a possible objector who should say that you cannot, with the imperfect instruments at your disposal, draw a mathematical straight line at all, and consequently (in the words of Aristotle, *Anal. post.* 1. 10, 76 b 41) that the geometer uses false hypotheses, since he calls a line a foot long when it is not or straight when it is not straight. It would seem (if Gherard's translation is right) that an-Nairīzī saw that one purpose of the Postulate was to refute this criticism: "the utility of the first three postulates is (to ensure) that the weakness of our equipment shall not prevent (scientific) demonstration" (ed. Curtze, p. 30). The fact is, as Aristotle says, that the geometer's demonstration is not concerned with the particular imperfect straight line which he has drawn, but with the ideal straight line of which it is the imperfect representation. Simplicius too indicates that the object of the Postulate is rather to enable the drawing of a mathematical straight line to be *imagined* than to assert that it can actually be realised in practice: "he would be a rash person who, taking things as they actually are, should postulate the drawing of a straight line from Aries to Libra."

There is still something more that must be inferred from the Postulate combined with the definition of a straight line, namely (3) that the straight line joining two points is *unique*: in other words that, *if two straight lines* ("rectilineal segments," as Veronese would call them) *have the same extremities, they must coincide throughout their length.* The omission of Euclid to state this in so many words, though he assumes it in 1. 4, is no doubt answerable for the interpolation in the text of the equivalent assumption that *two straight lines cannot enclose a space*, which has constantly appeared in MSS. and editions of Euclid, either among Axioms or Postulates. That Postulate 1 included it, by conscious implication, is even clear from Proclus' words in his note on 1. 4 (p. 239, 16): "therefore two straight lines do not enclose a space, and it was with knowledge of this fact that the writer of the Elements said in the first of his Postulates, *to draw a straight line from any point to any point*, implying that it is *one* straight line which would always join the two points, not *two*."

Proclus attempts in the same note (p. 239) to *prove* that two straight lines cannot enclose a space, using as his basis the definition of the diameter of a circle and the theorem, stated in it, that any diameter divides the circle into two equal parts.

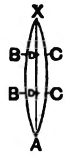
Suppose, he says, *ACB*, *ADB* to be two straight lines enclosing a space. Produce them (beyond *B*) indefinitely. With centre *B* and distance *AB* describe a circle, cutting the lines so produced in *F*, *E* respectively.



Then, since *ACBF*, *ADBE* are both diameters cutting off semi-circles, the arcs *AE*, *AEF* are equal: which is impossible. Therefore etc.

It will be observed, however, that the straight lines produced are assumed to meet the circle given in two *different* points *E*, *F*, whereas, for anything we know, *E*, *F* might coincide and the straight lines have *three* common points. The proof is therefore delusive.

Saccheri gives a different proof. From Euclid's definition of a straight line as that which lies evenly with its points he infers that, when such a line is turned about its two extremities, which remain fixed, all the points on it must remain throughout in the same position, and cannot take up different positions as the revolution proceeds. "In this view of the straight line the truth of the assertion that two straight lines do not enclose a space is obviously involved. In fact, if two lines are given which enclose a space, and of which the two points *A* and *X* are the common extremities, it is easily shown that neither, or else only one, of the two lines is straight."



It is however better to assume as a *postulate* the fact, inseparably connected with the idea of a straight line, that *there exists only one straight line containing two given points*, or, *if two straight lines have two points in common, they coincide throughout.*

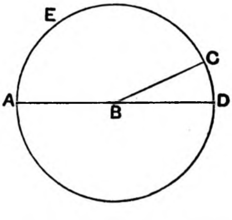
POSTULATE 2.

Καὶ πεπερασμένην εὐθεῖαν κατὰ τὸ συνεχὲς ἐπ' εὐθείας ἐκβαλεῖν.

To produce a finite straight line continuously in a straight line.

I translate πεπερασμένην by *finite*, because that is the received equivalent, and because any alternative word such as *limited*, *terminated*, if applied to a straight line, would equally fail to express what modern Italian geometers aptly call a *rectilineal segment*, that is, a straight line having *two* extremities.

Just as Post. 1 asserting the possibility of drawing a straight line from any one point to another must be held to declare at the same time that the straight line so drawn is unique, so Post. 2 maintaining the possibility of producing a finite straight line (a "rectilineal segment") continuously in a straight line must also be held to assert that the straight line can only be produced *in one way* at either end, or that the produced part in either direction is *unique*; in other words, that *two straight lines cannot have a common segment.* This latter assumption is not expressly appealed to by Euclid until XI. 1. But it is needed at the very beginning of Book I. Proclus (p. 214, 18) says that Zeno of Sidon, an Epicurean, maintained that the very first proposition I. 1 requires it to be admitted that "two straight lines cannot have the same segments"; otherwise *AC*, *BC* might meet before they arrive at *C* and have the rest of their length common, in which case the actual triangle formed by them and *AB* would not be equilateral. The assumption that two straight lines cannot have a common segment is certainly necessary in I. 4, where one side of one triangle is placed on that side of the other triangle which is equal to it, and it is inferred that the two coincide throughout their length: this would by no means follow if two straight lines could have a common segment. Proclus (p. 215, 24), while observing that Post. 2 clearly indicates that the produced portion must be *one*, attempts to prove it, but unsuccessfully. Both he and Simplicius practically use the same argument. Suppose, says Proclus, that the straight lines *AC*, *AD* have *AB* as a common segment. With centre *B* and radius *BA* describe a circle (Post. 3) meeting *AC*, *AD* in *C*, *D*. Then, since *ABC* is a straight line through the centre, *AEC* is a semi-circle. Similarly, *ABD* being a straight line through the centre, *AED* is a semi-circle. Therefore *AEC* is equal to *AED*: which is impossible.



Proclus observes that Zeno would object to this proof as really depending on the assumption that "two circumferences (of circles) cannot have one portion common"; for this, he would say, is assumed in the common proof by superposition of the fact that a circle is bisected by a diameter, since that proof takes it for granted that, if one part of the circumference cut off by the diameter, when applied to the other, does not coincide with it, it must necessarily fall either *entirely* outside or *entirely* inside it, whereas there is nothing to prevent their coinciding, not altogether, but in part only; and, until you really prove the bisection of a circle by its diameter, the above proof is not valid. Posidonius is represented as having derided Zeno for not seeing that the proof of the bisection of a circle by its diameter goes on just as well if the circumferences fail to coincide *in part* only. But the true objection to the proof above given is that the proof of the bisection of a circle by any diameter *itself* assumes that two straight lines cannot have a common segment; for, if we wish to draw the diameter of a circle which has its extremity at a given point of the circumference we have to join the latter point to the centre (Post. 1) and then to *produce* the straight line so drawn till it meets the circle again (Post. 2), and it is necessary for the proof that the produced part shall be *unique*.

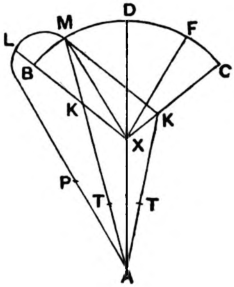
Saccheri adopted the proper order when he gave, first the proposition that two straight lines cannot have a common segment, and after that the proposition that any diameter of a circle bisects the circle and its circumference.

Saccheri's proof of the former is very interesting as showing the thoroughness of his method, if not at the end entirely convincing. It is in five stages which I shall indicate shortly, giving the full argument of the first only.

Suppose, if possible, that *AX* is a common segment of both the straight lines *AXB*, *AXC*, in one plane, produced beyond *X*. Then describe about *X* as centre, with radius *XB* or *XC*, the arc *BMC*, and draw through *X* to any point on it the straight line *XM*.

(i) I maintain that, with the assumption made, *the line* AXM *is also a straight line which is drawn from the point* A *to the point* X *and produced beyond* X.

For, if this line were not straight, we could draw another straight line *AM* which for its part would be straight. This straight line will either (*a*) cut one of the two straight lines *XB*, *XC* in a certain point *K* or (*b*) enclose one of them, for instance *XB*, in the area bounded by *AX*, *XM* and *APLM*.



But the first alternative (*a*) obviously contradicts the foregoing lemma [that two straight lines cannot enclose a space], since in that case the two lines *AXK*, *ATK*, which by hypothesis are straight, would enclose a space.

The second possibility (*b*) is at once seen to involve a similar absurdity. For the straight line *XB* must, when produced beyond *B*, ultimately meet *APLM* in a point *L*. Consequently the two lines *AXBL*, *APL*, which by hypothesis are straight, would again enclose a space. If however we were to assume that the straight line *XB* produced beyond *B* will ultimately meet either the straight line *XM* or the straight line *XA* in another point, we should in the same way arrive at a contradiction.

From this it obviously follows that, on the assumption made, the line *AXM* is itself the straight line which was drawn from the point *A* to the point *M*; and that is what was maintained.

The remaining stages are in substance these.

(ii) *If the straight line* AXB, *regarded as rigid, revolves about* AX *as axis, it cannot assume* two *more positions in the same plane, so that, for example, in one position* XB *should coincide with* XC, *and in the other with* XM.

[This is proved by considerations of symmetry. *AXB* cannot be altogether "similar or equal to" *AXC*, if viewed from the same side (left or right) of both: otherwise they would coincide, which by hypothesis they do not. But there is nothing to prevent *AXB* viewed from one side (say the left) being "similar or equal to" *AXC* viewed from the other side (i.e. the right), so that *AXB can*, without any change, be brought into the position *AXC*.

AXB cannot however take the position of the other straight line *AXM* as well. If they were like on one side, they would coincide; if they were like on opposite sides, *AXM*, *AXC* would be like on the same side and therefore coincide.]

(iii) The other positions of *AXB* during the revolution must be above or below the original plane.

(iv) It is next maintained that *there* is *a point* D *on the arc* BC *such that, if* XD *is drawn*, AXD *is not only a straight line but is such that viewed from the left side it is exactly "similar or equal" to what it is when viewed from the right side.*

[*First*, it is proved that points *M*, *F* can be found on the arc, corresponding in the same way as *B*, *C* do, but nearer together, and of course *AXM*, *AXF* are both straight lines.

Secondly, similar corresponding points can be found still nearer together, and so on continually, until either (*a*) we come to *one* point *D* such that *AXD is exactly like itself when the right and left sides are compared*, or (*b*) there are *two* ultimate points of this sort *M*, *F*, so that *both AXM*, *AXF have this property*.

Thirdly, (*b*) is ruled out by reference to the definition of a straight line.

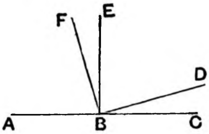
Hence (*a*) only is true, and there is only *one* point *D* such as described.]

(v) Lastly, Saccheri concludes that the straight line *AXD* so determined "is *alone* a straight line, and the *immediate* prolongation from *A* beyond *X* to *D*," relying again on the definition of a straight line as "lying evenly."

Simson deduced the proposition that *two straight lines cannot have a common segment* as a corollary from I. 11; but his argument is a complete *petitio principii*, as shown by Todhunter in his note on that proposition.

Proclus (p. 217, 10) records an ancient proof also based on the proposition I. 11. Zeno, he says, propounded this proof and then criticised it.

Suppose that two straight lines AC, AD have a common segment AB, and let BE be drawn at right angles to AC.



Then the angle EBC is right.

If then the angle EBD is also right, the two angles will be equal: which is impossible.

If the angle EBD is not right, draw BF at right angles to AD; therefore the angle FBA is right.

But the angle EBA is right.

Therefore the angles EBA, FBA are equal: which is impossible.

Zeno objected to this, says Proclus, because it assumed the later proposition I. 11 for its proof. Posidonius said that there was no trace of such a proof to be found in the text-books of Elements, and that it was only invented by Zeno for the purpose of slandering contemporary geometers. Posidonius maintains further that even this proof has something to be said for it. There must be some straight line at right angles to each of the two straight lines AC, AD (the very definition of right angles assumes this): "*suppose then it happens to be the straight line we have set up.*" Here then we have an ancient instance of a defence of *hypothetical construction*, but in such apologetic terms ("it is possible to say *something* even for this proof") that we may conclude that in general it would not have been accepted by geometers of that time as a legitimate means of proving a proposition.

Todhunter proposed to deduce that *two straight lines cannot have a common segment* from I. 13. But this will not serve either, since, as before mentioned, the assumption is really required for I. 4.

It is best to make it a postulate.

POSTULATE 3.

Καὶ παντὶ κέντρῳ καὶ διαστήματι κύκλον γράφεσθαι.

To describe a circle with any centre and distance.

In this case Euclid's text has the passive of the verb: "a circle can be drawn"; Proclus however has the active (γράψαι) as Euclid has in the first two Postulates.

Distance, διαστήματι. This word, meaning "distance" quite generally (cf. Arist. *Metaph.* 1055 a 9 "it is between extremities that distance is greatest," *ibid.* 1056 a 36 "things which have something between them, that is, a certain distance"), and also "distance" in the sense of "dimension" (as in "space has three dimensions, length, breadth and depth," Arist. *Physics* IV. 1, 209 a 4), was the regular word used for describing a circle with a certain *radius*, the idea being that each point of the circumference was at that *distance* from the centre (cf. Arist *Meteorologica* III. 5, 376 b 8: "if a circle be drawn...with distance MΠ"). The Greeks had no word corresponding to *radius*: if they had to express it, they said "(straight lines) drawn from the centre" (αἱ ἐκ τοῦ κέντρου, Eucl. III. Def. 1 and Prop. 26; *Meteorologica* II. 5, 362 b 1 has the full phrase αἱ ἐκ τοῦ κέντρου ἀγόμεναι γραμμαί).

Mr Frankland observes that it would be remarkable if, unlike Postulates 1 and 2, this Postulate implied *merely* what it says, that a circle can be drawn with any centre and distance. We may regard it, if we please, as helping to the complete delineation of the Space which Euclid's geometry is to investigate formally. The Postulate has the effect of removing any restriction upon the size of the circle. It may (1) be indefinitely small, and this implies that space is *continuous*, not discrete, with an irreducible minimum distance between contiguous points in it. (2) The circle may be indefinitely large, which implies the fundamental hypothesis of *infinitude* of space. This last assumed characteristic of space is essential to the proof of I. 16, a theorem not universally valid in a space which is unbounded in extent but finite in size. It would however be unsafe to suppose that Euclid foresaw the use to which his Postulate might thus be put, or formulated it with such an intention.

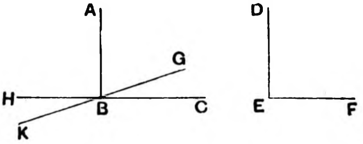
POSTULATE 4.

Καὶ πάσας τὰς ὀρθὰς γωνίας ἴσας ἀλλήλαις εἶναι.

That all right angles are equal to one another.

While this Postulate asserts the essential truth that a right angle is a *determinate magnitude* so that it really serves as an invariable standard by which other (acute and obtuse) angles may be measured, much more than this is implied, as will easily be seen from the following consideration. If the statement is to be *proved*, it can only be proved by the method of applying one pair of right angles to another and so arguing their equality. But this method would not be valid unless on the assumption of the *invariability of figures*, which would therefore have to be asserted as an antecedent postulate. Euclid preferred to assert as a postulate, directly, the fact that all right angles are equal; and hence his postulate must be taken as equivalent to the principle of *invariability of figures* or its equivalent, the *homogeneity of space*.

According to Proclus, Geminus held that this Postulate should not be classed as a postulate but as an axiom, since it does not, like the first three Postulates, assert the possibility of some *construction* but expresses an essential property of right angles. Proclus further observes (p. 188, 8) that it is not a postulate in Aristotle's sense either. (In this I think he is wrong, as explained above.) Proclus himself, while regarding the assumption as axiomatic ("the equality of right angles suggests itself even by virtue of our common notions"), is prepared with a proof, if such is asked for.



Let *ABC*, *DEF* be two right angles.

If they are not equal, one of them must be the greater, say *ABC*.

Then, if we apply *DE* to *AB*, *EF* will fall within *ABC*, as *BG*.

Produce *CB* to *H*. Then, since *ABC* is a right angle, so is *ABH*, and the two angles are equal (a right angle being by definition equal to its adjacent angle).

Therefore the angle *ABH* is *greater* than the angle *ABG*.

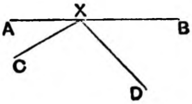
Producing *GB* to *K*, we have similarly the two angles *ABK*, *ABG* both right and equal to one another; whence the angle *ABH* is *less* than the angle *ABG*.

But it is also greater: which is impossible.

Therefore etc.

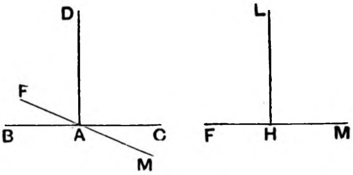
A defect in this proof is the assumption that *CB*, *GB* can each be produced only in one way, and that *BK* falls outside the angle *ABH*.

Saccheri's proof is more careful in that he premises a third lemma in addition to those asserting (1) that two straight lines cannot enclose a space and (2) that two straight lines cannot have a common segment. The third lemma is: *If two straight lines* AB, CXD *meet one another at an intermediate point* X, *they do not* touch *at that point, but* cut one another.



Suppose now that *DA* standing on *BAC* makes the two angles *DAB*, *DAC* equal, so that each is a right angle by the definition.

Similarly, let *LH* form with the straight line *FHM* the right angles *LHF*, *LHM*.



Let *DA*, *HL* be equal; and suppose the whole of the second figure so laid upon the first that the point *H* falls on *A*, and *L* on *D*.

Then the straight line *FHM* will (by the third lemma) not *touch* the straight line *BC* at *A*; it will either

(*a*) coincide exactly with *BC*, or

(*b*) *cut* it so that one of its extremities, as *F*, will fall above [*BC*] and the other, *M*, below it.

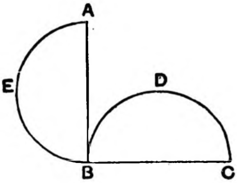
If the alternative (*a*) is true, we have already proved the exact equality of all rectilineal right angles.

Under alternative (*b*) we prove that the angle *LHF*, being equal to the angle *DAF*, is less than the angle *DAB* or *DAC*, and *a fortiori* less than the angle *DAM* or *LHM*: which is contrary to the hypothesis.

[Hence (*a*) is the only possible alternative, so that all right angles are equal.]

Saccheri adds that it makes no difference if the angle *DAF* diverges *infinitely little* from the angle *DAB*. This would equally lead to a conclusion contradicting the hypothesis.

It will be observed that Saccheri speaks of "the exact equality of all *rectilineal* right angles." He may have had in mind the remark of Pappus, quoted by Proclus (p. 189, 11), that the converse of this postulate, namely that an angle which is equal to a right angle is also right, is not necessarily true, unless the former angle is *rectilineal*. Suppose two equal straight lines *BA*, *BC* at right angles to one another, and semi-circles described on *BA*, *BC* respectively as *AEB*, *BDC* in the figure. Then, since the semi-circles are equal, they coincide if applied to one another. Hence the "angles" *EBA*, *DBC* are equal. Add to each the "angle" *ABD*; and it follows that the *lunular* angle *EBD* is equal to the right angle *ABC*. (Similarly, if *BA*, *BC* be inclined at an acute or obtuse angle, instead of at a right angle, we find a *lunular* angle equal to an acute or obtuse angle.) This is one of the curiosities which Greek commentators delighted in.



Veronese, Ingrami, and Enriques and Amaldi deduce the fact that *all right angles are equal* from the equivalent fact that *all flat angles are equal*, which is either itself assumed as a postulate or immediately deduced from some other postulate.

Hilbert takes quite a different line. He considers that Euclid did wrong in placing Post. 4 among "axioms." He himself, after his Group III. of Axioms containing six relating to congruence, proves several theorems about the congruence of triangles and angles, and then deduces our Postulate.

As to the *raison d'être* and the place of Post. 4 one thing is quite certain. It was essential from Euclid's point of view that it should come before Post. 5, since the condition in the latter that a certain pair of angles are together less than two right angles would be useless unless it were first made clear that right angles are angles of determinate and invariable magnitude.

POSTULATE 5.

Καὶ ἐὰν εἰς δύο εὐθείας εὐθεῖα ἐμπίπτουσα τὰς ἐντὸς καὶ ἐπὶ τὰ αὐτὰ μέρη γωνίας δύο ὀρθῶν ἐλάσσονας ποιῇ, ἐκβαλλομένας τὰς δύο εὐθείας ἐπ᾽ ἄπειρον συμπίπτειν, ἐφ᾽ ἃ μέρη εἰσὶν αἱ τῶν δύο ὀρθῶν ἐλάσσονες.

That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

Although Aristotle gives a clear idea of what he understood by a *postulate*, he does not give any instances from geometry; still less has he any allusion recalling the particular postulates found in Euclid. We naturally infer that the formulation of these postulates was Euclid's own work. There is a more positive indication of the originality of Postulate 5, since in the passage (*Anal. prior.* II. 16, 65 a 4) quoted above in the note on the definition of parallels he alludes to some *petitio principii* involved in the theory of parallels current in his time. This reproach was removed by Euclid when he laid down this epoch-making Postulate. When we consider the countless successive attempts made through more than twenty centuries to prove the Postulate, many of them by geometers of ability, we cannot but admire the genius of the man who concluded that such a hypothesis, which he found necessary to the validity of his whole system of geometry, was really indemonstrable.

From the very beginning, as we know from Proclus, the Postulate was attacked as such, and attempts were made to prove it as a theorem or to get rid of it by adopting some other definition of parallels; while in modern times the literature of the subject is enormous. Riccardi (*Saggio di una bibliografia Euclidea*, Part IV., Bologna, 1890) has twenty quarto pages of titles of monographs relating to Post. 5 between the dates 1607 and 1887. Max Simon (*Ueber die Entwicklung der Elementar-geometrie im XIX. Jahrhundert*, 1906) notes that he has seen three new attempts, as late as 1891 (a century after Gauss laid the foundation of non-Euclidean geometry), to prove the theory of parallels independently of the Postulate. Max Simon himself (pp. 53—61) gives a large number of references to books or articles on the subject and refers to the copious information, as to contents as well as names, contained in Schotten's *Inhalt und Methode des planimetrischen Unterrichts*, II. pp. 183—332.

This note will include some account of or allusion to a few of the most noteworthy attempts to prove the Postulate. Only those of ancient times, as being less generally accessible, will be described at any length; shorter references must suffice in the case of the modern geometers who have made the most important contributions to the discussion of the Postulate and have thereby, in particular, contributed most towards the foundation of the non-Euclidean geometries, and here I shall make use principally of the valuable Article 8, *Sulla teoria delle parallele e sulle geometrie non-euclidee* (by Roberto Bonola), in *Questioni riguardanti le matematiche elementari*, I. pp. 247—363.

Proclus (p. 191, 21 sqq.) states very clearly the nature of the first objections taken to the Postulate.

"This ought even to be struck out of the Postulates altogether; for it is a theorem involving many difficulties, which Ptolemy, in a certain book, set himself to solve, and it requires for the demonstration of it a number of definitions as well as theorems. And the converse of it is actually proved by Euclid himself as a theorem. It may be that some would be deceived and would think it proper to place even the assumption in question among the postulates as affording, in the lessening of the two right angles, ground for an instantaneous belief that the straight lines converge and meet. To such as these Geminus correctly replied that we have learned from the very pioneers of this science not to have any regard to mere plausible imaginings when it is a question of the reasonings to be included in our geometrical doctrine. For Aristotle says that it is as justifiable to ask scientific proofs of a rhetorician as to accept mere plausibilities from a geometer; and Simmias is made by Plato to say that he recognises as quacks those who fashion for themselves proofs from probabilities. So in this case the fact that, when the right angles are lessened, the straight lines converge is true and necessary; but the statement that, since they converge more and more as they are produced, they will sometime meet is plausible but not necessary, in the absence of some argument showing that this is true in the case of straight lines. For the fact that some lines exist which approach indefinitely, but yet remain non-secant (*ἀσύμπτωτοι*), although it seems improbable and paradoxical, is nevertheless true and fully ascertained with regard to other species of lines. May not then the same thing be possible in the case of straight lines which happens in the case of the lines referred to? Indeed, until the statement in the Postulate is clinched by proof, the facts shown in the case of other lines may direct our imagination the opposite way. And, though the controversial arguments against the meeting of the straight lines should contain much that is surprising, is there not all the more reason why we should expel from our body of doctrine this merely plausible and unreasoned (hypothesis)?

"It is then clear from this that we must seek a proof of the present theorem, and that it is alien to the special character of postulates. But how it should be proved, and by what sort of arguments the objections taken to it should be removed, we must explain at the point where the writer of the Elements is actually about to recall it and use it as obvious. It will be necessary at that stage to show that its obvious character does not appear independently of proof, but is turned by proof into matter of knowledge."

Before passing to the attempts of Ptolemy and Proclus to prove the Postulate, I should note here that Simplicius says (in an-Nairīzī, ed. Besthorn-Heiberg, p. 119, ed. Curtze, p. 65) that this Postulate is by no means manifest, but requires proof, and accordingly "Abthiniathus" and Diodorus had already proved it by means of many different propositions, while Ptolemy also had explained and proved it, using for the purpose Eucl. I. 13, 15 and 16 (or 18). The Diodorus here mentioned may be the author of the *Analemma* on which Pappus wrote a commentary. It is difficult even to frame a conjecture as to who "Abthiniathus" is. In one place in the Arabic text the name appears to be written "Anthisathus" (H. Suter in *Zeitschrift für Math. und Physik*, xxxviii., hist. litt. Abth. p. 194). It has occurred to me whether he might be Peithon, a friend of Serenus of Antinoeia (Antinoupolis) who was long known as Serenus of *Antissa*. Serenus says (*De sectione cylindri*, ed. Heiberg, p. 96): "Peithon the geometer, explaining parallels in a work of his, was not satisfied with what Euclid said, but showed their nature more cleverly by an example; for he says that parallel straight lines are such a thing as we see on walls or on the ground in the shadows of pillars which are made when either a torch or a lamp is burning behind them. And, although this has only been matter of merriment to every one, I at least must not deride it, for the respect I have for the author, who is my friend." If Peithon was known as "of Antinoeia" or "of Antissa," the two forms of the mysterious name might perhaps be an attempt at an equivalent; but this is no more than a guess.

Simplicius adds in full and word for word the attempt of his "friend" or his "master Aganis" to prove the Postulate.

Proclus returns to the subject (p. 365, 5) in his note on Eucl. I. 29. He says that before his time a certain number of geometers had classed as a theorem this Euclidean postulate and thought it matter for proof, and he then proceeds to give an account of Ptolemy's argument.

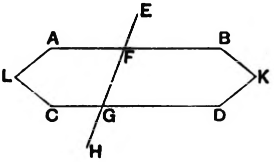
Noteworthy attempts to prove the Postulate.

Ptolemy.

We learn from Proclus (p. 365, 7—11) that Ptolemy wrote a book on the proposition that "straight lines drawn from angles less than two right angles meet if produced," and that he used in his "proof" many of the theorems in Euclid preceding I. 29. Proclus excuses himself from reproducing the early part of Ptolemy's argument, only mentioning as one of the propositions proved in it the theorem of Eucl. I. 28 that, if two straight lines meeting a transversal make the two interior angles on the same side equal to two right angles, the straight lines do not meet, however far produced.

I. From Proclus' note on I. 28 (p. 362, 14 sq.) we know that Ptolemy proved this somewhat as follows.

Suppose that there are two straight lines *AB*, *CD*, and that *EFGH*, meeting them, makes the angles *BFG*, *FGD* equal to two right angles. I say that *AB*, *CD* are parallel, that is, they are non-secant.



For, if possible, let *FB*, *GD* meet at *K*.

Now, since the angles *BFG*, *FGD* are equal to two right angles, while the four angles *AFG*, *BFG*, *FGD*, *FGC* are together equal to four right angles,

the angles *AFG*, *FGC* are equal to two right angles.

"*If therefore* FB, GD, *when the interior angles are equal to two right angles, meet at* K, *the straight lines* FA, GC *will also meet if produced;* for the angles *AFG*, *CGF* are also equal to two right angles.

"Therefore the straight lines will either meet in both directions or in neither direction, if the two pairs of interior angles are both equal to two right angles.

"Let, then, *FA*, *GC* meet at *L*.

"Therefore the straight lines *LABK*, *LCDK* enclose a space: which is impossible.

"Therefore it is not possible for two straight lines to meet when the interior angles are equal to two right angles. Therefore they are parallel."

[The argument in the words italicised would be clearer if it had been shown that the two interior angles on one side of *EH* are *severally* equal to the two interior angles on the other, namely *BFG* to *CGF* and *FGD* to *AFG*; whence, assuming *FB*, *GD* to meet in *K*, we can take the triangle *KFG* and place it (e.g. by rotating it in the plane about *O* the middle point of *FG*) so that *FG* falls where *GF* is in the figure and *GD* falls on *FA*, in which case *FB* must also fall on *GC*; hence, since *FB*, *GD* meet at *K*, *GC* and *FA* must meet at a corresponding point *L*. Or, as Mr Frankland does, we may substitute for *FG* a straight line *MN* through *O* the middle point of *FG* drawn perpendicular to one of the parallels, say *AB*. Then, since the two triangles *OMF*, *ONG* have two angles equal respectively, namely *FOM* to

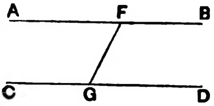
GON (I. 15) and *OFM* to *OGN*, and one side *OF* equal to one side *OG*, the triangles are congruent, the angle *ONG* is a right angle, and *MN* is perpendicular to both *AB* and *CD*. Then, by the same method of application, *MA*, *NC* are shown to form with *MN* a triangle *MALCN* congruent with the triangle *NDKBM*, and *MA*, *NC* meet at a point *L* corresponding to *K*. Thus the two straight lines would meet at the *two* points *K*, *L*. This is what happens under the Riemann hypothesis, where the axiom that two straight lines cannot enclose a space does not hold, but all straight lines meeting in one point have another point common also, and e.g. in the particular figure just used *K*, *L* are points common to all perpendiculars to *MN*. If we suppose that *K*, *L* are not distinct points, but *one* point, the axiom that two straight lines cannot enclose a space is *not* contradicted.]

II. Ptolemy now tries to prove I. 29 without using our Postulate, and then deduces the Postulate from it (Proclus, pp. 365, 14—367, 27).

The argument to prove I. 29 is as follows.

The straight line which cuts the parallels must make the sum of the interior angles on the same side equal to, greater than, or less than, two right angles.

"Let *AB*, *CD* be parallel, and let *FG* meet them. I say (1) that *FG* does not make the interior angles on the same side greater than two right angles.



"For, if the angles *AFG*, *CGF* are greater than two right angles, the remaining angles *BFG*, *DGF* are less than two right angles.

"But the same two angles are also greater than two right angles; *for* AF, CG *are no more parallel than* FB, GD, *so that, if the straight line falling on* AF, CG *makes the interior angles greater than two right angles, the straight line falling on* FB, GD *will also make the interior angles greater than two right angles*.

"But the same angles are also less than two right angles; for the four angles *AFG*, *CGF*, *BFG*, *DGF* are equal to four right angles: which is impossible.

"Similarly (2) we can show that the straight line falling on the parallels does not make the interior angles on the same side less than two right angles.

"But (3), if it makes them neither greater nor less than two right angles, it can only make the interior angles on the same side *equal* to two right angles."

III. Ptolemy deduces Post. 5 thus:

Suppose that the straight lines making angles with a transversal less than two right angles do not meet on the side on which those angles are.

Then, *a fortiori*, they will not meet on the other side on which are the angles *greater* than two right angles.

Hence the straight lines will not meet in either direction; they are therefore *parallel*.

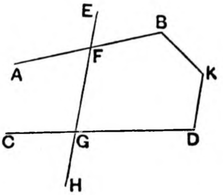
But, if so, the angles made by them with the transversal are equal to two right angles, by the preceding proposition (= I. 29).

Therefore the same angles will be both equal to and less than two right angles:
which is impossible.

Hence the straight lines will meet.

IV. Ptolemy lastly enforces his conclusion that the straight lines will meet *on the side on which are the angles less than two right angles* by recurring to the *a fortiori* step in the foregoing proof.

Let the angles *AFG*, *CGF* in the accompanying figure be together less than two right angles.



Therefore the angles *BFG*, *DGF* are greater than two right angles.

We have proved that the straight lines are not non-secant.

If they meet, they must meet either towards *A*, *C*, or towards *B*, *D*.

(1) Suppose they meet towards *B*, *D*, at *K*.

Then, since the angles *AFG*, *CGF* are less than two right angles, and the angles *AFG*, *GFB* are equal to two right angles, take away the common angle *AFG*, and

the angle *CGF* is less than the angle *BFG*;

that is, the exterior angle of the triangle *KFG* is less than the interior and opposite angle *BFG*:
which is impossible.

Therefore *AB*, *CD* do not meet towards *B*, *D*.

(2) But they do meet, and therefore they must meet in one direction or the other:

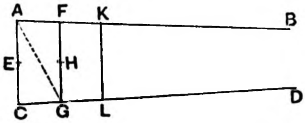
therefore they meet towards *A*, *B*, that is, on the side where are the angles less than two right angles.

The flaw in Ptolemy's argument is of course in the part of his proof of I. 29 which I have italicised. As Proclus says, he is not entitled to assume that, if *AB*, *CD* are parallel, whatever is true of the interior angles on one side of *FG* (i.e. that they are together equal to, greater than, or less than, two right angles) is necessarily true at the same time of the interior angles on the other side. Ptolemy justifies this by saying that *FA*, *GC* are no more parallel in one direction than *FB*, *GD* are in the other: which is equivalent to the assumption that *through any point only one parallel can be drawn to a given straight line.* That is, he assumes an equivalent of the very Postulate he is endeavouring to prove.

Proclus.

Before passing to his own attempt at a proof, Proclus (p. 368, 26 sqq.) examines an ingenious argument (recalling somewhat the famous one about Achilles and the tortoise) which appeared to show that it was *impossible* for the lines described in the Postulate to meet.

Let *AB*, *CD* make with *AC* the angles *BAC*, *ACD* together less than two right angles.



Bisect *AC* at *E* and along *AB*, *CD* respectively measure *AF*, *CG* so that each is equal to *AE*.

Bisect *FG* at *H* and mark off *FK*, *GL* each equal to *FH*; and so on.

Then *AF*, *CG* will not meet at any point on *FG*; for, if that were the case, two sides of a triangle would be together equal to the third: which is impossible.

Similarly, *AB*, *CD* will not meet at any point on *KL*; and "proceeding like this indefinitely, joining the non-coincident points, bisecting the lines so drawn, and cutting off from the straight lines portions equal to the half of these, they say they thereby prove that the straight lines *AB*, *CD* will not meet anywhere."

It is not surprising that Proclus does not succeed in exposing the fallacy here (the fact being that the process will indeed be endless, and yet the straight lines will intersect within a finite distance). But Proclus' criticism contains nevertheless something of value. He says that the argument will prove too much, since we have only to join *AG* in order to see that straight lines making *some* angles which are together less than two right angles do in fact meet, namely *AG*, *CG*. "Therefore it is not possible to assert, without some definite limitation, that the straight lines produced from angles less than two right angles do not meet. On the contrary, it is manifest that *some* straight lines, when produced from angles less than two right angles, do meet, although the argument seems to require it to be proved that this property belongs to *all* such straight lines. For one might say that, the lessening of the two right angles being subject to no limitation, *with such and such an amount of lessening the straight lines remain non-secant, but with an amount of lessening in excess of this they meet* (p. 371, 2—10)."

[Here then we have the germ of such an idea as that worked out by Lobachewsky, namely that the straight lines issuing from a point in a plane can be divided with reference to a straight line lying in that plane into two classes, "secant" and "non-secant," and that we may define as *parallel* the two straight lines which divide the secant from the non-secant class.]

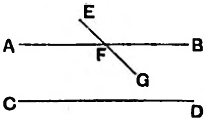
Proclus goes on (p. 371, 10) to base his own argument upon "an axiom such as Aristotle too used in arguing that the universe is finite. For, *if from one point two straight lines forming an angle be produced indefinitely, the distance* (διάστασις, Arist. διάστημα) *between the said straight lines produced indefinitely will exceed any finite magnitude.* Aristotle at all events showed that, if the straight lines drawn from the centre to the circumference are infinite, the interval between them is infinite. For, if it is finite, it is impossible to increase the distance, so that the straight lines (the radii) are not infinite. Hence the straight lines, when produced indefinitely, will be at a distance from one another greater than any assumed finite magnitude."

This is a fair representation of Aristotle's argument in *De caelo* I. 5, 271 b 28, although of course it is not a proof of what Proclus assumes as an axiom.

This being premised, Proclus proceeds (p. 371, 24):

I. "I say that, *if any straight line cuts one of two parallels, it will cut the other also.*

"For let *AB*, *CD* be parallel, and let *EFG* cut *AB*; I say that it will cut *CD* also.

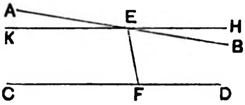


"For, since *BF*, *FG* are two straight lines from one point *F*, they have, when produced indefinitely, a distance greater than any magnitude, so that it will also be greater than the interval between the parallels. Whenever therefore they are at a distance from one another greater than the distance between the parallels, *FG* will cut *CD*.

"Therefore etc."

II. "Having proved this, we shall prove, as a deduction from it, the theorem in question.

"For let AB, CD be two straight lines, and let EF falling on them make the angles BEF, DFE less than two right angles.



"I say that the straight lines will meet on that side on which are the angles less than two right angles.

"For, since the angles BEF, DFE are less than two right angles, let the angle HEB be equal to the excess of two right angles (over them), and let HE be produced to K.

"Since then EF falls on KH, CD and makes the two interior angles HEF, DFE equal to two right angles,

the straight lines HK, CD are parallel.

"And AB cuts KH; therefore it will also cut CD, by what was before shown.

"Therefore AB, CD will meet on that side on which are the angles less than two right angles.

"Hence the theorem is proved."

Clavius criticised this proof on the ground that the axiom from which it starts, taken from Aristotle, itself requires proof. He points out that, just as you cannot assume that two lines which continually approach one another will meet (witness the hyperbola and its asymptote), so you cannot assume that two lines which continually diverge will ultimately be so far apart that a perpendicular from a point on one let fall on the other will be greater than any assigned distance; and he refers to the *conchoid* of Nicomedes, which continually approaches its asymptote, and therefore continually gets farther away from the tangent at the vertex; yet the perpendicular from any point on the curve to that tangent will always be less than the distance between the tangent and the asymptote. Saccheri supports the objection.

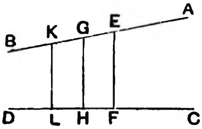
Proclus' first proposition is open to the objection that it assumes that two "parallels" (in the Euclidean sense) or, as we may say, two straight lines which have a common perpendicular, are (not necessarily equidistant, but) so related that, when they are produced indefinitely, the perpendicular from a point of one upon the other remains finite.

This last assumption is incorrect on the hyperbolic hypothesis; the "axiom" taken from Aristotle does not hold on the elliptic hypothesis.

Naṣīraddīn aṭ-Ṭūsī.

The Persian-born editor of Euclid, whose date is 1201—1274, has three lemmas leading up to the final proposition. Their content is substantially as follows, the first lemma being apparently assumed as evident.

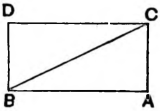
I. (*a*) If AB, CD be two straight lines such that successive perpendiculars, as EF, GH, KL, from points on AB to CD always make with AB unequal angles, which are always acute on the side towards B and always obtuse on the side towards A, then the lines AB, CD, so long as they do not cut, approach continually nearer in the direction of the acute angles and diverge continually in the direction of the obtuse angles, and the perpendiculars diminish towards B, D, and increase towards A, C.



(*b*) Conversely, if the perpendiculars so drawn continually become shorter in the direction of B, D, and longer in the direction of A, C, the straight lines AB, CD approach continually nearer in the direction of B, D and diverge continually in the other direction; also each perpendicular will make with AB two angles one of which is acute and the other is obtuse, and all the acute angles will lie in the direction towards B, D, and the obtuse angles in the opposite direction.

[Saccheri points out that even the first part (*a*) requires proof. As regards the converse (*b*) he asks, why should not the successive acute angles made by the perpendiculars with AB, while remaining acute, become greater and greater as the perpendiculars become smaller until we arrive at last at a perpendicular which is a common perpendicular to *both* lines? If that happens, all the author's efforts are in vain. And, if you are to assume the truth of the statement in the lemma without proof, would it not, as Wallis said, be as easy to assume as axiomatic the statement in Post. 5 without more ado?]

II. *If* AC, BD *be drawn from the extremities of* AB *at right angles to it and on the same side, and if* AC, BD *be made equal to one another and* CD *be joined, each of the angles* ACD, BDC *will be right, and* CD *will be equal to* AB.



The first part of this lemma is proved by *reductio ad absurdum* from the preceding lemma. If, e.g., the angle ACD is not right, it must either be acute or obtuse. Suppose it is acute; then, by lemma 1, AC is greater than BD, which is contrary to the hypothesis. And so on.

The angles ACD, BDC being proved to be right angles, it is easy to prove that AB, CD are equal.

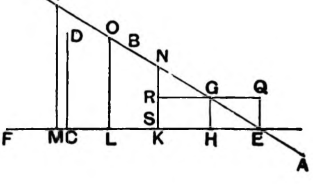
[It is of course assumed in this "proof" that, if the angle ACD is acute, the angle BDC is obtuse, and *vice versa*.]

III. *In any triangle the three angles are together equal to two right angles.*

This is proved for a *right-angled* triangle by means of the foregoing lemma, the four angles of the quadrilateral $ABCD$ of that lemma being all right angles. The proposition is then true for *any* triangle, since any triangle can be divided into two right-angled triangles.

IV. Here we have the final "proof" of Post. 5. Three cases are distinguished, but it is enough to show the case where one of the interior angles is right and the other acute.

Suppose AB, CD to be two straight lines met by FCE making the angle ECD a right angle and the angle CEB an acute angle.



Take any point G on EB, and draw GH perpendicular to EC.

Since the angle CEG is acute, the perpendicular GH will fall on the side of E towards D, and will either coincide with CD or not coincide with it. In the former case the proposition is proved.

If GH does not coincide with CD but falls on the side of it towards F, CD, being within the triangle formed by the perpendicular and by CE, EG, must cut EG. [An axiom is here used, namely that, if CD be produced far enough, it must pass *outside* the triangle and therefore cut *some* side, which must be EB, since it cannot be the perpendicular (I. 27), or CE.]

Lastly, let GH fall on the side of CD towards E.

Along HC set off HK, KL etc., each equal to EH, until we get the first point of division, as M, beyond C.

Along GB set off GN, NO etc., each equal to EG, until EP is the same multiple of EG that EM is of EH.

Then we can prove that the perpendiculars from N, O, P on EC fall on the points K, L, M respectively.

For take the first perpendicular, that from N, and call it NS.

Draw EQ at right angles to EH and equal to GH, and set off SR along SN also equal to GH. Join QG, GR.

Then (second lemma) the angles EQG, QGH are right, and $QG = EH$.

Similarly the angles SRG, RGH are right, and $RG = SH$.

Thus RGQ is one straight line, and the vertically opposite angles NGR, EGQ are equal. The angles NRG, EQG are both right, and $NG = GE$, by construction.

Therefore (I. 26) $RG = GQ$;

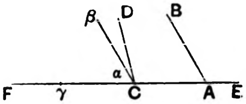
whence $SH = HE = KH$, and S coincides with K.

We may proceed similarly with the other perpendiculars.

Thus PM is perpendicular to FE. Hence CD, being parallel to MP and within the triangle PME, must cut EP, if produced far enough.

John Wallis.

As is well known, the argument of Wallis (1616—1703) assumed as a postulate that, *given a figure, another figure is possible which is similar to the given one and of any size whatever.* In fact Wallis assumed this for *triangles* only. He first proved (1) that, if a finite straight line is placed on an infinite straight line, and is then moved in its own direction as far as we please, it will always lie on the same infinite straight line, (2) that, if an angle be moved so that one leg always slides along an infinite straight line, the angle will remain the same, or equal, (3) that, if two straight lines, cut by a third, make the interior angles on the same side less than two right angles, each of the exterior angles is greater than the opposite interior angle (proved by means of I. 13).



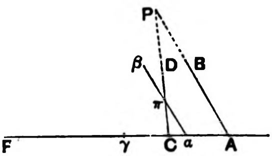
(4) If AB, CD make, with AC, the interior angles less than two right angles, suppose AC (with AB rigidly attached to it) to move along AF to the position $\alpha\gamma$, such that α coincides with C. If AB then takes the position $\alpha\beta$, $\alpha\beta$ *lies entirely outside* CD (proved by means of (3) above).

(5) With the same hypotheses, *the straight line* $\alpha\beta$, *or* AB, *during its motion, and before* α *reaches* C, *must* cut *the straight line* CD.

(6) Here is enunciated the postulate stated above.

(7) Postulate 5 is now proved thus.

Let AB, CD be the straight lines which make, with the infinite straight line ACF meeting them, the interior angles BAC, DCA together less than two right angles.



Suppose AC (with AB rigidly attached to it) to move along ACF until AB takes the position of $\alpha\beta$ cutting CD in π.

Then, $\alpha C\pi$ being a triangle, we can, by the above postulate, suppose a triangle drawn on the base CA similar to the triangle $\alpha C\pi$. Let it be ACP.

[Wallis here interposes a defence of the hypothetical construction.]

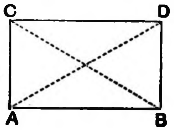
Thus CP and AP meet at P; and, as by the definition of similar figures the angles of the triangles PCA, πCa are respectively equal, the angle PCA being equal to the angle πCa and the angle PAC to the angle πaC or BAC, it follows that CP, AP lie on CD, AB produced respectively.

Hence AB, CD meet on the side on which are the angles less than two right angles.

[The whole gist of this proof lies in the assumed postulate as to the existence of similar figures; and, as Saccheri points out, this is equivalent to unconditionally assuming the "hypothesis of the right angle," and consequently Euclid's Postulate 5.]

Gerolamo Saccheri.

The book *Euclides ab omni naevo vindicatus* (1733) by Gerolamo Saccheri (1667—1733), a Jesuit, and professor at the University of Pavia, is now accessible (1) edited in German by Engel and Stäckel, *Die Theorie der Parallellinien von Euklid bis auf Gauss*, 1895, pp. 41—136, and (2) in an Italian version, abridged but annotated, *L'Euclide emendato del P. Gerolamo Saccheri*, by G. Boccardini (Hoepli, Milan, 1904). It is of much greater importance than all the earlier attempts to prove Post. 5 because Saccheri was the first to contemplate the possibility of hypotheses other than that of Euclid, and to work out a number of consequences of those hypotheses. He was therefore a true precursor of Legendre and of Lobachewsky, as Beltrami called him (1889), and, it might be added, of Riemann also. For, as Veronese observes (*Fondamenti di geometria*, p. 570), Saccheri obtained a glimpse of the theory of parallels in all its generality, while Legendre, Lobachewsky and G. Bolyai excluded *a priori*, without knowing it, the "hypothesis of the obtuse angle," or the Riemann hypothesis. Saccheri, however, was the victim of the preconceived notion of his time that the sole possible geometry was the Euclidean, and he presents the curious spectacle of a man laboriously erecting a structure upon new foundations for the very purpose of demolishing it afterwards; he sought for contradictions in the heart of the systems which he constructed, in order to prove thereby the falsity of his hypotheses.



For the purpose of formulating his hypotheses he takes a plane quadrilateral $ABDC$, two opposite sides of which, AC, BD, are equal and perpendicular to a third AB. Then the angles at C and D are easily proved to be equal. On the Euclidean hypothesis they are both right angles; but apart from this hypothesis they might be both obtuse or both acute. To the three possibilities, which Saccheri distinguishes by the names (1) *the hypothesis of the right angle*, (2) *the hypothesis of the obtuse angle* and (3) *the hypothesis of the acute angle* respectively, there corresponds a certain group of theorems; and Saccheri's point of view is that the Postulate will be completely proved if the consequences which follow from the last two hypotheses comprise results inconsistent with one another.

Among the most important of his propositions are the following:

(1) *If the hypothesis of the right angle, or of the obtuse angle, or of the acute angle is proved true in a single case, it is true in every other case.* (Props. v., vi., vii.)

(2) *According as the hypothesis of the right angle, the obtuse angle, or the acute angle is true, the sum of the three angles of a triangle is equal to, greater than, or less than two right angles.* (Prop. ix.)

(3) *From the existence of a single triangle in which the sum of the angles is equal to, greater than, or less than two right angles the truth of the hypothesis of the right angle, obtuse angle, or acute angle respectively follows.* (Prop. XV.)

These propositions involve the following: *If in a single triangle the sum of the angles is equal to, greater than, or less than two right angles, then* any *triangle has the sum of its angles equal to, greater than, or less than two right angles respectively*, which was proved about a century later by Legendre for the two cases only where the sum is *equal to* or *less than* two right angles.

The proofs are not free from imperfections, as when, in the proofs of Prop. XII. and the part of Prop. XIII. relating to the hypothesis of the *obtuse angle*, Saccheri uses Eucl. I. 18 depending on I. 16, a proposition which is only valid on the assumption that *straight lines are infinite in length*; for this assumption itself does not hold under the hypothesis of the obtuse angle (the Riemann hypothesis).

The hypothesis of the acute angle takes Saccheri much longer to dispose of, and this part of the book is less satisfactory; but it contains the following propositions afterwards established anew by Lobachewsky and Bolyai, viz.:

(4) *Two straight lines in a plane* (even on the hypothesis of the acute angle) *either have a common perpendicular, or must, if produced in one and the same direction, either intersect once at a finite distance or at least continually approach one another.* (Prop. XXIII.)

(5) *In a cluster of rays issuing from a point there exist always* (on the hypothesis of the acute angle) *two determinate straight lines which separate the straight lines which intersect a fixed straight line from those which do not intersect it, ending with and including the straight line which has a common perpendicular with the fixed straight line.* (Props. XXX., XXXI., XXXII.)

Lambert.

A dissertation by G. S. Klügel, *Conatuum praecipuorum theoriam parallelarum demonstrandi recensio* (1763), contained an examination of some thirty "demonstrations" of Post. 5 and is remarkable for its conclusion expressing, apparently for the first time, *doubt as to its demonstrability* and observing that the certainty which we have in us of the truth of the Euclidean hypothesis is not the result of a series of rigorous deductions but rather of experimental observations. It also had the greater merit that it called the attention of Johann Heinrich Lambert (1728—1777) to the theory of parallels. His *Theory of Parallels* was written in 1766 and published after his death by G. Bernoulli and C. F. Hindenburg; it is reproduced by Engel and Stäckel (*op. cit.* pp. 152—208).

The third part of Lambert's tract is devoted to the discussion of the same three hypotheses as Saccheri's, the hypothesis of the *right angle* being for Lambert the *first*, that of the *obtuse angle* the *second*, and that of the *acute angle* the *third*, hypothesis; and, with reference to a quadrilateral with *three right angles* from which Lambert starts (that is, one of the halves into which the median divides Saccheri's quadrilateral), the three hypotheses are the assumptions that the fourth angle is a right angle, an obtuse angle, or an acute angle respectively.

Lambert goes much further than Saccheri in the deduction of new propositions from the *second* and *third* hypotheses. The most remarkable is the following.

The area of a plane triangle, under the second *and* third *hypotheses, is proportional to the difference between the sum of the three angles and two right angles.*

Thus the numerical expression for the area of a triangle is, under the *third* hypothesis

$$\Delta = k(\pi - A - B - C) \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

and under the *second* hypothesis

$$\Delta = k(A + B + C - \pi) \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

where k is a positive constant.

A remarkable observation is appended (§ 82): "In connexion with this it seems to be remarkable that the *second* hypothesis holds if *spherical* instead of plane triangles are taken, because in the former also the sum of the angles is greater than two right angles, and the excess is proportional to the area of the triangle.

"It appears still more remarkable that what I here assert of spherical triangles can be proved independently of the difficulty of parallels."

This discovery that the *second* hypothesis is realised on the surface of a sphere is important in view of the development, later, of the Riemann hypothesis (1854).

Still more remarkable is the following prophetic sentence: "*I am almost inclined to draw the conclusion that the* third *hypothesis arises with an imaginary spherical surface*" (cf. Lobachewsky's *Géométrie imaginaire*, 1837).

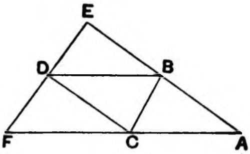
No doubt Lambert was confirmed in this by the fact that, in the formula (2) above, which, for $k = r^2$, represents the area of a spherical triangle, if $r\sqrt{-1}$ is substituted for r, and $r^2 = k$, we obtain the formula (1).

Legendre.

No account of our present subject would be complete without a full reference to what is of permanent value in the investigations of Adrien Marie Legendre (1752—1833) relating to the theory of parallels, which extended over the space of a generation. His different attempts to prove the Euclidean hypothesis appeared in the successive editions of his *Éléments de Géométrie* from the first (1794) to the twelfth (1823), which last may be said to contain his last word on the subject. Later, in 1833, he published, in the *Mémoires de l'Académie Royale des Sciences*, XII. p. 367 sqq., a collection of his different proofs under the title *Réflexions sur différentes manières de démontrer la théorie des parallèles.* His exposition brought out clearly, as Saccheri had done, and kept steadily in view, the essential connexion between the theory of parallels and the sum of the angles of a triangle. In the first edition of the *Éléments* the proposition that *the sum of the angles of a triangle is equal to two right angles* was proved analytically on the basis of the assumption that the choice of a *unit of length* does not affect the correctness of the proposition to be proved, which is of course equivalent to Wallis' assumption of *the existence of similar figures.* A similar analytical proof is given in the notes to the twelfth edition. In his second edition Legendre proved Postulate 5 by means of the assumption that, *given three points not in a straight line, there exists a circle passing through all three.* In the third edition (1800) he gave the proposition that *the sum of the angles of a triangle is not greater than two right angles*; this proof, which was geometrical, was replaced later by another, the best known, depending on a construction like that of Euclid I. 16, the continued application of which enables any number of successive triangles to be evolved in which, while the sum of the angles in each remains always equal to the sum of the angles of the original triangle, one of the angles increases and the sum of the other two diminishes continually. But Legendre found the proof of the equally necessary proposition that the sum of the angles of a triangle is *not less* than two right angles to present great difficulties. He first observed that, as in the case of spherical triangles (in which the sum of the angles is greater than two right angles) the excess of the sum of the angles over two right angles is proportional to the area of the triangle, so in the case of rectilineal triangles, if the sum of the angles is less than two right angles by a certain *deficit*, the *deficit* will be proportional to the area of the triangle. Hence if, starting from a given triangle, we could construct another triangle in which the original triangle is contained at least *m* times, the *deficit* of this new triangle will be equal to at least *m* times that of the original triangle, so that the sum of the angles of the greater triangle will diminish progressively as *m* increases, until it becomes zero or negative: which is absurd. The whole difficulty was thus reduced to that of the construction of a triangle containing the given triangle at least twice; but the solution of even this simple problem requires it to be assumed (or proved) that *through a given point within a given angle less than two-thirds of a right angle we can always draw a straight line which shall meet both sides of the angle.* This is however really equivalent to Euclid's Postulate. The proof in the course of which the necessity for the assumption appeared is as follows.

It is required to prove that the sum of the angles of a triangle cannot be *less* than two right angles.

Suppose A is the least of the three angles of a triangle ABC. Apply to the opposite side BC a triangle DBC, equal to the triangle ACB, and such that the angle DBC is equal to the angle ACB, and the angle DCB to the angle ABC; and *draw any straight line through D cutting AB, AC produced in E, F.*



If now the sum of the angles of the triangle ABC is less than two right angles, being equal to $2R - \delta$ say, the sum of the angles of the triangle DBC, equal to the triangle ABC, is also $2R - \delta$.

Since the sum of the three angles of the remaining triangles DEB, FDC respectively cannot at all events be *greater* than two right angles [for Legendre's proofs of this see below], the sum of the twelve angles of the four triangles in the figure *cannot be greater* than

$$4R + (2R - \delta) + (2R - \delta), \text{ i.e. } 8R - 2\delta.$$

Now the sum of the three angles at each of the points B, C, D is $2R$.

Subtracting these nine angles, we have the result that the three angles of the triangle AEF *cannot be greater* than $2R - 2\delta$.

Hence, if the sum of the angles of the triangle ABC is less than two right angles by δ, the sum of the angles of the larger triangle AEF is less than two right angles by *at least* 2δ.

We can continue the construction, making a still larger triangle from AEF, and so on.

But, however small δ is, we can arrive at a multiple $2^n\delta$ which shall exceed any given angle and therefore $2R$ itself; so that the sum of the three angles of a triangle sufficiently large would be zero or even less than zero: which is absurd.

Therefore etc.

The difficulty caused by the necessity of making the above-mentioned assumption made Legendre abandon, in his ninth edition, the method of the editions from the third to the eighth and return to Euclid's method pure and simple.

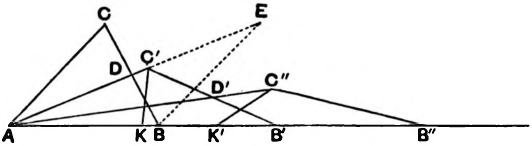
But again, in the twelfth, he returned to the plan of constructing any number of successive triangles such that the sum of the three angles in all of them remains equal to the sum of the three angles of the original triangle, but two of the angles of the new triangles become smaller and smaller, while the third becomes larger and larger; and this time he claims to prove in one proposition that the sum of the three angles of the original triangle is *equal* to two right angles by continuing the construction of new triangles *indefinitely* and compressing the two smaller angles of the ultimate triangle into nothing, while the third angle is made to become a *flat* angle at the same time. The construction and attempted proof are as follows.

Let ABC be the given triangle; let AB be the greatest side and BC the least; therefore C is the greatest angle and A the least.

From A draw AD to the middle point of BC, and produce AD to C', making AC' equal to AB.

Produce AB to B', making AB' equal to twice AD.

The triangle $AB'C'$ is then such that the sum of its three angles is equal to the sum of the three angles of the triangle ABC.



For take AK along AB equal to AD, and join $C'K$.

Then the triangles ABD, $AC'K$ have two sides and the included angles respectively equal, and are therefore equal in all respects; and $C'K$ is equal to BD or DC.

Next, in the triangles $B'C'K$, ACD, the angles $B'KC'$, ADC are equal, being respectively supplementary to the equal angles AKC', ADB; and the two sides about the equal angles are respectively equal;

therefore the triangles $B'C'K$, ACD are equal in all respects.

Thus the angle $AC'B'$ is the sum of two angles respectively equal to the angles B, C of the original triangle; and the angle A in the original triangle is the sum of two angles respectively equal to the angles at A and B' in the triangle $AB'C'$.

It follows that the sum of the three angles of the new triangle $AB'C'$ is equal to the sum of the angles of the triangle ABC.

Moreover, the side AC', being equal to AB, and therefore greater than AC, is greater than $B'C'$ which is equal to AC.

Hence the angle $C'AB'$ is less than the angle $AB'C'$; so that the angle $C'AB'$ is less than $\frac{1}{2}A$, where A denotes the angle CAB of the original triangle.

[It will be observed that the triangle $AB'C'$ is really the same triangle as the triangle AEB obtained by the construction of Eucl. I. 16, but differently placed so that the longest side lies along AB.]

By taking the middle point D' of the side $B'C'$ and repeating the same construction, we obtain a triangle $AB''C''$ such that (1) the sum of its three angles is equal to the sum of the three angles of ABC, (2) the sum of the two angles $C''AB''$, $AB''C''$ is equal to the angle $C'AB'$ in the preceding triangle, and is therefore less than $\frac{1}{2}A$, and (3) the angle $C''AB''$ is less than half the angle $C'AB'$, and therefore less than $\frac{1}{4}A$.

Continuing in this way, we shall obtain a triangle Abc such that the sum of two angles, those at A and b, is less than $\frac{1}{2^n}A$, and the angle at c is greater than the corresponding angle in the preceding triangle.

If, Legendre argues, the construction be continued indefinitely so that $\frac{1}{2^n}A$ becomes smaller than any assigned angle, the point c ultimately lies on Ab, and the sum of the three angles of the triangle (which is equal to the sum of the three angles of the original triangle) becomes identical with the angle at c, which is then a *flat* angle, and therefore equal to two right angles.

This proof was however shown to be unsound (in respect of the final inference) by J. P. W. Stein in Gergonne's *Annales de Mathématiques* XV., 1824, pp. 77—79.

We will now reproduce shortly the substance of the theorems of Legendre which are of the most permanent value as not depending on a particular hypothesis as regards parallels.

I. *The sum of the three angles of a triangle cannot be* greater *than two right angles.*

This Legendre proved in two ways.

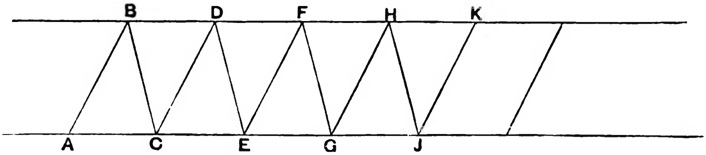
(1) *First proof* (in the third edition of the *Éléments*).

Let ABC be the given triangle, and ACJ a straight line.

Make CE equal to AC, the angle DCE equal to the angle BAC, and DC equal to AB. Join DE.

Then the triangle DCE is equal to the triangle BAC in all respects.

If then the sum of the three angles of the triangle ABC is greater than $2R$, the said sum must be greater than the sum of the angles BCA, BCD, DCE, which sum is *equal* to $2R$.



Subtracting the equal angles on both sides, we have the result that

the angle ABC is *greater* than the angle BCD.

But the two sides AB, BC of the triangle ABC are respectively equal to the two sides DC, CB of the triangle BCD.

Therefore the base AC is *greater* than the base BD (Eucl. I. 24).

Next, make the triangle FEG (by the same construction) equal in all respects to the triangle BAC or DCE; and we prove in the same way that CE (or AC) is *greater* than DF.

And, at the same time, BD is equal to DF, because the angles BCD, DEF are equal.

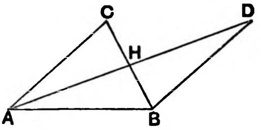
Continuing the construction of further triangles, however small the difference between AC and BD is, we shall ultimately reach some multiple of this difference, represented in the figure by (say) the difference between the straight line AJ and the composite line $BDFHK$, which will be greater than any assigned length, and greater therefore than the sum of AB and JK.

Hence, on the assumption that the sum of the angles of the triangle ABC is greater than $2R$, the broken line $ABDFHKJ$ may be less than the straight line AJ: which is impossible.

Therefore etc.

(2) *Proof substituted later.*

If possible, let $2R + \alpha$ be the sum of the three angles of the triangle ABC, of which A is not greater than either of the others.



Bisect BC at H, and produce AH to D, making HD equal to AH; join BD.

Then the triangles AHC, DHB are equal in all respects (I. 4); and the angles CAH, ACH are respectively equal to the angles BDH, DBH.

It follows that the sum of the angles of the triangle ABD is equal to the sum of the angles of the original triangle, i.e. to $2R + \alpha$.

And one of the angles DAB, ADB is either equal to or less than half the angle CAB.

Continuing the same construction with the triangle ADB, we find a third triangle in which the sum of the angles is still $2R + \alpha$, while one of them is equal to or less than $(\angle CAB)/4$.

Proceeding in this way, we arrive at a triangle in which the sum of the angles is $2R + \alpha$, and one of them is not greater than $(\angle CAB)/2^n$.

And, if n is sufficiently large, this will be less than α; in which case we should have a triangle in which two angles are together greater than two right angles: which is absurd.

Therefore α is equal to or less than zero.

(It will be noted that in both these proofs, as in Eucl. I. 16, it is taken for granted that *a straight line is infinite in length* and does not return into itself, which is not true under the Riemann hypothesis.)

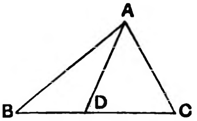
II. On the assumption that the sum of the angles of a triangle is *less* than two right angles, *if a triangle is made up of two others, the "deficit" of the former is equal to the sum of the "deficits" of the others.*

In fact, if the sums of the angles of the component triangles are $2R - \alpha$, $2R - \beta$ respectively, the sum of the angles of the whole triangle is

$$(2R - \alpha) + (2R - \beta) - 2R = 2R - (\alpha + \beta).$$

III. *If the sum of the three angles of a triangle is* equal *to two right angles, the same is true of all triangles obtained by subdividing it by straight lines drawn from a vertex to meet the opposite side.*

Since the sum of the angles of the triangle ABC is equal to $2R$, if the sum of the angles of the triangle ABD were $2R - \alpha$, it would follow that the sum of the angles of the triangle ADC must be $2R + \alpha$, which is absurd (by I. above).



IV. *If in a triangle the sum of the three angles is* equal *to two right angles, a quadrilateral can always be constructed with four right angles and four equal sides exceeding in length any assigned rectilineal segment.*

Let ABC be a triangle in which the sum of the angles is equal to two right angles. We can assume ABC to be an *isosceles right-angled* triangle because we can reduce the case to this by making subdivisions of ABC by straight lines through vertices (as in Prop. III. above).

Taking two equal triangles of this kind and placing their hypotenuses together, we obtain a quadrilateral with four right angles and four equal sides.

Putting four of these quadrilaterals together, we obtain a new quadrilateral of the same kind but with its sides double of those of the first quadrilateral.

After n such operations we have a quadrilateral with four right angles and four equal sides, each being equal to 2^n times the side AB.

The diagonal of this quadrilateral divides it into two equal isosceles right-angled triangles in each of which the sum of the angles is equal to two right angles.

Consequently, from the existence of *one* triangle in which the sum of the three angles is equal to two right angles it follows that there exists an isosceles right-angled triangle with sides greater than any assigned rectilineal segment and such that the sum of its three angles is also equal to two right angles.

V. *If the sum of the three angles of* one *triangle is equal to two right angles, the sum of the three angles of* any other *triangle is also equal to two right angles.*

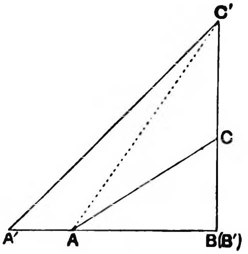
It is enough to prove this for a *right-angled* triangle, since any triangle can be divided into two right-angled triangles.

Let ABC be any right-angled triangle.

If then the sum of the angles of any one triangle is equal to two right angles, we can construct (by the preceding Prop.) an isosceles right-angled triangle with the same property and with its perpendicular sides greater than those of ABC.

Let $A'B'C'$ be such a triangle, and let it be applied to ABC, as in the figure.

Applying then Prop. III. above, we deduce first that the sum of the three angles of the triangle $AB'C'$ is equal to two right angles, and next, for the same reason, that the sum of the three angles of the original triangle ABC is equal to two right angles.



VI. *If in* any one *triangle the sum of the three angles is* less *than two right angles, the sum of the three angles of* any other *triangle is also less than two right angles.*

This follows from the preceding theorem.

(It will be observed that the last two theorems are included among those of Saccheri, which contain however in addition the corresponding theorem touching the case where the sum of the angles is *greater* than two right angles.)

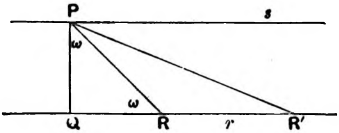
We come now to the bearing of these propositions upon Euclid's Postulate 5; and the next theorem is

VII. *If the sum of the three angles of a triangle is equal to two right angles, through any point in a plane there can only be drawn one parallel to a given straight line.*

For the proof of this we require the following

LEMMA. *It is always possible, through a point* P, *to draw a straight line which shall make, with a given straight line* (r), *an angle less than any assigned angle.*

Let Q be the foot of the perpendicular from P upon r.



Let a segment QR be taken on r, on either side of Q, such that QR is equal to PQ.

Join PR, and mark off the segment RR' equal to PR; join PR'.

If ω represents the angle QPR or the angle QRP, each of the equal angles RPR', $RR'P$ is not greater than $\omega/2$.

Continuing the construction, we obtain, after the requisite number of operations, a triangle $PR_{n-1}R_n$ in which each of the equal angles is equal to or less than $\omega/2^n$.

Hence we shall arrive at a straight line PR_n which, starting from P and meeting r, makes with r an angle as small as we please.

To return now to the Proposition. Draw from P the straight line s perpendicular to PQ.

Then any straight line drawn from P which meets r in R will form equal angles with r and s, since, by hypothesis, the sum of the angles of the triangle PQR is equal to two right angles.

And since, by the Lemma, it is always possible to draw through P straight lines which form with r angles as small as we please, it follows that all the straight lines through P, except s, will meet r. Hence s is the only parallel to r that can be drawn through P.

The history of the attempts to prove Postulate 5 or something equivalent has now been brought down to the parting of the ways. The further developments on lines independent of the Postulate, beginning with Schweikart (1780—1857), Taurinus (1794—1874), Gauss (1777—1855), Lobachewsky (1793—1856), J. Bolyai (1802—1860), Riemann (1826—1866), belong to the history of non-Euclidean geometry, which is outside the scope of this work. I may refer the reader to the full article *Sulla teoria delle parallele e sulle geometrie non-euclidee* by R. Bonola in *Questioni riguardanti le matematiche elementari*, I., of which I have made considerable use in the above, to the same author's *La geometria non-euclidea*, Bologna, 1906, to the first volume of Killing's *Einführung in die Grundlagen der Geometrie*, Paderborn, 1893, to P. Mansion's *Premiers principes de métagéométrie*, and P. Barbarin's *La géométrie non-Euclidienne*, Paris, 1902, to the historical summary in Veronese's *Fondamenti di geometria*, 1891, p. 565 sqq., and (for original sources) to Engel and Stäckel, *Die Theorie der Parallellinien von Euklid bis auf Gauss*, 1895, and *Urkunden zur Geschichte der nicht-Euklidischen Geometrie*, I. (Lobachewsky), 1899, and II. (Wolfgang und Johann Bolyai). I will only add that it was Gauss who first expressed a conviction that the Postulate could never be proved; he indicated this in reviews in the *Göttingische gelehrte Anzeigen*, 20 Apr. 1816 and 28 Oct. 1822, and affirmed it in a letter to Bessel of 27 January, 1829. The actual indemonstrability of the Postulate was proved by Beltrami (1868) and by Hoüel (*Note sur l'impossibilité de démontrer par une construction plane le principe de la théorie des parallèles dit Postulatum d'Euclide* in Battaglini's *Giornale di matematiche*, VIII., 1870, pp. 84—89).

Alternatives for Postulate 5.

It may be convenient to collect here a few of the more noteworthy substitutes which have from time to time been formally suggested or tacitly assumed.

(1) *Through a given point only one parallel can be drawn to a given straight line* or, *Two straight lines which intersect one another cannot both be parallel to one and the same straight line.*

This is commonly known as "Playfair's Axiom," but it was of course not a new discovery. It is distinctly stated in Proclus' note to Eucl. I. 31.

(1 *a*) *If a straight line intersect one of two parallels, it will intersect the other also* (Proclus).

(1 *b*) *Straight lines parallel to the same straight line are parallel to one another.*

The forms (1 *a*) and (1 *b*) are exactly equivalent to (1).

(2) *There exist straight lines everywhere equidistant from one another* (Posidonius and Geminus); with which may be compared Proclus' tacit assumption that *Parallels remain, throughout their length, at a finite distance from one another.*

(3) *There exists a triangle in which the sum of the three angles is equal to two right angles* (Legendre).

(4) *Given any figure, there exists a figure similar to it of any size we please* (Wallis, Carnot, Laplace).

Saccheri points out that it is not necessary to assume so much, and that it is enough to postulate that *there exist two unequal triangles with equal angles.*

(5) *Through any point within an angle less than two-thirds of a right angle a straight line can always be drawn which meets both sides of the angle* (Legendre).

With this may be compared the similar axiom of Lorenz (*Grundriss der reinen und angewandten Mathematik*, 1791): *Every straight line through a point within an angle must meet one of the sides of the angle.*

(6) *Given any three points not in a straight line, there exists a circle passing through them* (Legendre, W. Bolyai).

(7) "*If I could prove that a rectilineal triangle is possible the content of which is greater than any given area, I am in a position to prove perfectly rigorously the whole of geometry*" (Gauss, in a letter to W. Bolyai, 1799).

Cf. the proposition of Legendre numbered IV. above, and the axiom of Worpitzky: *There exists no triangle in which every angle is as small as we please.*

(8) *If in a quadrilateral three angles are right angles, the fourth angle is a right angle also* (Clairaut, 1741).

(9) *If two straight lines are parallel, they are figures opposite to* (*or the reflex of*) *one another with respect to the middle points of all their transversal segments* (Veronese, *Elementi*, 1904).

Or, *Two parallel straight lines intercept, on every transversal which passes through the middle point of a segment included between them, another segment the middle point of which is the middle point of the first* (Ingrami, *Elementi*, 1904).

Veronese and Ingrami deduce immediately Playfair's Axiom.

AXIOMS OR *COMMON NOTIONS*.

In a paper *Sur l'authenticité des axiomes d'Euclide* in the *Bulletin des sciences math. et astron.* 1884, p. 162 sq. (*Mémoires scientifiques*, II., pp. 48—63), Paul Tannery maintained that the *Common Notions* (including the first three) were not in Euclid's work but were interpolated later. The following are his main arguments. (1) If Euclid had set about distinguishing between indemonstrable principles (*a*) common to all demonstrative sciences and (*b*) peculiar to geometry, he would, says Tannery, certainly not have placed the common principles second and the special principles (the Postulates) first. (2) If the *Common Notions* are Euclid's, this designation of them must be his too; for he must have used *some* name to distinguish them from the Postulates and, if he had used another name, such as *Axioms*, it is impossible to imagine why that name was changed afterwards for a less suitable one. The word ἔννοια (*notion*), says Tannery, never signified a notion in the sense of a *proposition*, but a notion of some *object*; nor is it found in any technical sense in Plato and Aristotle. (3) Tannery's own view was that the formulation of the *Common Notions* dates from the time of Apollonius, and that it was inspired by his work relating to the Elements (we know from Proclus that Apollonius tried to prove the *Common Notions*). This idea, Tannery thought, was confirmed by a "fortunate coincidence" furnished by the occurrence of the word ἔννοια (*notion*) in a quotation by Proclus (p. 100, 6): "we shall agree with Apollonius when he says that we have a *notion* (ἔννοιαν) of a line when we order the lengths, only, of roads or walls to be measured."

In reply to argument (1) that it is an unnatural order to place the purely geometrical Postulates first, and the *Common Notions*, which are not peculiar to geometry, last, it may be pointed out that it would surely have been a still more awkward arrangement to give the Definitions first and then to separate from them, by the interposition of the *Common Notions*, the Postulates, which are so closely connected with the Definitions in that they proceed to postulate the *existence* of certain of the things defined, namely straight lines and circles.

(2) Though it is true that ἔννοια in Plato and Aristotle is generally a notion of an *object*, not of a *fact* or proposition, there are instances in Aristotle where it does mean a notion of a fact: thus in the *Eth. Nic.* IX. 11, 1171 a 32 he speaks of "the notion (or consciousness) *that friends sympathise*" (ἡ ἔννοια τοῦ συναλγεῖν τοὺς φίλους) and again, b 14, of "the *notion* (or consciousness) *that they are pleased* at his good fortune." It is true that Plato and Aristotle do not use the word in a technical sense; but neither was there apparently in Aristotle's time any fixed technical term for what we call "axioms," since he speaks of them variously as "the so-called axioms in mathematics," "the so-called common axioms," "the common (things)" (τὰ κοινά), and even "the common *opinions*" (κοιναὶ δόξαι). I see therefore no reason why Euclid should not himself have given a technical sense to "Common Notions," which is at least a distinct improvement upon "common opinions."

(3) The use of ἔννοια in Proclus' quotation from Apollonius seems to me to be an unfortunate, rather than a fortunate, coincidence from Tannery's point of view, for it is there used precisely in the old sense of the notion of an *object* (in that case a line).

No doubt it is difficult to feel certain that Euclid did himself use the term *Common Notions*, seeing that Proclus' commentary generally speaks of *Axioms*. But even Proclus (p. 194, 8), after explaining the meaning of the word "axiom," first as used by the Stoics, and secondly as used by "Aristotle and the geometers," goes on to say: "For in their view (that of Aristotle and the geometers) *axiom* and *common notion* are the same thing." This, as it seems to me, may be a sort of apology for using the word "axiom" exclusively in what has gone before, as if Proclus had suddenly bethought himself that he had described both Aristotle and the geometers as using the one term "axiom," whereas he should have said that Aristotle spoke of "axioms," while "the geometers" (in fact Euclid), though meaning the same thing, called them *Common Notions*. It may be for a like reason that in another passage (p. 76, 16), after quoting Aristotle's view of an "axiom," as distinct from a postulate and a hypothesis, he proceeds: "For it is not by virtue of a *common notion* that, without being taught, we preconceive the circle to be such and such a figure." If this view of the two passages just quoted is correct, it would strengthen rather than weaken the case for the genuineness of *Common Notions* as the Euclidean term.

Again, it is clear from Aristotle's allusions to the "common axioms in mathematics" that more than one axiom of this kind had a place in the text-books of his day; and as he constantly quotes the particular axiom that, *if equals be taken from equals, the remainders are equal*, which is Euclid's *Common Notion* 3, it would seem that at least the first three *Common Notions* were adopted by Euclid from earlier text-books. It is, besides, scarcely credible that, if the *Common Notions* which Apollonius tried to prove had not been introduced earlier (e.g. by Euclid), they would then have been interpolated as axioms and not as propositions to be proved. The line taken by Apollonius is much better explained on the assumption that he was directly attacking axioms which he found already admitted into the *Elements*.

Proclus, who recognised the five *Common Notions* given in the text, warns us, not only against the error of unnecessarily multiplying the axioms, but against the contrary error of reducing their number unduly (p. 196, 15), "as Heron does in enunciating three only; for it is also an axiom that *the whole is greater than the part*, and indeed the geometer employs this in many places for his demonstrations, and again that *things which coincide are equal*."

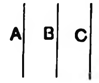
Thus Heron recognised the first three of the *Common Notions*; and this fact, together with Aristotle's allusions to "common axioms" (in the plural), and in particular to our *Common Notion* 3, may satisfy us that at least the first three *Common Notions* were contained in the *Elements* as they left Euclid's hands.

Common Notion 1.

Τὰ τῷ αὐτῷ ἴσα καὶ ἀλλήλοις ἐστὶν ἴσα.

Things which are equal to the same thing are also equal to one another.

Aristotle throughout emphasises the fact that axioms are self-evident truths, which it is impossible to demonstrate. If, he says, any one should attempt to prove them, it could only be through ignorance. Aristotle therefore would undoubtedly have agreed in Proclus' strictures on Apollonius for attempting to prove the axioms. Proclus gives (p. 194, 25), as a specimen of these attempted proofs by Apollonius, that of the first of the *Common Notions*. "Let A be equal to B, and the latter to C; I say that A is also equal to C. For, since A is equal to B, it occupies the same space with it; and since B is equal to C, it occupies the same space with it.



Therefore A also occupies the same space with C."

Proclus rightly remarks (p. 194, 22) that "the middle term is no more intelligible (better known, γνωριμώτερον) than the conclusion, if it is not actually more disputable." Again (p. 195, 6), the proof assumes two things, (1) that things which "occupy the same space" (τόπος) are equal to one another, and (2) that things which occupy the same space with one and the same thing occupy the same space with one another; which is to explain the obvious by something much more obscure, for space is an entity more unknown to us than the things which exist in space.

Aristotle would also have objected to the proof that it is partial and not general (καθόλου), since it refers only to things which can be supposed to occupy a space (or take up room), whereas the axiom is, as Proclus says (p. 196, 1), true of numbers, speeds, and periods of time as well, though of course each science uses axioms in relation to its own subject-matter only.

COMMON NOTIONS 2, 3.

2. Καὶ ἐὰν ἴσοις ἴσα προστεθῇ, τὰ ὅλα ἐστὶν ἴσα.
3. Καὶ ἐὰν ἀπὸ ἴσων ἴσα ἀφαιρεθῇ, τὰ καταλειπόμενά ἐστιν ἴσα.

2. *If equals be added to equals, the wholes are equal.*
3. *If equals be subtracted from equals, the remainders are equal.*

These two Common Notions are recognised by Heron and Proclus as genuine. The latter is the axiom which is so favourite an illustration with Aristotle.

Following them in the MSS. and editions there came four others of the same type as 1—3. Three of these are given by Heiberg in brackets; the fourth he omits altogether.

The three are:

(*a*) *If equals be added to unequals, the wholes are unequal.*

(*b*) *Things which are double of the same thing are equal to one another.*

(*c*) *Things which are halves of the same thing are equal to one another.*

The fourth, which was placed between (*a*) and (*b*), was:

(*d*) *If equals be subtracted from unequals, the remainders are unequal.*

Proclus, in observing that axioms ought not to be multiplied, indicates that all should be rejected which follow from the five admitted by him and appearing in the text above (p. 155). He mentions the second of those just quoted (*b*) as one of those to be excluded, since it follows from *Common Notion* 1. Proclus does not mention (*a*), (*c*) or (*d*); an-Nairīzī gives (*a*), (*d*), (*b*) and (*c*), in that order, as Euclid's, adding a note of Simplicius that "three axioms (sententiae acceptae) only are extant in the ancient manuscripts, but the number was increased in the more recent."

(*a*) stands self-condemned because "unequal" tells us nothing. It is easy to see what is wanted if we refer to I. 17, where the same angle is added to a *greater* and a *less*, and it is inferred that the first sum is greater than the second. So far however as the wording of (*a*) is concerned, the addition of equal to *greater* and *less* might be supposed to produce *less* and *greater* respectively. If therefore such an axiom were given at all, it should be divided into two. Heiberg conjectures that this axiom may have been taken from the commentary of Pappus, who had the axiom about equals added to unequals quoted below (*e*); if so, it can only be an unskilful adaptation of some remark of Pappus, for his axiom (*e*) has some point, whereas (*a*) is useless.

As regards (*b*), I agree with Tannery in seeing no sufficient reason why, if we reject it (as we certainly must), the words in I. 47 "But things which are double of equals are equal to one another" should be condemned as an interpolation. If they were interpolated, we should have expected to find the same interpolation in I. 42, where the axiom is *tacitly* assumed. I think it quite possible that Euclid may have inserted such words in one case and left them out in another, without necessarily implying either that he was quoting a formal *Common Notion* of his own or that he had *not* included among his Common Notions the particular fact stated as obvious.

The corresponding axiom (*c*) about the *halves* of equals can hardly be genuine if (*b*) is not, and Proclus does not mention it. Tannery acutely observes however that, when Heiberg, in I. 37, 38, brackets words stating that "the halves of equal things are equal to one another" on the ground that axiom (*c*) was interpolated (although before Theon's time), and explains that Euclid used *Common Notion* 3 in making his inference, he is clearly mistaken. For, while axiom (*b*) is an obvious inference from *Common Notion* 2, axiom (*c*) is not an inference from *Common Notion* 3. Tannery says, in a note, that (*c*) would have to be established by *reductio ad absurdum* with the help of axiom (*b*), that is to say, of *Common Notion* 2. But, as the hypothesis in the *reductio ad absurdum* would be that one of the halves is *greater* than the other, and it would therefore be necessary to prove that the one whole is *greater* than the other, while axiom (*b*) or *Common Notion* 2 only refers to *equals*, a little argument would be necessary in addition to the reference to *Common Notion* 2. I think Euclid would not have gone through this process in order to prove (*c*), but would have assumed it as equally obvious with (*b*).

Proclus (pp. 197, 6—198, 5) definitely rejects two other axioms of the above kind given by Pappus, observing that, as they follow from the genuine axioms, they are rightly omitted in most copies, although Pappus said that they were "on record" with the others (συναναγράφεσθαι):

(*e*) *If unequals be added to equals, the difference between the wholes is equal to the difference between the added parts*; and

(*f*) *If equals be added to unequals, the difference between the wholes is equal to the difference between the original unequals.*

Proclus and Simplicius (in an-Nairīzī) give proofs of both. The proof of the former, as given by Simplicius, is as follows:

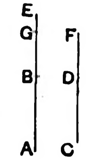
Let AB, CD be equal magnitudes; and let EB, FD be added to them respectively, EB being greater than FD.

I say that AE exceeds CF by the same difference as that by which BE exceeds DF.

Cut off from BE the magnitude BG equal to DF.

Then, since AE exceeds AG by GE, and AG is equal to CF and BG to DF,

AE exceeds CF by the same difference as that by which BE exceeds DF.



COMMON NOTION 4.

Καὶ τὰ ἐφαρμόζοντα ἐπ' ἄλληλα ἴσα ἀλλήλοις ἐστίν.

Things which coincide with one another are equal to one another.

The word ἐφαρμόζειν, as a geometrical term, has a different meaning according as it is used in the active or in the passive. In the passive, ἐφαρμόζεσθαι, it means "to be *applied* to" without any implication that the applied figure will exactly fit, or coincide with, the figure to which it is applied; on the other hand the active ἐφαρμόζειν is used intransitively and means "to fit exactly," "to coincide with." In Euclid and Archimedes ἐφαρμόζειν is constructed with ἐπί and the accusative, in Pappus with the dative.

On *Common Notion* 4 Tannery observes that it is incontestably geometrical in character, and should therefore have been excluded from the *Common Notions*; again, it is difficult to see why it is not accompanied by its converse, at all events for straight lines (and, it might be added, angles also), which Euclid makes use of in I. 4. As it is, says Tannery, we have here a definition of geometrical equality more or less sufficient, but not a real axiom.

It is true that Proclus seems to recognise this *Common Notion* and the next as proper axioms in the passage (p. 196, 15—21) where he says that we should not cut down the axioms to the minimum, as Heron does in giving only three axioms; but the statement seems to rest, not upon authority, but upon an assumption that Euclid would state explicitly at the beginning all axioms subsequently used and not reducible to others unquestionably included. Now in I. 4 this *Common Notion* is not quoted; it is simply inferred that "the base *BC* will coincide with *EF*, *and will be equal* to it." The position is therefore the same as it is in regard to the statement in the same proposition that, "if... the base *BC* does not coincide with *EF*, *two straight lines will enclose a space*: which is impossible"; and, if we do not admit that Euclid had the axiom that "two straight lines cannot enclose a space," neither need we infer that he had *Common Notion* 4. I am therefore inclined to think that the latter is more likely than not to be an interpolation.

It seems clear that the Common Notion, as here formulated, is intended to assert that superposition is a legitimate way of proving the equality of two figures which have the necessary parts respectively equal, or, in other words, to serve as an *axiom of congruence*.

The phraseology of the propositions, e.g. I. 4 and I. 8, in which Euclid employs the method indicated, leaves no room for doubt that he regarded one figure as actually *moved* and *placed upon* the other. Thus in I. 4 he says, "The triangle *ABC* being applied (ἐφαρμοζομένου) to the triangle *DEF*, and the point *A* being *placed* (τιθεμένου) upon the point *D*, and the straight line *AB* on *DE*, the point *B* will also coincide with *E* because *AB* is equal to *DE*"; and in I. 8, "If the sides *BA*, *AC* do not coincide with *ED*, *DF*, but *fall beside them* (take a different position, παραλλάξουσιν), then" etc. At the same time, it is clear that Euclid disliked the method and avoided it wherever he could, e.g. in I. 26, where he proves the equality of two triangles which have two angles respectively equal to two angles and one side of the one equal to the corresponding side of the other. It looks as though he found the method handed down by tradition (we can hardly suppose that, if Thales proved that the diameter of a circle divides it into two equal parts, he would do so by any other method than that of superposition), and followed it, in the few cases where he does so, only because he had not been able to see his way to a satisfactory substitute. But seeing how much of the *Elements* depends on I. 4, directly or indirectly, the method can hardly be regarded as being, in Euclid, of only subordinate importance; on the contrary, it is fundamental. Nor, as a matter of fact, do we find in the ancient geometers any expression of doubt as to the legitimacy of the method. Archimedes uses it to prove that any spheroidal figure cut by a plane through the centre is divided into two equal parts in respect of both its surface and its volume; he also postulates in *Equilibrium of Planes* I. that "when equal and similar plane figures coincide if applied to one another, their centres of gravity coincide also."

Killing (*Einführung in die Grundlagen der Geometrie*, II. pp. 4, 5) contrasts the attitude of the Greek geometers with that of the philosophers, who, he says, appear to have agreed in banishing motion from geometry altogether. In support of this he refers to the view frequently expressed by Aristotle that mathematics has to do with *immovable* objects (ἀκίνητα), and that only where astronomy is admitted as part of mathematical science is motion mentioned as a subject for mathematics. Cf. *Metaph.* 989 b 32 "For mathematical objects are among things which exist apart from motion, except such as relate to astronomy"; *Metaph.* 1064 a 30 "Physics deals with things which have in themselves the principle of motion; mathematics is a theoretical science and one concerned with things which are *stationary* (μένοντα) but not separable" (sc. from matter); in *Physics* II. 2, 193 b 34 he speaks of the subjects of mathematics as "in thought separable from motion."

But I doubt whether in Aristotle's use of the words "immovable," "without motion" etc. as applied to the subjects of mathematics there is any implication such as Killing supposes. We arrive at mathematical concepts by abstraction from material objects; and just as we, in thought, eliminate the matter, so according to Aristotle we eliminate the attributes of matter as such, e.g. qualitative change and *motion*. It does not appear to me that the use of "immovable" in the passages referred to means more than this. I do not think that Aristotle would have regarded it as illegitimate to *move* a geometrical figure from one position to another; and I infer this from a passage in *De caelo* III. 1 where he is criticising "those who make up every body that has an origin by putting together *planes*, and resolve it again into *planes*." The reference must be to the *Timaeus* (54 B sqq.) where Plato evolves the four elements in this way. He begins with a right-angled triangle in which the hypotenuse is double of the smaller side; six of these put together in the proper way produce one equilateral triangle. Making solid angles with (*a*) three, (*b*) four, and (*c*) five of these equilateral triangles respectively, and taking the requisite number of these solid angles, namely four of (*a*), six of (*b*) and twelve of (*c*) respectively, and putting them together so as to form regular solids, he obtains (α) a tetrahedron, (β) an octahedron, (γ) an icosahedron respectively. For the fourth element (earth), four isosceles right-angled triangles are first put together so as to form a square, and then six of these squares are put together to form a cube. Now, says Aristotle (299 b 23), "it is absurd that planes should only admit of being put together so as to touch in a *line*; for just as a line and a line are put together in both ways, lengthwise and breadthwise, so must a plane and a plane. A line can be combined with a line in the sense of being a line *superposed*, and not *added*"; the inference being that a *plane* can be superposed on a *plane*. Now this is precisely the sort of motion in question here; and Aristotle, so far from denying its permissibility, seems to blame Plato for not using it. Cf. also *Physics* V. 4, 228 b 25, where Aristotle speaks of "the spiral or other magnitude in which any part will not coincide with any other part," an where superposition is obviously contemplated.

Motion without deformation.

It is well known that Helmholtz maintained that geometry requires us to assume the actual existence of rigid bodies and their free mobility in space, whence he inferred that geometry is dependent on mechanics.

Veronese exposed the fallacy in this (*Fondamenti di geometria*, pp. xxxv—xxxvi, 239—240 note, 615—7), his argument being as follows. Since geometry is concerned with empty space, which is immovable, it would be at least strange if it was necessary to have recourse to the real motion of bodies for a definition, and for the proof of the properties, of immovable space. We must distinguish the intuitive principle of motion in itself from that of motion *without deformation*. Every point of a figure which moves is transferred to another point in space. "Without deformation" means that the mutual relations between the points of the figure do not change, but the relations between them and other figures do change (for if they did not, the figure could not move). Now consider what we mean by saying that, when the figure A has moved from the position A_1 to the position A_2, the relations between the points of A in the position A_2 are unaltered from what they were in the position A_1, are the same in fact as if A had not moved but remained at A_1. We can only say that, judging of the figure (or the body with its physical qualities eliminated) by the impressions it produces in us during its movement, the impressions produced in us in the two different positions (which are in time distinct) *are equal*. In fact, we are making use of the notion of *equality* between two distinct figures. Thus, if we say that two bodies are equal when they can be superposed by means of *movement without deformation*, we are committing a *petitio principii*. The notion of the equality of spaces is really prior to that of rigid bodies or of motion without deformation. Helmholtz supported his view by reference to the process of measurement in which the measure must be, at least approximately, a rigid body, but the existence of a rigid body as a standard to measure by, and the question how we discover two equal spaces to be equal, are matters of no concern to the geometer. The method of superposition, depending on motion without deformation, is only of use as a *practical* test; it has nothing to do with the *theory* of geometry.

Compare an acute observation of Schopenhauer (*Die Welt als Wille*, 2 ed. 1844, II. p. 130) which was a criticism in advance of Helmholtz' theory: "I am surprised that, instead of the eleventh axiom [the Parallel-Postulate], the eighth is not rather attacked: 'Figures which coincide (sich decken) are equal to one another.' For *coincidence* (das Sichdecken) is either mere tautology, or something entirely empirical, which belongs, not to pure intuition (Anschauung), but to external sensuous experience. It presupposes in fact the mobility of figures; but that which is movable in space is matter and nothing else. Thus this appeal to coincidence means leaving pure space, the sole element of geometry, in order to pass over to the material and empirical."

Mr Bertrand Russell observes (*Encyclopaedia Britannica*, Suppl. Vol. 4, 1902, Art. "Geometry, non-Euclidean") that the apparent use of motion here is deceptive; what in geometry is called a motion is merely the transference of our attention from one figure to another. Actual superposition, which is nominally employed by Euclid, is not required; all that is required is the transference of our attention from the original figure to a new one defined by the position of some of its elements and by certain properties which it shares with the original figure.

If the method of superposition is given up as a means of defining theoretically the equality of two figures, some other definition of equality is necessary. But such a definition can be evolved out of *empirical* or *practical* observation of the result of superposing two material representations of figures. This is done by Veronese (*Elementi di geometria*, 1904) and Ingrami (*Elementi di geometria*, 1904). Ingrami says, namely (p. 66):

"If a sheet of paper be folded double, and a triangle be drawn upon it and then cut out, we obtain two triangles *superposed* which we in practice call *equal*. If points A, B, C, D ... be marked on one of the triangles, then, when we place this triangle upon the other (so as to coincide with it), we see that *each* of the particular points taken on the first is superposed on one particular point of the second in such a way that the segments AB, AC, AD, BC, BD, CD, ... are respectively superposed on as many segments in the second triangle and are therefore equal to them respectively. In this way we justify the following

"Definition of equality.

"Any two figures whatever will be called *equal* when to the points of one the points of the other can be made to correspond *univocally* [i.e. every *one* point in one to *one distinct* point in the other and *vice versa*] in such a way that the segments which join the points, two and two, in one figure are respectively equal to the segments which join, two and two, the corresponding points in the other."

Ingrami has of course previously postulated as known the signification of the phrase *equal* (*rectilineal*) *segments*, of which we get a *practical* notion when we can place one upon the other or can place a third movable segment successively on both.

New systems of Congruence-Postulates.

In the fourth Article of *Questioni riguardanti le matematiche elementari*, I., pp. 93—122, a review is given of three different systems: (1) that of Pasch in *Vorlesungen über neuere Geometrie*, 1882, p. 101 sqq., (2) that of Veronese according to the *Fondamenti di geometria*, 1891, and the *Elementi* taken together, (3) that of Hilbert (see *Grundlagen der Geometrie*, 1903, pp. 7—15).

These systems differ in the particular conceptions taken by the three authors as primary. (1) Pasch considers as primary the notion of *congruence* or *equality* between *any figures which are made up of a finite number of points only*. The definitions of congruent *segments* and of congruent *angles* have to be *deduced* in the way shown on pp. 102—103 of the Article referred to, after which Eucl. I. 4 follows immediately, and Eucl. I. 26 (1) and I. 8 by a method recalling that in Eucl. I. 7, 8.

(2) Veronese takes as primary the conception of congruence between *segments* (rectilineal). The transition to congruent *angles*, and thence to *triangles* is made by means of the following postulate:

"Let AB, AC and $A'B'$, $A'C'$ be two pairs of straight lines intersecting at A, A', and let there be determined upon them the congruent segments AB, $A'B'$ and the congruent segments AC, $A'C'$;

then, if BC, $B'C'$ are congruent, the two *pairs of straight lines* are congruent."

(3) Hilbert takes as primary the notions of congruence between *both segments and angles*.

It is observed in the Article referred to that, from the theoretical standpoint, Veronese's system is an advance upon that of Pasch, since the idea of congruence between *segments* is more simple than that of congruence between *any figures*; but, didactically, the development of the theory is more complicated when we start from Veronese's system than when we start from that of Pasch.

The system of Hilbert offers advantages over both the others from the point of view of the teaching of geometry, and I shall therefore give a short account of his system only, following the Artiole above quoted.

Hilbert's system.

The following are substantially the Postulates laid down.

(1) *If one segment is congruent with another, the second is also congruent with the first.*

(2) *If an angle is congruent with another angle, the second angle is also congruent with the first.*

(3) *Two segments congruent with a third are congruent with one another.*

(4) *Two angles congruent with a third are congruent with one another.*

(5) *Any segment* AB *is congruent with itself, independently of its sense.* This we may express symbolically thus:

$$AB \equiv AB \equiv BA.$$

(6) *Any angle* (ab) *is congruent with itself, independently of its sense.* This we may express symbolically thus:

$$(ab) \equiv (ab) \equiv (ba).$$

(7) *On any straight line* r′, *starting from any one of its points* A′, *and on each side of it respectively, there exists one and only one segment congruent with a segment* AB *belonging to the straight line* r.

(8) *Given a ray* a, *issuing from a point* O, *in any plane which contains it and on each of the two sides of it, there exists one and only one ray* b *issuing from* O *such that the angle* (ab) *is congruent with a given angle* (a′b′).

(9) *If* AB, BC *are two consecutive segments of the same straight line* r (*segments, that is, having an extremity and no other point common*), *and* A′B′, B′C′ *two consecutive segments on another straight line* r′, *and if* AB ≡ A′B′, BC ≡ B′C′, *then*

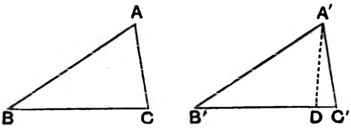
$$AC \equiv A'C'.$$

(10) *If* (ab), (bc) *are two consecutive angles in the same plane* π (*angles, that is, having the vertex and one side common*), *and* (a′b′), (b′c′) *two consecutive angles in another plane* π', *and if* (ab) ≡ (a′b′), (bc) = (b′c′), *then*

$$(ac) = (a'c').$$

(11) *If two triangles have two sides and the included angles respectively congruent, they have also their third sides congruent as well as the angles opposite to the congruent sides respectively.*

As a matter of fact, Hilbert's postulate corresponding to (11) does not assert the equality of the third sides in each, but only the equality of the two remaining angles in one triangle to the two remaining angles in the other respectively. He proves the equality of the third sides (thereby completing the theorem of Eucl. I. 4) by *reductio ad absurdum* thus. Let ABC, $A'B'C'$ be the two triangles which have the sides AB, AC respectively congruent with the sides $A'B'$, $A'C'$ and the included angle at A congruent with the included angle at A'.



Then, by Hilbert's own postulate, the angles ABC, $A'B'C'$ are congruent, as also the angles ACB, $A'C'B'$.

If BC is not congruent with $B'C'$, let D be taken on $B'C'$ such that BC, $B'D$ are congruent, and join $A'D$.

Then the two triangles ABC, $A'B'D$ have two sides and the included angles congruent respectively; therefore, by the same postulate, the angles BAC, $B'A'D$ are congruent.

But the angles BAC, $B'A'C'$ are congruent; therefore, by (4) above, the angles $B'A'C'$, $B'A'D$ are congruent: which is impossible, since it contradicts (8) above.

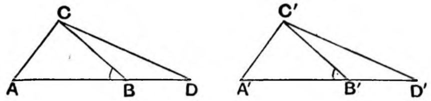
Hence BC, $B'C'$ cannot but be congruent.

Eucl. I. 4 is thus proved; but it seems to be as well to include all of that theorem in the postulate, as is done in (11) above, since the two parts of it are equally suggested by empirical observation of the result of one superposition.

A proof similar to that just given immediately establishes Eucl. I. 26 (1), and Hilbert next proves that

If two angles ABC, A'B'C' *are congruent with one another, their supplementary angles* CBD, C'B'D' *are also congruent with one another.*

We choose A, D on one of the straight lines forming the first angle, and A', D' on one of those forming the second angle, and again C, C' on the other straight lines forming the angles, so that $A'B'$ is congruent with AB, $C'B'$ with CB, and $D'B'$ with DB.



The triangles ABC, $A'B'C'$ are congruent, by (11) above; and AC is congruent with $A'C'$, and the angle CAB with the angle $C'A'B'$.

Thus, AD, $A'D'$ being congruent, by (9), the triangles CAD, $C'A'D'$ are also congruent, by (11);

whence CD is congruent with $C'D'$, and the angle ADC with the angle $A'D'C'$.

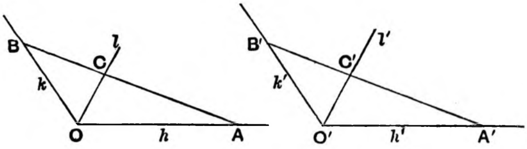
Lastly, by (11), the triangles CDB, $C'D'B'$ are congruent, and the angles CBD, $C'B'D'$ are thus congruent.

Hilbert's next proposition is that

Given that the angle (h, k) *in the plane* α *is congruent with the angle* (h', k') *in the plane* α', *and that* l *is a half-ray in the plane* α *starting from the vertex of the angle* (h, k) *and lying within that angle, there always exists a half-ray* l' *in the second plane* α', *starting from the vertex of the angle* (h', k') *and lying within that angle, such that*

$$(h, l) \equiv (h', l'), \text{ and } (k, l) \equiv (k', l').$$

If O, O' are the vertices, we choose points A, B on h, k, and points A', B' on h', k' respectively, such that OA, $O'A'$ are congruent and also OB, $O'B'$.



The triangles OAB, $O'A'B'$ are then congruent; and, if l meets AB in C, we can determine C' on $A'B'$ such that $A'C'$ is congruent with AC.

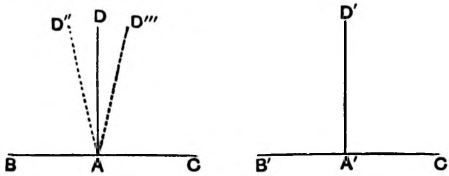
Then l' drawn from O' through C' is the half-ray required.

The congruence of the angles (h, l), (h', l') follows from (11) directly, and that of (k, l) and (k', l') follows in the same way after we have inferred by means of (9) that, AB, AC being respectively congruent with $A'B'$, $A'C'$, the difference BC is congruent with the difference $B'C'$.

It is by means of the two propositions just given that Hilbert proves that

All right angles are congruent with one another.

Let the angle BAD be congruent with its adjacent angle CAD, and likewise the angle $B'A'D'$ congruent with its adjacent angle $C'A'D'$ All four angles are then right angles.



If the angle $B'A'D'$ is not congruent with the angle BAD, let the angle with AB for one side and congruent with the angle $B'A'D'$ be the angle BAD'', so that AD'' falls either within the angle BAD or within the angle DAC. Suppose the former.

By the last proposition but one (about adjacent angles), the angles $B'A'D'$, BAD'' being congruent, the angles $C'A'D'$, CAD'' are congruent.

Hence, by the hypothesis and postulate (4) above, the angles BAD'', CAD'' are also congruent.

And, since the angles BAD, CAD are congruent, we can find within the angle CAD a half-ray CAD''' such that the angles BAD'', CAD''' are congruent, and likewise the angles DAD'', DAD''' (by the last proposition).

But the angles BAD'', CAD'' were congruent (see above); and it follows, by (4), that the angles CAD'', CAD''' are congruent: which is impossible, since it contradicts postulate (8).

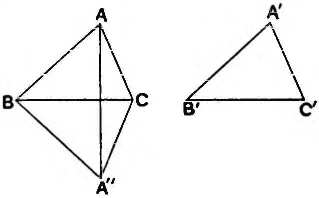
Therefore etc.

Euclid I. 5 follows directly by applying the postulate (11) above to ABC, ACB as distinct triangles.

Postulates (9), (10) above give in substance the proposition that "the sums or differences of segments, or of angles, respectively equal, are equal."

Lastly, Hilbert proves Eucl. I. 8 by means of the theorem of Eucl. I. 5 and the proposition just stated as applied to angles.

ABC, $A'B'C$ being the given triangles with three sides respectively congruent, we suppose an angle CBA'' to be determined, on the side of BC opposite to A, congruent with the angle $A'B'C'$, and we make BA'' equal to $A'B'$.



The proof is obvious, being equivalent to the alternative proof often given in our text-books for Eucl. I. 8.

COMMON NOTION 5.

καὶ τὸ ὅλον τοῦ μέρους μεῖζόν [*ἐστιν*].
The whole is greater than the part.

Proclus includes this "axiom" on the same ground as the preceding one. I think however there is force in the objection which Tannery takes to it, namely that it replaces a *different* expression in Eucl. I. 6, where it is stated that "the triangle *DBC* will be equal to the triangle *ACB*, *the less to the greater: which is absurd*." The axiom appears to be an abstraction or generalisation substituted for an immediate inference from a geometrical figure, but it takes the form of a sort of definition of whole and part. The probabilities seem to be against its being genuine, notwithstanding Proclus' approval of it.

Clavius added the axiom that *the whole is the equal to the sum of its parts.*

OTHER AXIOMS INTRODUCED AFTER EUCLID'S TIME.

[9] *Two straight lines do not enclose* (or *contain*) *a space.*

Proclus (p. 196, 21) mentions this in illustration of the undue multiplication of axioms, and he points out, as an objection to it, that it belongs to the subject matter of geometry, whereas axioms are of a general character, and not peculiar to any one science. The real objection to the axiom is that it is unnecessary, since the fact which it states is included in the meaning of Postulate I. It was no doubt taken from the passage in I. 4, "if...the base *BC* does not coincide with the base *EF*, *two straight lines will enclose a space: which is impossible*"; and we must certainly regard it as an interpolation, notwithstanding that two of the best MSS. have it after Postulate 5, and one gives it as *Common Notion* 9.

Pappus added some others which Proclus objects to (p. 198, 5) because they are either anticipated in the definitions or follow from them.

(*g*) *All the parts of a plane, or of a straight line, coincide with one another.*

(*h*) *A point divides a line, a line a surface, and a surface a solid*; on which Proclus remarks that everything is *divided* by the same things as those by which it is *bounded.*

An-Nairīzī (ed. Besthorn-Heiberg, p. 31, ed. Curtze, p. 38) in his version of this axiom, which he also attributes to Pappus, omits the reference to solids, but mentions planes as a particular case of surfaces.

"(α) *A surface cuts a surface in a line;*

(β) *If two surfaces which cut one another are plane, they cut one another in a straight line;*

(γ) *A line cuts a line in a point* (this last we need in the first proposition)."

(*k*) *Magnitudes are susceptible of the infinite* (*or unlimited*) *both by way of addition and by way of successive diminution, but in both cases potentially only* (τὸ ἄπειρον ἐν τοῖς μεγέθεσίν ἐστιν καὶ τῇ προσθέσει καὶ τῇ ἐπικαθαιρέσει, δυνάμει δὲ ἑκάτερον).

An-Nairīzī's version of this refers to straight lines and plane surfaces only: "*as regards the straight line and the plane surface, in consequence of their evenness, it is possible to produce them indefinitely.*

This "axiom" of Pappus, as quoted by Proclus, seems to be taken directly from the discussion of τὸ ἄπειρον in Aristotle, *Physics* III. 5—8, even to the wording, for, while Aristotle uses the term *division* (διαίρεσις) most frequently as the antithesis of *addition* (σύνθεσις), he occasionally speaks of *subtraction* (ἀφαίρεσις) and *diminution* (καθαίρεσις). Hankel (*Zur Geschichte der Mathematik im Alterthum und Mittelalter*, 1874, pp. 119—120) gave an admirable summary of Aristotle's views on this subject; and they are stated in greater detail in Görland, *Aristoteles und die Mathematik*, Marburg, 1899, pp. 157—183. The infinite or unlimited (ἄπειρον) only exists potentially (δυνάμει), not in actuality (ἐνεργείᾳ). The infinite is so in virtue of its endlessly changing into something else, like day or the Olympic Games (*Phys.* III. 6, 206 a 15—25). The infinite is manifested in different forms in time, in Man, and in the division of magnitudes. For, in general, the infinite consists in something new being continually taken, that something being itself always finite but always different. Therefore the infinite must not be regarded as a particular thing (τόδε τι), as man, house, but as being always in course of becoming or decay, and, though finite at any moment, always different from moment to moment. But there is the distinction between the forms above referred to that, whereas in the case of magnitudes what is once taken remains, in the case of time and Man it passes or is destroyed but the succession is unbroken. The case of addition is in a sense the same as that of division; in the finite magnitude the former takes place in the converse way to the latter; for, as we see the finite magnitude divided *ad infinitum*, so we shall find that addition gives a sum tending to a definite limit. I mean that, in the case of a finite magnitude, you may take a definite fraction of it and add to it (continually) in the same ratio; if now the successive added terms do not include one and the same magnitude whatever it is [i.e. if the successive terms diminish in geometrical progression], you will not come to the end of the finite magnitude, but, if the ratio is increased so that each term does include one and the same magnitude whatever it is, you will come to the end of the finite magnitude, for every finite magnitude is exhausted by continually taking from it any definite fraction whatever. Thus in no other sense does the infinite exist, but only in the sense just mentioned, that is, potentially and by way of diminution (206 a 25—b 13). And in this sense you may have potentially infinite addition, the process being, as we say, in a manner, the same as with division *ad infinitum*: for in the case of addition you will always be able to find something outside the total for the time being, but the total will never exceed every definite (or assigned) magnitude in the way that, in the direction of division, the result will pass every definite magnitude, that is, by becoming smaller than it. The infinite therefore cannot exist even potentially in the sense of exceeding every finite magnitude as the result of successive addition (206 b 16—22). It follows that the correct view of the infinite is the opposite of that commonly held: it is not that which has nothing outside it, but that which always has something outside it (206 b 33—207 a 1).

Contrasting the case of number and magnitude, Aristotle points out that (1) in number there is a limit in the direction of smallness, namely unity, but none in the other direction: a number may exceed any assigned number however great; but (2) with magnitude the contrary is the case: you can find a magnitude smaller than any assigned magnitude, but in the other direction there is no such thing as an infinite magnitude (207 b 1—5). The latter assertion he justified by the following argument. However large a thing can be potentially, it can be as large actually. But there is no magnitude perceptible to sense that is infinite. Therefore excess over every assigned magnitude is an impossibility; otherwise there would be something larger than the universe (οὐρανός) (207 b 17—21).

Aristotle is aware that it is essentially of physical magnitudes that he is speaking. He had observed in an earlier passage (*Phys.* III. 5, 204 a 34) that it is perhaps a more general inquiry that would be necessary to determine whether the infinite is possible in mathematics, and in the domain of thought and of things which have no magnitude; but he excuses himself from entering upon this inquiry on the ground that his subject is physics and sensible objects. He returns however to the bearing of his conclusions on mathematics in III. 7, 207 b 27: "my argument does not even rob mathematicians of their study, although it denies the existence of the infinite in the sense of actual existence as something increased to such an extent that it cannot be gone through (ἀδιεξίτητον); for, as it is, they do not even need the infinite or use it, but only require that the finite (straight line) shall be as long *as they please*; and another magnitude of any size whatever can be cut in the same ratio as the greatest magnitude. Hence it will make no difference to them for the purpose of demonstration."

Lastly, if it should be urged that the infinite exists in *thought*, Aristotle replies that this does not involve its existence in *fact*. A thing is not greater than a certain size because it is conceived to be so, but because it *is*; and magnitude is not infinite in virtue of increase in thought (208 a 16—22).

Hankel and Görland do not quote the passage about an infinite series of magnitudes (206 b 3—13) included in the above paraphrase; but I have thought that mathematicians would be interested in the distinct expression of Aristotle's view that the existence of an infinite series the terms of which are *magnitudes* is impossible unless it is convergent, and (with reference to Riemann's developments) in the statement that it does not matter to geometry if the straight line is not infinite in length, provided that it is as long as we please.

Aristotle's denial of even the potential existence of a sum of magnitudes which shall exceed every definite magnitude was, as he himself implies, in conflict with the lemma or assumption used by Eudoxus (as we infer from Archimedes) to prove the theorem about the volume of a pyramid. The lemma is thus stated by Archimedes (*Quadrature of a parabola*, preface): "The excess by which the greater of two unequal areas exceeds the less can, if it be continually added to itself, be made to exceed any assigned finite area." We can therefore well understand why, a century later, Archimedes felt it necessary to justify his own use of the lemma as he does in the same preface: "The earlier geometers too have used this lemma: for it is by its help that they have proved that circles have to one another the duplicate ratio of their diameters, that spheres have to one another the triplicate ratio of their diameters, and so on. And, in the result, each of the said theorems has been accepted no less than those proved without the aid of this lemma."

Principle of continuity.

The use of actual construction as a method of proving the existence of figures having certain properties is one of the characteristics of the *Elements*. Now constructions are effected by means of straight lines and circles drawn in accordance with Postulates 1—3; the essence of them is that such straight lines and circles determine by their intersections other points in addition to those given, and these points again are used to determine new lines, and so on. This being so, the *existence* of such points of intersection must be postulated or proved in the same way as that of the lines which determine them. Yet there is no postulate of this character expressed in Euclid except Post. 5. This postulate asserts that two straight lines meet if they satisfy a certain condition. The condition is of the nature of a διορισμός (*discrimination*, or condition of possibility) in a problem; and, if the existence of the point of intersection were not granted, the solutions of problems in which the points of intersection of straight lines are used would not in general furnish the required proofs of the existence of the figures to be constructed.

But, equally with the intersections of straight lines, the intersections of circle with straight line, and of circle with circle, are used in constructions. Hence, in addition to Postulate 5, we require postulates asserting the actual existence of points of intersection of circle with straight line and of circle with circle. In the very first proposition the vertex of the required equilateral triangle is determined as one of the intersections of two circles, and we need therefore to be assured that the circles will intersect. Euclid seems to assume it as obvious, although it is not so; and he makes a similar assumption in I. 22. It is true that in the latter case Euclid adds to the enunciation that two of the given straight lines must be together greater than the third; but there is nothing to show that, if this condition is satisfied, the construction is always possible. In I. 12, in order to be sure that the circle with a given centre will intersect a given straight line, Euclid makes the circle pass through a point on the side of the line opposite to that where the centre is. It appears therefore as if, in this case, he based his inference in some way upon the definition of a circle combined with the fact that the point within it called the centre is on one side of the straight line and one point of the circumference on the other, and, in the case of two intersecting circles, upon similar considerations. But not even in Book III., where there are several propositions about the relative positions of two circles, do we find any discussion of the conditions under which two circles have two, one, or no point common.

The deficiency can only be made good by the *Principle of Continuity*.

Killing (*Einführung in die Grundlagen der Geometrie*, II. p. 43) gives the following forms as sufficient for most purposes.

(*a*) Suppose a line belongs entirely to a figure which is divided into two parts; then, if the line has at least one point common with each part, it must also meet the boundary between the parts; or

(*b*) If a point moves in a figure which is divided into two parts, and if it belongs at the beginning of the motion to one part and at the end of the motion to the other part, it must during the motion arrive at the boundary between the two parts.

In the *Questioni riguardanti le matematiche elementari*, I., Art. 5, pp. 123—143, the principle of continuity is discussed with special reference to the Postulate of Dedekind, and it is shown, first, how the Postulate may be led up to and, secondly, how it may be applied for the purposes of elementary geometry.

Suppose that in a segment AB of a straight line a point C determines two segments AC, CB. If we consider the point C as belonging to only one of the two segments AC, CB, we have a division of the segment AB into two parts with the following properties.

1. Every point of the segment AB belongs to *one* of the two parts.
2. The point A belongs to one of the two parts (which we will call the *first*) and the point B to the other; the point C may belong indifferently to one or the other of the two parts according as we choose to premise.
3. Every point of the first part precedes every point of the second in the order AB of the segment.

(For generality we may also suppose the case in which the point C falls at A or at B. Considering C, in these cases respectively, as belonging to the first or second part, we still have a division into parts which have the properties above enunciated, one part being then a single point A or B.)

Now, considering carefully the inverse of the above proposition, we see that it agrees with the idea which we have of the continuity of the straight line. Consequently we are induced to admit as a *postulate* the following.

If a segment of a straight line AB *is divided into two parts so that*

(1) *every point of the segment* AB *belongs to one of the parts,*

(2) *the extremity* A *belongs to the first part and* B *to the second, and*

(3) *any point whatever of the first part precedes any point whatever of the second part, in the order* AB *of the segment,*

there exists a point C *of the segment* AB (*which may belong either to one part or to the other*) *such that every point of* AB *that precedes* C *belongs to the first part, and every point of* AB *that follows* C *belongs to the second part in the division originally assumed.*

(If one of the two parts consists of the single point A or B, the point C is the said extremity A or B of the segment.)

This is the Postulate of Dedekind, which was enunciated by Dedekind himself in the following slightly different form (*Stetigkeit und irrationale Zahlen*, 1872, new edition 1905, p. 11).

"*If all points of a straight line fall into two classes such that every point of the first class lies to the left of every point of the second class, there exists one and only one point which produces this division of all the points into two classes, this division of the straight line into two parts.*"

The above enunciation may be said to correspond to the intuitive notion which we have that, if in a segment of a straight line two points start from the ends and describe the segment in opposite senses, they meet in a point. The point of meeting might be regarded as belonging to both parts, but for the present purpose we must regard it as belonging to one only and subtracted from the other part.

Application of Dedekind's postulate to angles.

If we consider an angle less than two right angles bounded by two rays a, b, and draw the straight line connecting A, a point on a, with B, a point on b, we see that all points on the finite segment AB correspond univocally to all the rays of the angle, the point corresponding to any ray being the point in which the ray cuts the segment AB; and if a ray be supposed to move about the vertex of the angle from the position a to the position b, the corresponding points of the segment AB are seen to follow in the same order as the corresponding rays of the angle (ab).

Consequently, if the angle (ab) is divided into two parts so that

(1) each ray of the angle (ab) belongs to one of the two parts,

(2) the outside ray a belongs to the first part and the ray b to the second,

(3) any ray whatever of the first part precedes any ray whatever of the second part,

the corresponding points of the segment AB determine two parts of the segments such that

(1) every point of the segment AB belongs to one of the two parts,

(2) the extremity A belongs to the first part and B to the second,

(3) any point whatever of the first part precedes any point whatever of the second.

But in that case there exists a point C of AB (which may belong to one or the other of the two parts) such that every point of AB that precedes C belongs to the first part and every point of AB that follows C belongs to the second part.

Thus exactly the same thing holds of c, the ray corresponding to C, with reference to the division of the angle (ab) into two parts.

It is not difficult to extend this to an angle (ab) which is either *flat* or greater than two right angles; this is done (Vitali, *op. cit.* pp. 126—127) by supposing the angle to be divided into two, (ad), (db), each less than two right angles, and considering the three cases in which

(1) the ray d is such that all the rays that precede it belong to the first part and those which follow it to the second part,
(2) the ray d is followed by some rays of the first part,
(3) the ray d is preceded by some rays of the second part.

Application to circular arcs.

If we consider an arc AB of a circle with centre O, the points of the arc correspond univocally, and in the same order, to the rays from the point O passing through those points respectively, and the same argument by which we passed from a segment of a straight line to an angle can be used to make the transition from an angle to an arc.

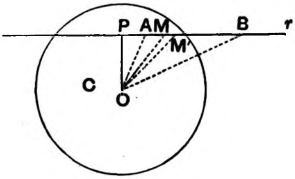
Intersections of a straight line with a circle.

It is possible to use the Postulate of Dedekind to prove that

If a straight line has one point inside and one point outside a circle, it has two points common with the circle.

For this purpose it is necessary to assume (1) the proposition with reference to the perpendicular and obliques drawn from a given point to a given straight line, namely that of all straight lines drawn from a given point to a given straight line the perpendicular is the shortest, and of the rest (the obliques) that is the longer which has the longer projection upon the straight line, while those are equal the projections of which are equal, so that for any given length of projection there are two equal obliques and two only, one on each side of the perpendicular, and (2) the proposition that any side of a triangle is less than the sum of the other two.

Consider the circle (C) with centre O, and a straight line (r) with one point A inside and one point B outside the circle.



By the definition of the circle, if R is the radius,

$$OA < R, \quad OB > R.$$

Draw OP perpendicular to the straight line r.

Then $OP < OA$, so that OP is always less than R, and P is therefore within the circle C.

Now let us fix our attention on the finite segment AB of the straight line r. It can be divided into two parts, (1) that containing all the points H for which $OH < R$ (i.e. points inside C), and (2) that containing all the points K for which $OK \geqq R$ (points outside C or on the circumference of C).

Thus, remembering that, of two obliques from a given point to a given straight line, that is greater the projection of which is greater, we can assert that all the points of the segment PB which *precede* a point inside C are inside C, and those which *follow* a point on the circumference of C or outside C are outside C.

Hence, by the Postulate of Dedekind, there exists on the segment PB a point M such that all the points which precede it belong to the first part and those which follow it to the second part.

I say that M is common to the straight line r and the circle C, or

$$OM = R.$$

For suppose, e.g., that $OM < R$.

There will then exist a segment (or length) σ less than the difference between R and OM.

Consider the point M', one of those which *follow* M, such that MM' is equal to σ.

Then, because any side of a triangle is less than the sum of the other two,

$$OM' < OM + MM'.$$

But $$OM + MM' = OM + \sigma < R,$$

whence $$OM' < R,$$

which is absurd.

A similar absurdity would follow if we suppose that $OM > R$.

Therefore OM must be equal to R.

It is immediately obvious that, corresponding to the point M on the segment PB which is common to r and C, there is another point on r which has the same property, namely that which is symmetrical to M with respect to P.

And the proposition is proved.

Intersections of two circles.

We can likewise use the Postulate of Dedekind to prove that

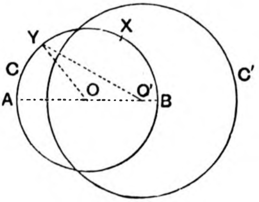
If in a given plane a circle C *has one point* X *inside and one point* Y *outside another circle* C′, *the two circles intersect in two points.*

We must first prove the following

Lemma.

If O, O' are the centres of two circles C, C', and R, R' their radii respectively, the straight line OO' meets the circle C in two points A, B, one of which is inside C' and the other outside it.

Now one of these points must fall (1) on the prolongation of $O'O$ beyond O or (2) on OO' itself or (3) on the prolongation of OO' beyond O'.



(1) First, suppose A to lie on $O'O$ produced.

Then $AO' = AO + OO' = R + OO'$ ......(α).

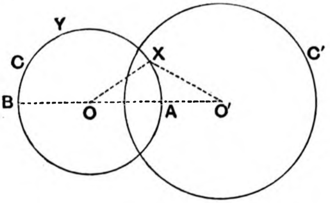
But, in the triangle $OO'Y$,

$$O'Y < OY + OO',$$

and, since $O'Y > R'$, $OY = R$,

$$R' < R + OO'.$$

It follows from (α) that $AO' > R'$; and A therefore lies *outside* C'.



(2) Secondly, suppose A to lie on OO'.

Then $OO' = OA + AO' = R + AO'$...(β).

From the triangle $OO'X$ we have

$$OO' < OX + O'X,$$

and, since $OX = R$, $O'X < R'$, it follows that

$$OO' < R + R',$$

whence, by (β), $AO' < R'$, so that A lies *inside* C'.

(3) Thirdly, suppose A to lie on OO' produced.

Then $\quad R = OA = OO' + O'A \,.........(\gamma)$.

And, in the triangle $OO'X$,

$$OX < OO' + O'X,$$

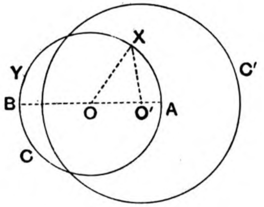
that is $\quad R < OO' + O'X,$

whence, by (γ),

$$OO' + O'A < OO' + O'X,$$

or $\quad O'A < O'X,$

and A lies *inside* C'.



It is to be observed that one of the two points A, B is in the position of case (1) and the other in the position of either case (2) or case (3): whence we must conclude that one of the two points A, B is *inside* and the other *outside* the circle C'.

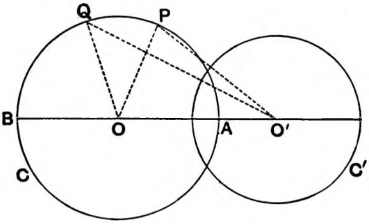
Proof of theorem.

The circle C is divided by the points A, B into two semicircles. Consider one of them, and suppose it to be described by a point moving from A to B.

Take two separate points P, Q on it and, to fix our ideas, suppose that P precedes Q.

Comparing the triangles $OO'P$, $OO'Q$, we observe that one side OO' is common, OP is equal to OQ, and the angle POO' is less than the angle QOO'.

Therefore $O'P < O'Q$.



Now, considering the semicircle $APQB$ as divided into two parts, so that the points of the first part are inside the circle C', and those of the second part on the circumference of C' or outside it, we have the conditions necessary for the applicability of the Postulate of Dedekind (which is true for arcs of circles as for straight lines); whence *there exists a point* M *separating the two parts.*

I say that $O'M = R'$.

For, if not, suppose $O'M < R'$.

If then σ signifies the difference between R' and $O'M$, suppose a point M', which *follows* M, taken on the semicircle such that the chord MM' is not greater than σ (for a way of doing this see below).

Then, in the triangle $O'MM'$,

$$O'M' < O'M + MM' < O'M + \sigma,$$

and therefore $\quad O'M' < R'$.

It follows that M', a point on the arc MB, is inside the circle C': which is absurd.

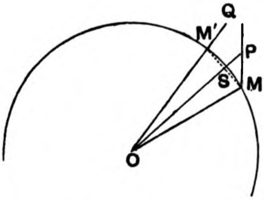
Similarly it may be proved that $O'M$ is not greater than R.

Hence $O'M = R$.

[To find a point M' such that the chord MM' is not greater than σ, we may proceed thus.

Draw from M a straight line MP distinct from OM, and cut off MP on it equal to $\sigma/2$.

Join OP, and draw another radius OQ such that the angle POQ is equal to the angle MOP.



The intersection, M', of OQ with the circle satisfies the required condition.

For MM' meets OP at right angles in S.

Therefore, in the right-angled triangle MSP, MS is not greater than MP (it is less, unless MP coincides with MS, when it is equal).

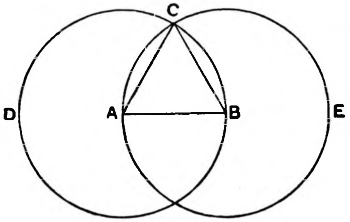
Therefore MS is not greater than $\sigma/2$, so that MM' is not greater than σ.]

BOOK I. PROPOSITIONS.

PROPOSITION 1.

On a given finite straight line to construct an equilateral triangle.

Let AB be the given finite straight line.



Thus it is required to construct an equilateral triangle on the straight line AB.

With centre A and distance AB let the circle BCD be described; [Post. 3]

again, with centre B and distance BA let the circle ACE be described; [Post. 3]

and from the point C, in which the circles cut one another, to the points A, B let the straight lines CA, CB be joined. [Post. 1]

Now, since the point A is the centre of the circle CDB,

AC is equal to AB. [Def. 15]

Again, since the point B is the centre of the circle CAE,

BC is equal to BA. [Def. 15]

But CA was also proved equal to AB;

therefore each of the straight lines CA, CB is equal to AB.

And things which are equal to the same thing are also equal to one another; [*C. N.* 1]

therefore CA is also equal to CB.

Therefore the three straight lines CA, AB, BC are equal to one another.

Therefore the triangle *ABC* is equilateral; and it has been constructed on the given finite straight line *AB*.

(Being) what it was required to do.

1. **On a given finite straight line.** The Greek usage differs from ours in that the definite article is employed in such a phrase as this where we have the indefinite. ἐπὶ τῆς δοθείσης εὐθείας πεπερασμένης, "on *the* given finite straight line," i.e. the finite straight line which we choose to take.

3. **Let AB be the given finite straight line.** To be strictly literal we should have to translate in the reverse order "let the given finite straight line be the (straight line) *AB*"; but this order is inconvenient in other cases where there is more than one datum, e.g. in the *setting-out* of I. 2, "let the given point be *A*, and the given straight line *BC*," the awkwardness arising from the omission of the verb in the second clause. Hence I have, for clearness' sake, adopted the other order throughout the book.

8. **let the circle BCD be described.** Two things are here to be noted, (1) the elegant and practically universal use of the perfect passive imperative in constructions, γεγράφθω meaning of course "let it *have been* described" or "suppose it described," (2) the impossibility of expressing shortly in a translation the force of the words in their original order. κύκλος γεγράφθω ὁ ΒΓΔ means literally "let a circle have been described, the (circle, namely, which I denote by) *BCD*." Similarly we have lower down "let straight lines, (namely) the (straight lines) *CA*, *CB*, be joined," ἐπεζεύχθωσαν εὐθεῖαι αἱ ΓΑ, ΓΒ. There seems to be no practicable alternative, in English, but to translate as I have done in the text.

13. **from the point C....** Euclid is careful to adhere to the phraseology of Postulate 1 except that he speaks of "joining" (ἐπεζεύχθωσαν) instead of "drawing" (γράφειν). He does not allow himself to use the shortened expression "let the straight line *FC* be joined" (without mention of the points *F*, *C*) until I. 5.

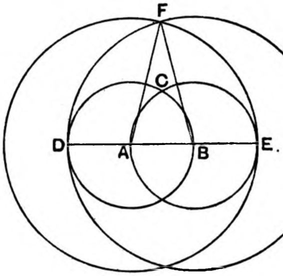
20. **each of the straight lines CA, CB,** ἑκατέρα τῶν ΓΑ, ΓΒ and 24. **the three straight lines CA, AB, BC,** αἱ τρεῖς αἱ ΓΑ, ΑΒ, ΒΓ. I have, here and in all similar expressions, inserted the words "straight lines" which are not in the Greek. The possession of the inflected definite article enables the Greek to omit the words, but this is not possible in English, and it would scarcely be English to write "each of *CA*, *CB*" or "the three *CA*, *AB*, *BC*."

It is a commonplace that Euclid has no right to assume, without premising some postulate, that the two circles *will* meet in a point *C*. To supply what is wanted we must invoke the Principle of Continuity (see note thereon above, p. 235). It is sufficient for the purpose of this proposition and of I. 22, where there is a similar tacit assumption, to use the form of postulate suggested by Killing. "*If a line* [in this case e.g. the circumference *ACE*] *belongs entirely to a figure* [in this case a plane] *which is divided into two parts* [namely the part enclosed within the circumference of the circle *BCD* and the part outside that circle], *and if the line has at least one point common with each part, it must also meet the boundary between the parts* [i.e. the circumference *ACE* must meet the circumference *BCD*]."

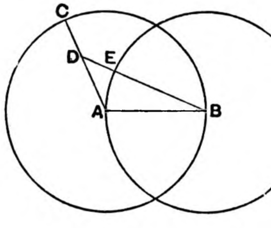
Zeno's remark that the problem is not solved unless it is taken for granted that two straight lines cannot have a common segment has already been mentioned (note on Post. 2, p. 196). Thus, if *AC*, *BC* meet at *F* before reaching *C*, and have the part *FC* common, the triangle obtained, namely *FAB*, will not be equilateral, but *FA*, *FB* will each be less than *AB*. But Post. 2 has already laid it down that two straight lines cannot have a common segment.

Proclus devotes considerable space to this part of Zeno's criticism, but satisfies himself with the bare mention of the other part, to the effect that it is also necessary to assume that two *circumferences* (with different centres) cannot have a common part. That is, for anything we know, there may be any number of points *C* common to the two circumferences *ACE*, *BCD*. It is not until III. 10 that it is proved that two circles cannot intersect in more points than two, so that we are not entitled to assume it here. The most we can say is that it is enough for the purpose of this proposition if *one* equilateral triangle can be found with the given base; that the construction only gives *two* such triangles has to be left over to be proved subsequently. And indeed we have not long to wait; for I. 7 clearly shows that on either side of the base *AB* only *one* equilateral triangle can be described. Thus I. 7 gives us the *number of solutions* of which the present problem is susceptible, and it supplies the same want in I. 22 where a triangle has to be described with three sides of given length; that is, I. 7 furnishes us, in both cases, with one of the essential parts of a complete διορισμός, which includes not only the determination of the conditions of possibility but also the number of solutions (ποσαχῶς ἐγχωρεῖ, Proclus, p. 202, 5). This view of I. 7 as supplying an equivalent for III. 10 absolutely needed in I. 1 and I. 22 should serve to correct the idea so common among writers of text-books that I. 7 is merely of use as a lemma to Euclid's proof of I. 8, and therefore may be left out if an alternative proof of that proposition is adopted.

Agreeably to his notion that it is from I. 1 that we must satisfy ourselves that isosceles and scalene triangles actually exist, as well as equilateral triangles, Proclus shows how to draw, first a particular isosceles triangle, and then a scalene triangle, by means of the figure of the proposition. To make an isosceles triangle he produces *AB* in both directions to meet the respective circles in *D*, *E*, and then describes circles with *A*, *B* as centres and *AE*, *BD* as radii respectively. The result is an isosceles triangle with each of two sides double of the third side. To make an isosceles triangle in which the equal sides are not so related to the third side but have any given length would require the use of I. 3; and there is no object in treating the question at all in advance of I. 22. An easier way of satisfying ourselves of the existence of some isosceles triangles would surely be to conceive any two radii of a circle drawn and their extremities joined.



There is more point in Proclus' construction of a *scalene* triangle. Suppose *AC* to be a radius of one of the two circles, and *D* a point on *AC* lying in that portion of the circle with centre *A* which is outside the circle with centre *B*. Then, joining *BD*, as in the figure, we have a triangle which obviously has all its sides unequal, that is, a *scalene* triangle.



The above two constructions appear in an-Nairīzī's commentary under the name of Heron; Proclus does not mention his source.

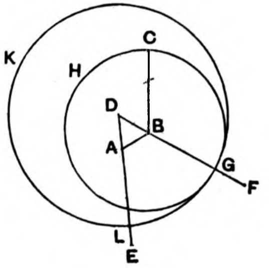
In addition to the above construction for a scalene triangle (producing a triangle in which the "given" side is greater than one and less than the other of the two remaining sides), Heron has two others showing the other two possible cases, in which the "given" side is (1) less than, (2) greater than, either of the other two sides.

PROPOSITION 2.

To place at a given point (as an extremity) a straight line equal to a given straight line.

Let *A* be the given point, and *BC* the given straight line.

Thus it is required to place at the point *A* (as an extremity) a straight line equal to the given straight line *BC*.



From the point *A* to the point *B* let the straight line *AB* be joined; [Post. 1]

and on it let the equilateral triangle *DAB* be constructed. [I. 1]

Let the straight lines *AE*, *BF* be produced in a straight line with *DA*, *DB*; [Post. 2]

with centre *B* and distance *BC* let the circle *CGH* be described; [Post. 3]

and again, with centre *D* and distance *DG* let the circle *GKL* be described. [Post. 3]

Then, since the point *B* is the centre of the circle *CGH*,

BC is equal to *BG*.

Again, since the point *D* is the centre of the circle *GKL*,

DL is equal to *DG*.

And in these *DA* is equal to *DB*;

therefore the remainder *AL* is equal to the remainder *BG*. [*C.N.* 3]

But *BC* was also proved equal to *BG*;

therefore each of the straight lines *AL*, *BC* is equal to *BG*.

And things which are equal to the same thing are also equal to one another; [*C.N.* 1]

therefore *AL* is also equal to *BC*.

Therefore at the given point *A* the straight line *AL* is placed equal to the given straight line *BC*.

(Being) what it was required to do.

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1. **(as an extremity).** I have inserted these words because "to place a straight line *at* a given point" (πρὸς τῷ δοθέντι σημείῳ) is not quite clear enough, at least in English.

11. **Let the straight lines AE, BF be produced....** It will be observed that in this first application of Postulate 2, and again in I. 5, Euclid speaks of the *continuation* of the straight line as that which is produced in such cases, ἐκβεβλήσθωσαν and προσεκβεβλήσθωσαν meaning little more than *drawing* straight lines "in a straight line with" the given straight lines. The first place in which Euclid uses phraseology exactly corresponding to ours when

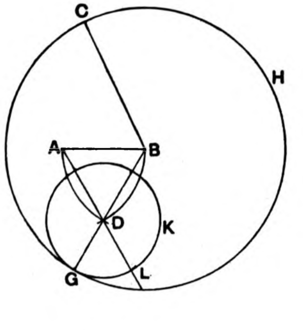
speaking of a straight line being produced is in I. 16: "let one side of it, *BC*, be produced to *D*" (προσεκβεβλήσθω αὐτοῦ μία πλευρὰ ἡ ΒΓ ἐπὶ τὸ Δ).

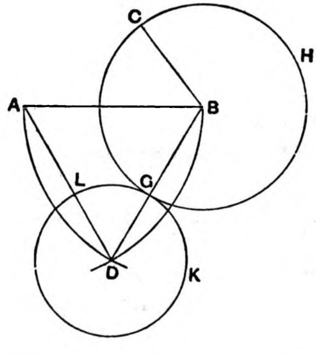
23. **the remainder AL...the remainder BG.** The Greek expressions are λοιπὴ ἡ ΑΛ and λοιπῇ τῇ ΒΗ, and the literal translation would be "*AL* (or *BG*) *remaining*," but the shade of meaning conveyed by the position of the definite article can hardly be expressed in English.

This proposition gives Proclus an opportunity, such as the Greek commentators revelled in, of distinguishing a multitude of *cases*. After explaining that those theorems and problems are said to have *cases* which have the same force, though admitting of a number of different figures, and preserve the same method of demonstration while admitting variations of position, and that cases reveal themselves in the *construction*, he proceeds to distinguish the cases in this problem arising from the different positions which the given point may occupy relatively to the given straight line. It may be (he says) either (1) outside the line or (2) on the line, and, if (1), it may be either (*a*) on the line produced or (*b*) situated obliquely with regard to it; if (2), it may be either (*a*) one of the extremities of the line or (*b*) an intermediate point on it. It will be seen that Proclus' anxiety to subdivide leads him to give a "case," (2) (*a*), which is useless, since in that "case" we are given what we are required to find, and there is really no problem to solve. As Savile says, "qui quaerit ad β punctum ponere rectam aequalem τῇ βγ rectae, quaerit quod datum est, quod nemo faceret nisi forte insaniat."

Proclus gives the construction for (2) (*b*) following Euclid's way of taking *G* as the point in which the circle with centre *B* intersects *DB produced*, and then proceeds to "cases," of which there are still more, which result from the different ways of drawing the equilateral triangle and of producing its sides.

This last class of "cases" he subdivides into three according as *AB* is (1) equal to, (2) greater than or (3) less than *BC*. Here again "case" (1) serves no purpose, since, if *AB* is equal to *BC*, the problem is already solved. But Proclus' figures for the other two cases are worth giving, because in one of them the point *G* is on *BD* produced beyond *D*, and in the other it lies on *BD* itself and there is no need to produce any side of the equilateral triangle.





A glance at these figures will show that, if they were used in the proposition, each of them would require a slight modification in the wording (1) of the construction, since *BD* is in one case produced beyond *D* instead of *B* and in the other case not produced at all, (2) of the proof, since *BG*, instead of being the difference between *DG* and *DB*, is in one case the sum of *DG* and *DB* and in the other the difference between *DB* and *DG*.

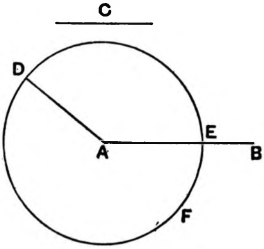
Modern editors generally seem to classify the cases according to the possible variations in the construction rather than according to differences in the data. Thus Lardner, Potts, and Todhunter distinguish eight cases due to the three possible alternatives, (1) that the given point may be joined to either end of the given straight line, (2) that the equilateral triangle may then be described on either side of the joining line, and (3) that the side of the equilateral triangle which is produced may be produced in either direction. (But it should have been observed that, where *AB* is greater than *BC*, the third alternative is between producing *DB* and not producing it at all.) Potts adds that, when the given point lies either on the line or on the line produced, the distinction which arises from joining the two ends of the line with the given point no longer exists, and there are only four cases of the problem (I think he should rather have said *solutions*).

To distinguish a number of cases in this way was foreign to the really classical manner. Thus, as we shall see, Euclid's method is to give one case only, for choice the most difficult, leaving the reader to supply the rest for himself. Where there was a real distinction between cases, sufficient to necessitate a substantial difference in the proof, the practice was to give separate *enunciations* and proofs altogether, as we may see, e.g., from the *Conics* and the *De sectione rationis* of Apollonius.

Proclus alludes, in conclusion, to the error of those who proposed to solve I. 2 by describing a circle with the given point as centre and with a distance *equal* to *BC*, which, as he says, is a *petitio principii*. De Morgan puts the matter very clearly (*Supplementary Remarks on the first six Books of Euclid's Elements* in the *Companion to the Almanac*, 1849, p. 6). We should "insist," he says, "here upon the restrictions imposed by the first three postulates, which do not allow a circle to be drawn with a compass-carried distance; suppose the compasses to close of themselves the moment they cease to touch the paper. These two propositions [I. 2, 3] extend the power of construction to what it would have been if *all* the usual power of the compasses had been assumed; they are mysterious to all who do not see that postulate iii does not ask for *every use of the compasses*."

PROPOSITION 3.

Given two unequal straight lines, to cut off from the greater a straight line equal to the less.



Let *AB*, *C* be the two given unequal straight lines, and let *AB* be the greater of them.

Thus it is required to cut off from *AB* the greater a straight line equal to *C* the less.

At the point *A* let *AD* be placed equal to the straight line *C*; [I. 2]
and with centre *A* and distance *AD* let the circle *DEF* be described. [Post. 3]

Now, since the point *A* is the centre of the circle *DEF*,

AE is equal to *AD*. [Def. 15]

But *C* is also equal to *AD*.

Therefore each of the straight lines *AE*, *C* is equal to *AD*; so that *AE* is also equal to *C*. [*C.N.* 1]

Therefore, given the two straight lines *AB*, *C*, from *AB* the greater *AE* has been cut off equal to *C* the less.

(Being) what it was required to do.

Proclus contrives to make a number of "cases" out of this proposition also, and gives as many as eight figures. But he only produces this variety by practically incorporating the construction of the preceding proposition, instead of assuming it as we are entitled to do. If Prop. 2 is assumed, there is really only one "case" of the present proposition, for Potts distinction between two cases according to the particular extremity of the straight line from which the given length has to be cut off scarcely seems to be worth making.

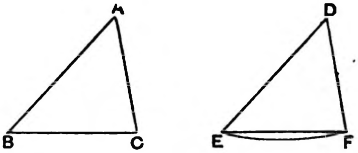
Proposition 4.

If two triangles have the two sides equal to two sides respectively, and have the angles contained by the equal straight lines equal, they will also have the base equal to the base, the triangle will be equal to the triangle, and the remaining angles will be equal to the remaining angles respectively, namely those which the equal sides subtend.

Let *ABC*, *DEF* be two triangles having the two sides *AB*, *AC* equal to the two sides *DE*, *DF* respectively, namely *AB* to *DE* and *AC* to *DF*, and the angle *BAC* equal to the angle *EDF*.

I say that the base *BC* is also equal to the base *EF*, the triangle *ABC* will be equal to the triangle *DEF*, and the remaining angles will be equal to the remaining angles respectively, namely those which the equal sides subtend, that is, the angle *ABC* to the angle *DEF*, and the angle *ACB* to the angle *DFE*.

For, if the triangle *ABC* be applied to the triangle *DEF*, and if the point *A* be placed on the point *D* and the straight line *AB* on *DE*, then the point *B* will also coincide with *E*, because *AB* is equal to *DE*.



Again, *AB* coinciding with *DE*,

the straight line *AC* will also coincide with *DF*, because the angle *BAC* is equal to the angle *EDF*;

hence the point *C* will also coincide with the point *F*, because *AC* is again equal to *DF*.

But *B* also coincided with *E*;

hence the base *BC* will coincide with the base *EF*.

[For if, when *B* coincides with *E* and *C* with *F*, the base *BC* does not coincide with the base *EF*, two straight lines will enclose a space: which is impossible.

Therefore the base *BC* will coincide with

EF] and will be equal to it. [*C.N.* 4]

Thus the whole triangle *ABC* will coincide with the whole triangle *DEF*,

and will be equal to it.

And the remaining angles will also coincide with the remaining angles and will be equal to them,

the angle *ABC* to the angle *DEF*,

and the angle *ACB* to the angle *DFE*.

Therefore etc.

(Being) what it was required to prove.

1—3. It is a fact that Euclid's enunciations not infrequently leave something to be desired in point of clearness and precision. Here he speaks of the triangles having "the angle equal to the angle, namely the angle contained by the equal straight lines" (τὴν γωνίαν τῇ γωνίᾳ ἴσην ἔχῃ τὴν ὑπὸ τῶν ἴσων εὐθειῶν περιεχομένην), only one of the two angles being described in the latter expression (in the accusative), and a similar expression in the dative being left to be understood of the other angle. It is curious too that, after mentioning two "*sides*," he speaks of the angles contained by the equal "*straight lines*," not "*sides*." It may be that he wished to adhere scrupulously, at the outset, to the phraseology of the definitions, where the angle is the inclination to one another of two *lines* or *straight lines*. Similarly in the enunciation of I. 5 he speaks of producing the equal "straight lines" as if to keep strictly to the wording of Postulate 2.

2. **respectively.** I agree with Mr H. M. Taylor (*Euclid*, p. ix) that it is best to abandon the traditional translation of "each to each," which would naturally seem to imply that all the four magnitudes are equal rather than (as the Greek ἑκατέρα ἑκατέρᾳ does) that one is equal to one and the other to the other.

3. **the base.** Here we have the word *base* used for the first time in the *Elements*. Proclus explains it (p. 236, 12—15) as meaning (1), when no side of a triangle has been mentioned before, the side "which is on a level with the sight" (τὴν πρὸς τῇ ὄψει κειμένην), and (2), when two sides have already been mentioned, the third side. Proclus thus avoids the mistake made by some modern editors who explain the term exclusively with reference to the case where two sides have been mentioned before. That this is an error is proved (1) by the occurrence of the term in the enunciations of I. 37 etc. about triangles on the same base and equal bases, (2) by the application of the same term to the bases of parallelograms in I. 35 etc. The truth is that the use of the term must have been suggested by the practice of drawing the particular side horizontally, as it were, and the rest of the figure above it. The *base* of a figure was therefore spoken of, primarily, in the same sense as the base of anything

else, e.g. of a pedestal or column; but when, as in I. 5, two triangles were compared occupying other than the normal positions which gave rise to the name, and when two sides had been previously mentioned, the base was, as Proclus says, necessarily the third side.

6. **subtend.** *ὑποτείνειν ὑπό*, "to stretch under," with accusative.

9. **the angle BAC.** The full Greek expression would be *ἡ ὑπὸ τῶν* ΒΑ, ΑΓ *περιεχομένη γωνία*, "the angle contained by the (straight lines) *BA*, *AC*." But it was a common practice of Greek geometers, e.g. of Archimedes and Apollonius (and Euclid too in Books X.—XIII.), to use the abbreviation *αἱ* ΒΑΓ for *αἱ* ΒΑ, ΑΓ, "the (straight lines) *BA*, *AC*." Thus, on *περιεχομένη* being dropped, the expression would become first *ἡ ὑπὸ τῶν* ΒΑΓ *γωνία*, then *ἡ ὑπὸ* ΒΑΓ *γωνία*, and finally *ἡ ὑπὸ* ΒΑΓ, without *γωνία*, as we regularly find it in Euclid.

17. **if the triangle be applied to..., 23. coincide.** The difference between the technical use of the passive *ἐφαρμόζεσθαι* "to be *applied* (to)," and of the active *ἐφαρμόζειν* "to *coincide* (with)" has been noticed above (note on *Common Notion* 4, pp. 224—5).

32. **[For if, when B coincides...36. coincide with EF].** Heiberg (*Paralipomena zu Euklid* in *Hermes*, XXXVIII., 1903, p. 56) has pointed out, as a conclusive reason for regarding these words as an early interpolation, that the text of an-Nairīzī (*Codex Leidensis* 399, 1, ed. Besthorn-Heiberg, p. 55) does not give the words in this place but after the conclusion Q.E.D., which shows that they constitute a *scholium* only. They were doubtless added by some commentator who thought it necessary to explain the immediate inference that, since *B* coincides with *E* and *C* with *F*, the straight line *BC* coincides with the straight line *EF*, an inference which really follows from the definition of a straight line and Post. 1; and no doubt the Postulate that "Two straight lines cannot enclose a space" (afterwards placed among the *Common Notions*) was interpolated at the same time.

44. **Therefore etc.** Where (as here) Euclid's *conclusion* merely repeats the enunciation word for word, I shall avoid the repetition and write "Therefore etc." simply.

In the note on *Common Notion* 4 I have already mentioned that Euclid obviously used the method of superposition with reluctance, and I have given, after Veronese for the most part, the reason for holding that that method is not admissible as a *theoretical* means of proving equality, although it may be of use as a *practical* test, and may thus furnish an empirical basis on which to found a postulate. Mr Bertrand Russell observes (*Principles of Mathematics* I. p. 405) that Euclid would have done better to assume I. 4 as an axiom, as is practically done by Hilbert (*Grundlagen der Geometrie*, p. 9). It may be that Euclid himself was as well aware of the objections to the method as are his modern critics; but at all events those objections were stated, with almost equal clearness, as early as the middle of the 16th century. Peletarius (Jacques Peletier) has a long note on this proposition (*In Euclidis Elementa geometrica demonstrationum libri sex*, 1557), in which he observes that, if superposition of lines and figures could be assumed as a method of proof, the whole of geometry would be full of such proofs, that it could equally well have been used in I. 2, 3 (thus in I. 2 we could simply have supposed the line taken up and *placed* at the point), and that in short it is obvious how far removed the method is from the dignity of geometry. The theorem, he adds, is obvious in itself and does not require proof; although it is introduced as a theorem, it would seem that Euclid intended it rather as a *definition* than a theorem, "for I cannot think that two angles are equal unless I have a conception of what equality of angles is." Why then did Euclid include the proposition among theorems, instead of placing it among the axioms? Peletarius makes the best excuse he can, but concludes thus: "Huius itaque propositionis veritatem non aliunde quam a communi iudicio petemus; cogitabimusque figuras figuris superponere, Mechanicum quippiam esse: intelligere verò, id demum esse Mathematicum."

Expressed in terms of the modern systems of Congruence-Axioms referred to in the note on *Common Notion* 4, what Euclid really assumes amounts to the following:

(1) On the line *DE*, there is a point *E*, on either side of *D*, such that *AB* is equal to *DE*.

(2) On either side of the ray *DE* there is a ray *DF* such that the angle *EDF* is equal to the angle *BAC*.

It now follows that on *DF* there is a point *F* such that *DF* is equal to *AC*.

And lastly (3), we require an axiom from which to infer that the two remaining angles of the triangles are respectively equal and that the bases are equal.

I have shown above (pp. 229—230) that Hilbert has an axiom stating the equality of the remaining angles simply, but proves the equality of the bases.

Another alternative is that of Pasch (*Vorlesungen über neuere Geometrie*, p. 109) who has the following "Grundsatz":

If two figures *AB* and *FGH* are given (*FGH* not being contained in a straight length), and *AB*, *FG* are congruent, and if a plane surface be laid through *A* and *B*, we can specify in this plane surface, produced if necessary, two points *C*, *D*, neither more nor less, such that the figures *ABC* and *ABD* are congruent with the figure *FGH*, and the straight line *CD* has with the straight line *AB* or with *AB* produced one point common.

I pass to two points of detail in Euclid's proof:

(1) The inference that, since *B* coincides with *E*, and *C* with *F*, the bases of the triangles are wholly coincident rests, as expressly stated, on the impossibility of two straight lines enclosing a space, and therefore presents no difficulty.

But (2) most editors seem to have failed to observe that at the very beginning of the proof a much more serious assumption is made without any explanation whatever, namely that, if *A* be placed on *D*, and *AB* on *DE*, the point *B* will coincide with *E*, because *AB* is equal to *DE*. That is, the *converse* of *Common Notion* 4 is assumed for straight lines. Proclus merely observes, with regard to the converse of this Common Notion, that it is only true in the case of things "of the same form" (ὁμοειδῆ), which he explains as meaning straight lines, arcs of one and the same circle, and angles "contained by lines similar and similarly situated" (p. 241, 3—8).

Savile however saw the difficulty and grappled with it in his note on the Common Notion. After stating that all straight lines with two points common are congruent between them (for otherwise two straight lines would enclose a space), he argues thus. Let there be two straight lines *AB*, *DE*, and let *A* be placed on *D*, and *AB* on *DE*. Then *B* will coincide with *E*. For, if not, let *B* fall somewhere short of *E* or beyond *E*; and in either case it will follow that the less is equal to the greater, which is impossible.

Savile seems to assume (and so apparently does Lardner who gives the same proof) that, if the straight lines be "applied," *B* will fall somewhere on *DE* or *DE* produced. But the ground for this assumption should surely be stated; and it seems to me that it is necessary to use, not Postulate 1 alone, nor Postulate 2 alone, but both, for this purpose (in other words to assume, not only that *two straight lines cannot enclose a space*, but also that *two straight lines cannot have a common segment*). For the only safe course is to place *A* upon *D* and then turn *AB* about *D* until *some* point on *AB* intermediate between *A* and *B* coincides with *some* point on *DE*. In this position *AB* and *DE* have two points common. Then Postulate 1 enables us to infer that the straight lines coincide *between* the two common points, and Postulate 2 that they coincide beyond the second common point towards *B* and *E*. Thus the straight lines coincide throughout so far as *both* extend; and Savile's argument then proves that *B* coincides with *E*.

PROPOSITION 5.

In isosceles triangles the angles at the base are equal to one another, and, if the equal straight lines be produced further, the angles under the base will be equal to one another.

Let *ABC* be an isosceles triangle having the side *AB* equal to the side *AC*;

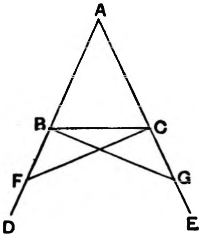
and let the straight lines *BD*, *CE* be produced further in a straight line with *AB*, *AC*. [Post. 2]

I say that the angle *ABC* is equal to the angle *ACB*, and the angle *CBD* to the angle *BCE*.

Let a point *F* be taken at random on *BD*;

from *AE* the greater let *AG* be cut off equal to *AF* the less; [I. 3]

and let the straight lines *FC*, *GB* be joined. [Post. 1]



Then, since *AF* is equal to *AG* and *AB* to *AC*,

the two sides *FA*, *AC* are equal to the two sides *GA*, *AB*, respectively;

and they contain a common angle, the angle *FAG*.

Therefore the base *FC* is equal to the base *GB*,

and the triangle *AFC* is equal to the triangle *AGB*,

and the remaining angles will be equal to the remaining angles respectively, namely those which the equal sides subtend,

that is, the angle *ACF* to the angle *ABG*,

and the angle *AFC* to the angle *AGB*. [I. 4]

And, since the whole *AF* is equal to the whole *AG*,

and in these *AB* is equal to *AC*,

the remainder *BF* is equal to the remainder *CG*.

But *FC* was also proved equal to *GB*;

therefore the two sides *BF*, *FC* are equal to the two sides *CG*, *GB* respectively;

and the angle *BFC* is equal to the angle *CGB*,

while the base *BC* is common to them;

therefore the triangle *BFC* is also equal to the triangle *CGB*, and the remaining angles will be equal to the remaining angles respectively, namely those which the equal sides subtend;

therefore the angle FBC is equal to the angle GCB,

and the angle BCF to the angle CBG.

Accordingly, since the whole angle ABG was proved equal to the angle ACF,

and in these the angle CBG is equal to the angle BCF,

the remaining angle ABC is equal to the remaining angle ACB;

and they are at the base of the triangle ABC.

But the angle FBC was also proved equal to the angle GCB;

and they are under the base.

Therefore etc. Q. E. D.

2. **the equal straight lines** (meaning the equal *sides*). Cf. note on the similar expression in Prop. 4, lines 2, 3.

10. **Let a point F be taken at random on BD,** εἰλήφθω ἐπὶ τῆς ΒΔ τυχὸν σημεῖον τὸ Ζ, where τυχὸν σημεῖον means "a chance point."

17. **the two sides FA, AC are equal to the two sides GA, AB respectively,** δύο αἱ ΖΑ, ΑΓ δυσὶ ταῖς ΗΑ, ΑΒ ἴσαι εἰσὶν ἑκατέρα ἑκατέρᾳ. Here, and in numberless later passages, I have inserted the word "sides" for the reason given in the note on I. 1, line 20. It would have been permissible to supply either "straight lines" or "sides"; but on the whole "sides" seems to be more in accordance with the phraseology of I. 4.

33. **the base BC is common to them,** i.e., apparently, common to the *angles*, as the αὐτῶν in βάσις αὐτῶν κοινή can only refer to γωνία and γωνίᾳ preceding. Simson wrote "and the base BC is common to the two triangles BFC, CGB"; Todhunter left out these words as being of no use and tending to perplex a beginner. But Euclid evidently chose to quote the conclusion of I. 4 exactly; the first phrase of that conclusion is that the bases (of the two triangles) are equal, and, as the equal bases are here the *same* base, Euclid naturally substitutes the word "common" for "equal."

48. As "(Being) what it was required to prove" (or "do") is somewhat long, I shall henceforth write the time-honoured "Q. E. D." and "Q. E. F." for ὅπερ ἔδει δεῖξαι and ὅπερ ἔδει ποιῆσαι.

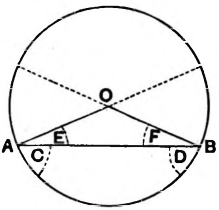
According to Proclus (p. 250, 20) the discoverer of the fact that in any isosceles triangle the angles at the base are equal was Thales, who however is said to have spoken of the angles as being *similar*, and not as being *equal*. (Cf. Arist. *De caelo* IV. 4, 311 b 34 πρὸς ὁμοίας γωνίας φαίνεται φερόμενον where *equal* angles are meant.)

A pre-Euclidean proof of I. 5.

One of the most interesting of the passages in Aristotle indicating differences between Euclid's proofs and those with which Aristotle was familiar, in other words, those of the text-books immediately preceding Euclid's, has reference to the theorem of I. 5. The passage (*Anal. Prior.* I. 24, 41 b 13—22) is so important that I must quote it in full. Aristotle is illustrating the fact that in any syllogism one of the propositions must be affirmative and *universal* (καθόλου). "This," he says, "is better shown in the case of geometrical propositions" (ἐν τοῖς διαγράμμασιν), e.g. the proposition that *the angles at the base of an isosceles triangle are equal.*

"For let A, B be drawn [i.e. joined] to the centre.

"If, then, we assumed (1) that the angle AC [i.e. $A + C$] is equal to the angle BD [i.e. $B + D$] without asserting generally that *the angles of semicircles are equal*, and again (2) that the angle C is equal to the angle D without making the further assumption that *the two angles of all segments are equal*, and if we then inferred, lastly, that, since the whole angles are equal, and equal angles are subtracted from them, the angles which remain, namely E, F, are equal, we should commit a *petitio principii*, unless we assumed [generally] that, *when equals are subtracted from equals, the remainders are equal*."



The language is noteworthy in some respects.

(1) A, B are said to be *drawn* (ἦχθαι) to the centre (of the circle of which the two equal sides are radii) as if A, B were not the angular points but the sides or the radii themselves. (There is a parallel for this in Eucl. IV. 4.)

(2) "The angle AC" is the angle which is the sum of A and C, and A means here the angle at A of the *isosceles triangle* shown in the figure, and afterwards spoken of by Aristotle as E, while C is the "mixed" angle between AB and the circumference of the smaller segment cut off by it.

(3) The "angle *of* a semicircle" (i.e. the "angle" between the diameter and the circumference, at the extremity of the diameter) and the "angle *of* a segment" appear in Euclid III. 16 and III. Def. 7 respectively, obviously as survivals from earlier text-books.

But the most significant facts to be gathered from the extract are that in the text-books which preceded Euclid's "mixed" angles played a much more important part than they do with Euclid, and, in particular, that at least two propositions concerning such angles appeared quite at the beginning, namely the propositions that *the* (*mixed*) *angles of semicircles are equal* and that *the two* (*mixed*) *angles of any segment of a circle are equal*. The wording of the first of the two propositions is vague, but it does not necessarily mean more than that the two (mixed) angles in *one* semicircle are equal, and I know of no evidence going to show that it asserts that the angle of any one semicircle is equal to the angle of any other semicircle (of different size). It is quoted in the same form, "because the angles of semicircles are equal," in the Latin translation from the Arabic of Heron's *Catoptrica*, Prop. 9 (Heron, Vol. II., Teubner, p. 334), but it is only inferred that the different radii of *one* circle make equal "angles" with the circumference; and in the similar proposition of the Pseudo-Euclidean *Catoptrica* (Euclid, Vol. VII., p. 294) angles of the same sort in *one* circle are said to be equal "because they are (angles) of a semicircle." Therefore the first of the two propositions may be only a particular case of the second.

But it is remarkable enough that the second proposition (that *the two "angles of" any segment of a circle are equal*) should, in earlier text-books, have been placed before the theorem of Eucl. I. 5. We can hardly suppose it to have been proved otherwise than by the superposition of the semicircles into which the circle is divided by the diameter which bisects at right angles the base of the segment; and no doubt the proof would be closely connected with that of Thales' other proposition that any diameter of a circle bisects it, which must also (as Proclus indicates) have been proved by superposing one of the two parts upon the other.

It is a natural inference from the passage of Aristotle that Euclid's proof of

I. 5 was his own, and it would thus appear that his innovations as regards order of propositions and methods of proof began at the very threshold of the subject.

Proof without producing the sides.

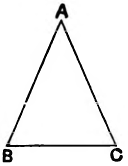
In this proof, given by Proclus (pp. 248, 22—249, 19), *D* and *E* are taken on *AB*, *AC*, instead of on *AB*, *AC produced*, so that *AD*, *AE* are equal. The method of proof is of course exactly like Euclid's, but it does not establish the equality of the angles beyond the base as well.

Pappus' proof.

Proclus (pp. 249, 20—250, 12) says that Pappus proved the theorem in a still shorter manner without the help of any construction whatever.

This very interesting proof is given as follows:

"Let *ABC* be an isosceles triangle, and *AB* equal to *AC*.



Let us conceive this one triangle as two triangles, and let us argue in this way.

Since *AB* is equal to *AC*, and *AC* to *AB*,

the two sides *AB*, *AC* are equal to the two sides *AC*, *AB*.

And the angle *BAC* is equal to the angle *CAB*, for it is the same.

Therefore all the corresponding parts (in the triangles) are equal, namely

BC to *BC*,
the triangle *ABC* to the triangle *ABC* (i.e. *ACB*),
the angle *ABC* to the angle *ACB*,
and the angle *ACB* to the angle *ABC*,

(for these are the angles subtended by the equal sides *AB*, *AC*.

Therefore in isosceles triangles the angles at the base are equal."

This will no doubt be recognised as the foundation of the alternative proof frequently given by modern editors, though they do not refer to Pappus. But they state the proof in a different form, the common method being to suppose the triangle to be taken up, turned over, and placed again upon *itself*, after which the same considerations of congruence as those used by Euclid in I. 4 are used over again. There is the obvious difficulty that it supposes the triangle to be taken up and at the same time to remain where it is. (Cf. Dodgson's humorous remark upon this, *Euclid and his modern Rivals*, p. 47.) Whatever we may say in justification of the proceeding (e.g. that the triangle may be supposed to leave a *trace*), it is really equivalent to assuming the construction (hypothetical, if you will) of another triangle equal in all respects to the given triangle; and such an assumption is not in accordance with Euclid's principles and practice.

It seems to me that the form given to the proof by Pappus himself is by far the best, for the reasons (1) that it assumes no construction of a second triangle, real or hypothetical, (2) that it avoids the distinct awkwardness involved by a proof which, instead of merely quoting and applying the *result* of a previous proposition, repeats, with reference to a new set of data, the *process* by which that result was established. If it is asked how we are to realise Pappus' idea of *two* triangles, surely we may answer that we keep to one triangle and merely view it in two aspects. If it were a question of helping a beginner to understand this, we might say that one triangle is the triangle looked at in front and that the other triangle is the *same* triangle looked at from *behind*; but even this is not really necessary.

Pappus' proof, of course, does not include the proof of the second part of the proposition about the angles under the base, and we should still have to establish this much in the same way as Euclid does.

Purpose of the second part of the theorem.

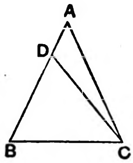
An interesting question arises as to the reason for Euclid's insertion of the second part, to which, it will be observed, the converse proposition I. 6 has nothing corresponding. As a matter of fact, it is not necessary for any subsequent demonstration that is to be found in the original text of Euclid, but only for the interpolated second case of I. 7; and it was perhaps not unnatural that the undoubted genuineness of the second part of I. 5 convinced many editors that the second case of I. 7 must necessarily be Euclid's also. Proclus' explanation, which must apparently be the right one, is that the second part of I. 5 was inserted for the purpose of fore-arming the learner against a possible *objection* (ἔνστασις), as it was technically called, which might be raised to I. 7 as given in the text, with one case only. The *objection* would, as we have seen, take the specific ground that, as demonstrated, the theorem was not conclusive, since it did not cover all possible cases. From this point of view, the second part of I. 5 is useful not only for I. 7 but, according to Proclus, for I. 9 also. Simson does not seem to have grasped Proclus' meaning, for he says: "And Proclus acknowledges, that the second part of Prop. 5 was added upon account of Prop. 7 but gives a ridiculous reason for it, 'that it might afford an answer to objections made against the 7th,' as if the case of the 7th which is left out were, as he expressly makes it, an objection against the proposition itself."

PROPOSITION 6.

If in a triangle two angles be equal to one another, the sides which subtend the equal angles will also be equal to one another.

Let ABC be a triangle having the angle ABC equal to the angle ACB;

I say that the side AB is also equal to the side AC.



For, if AB is unequal to AC, one of them is greater.

Let AB be greater; and from AB the greater let DB be cut off equal to AC the less;

let DC be joined.

Then, since DB is equal to AC,

and BC is common,

the two sides DB, BC are equal to the two sides AC, CB respectively;

and the angle DBC is equal to the angle ACB;
therefore the base DC is equal to the base AB,
and the triangle DBC will be equal to the triangle ACB,
the less to the greater:
which is absurd.
Therefore AB is not unequal to AC;
it is therefore equal to it.
Therefore etc.

Q. E. D.

Euclid assumes that, because D is between A and B, the triangle DBC is less than the triangle ABC. Some postulate is necessary to justify this tacit assumption; considering an angle less than two right angles, say the angle ACB in the figure of the proposition, as a cluster of rays issuing from C and bounded by the rays CA, CB, and joining AB (where A, B are any two points on CA, CB respectively), we see that to each successive ray taken in the direction from CA to CB there corresponds one point on AB in which the said ray intersects AB, and that all the points on AB taken in order from A to B correspond univocally to all the rays taken in order from CA to CB, each point namely to the ray intersecting AB in the point.

We have here used, for the first time in the *Elements*, the method of *reductio ad absurdum*, as to which I would refer to the section above (pp. 136, 140) dealing with this among other technical terms.

This proposition also, being the *converse* of the preceding proposition, brings us to the subject of

Geometrical Conversion.

This must of course be distinguished from the *logical* conversion of a proposition. Thus, from the proposition that all isosceles triangles have the angles opposite to the equal sides equal, *logical* conversion would only enable us to conclude that *some* triangles with two angles equal are isosceles. Thus I. 6 is the geometrical, but not the logical, converse of I. 5. On the other hand, as De Morgan points out (*Companion to the Almanac*, 1849, p. 7), I. 6 is a purely *logical* deduction from I. 5 and I. 18 taken together, as is I. 19 also. For the general argument see the note on I. 19. For the present proposition it is enough to state the matter thus. Let X denote the class of triangles which have the two sides other than the base equal, Y the class of triangles which have the base angles equal; then we may call non-X the class of triangles having the sides other than the base unequal, non-Y the class of triangles having the base angles unequal.

Thus we have

All X is Y, [I. 5]
All non-X is non-Y; [I. 18]

and it is a purely logical deduction that

All Y is X. [I. 6]

According to Proclus (p. 252, 5 sqq.) two forms of *geometrical conversion* were distinguished.

(1) The leading form (προηγουμένη), the conversion *par excellence* (ἡ κυρίως

ἀντιστροφή), is the complete or simple conversion in which the hypothesis and the conclusion of a theorem change places exactly, the conclusion of the theorem being the hypothesis of the converse theorem, which again establishes, as its conclusion, the hypothesis of the original theorem. The relation between the first part of I. 5 and I. 6 is of this character. In the former the hypothesis is that two sides of a triangle are equal and the conclusion is that the angles at the base are equal, while the converse (I. 6) starts from the hypothesis that two angles are equal and proves that the sides subtending them are equal.

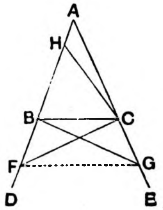
(2) The other form of conversion, which we may call *partial*, is seen in cases where a theorem starts from two or more hypotheses combined into one enunciation and leads to a certain conclusion, after which the converse theorem takes this conclusion in substitution for one of the hypotheses of the original theorem and from the said conclusion along with the rest of the original hypotheses obtains, as its conclusion, the omitted hypothesis of the original theorem. I. 8 is in this sense a converse proposition to I. 4; for I. 4 takes as hypotheses (1) that two sides in two triangles are respectively equal, (2) that the included angles are equal, and proves (3) that the bases are equal, while I. 8 takes (1) and (3) as hypotheses and proves (2) as its conclusion. It is clear that a conversion of the *leading* type must be unique, while there may be many *partial* conversions of a theorem according to the number of hypotheses from which it starts.

Further, of convertible theorems, those which took as their hypothesis the *genus* and proved a *property* were distinguished as the *leading* theorems (προηγούμενα), while those which started from the property as hypothesis and described, as the conclusion, the genus possessing that property were the *converse* theorems. I. 5 is thus the leading theorem and I. 6 its converse, since the genus is in this case taken to be the isosceles triangle.

Converse of second part of I. 5.

Why, asks Proclus, did not Euclid convert the *second* part of I. 5 as well? He suggests, properly enough, two reasons: (1) that the second part of I. 5 itself is not wanted for any proof occurring in the original text, but is only put in to enable *objections* to the existing form of later propositions to be met, whereas the converse is not even wanted for this purpose; (2) that the converse could be deduced from I. 6, if wanted, at any time after we have passed I. 13, which can be used to prove that, if the angles formed by producing two sides of a triangle beyond the base are equal, the base angles themselves are equal.

Proclus adds a proof of the converse of the second part of I. 5, i.e. of the proposition that, if the angles formed by producing two sides of a triangle beyond the base are equal, the triangle is isosceles; but it runs to some length and then only effects a reduction to the theorem of I. 6 as we have it. As the result of this should hardly be assumed, a better proof would be an independent one *adapting* Euclid's own method in I. 6. Thus, with the construction of I. 5, we first prove by means of I. 4 that the triangles BFC, CGB are equal in all respects, and therefore that FC is equal to GB, and the angle BFC equal to the angle CGB. Then we have to prove that AF, AG are equal. If they are not, let AF be the greater, and from FA cut off FH equal to GA. Join CH.



Then we have, in the two triangles *HFC*, *AGB*,

two sides *HF*, *FC* equal to two sides *AG*, *GB*

and the angle *HFC* equal to the angle *AGB*.

Therefore (I. 4) the triangles *HFC*, *AGB* are equal. But the triangles *BFC*, *CGB* are also equal.

Therefore (if we take away these equals respectively) the triangles *HBC*, *ACB* are equal: which is impossible.

Therefore *AF*, *AG* are not unequal.

Hence *AF* is equal to *AG* and, if we subtract the equals *BF*, *CG* respectively, *AB* is equal to *AC*.

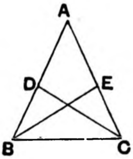
This proof is found in the commentary of an-Nairīzī (ed. Besthorn-Heiberg, p. 61; ed. Curtze, p. 50).

Alternative proofs of I. 6.

Todhunter points out that I. 6, not being wanted till II. 4, could be postponed till later and proved by means of I. 26. Bisect the angle *BAC* by a straight line meeting the base at *D*. Then the triangles *ABD*, *ACD* are equal in all respects.

Another method depending on I. 26 is given by an-Nairīzī after that proposition.

Measure equal lengths *BD*, *CE* along the sides *BA*, *CA*. Join *BE*, *CD*.



Then [I. 4] the triangles *DBC*, *ECB* are equal in all respects;

therefore *EB*, *DC* are equal, and the angles *BEC*, *CDB* are equal.

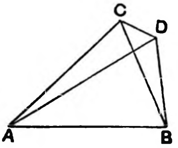
The supplements of the latter angles are equal [I. 13], and hence the triangles *ABE*, *ACD* have two angles equal respectively and the side *BE* equal to the side *CD*.

Therefore [I. 26] *AB* is equal to *AC*.

PROPOSITION 7.

Given two straight lines constructed on a straight line (from its extremities) and meeting in a point, there cannot be constructed on the same straight line (from its extremities), and on the same side of it, two other straight lines meeting in another point and equal to the former two respectively, namely each to that which has the same extremity with it.

For, if possible, given two straight lines *AC*, *CB* constructed on the straight line *AB* and meeting at the point *C*, let two other straight lines *AD*, *DB* be constructed on the same straight line *AB*, on the same side of it, meeting in another point *D* and equal to the former two respectively, namely each to that which has the same extremity with it, so that *CA* is equal to *DA* which has the same extremity *A* with it, and



CB to *DB* which has the same extremity *B* with it; and let *CD* be joined.

Then, since *AC* is equal to *AD*,

the angle *ACD* is also equal to the angle *ADC*; [I. 5]

therefore the angle *ADC* is greater than the angle *DCB*; therefore the angle *CDB* is much greater than the angle *DCB*.

Again, since *CB* is equal to *DB*,

the angle *CDB* is also equal to the angle *DCB*.

But it was also proved much greater than it:

which is impossible.

Therefore etc. Q. E. D.

1—6. In an English translation of the enunciation of this proposition it is absolutely necessary, in order to make it intelligible, to insert some words which are not in the Greek. The reason is partly that the Greek enunciation is itself very elliptical, and partly that some words used in it conveyed more meaning than the corresponding words in English do. Particularly is this the case with οὐ συσταθήσονται ἐπὶ "there shall not be constructed upon," since συνίστασθαι is the regular word for constructing a *triangle* in particular. Thus a Greek would easily understand συσταθήσονται ἐπὶ as meaning the construction of two lines *forming a triangle on* a given straight line as base; whereas to "construct two straight lines on a straight line" is not in English sufficiently definite unless we explain that they are drawn from the *ends* of the straight line to *meet* at a point. I have had the less hesitation in putting in the words "from its extremities" because they are actually used by Euclid in the somewhat similar enunciation of I. 21.

How impossible a literal translation into English is, if it is to convey the meaning of the enunciation intelligibly, will be clear from the following attempt to render literally: "On the same straight line there shall not be constructed two other straight lines equal, each to each, to the same two straight lines, (terminating) at different points on the same side, having the same extremities as the original straight lines" (ἐπὶ τῆς αὐτῆς εὐθείας δύο ταῖς αὐταῖς εὐθείαις ἄλλαι δύο εὐθεῖαι ἴσαι ἑκατέρα ἑκατέρᾳ οὐ συσταθήσονται πρὸς ἄλλῳ καὶ ἄλλῳ σημείῳ ἐπὶ τὰ αὐτὰ μέρη τὰ αὐτὰ πέρατα ἔχουσαι ταῖς ἐξ ἀρχῆς εὐθείαις).

The reason why Euclid allowed himself to use, in this enunciation, language apparently so obscure is no doubt that the phraseology was traditional and therefore, vague as it was, had a conventional meaning which the contemporary geometer well understood. This is proved, I think, by the occurrence in Aristotle (*Meteorologica* III. 5, 376 a 2 sqq.) of the very same, evidently technical, expressions. Aristotle is there alluding to the theorem given by Eutocius from Apollonius' *Plane Loci* to the effect that, if *H*, *K* be two fixed points and *M* such a variable point that the ratio of *MH* to *MK* is a given ratio (not one of equality), the locus of *M* is a circle. (For an account of this theorem see note on VI. 3 below.) Now Aristotle says "The lines drawn up from *H*, *K* in this ratio cannot be constructed to two different points of the semicircle *A*" (αἱ οὖν ἀπὸ τῶν HK ἀναγόμεναι γραμμαὶ ἐν τούτῳ τῷ λόγῳ οὐ συσταθήσονται τοῦ ἐφ' ᾧ A ἡμικυκλίου πρὸς ἄλλο καὶ ἄλλο σημεῖον).

If a paraphrase is allowed instead of a translation adhering as closely as possible to the original, Simson's is the best that could be found, since the fact that the straight lines form *triangles* on the same base is really conveyed in the Greek. Simson's enunciation is, *Upon the same base, and on the same side of it, there cannot be two triangles that have their sides which are terminated in one extremity of the base equal to one another, and likewise those which are terminated at the other extremity.* Th. Taylor (the translator of Proclus) attacks Simson's alteration as "indiscreet" and as detracting from the beauty and accuracy of Euclid's enunciation which are enlarged upon by Proclus in his commentary. Yet, when Taylor says "Whatever difficulty learners may find in conceiving this proposition abstractedly is easily removed by its exposition in the figure," he really gives his case away. The fact is that Taylor, always enthusiastic over his author, was nettled by Simson's slighting remarks on Proclus' comments on the proposition. Simson had said, with reference to Proclus' explanation of the bearing of the second part of I. 5 on I. 7, that it was not "worth while to relate his trifles at full length," to which Taylor retorts "But Mr Simson was no philosopher; and therefore the greatest part of these Commentaries must be considered by him as trifles, from the want of a philosophic genius to comprehend their meaning, and a taste superior to that of a *mere mathematician*, to discover their beauty and elegance."

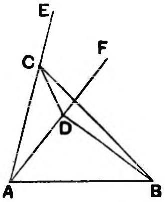
20. It would be natural to insert here the step "but the angle *ACD* is greater than the angle *BCD*. [*C. N.* 5.]"

21. **much greater**, literally "greater by much" (πολλῷ μείζων). Simson and those who follow him translate: "*much more then* is the angle *BDC greater* than the angle *BCD*," but the Greek for this would have to be πολλῷ (or πολὺ) μᾶλλόν ἐστι...μείζων. πολλῷ μᾶλλον, however, though used by Apollonius, is not, apparently, found in Euclid or Archimedes.

Just as in I. 6 we need a Postulate to justify theoretically the statement that *CD* falls within the angle *ACB*, so that the triangle *DBC* is less than the triangle *ABC*, so here we need Postulates which shall satisfy us as to the relative positions of *CA*, *CB*, *CD* on the one hand and of *DC*, *DA*, *DB* on the other, in order that we may be able to infer that the angle *BDC* is greater than the angle *ADC*, and the angle *ACD* greater than the angle *BCD*.

De Morgan (*op. cit.* p. 7) observes that I. 7 would be made easy to beginners if they were first familiarised, as a common notion, with "if two magnitudes be equal, any magnitude greater than the one is greater than any magnitude less than the other." I doubt however whether a beginner would follow this easily; perhaps it would be more easily apprehended in the form "if any magnitude *A* is greater than a magnitude *B*, the magnitude *A* is greater than any magnitude equal to *B*, and (*a fortiori*) greater than any magnitude less than *B*."

It has been mentioned already (note on I. 5) that the second case of I. 7 given by Simson and in our text-books generally is not in the original text (the omission being in accordance with Euclid's general practice of giving only one case, and that the most difficult, and leaving the others to be worked out by the reader for himself). The second case is given by Proclus as the answer to a possible *objection* to Euclid's proposition, which should assert that the proposition is not proved to be universally true, since the proof given does not cover all possible cases. Here the objector is supposed to contend that what Euclid declares to be impossible may still be possible if one pair of lines lie wholly within the other pair of lines; and the second part of I. 5 enables the objection to be refuted.



If possible, let *AD*, *DB* be entirely within the triangle formed by *AC*, *CB* with *AB*, and let *AC* be equal to *AD* and *BC* to *BD*.

Join *CD*, and produce *AC*, *AD* to *E* and *F*.

Then, since *AC* is equal to *AD*,

the triangle *ACD* is isosceles,

and the angles *ECD*, *FDC* under the base are equal.

But the angle *ECD* is greater than the angle *BCD*,

therefore the angle *FDC* is also greater than the angle *BCD*.

Therefore the angle *BDC* is greater by far than the angle *BCD*.

Again, since *DB* is equal to *CB*,

the angles at the base of the triangle *BDC* are equal, [I. 5]

that is, the angle *BDC* is equal to the angle *BCD*.

Therefore the same angle *BDC* is both greater than and equal to the angle *BCD*: which is impossible.

The case in which *D* falls on *AC* or *BC* does not require proof.

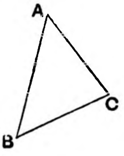
I have already referred (note on I. 1) to the mistake made by those editors who regard I. 7 as being of no use except to prove I. 8. What I. 7 proves is that if, in addition to the base of a triangle, the length of the side terminating at each extremity of the base is given, only one triangle satisfying these conditions can be constructed on one and the same side of the given base. Hence not only does I. 7 enable us to prove I. 8, but it supplements I. 1 and I. 22 by showing that the constructions of those propositions give one triangle only on one and the same side of the base. But for I. 7 this could not be proved except by anticipating III. 10, of which therefore I. 7 is the equivalent for Book I. purposes. Dodgson (*Euclid and his modern Rivals*, pp. 194—5) puts it in another way. "It [I. 7] shows that, of all plane figures that can be made by hingeing rods together, the *three*-sided ones (and these only) are *rigid* (which is another way of stating the fact that there cannot be *two* such figures on the same base). This is analogous to the fact, in relation to solids contained by plane surfaces hinged together, that *any* such solid is rigid, there being no maximum number of sides. And there is a close analogy between I. 7, 8 and III. 23, 24. These analogies give to geometry much of its beauty, and I think that they ought not to be lost sight of." It will therefore be apparent how ill-advised are those editors who eliminate I. 7 altogether and rely on Philo's proof for I. 8.

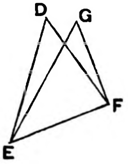
Proclus, it may be added, gives (pp. 268, 19—269, 10) another explanation of the retention of I. 7, notwithstanding that it was apparently only required for I. 8. It was said that astronomers used it to prove that three successive eclipses could not occur at equal intervals of time, i.e. that the third could not follow the second at the same interval as the second followed the first; and it was argued that Euclid had an eye to this astronomical application of the proposition. But, as we have seen, there are other grounds for retaining the proposition which are quite sufficient of themselves.

PROPOSITION 8.

If two triangles have the two sides equal to two sides respectively, and have also the base equal to the base, they will also have the angles equal which are contained by the equal straight lines.

Let *ABC*, *DEF* be two triangles having the two sides *AB*, *AC* equal to the two sides *DE*, *DF* respectively, namely *AB* to *DE*, and *AC* to *DF*; and let them have the base *BC* equal to the base *EF*;





I say that the angle *BAC* is also equal to the angle *EDF*.

For, if the triangle *ABC* be applied to the triangle *DEF*, and if the point *B* be placed on the point *E* and the straight line *BC* on *EF*,

the point *C* will also coincide with *F*,

because *BC* is equal to *EF*.

Then, BC coinciding with EF,

BA, AC will also coincide with ED, DF;

for, if the base BC coincides with the base EF, and the sides BA, AC do not coincide with ED, DF but fall beside them as EG, GF,

then, given two straight lines constructed on a straight line (from its extremities) and meeting in a point, there will have been constructed on the same straight line (from its extremities), and on the same side of it, two other straight lines meeting in another point and equal to the former two respectively, namely each to that which has the same extremity with it.

But they cannot be so constructed. [I. 7]

Therefore it is not possible that, if the base BC be applied to the base EF, the sides BA, AC should not coincide with ED, DF;

they will therefore coincide,

so that the angle BAC will also coincide with the angle EDF, and will be equal to it.

If therefore etc. Q. E. D.

19. **BA, AC.** The text has here "BA, CA."

21. **fall beside them.** The Greek has the future, παραλλάξουσι. παραλλάττω means "to pass by without touching," "to miss" or "to deviate."

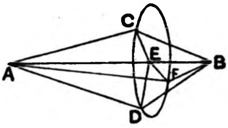
As pointed out above (p. 257) I. 8 is a *partial* converse of I. 4.

It is to be observed that in I. 8 Euclid is satisfied with proving the equality of the vertical angles and does not, as in I. 4, add that the triangles are equal, and the remaining angles are equal respectively. The reason is no doubt (as pointed out by Proclus and by Savile after him) that, when once the vertical angles are proved equal, the rest follows from I. 4, and there is no object in proving again what has been proved already.

Aristotle has an allusion to the theorem of this proposition in *Meteorologica* III. 3, 373 a 5—16. He is speaking of the rainbow and observes that, if equal rays be reflected from one and the same point to one and the same point, the points at which reflection takes place are on the circumference of a circle. "For let the broken lines ACB, AFB, ADB be all reflected from the point A to the point B (in such a way that) AC, AF, AD are all equal to one another, and the lines (terminating) at B, i.e. CB, FB, DB, are likewise all equal; and let AEB be joined. It follows that *the triangles are equal*; for they are upon the equal (base) AEB."

Heiberg (*Mathematisches zu Aristoteles*, p. 18) thinks that the form of the conclusion quoted is an indication that in the corresponding proposition to Eucl. I. 8, as it lay before Aristotle, it was maintained that the *triangles* were equal, and not only the angles, and "we see here therefore, in a clear example, how the stones of the ancient fabric were recut for the rigid structure of his

Elements." I do not, however, think that this inference from Aristotle's language as to the form of the pre-Euclidean proposition is safe. Thus if we, nowadays, were arguing from the data in the passage of Aristotle, we should doubtless infer directly that the triangles are equal in all respects, quoting I. 8 alone. Besides, Aristotle's language is rather careless, as the next sentences of the same passage show. "Let perpendiculars," he says, "be drawn to *AEB* from the angles, *CE* from *C*, *FE* from *F* and *DE* from *D*. These, then, are equal; for they are all in equal triangles, and in one plane; for all of them are perpendicular to *AEB*, and they meet at one point *E*. Therefore the (line) drawn (through *C*, *F*, *D*) will be a circle, and its centre (will be) *E*." Aristotle should obviously have proved that the three perpendiculars *will* meet at one point *E* on *AEB* before he spoke of drawing the perpendiculars *CE*, *FE*, *DE*. This of course follows from their being "in equal triangles" (by means of Eucl. I. 26); and then, from the fact that the perpendiculars meet at one point on *AB*, it can be inferred that all three are in one plane.



Philo's proof of I. 8.

This alternative proof avoids the use of I. 7, and it is elegant; but it is inconvenient in one respect, since three cases have to be distinguished. Proclus gives the proof in the following order (pp. 266, 15—268, 14).

Let *ABC*, *DEF* be two triangles having the sides *AB*, *AC* equal to the sides *DE*, *DF* respectively, and the base *BC* equal to the base *EF*.

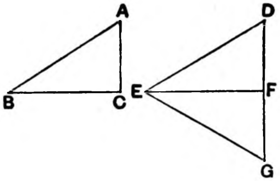
Let the triangle *ABC* be applied to the triangle *DEF*, so that *B* is placed on *E* and *BC* on *EF*, but so that *A* falls on the opposite side of *EF* from *D*, taking the position *G*. Then *C* will coincide with *F*, since *BC* is equal to *EF*.

Now *FG* will either be in a straight line with *DF*, or make an angle with it, and in the latter case the angle will either be *interior* (κατὰ τὸ ἐντός) to the figure or *exterior* (κατὰ τὸ ἐκτός).

I. Let *FG* be in a straight line with *DF*.

Then, since *DE* is equal to *EG*, and *DFG* is a straight line,

DEG is an isosceles triangle, and the angle at *D* is equal to the angle at *G*. [I. 5].



II. Let *DF*, *FG* form an angle *interior* to the figure.

Let *DG* be joined.

Then, since *DE*, *EG* are equal,

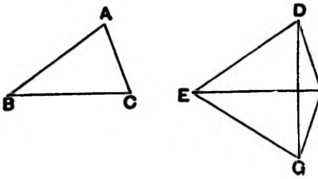
the angle *EDG* is equal to the angle *EGD*.

Again, since *DF* is equal to *FG*,

the angle *FDG* is equal to the angle *FGD*.

Therefore, by addition,

the whole angle *EDF* is equal to the whole angle *EGF*.

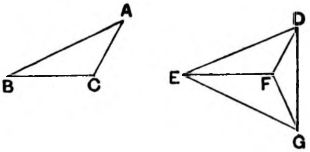


III. Let *DF*, *FG* form an angle *exterior* to the figure.

Let *DG* be joined.

The proof proceeds as in the last case, except that subtraction takes the place of addition, and

the remaining angle *EDF* is equal to the remaining angle *EGF*.



Therefore in all three cases the angle *EDF* is equal to the angle *EGF*, that is, to the angle *BAC*.

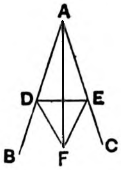
It will be observed that, in accordance with the practice of the Greek geometers in not recognising as an "angle" any angle not less than two right angles, the re-entrant angle of the quadrilateral *DEGF* is ignored and the angle *DFG* is said to be *outside* the figure.

PROPOSITION 9.

To bisect a given rectilineal angle.

Let the angle *BAC* be the given rectilineal angle.

Thus it is required to bisect it.



Let a point *D* be taken at random on *AB*;

let *AE* be cut off from *AC* equal to *AD*; [I. 3]

let *DE* be joined, and on *DE* let the equilateral triangle *DEF* be constructed;

let *AF* be joined.

I say that the angle *BAC* has been bisected by the straight line *AF*.

For, since *AD* is equal to *AE*,

and *AF* is common,

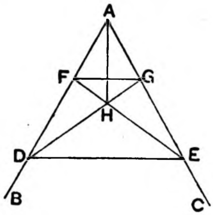
the two sides *DA*, *AF* are equal to the two sides *EA*, *AF* respectively.

And the base *DF* is equal to the base *EF*;

therefore the angle *DAF* is equal to the angle *EAF*. [I. 8]

Therefore the given rectilineal angle *BAC* has been bisected by the straight line *AF*. Q. E. F.

It will be observed from the translation of this proposition that Euclid does not say, in his description of the construction, that the equilateral triangle should be constructed on the side of *DE* opposite to *A*; he leaves this to be inferred from his figure. There is no particular value in Proclus' explanation as to how we should proceed in case any one should assert that he could not recognise the existence of any space below *DE*. He supposes, then, the equilateral triangle described on the side of *DE* towards *A*, and hence has to consider three cases according as the vertex of the equilateral triangle falls on *A*, above *A* or below it. The second and third cases do not differ substantially from Euclid's. In the first case, where *ADE* is the equilateral triangle constructed on *DE*, take any point *F* on *AD*, and from *AE* cut off *AG* equal to *AF*. Join *DG*, *EF* meeting in *H*; and join *AH*. Then *AH* is the bisector required.



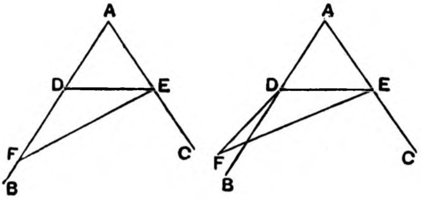
Proclus also answers the possible *objection* that might be raised to Euclid's proof on the ground that it assumes that, if the equilateral triangle be described on the side of *DE* opposite to *A*, its vertex *F* will lie within the angle *BAC*. The objector is supposed to argue that this is not necessary, but that *F* might fall either on one of the lines forming the angle or outside it altogether. The two cases are disposed of thus.

Suppose *F* to fall as shown in the two figures below respectively.

Then, since *FD* is equal to *FE*,
the angle *FDE* is equal to the angle *FED*.

Therefore the angle *CED* is greater than the angle *FDE*; and, in the second figure, *a fortiori*, the angle *CED* is greater than the angle *BDE*.

But, since *ADE* is an isosceles triangle, and the equal sides are produced, the angles under the base are equal,
i.e., the angle *CED* is equal to the angle *BDE*.



But the angle *CED* was proved greater: which is impossible.

Here then is the second case in which, in Proclus' view, the second part of I. 5 is useful for refuting objections.

On this proposition Proclus takes occasion (p. 271, 15—19) to emphasize the fact that the given angle must be *rectilineal*, since the bisection of any sort of angle (including angles made by curves with one another or with straight lines) is not matter for an elementary treatise, besides which it is questionable whether such bisection is always possible. "Thus it is difficult to say whether it is possible to bisect the so-called *horn-like* angle" (formed by the circumference of a circle and a tangent to it).

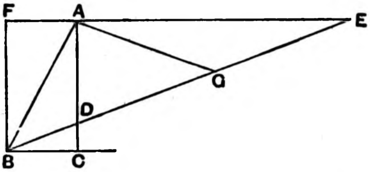
Trisection of an angle.

Further it is here that Proclus gives us his valuable historical note about the *trisection* of any acute angle, which (as well as the division of an angle in any given ratio) requires resort to other curves than circles, i.e. curves of the species which, after Geminus, he calls "mixed." "This," he says (p. 272, 1—12), "is shown by those who have set themselves the task of trisecting such a given rectilineal angle. For Nicomedes trisected any rectilineal angle by means of the *conchoidal* lines, the origin, order, and properties of which he has handed down to us, being himself the discoverer of their peculiarity. Others have done the same thing by means of the *quadratrices* of Hippias and Nicomedes, thereby again using 'mixed' curves. But others, starting from the Archimedean spirals, cut a given rectilineal angle in a given ratio."

(*a*) Trisection by means of the *conchoid.*

I have already spoken of the *conchoid* of Nicomedes (note on Def. 2, pp. 160—1); it remains to show how it could be used for trisecting an angle. Pappus explains this (IV. pp. 274—5) as follows.

Let ABC be the given acute angle, and from any point A in AB draw AC perpendicular to BC.



Complete the parallelogram $FBCA$ and produce FA to a point E such that, if BE be joined, *BE intercepts between AC and AE a length DE equal to twice AB.*

I say that the angle EBC is one-third of the angle ABC.

For, joining A to G, the middle point of DE, we have the three straight lines AG, DG, EG equal, and the angle AGD is double of the angle AED or EBC.

But DE is double of AB;

therefore AG, which is equal to DG, is equal to AB.

Hence the angle AGD is equal to the angle ABG.

Therefore the angle ABD is also double of the angle EBC;

so that the angle EBC is one-third of the angle ABC.

So far Pappus, who reduces the construction to the drawing of BE so that DE shall be equal to twice AB.

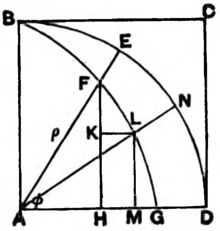
This is what the conchoid constructed with B as *pole*, AC as *directrix*, and *distance* equal to twice AB enables us to do; for that conchoid cuts AE in the required point E.

(*b*) Use of the *quadratrix.*

The plural *quadratrices* in the above passage is a Hellenism for the singular *quadratrix*, which was a curve discovered by Hippias of Elis about 420 B.C. According to Proclus (p. 356, 11) Hippias proved its properties; and we are told (1) in the passage quoted above that Nicomedes also investigated it and that it was used for trisecting an angle, and (2) by Pappus (IV. pp. 250, 33—252, 4) that it was used by Dinostratus and Nicomedes and some more recent writers for squaring the circle, whence its name. It is described thus (Pappus IV. p. 252).

Suppose that $ABCD$ is a square and BED a quadrant of a circle with centre A.

Suppose (1) that a radius of the circle moves uniformly about A from the position AB to the position AD, and (2) that *in the same time* the line BC moves uniformly, always parallel to itself, and with its extremity B moving along BA, from the position BC to the position AD.



Then the radius AE and the moving line BC determine at any instant by their intersection a point F.

The locus of F is the *quadratrix.*

The property of the curve is that, if F is any point, the arc BED is to the arc ED as AB is to FH.

In other words, if ϕ is the angle FAD, ρ the radius vector AF and a the side of the square,

$$(\rho \sin \phi)/a = \phi/\tfrac{1}{2}\pi.$$

Now the angle EAD can not only be *trisected* but *divided in any given ratio* by means of the quadratrix (Pappus IV. p. 286).

For let FH be divided at K in the given ratio.

Draw KL parallel to AD, meeting the curve in L; join AL and produce it to meet the circle in N.

Then the angles EAN, NAD are in the ratio of FK to KH, as is easily proved.

(*c*) Use of the *spiral of Archimedes.*

The trisection of an angle, or the division of an angle in any ratio, by means of the *spiral of Archimedes* is of course an equally simple matter. Suppose any angle included between the two radii vectores OA and OB of the spiral, and let it be required to cut the angle AOB in a given ratio. Since the radius vector increases proportionally with the angle described by the vector which generates the curve (reckoned from the original position of the vector coinciding with the initial line to the particular position assumed), we have only to take the radius vector OB (the greater of the two OA, OB), mark off OC along it equal to OA, cut CB in the given ratio (at D say), and then draw the circle with centre O and radius OD cutting the spiral in E. Then OE will divide the angle AOB in the required manner.

PROPOSITION 10.

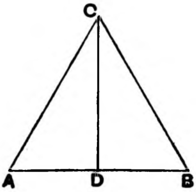
To bisect a given finite straight line.

Let AB be the given finite straight line.

Thus it is required to bisect the finite straight line AB.

Let the equilateral triangle ABC be constructed on it, [I. 1]

and let the angle ACB be bisected by the straight line CD; [I. 9]



I say that the straight line AB has been bisected at the point D.

For, since AC is equal to CB, and CD is common,

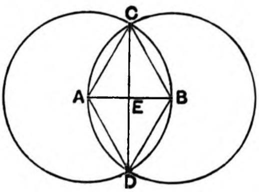
the two sides AC, CD are equal to the two sides BC, CD respectively;

and the angle ACD is equal to the angle BCD;

therefore the base AD is equal to the base BD. [I. 4]

Therefore the given finite straight line AB has been bisected at D. Q. E. F.

Apollonius, we are told (Proclus, pp. 279, 16—280, 4), bisected a straight line AB by a construction like that of I. 1. With centres A, B, and radii AB, BA respectively, two circles are described, intersecting in C, D. Joining CD, AC, CB, AD, DB, Apollonius proves in two steps that CD bisects AB.



(1) Since, in the triangles ACD, BCD,
two sides AC, CD are equal to two sides BC, CD,
and the bases AD, BD are equal,
the angle ACD is equal to the angle BCD. [I. 8]

(2) The latter angles being equal, and AC being equal to CB, while CE is common,
the equality of AE, EB follows by I. 4.

The objection to this proof is that, instead of *assuming* the bisection of the angle ACB, as already effected by I. 9, Apollonius goes a step further back and embodies a construction for bisecting the angle. That is, he unnecessarily does over again what has been done before, which is open to objection from a theoretical point of view.

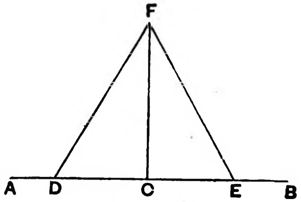
Proclus (pp. 277, 25—279, 4) warns us against being moved by this proposition to conclude that geometers assumed, as a preliminary hypothesis, that a line is not made up of indivisible parts (ἐξ ἀμερῶν). This might be argued thus. If a line is made up of indivisibles, there must be in a finite line either an odd or an even number of them. If the number were odd, it would be necessary in order to bisect the line to bisect an indivisible (the odd one). In that case therefore it would not be possible to bisect a straight line, if it is a magnitude made up of indivisibles. But, if it is not so made up, the straight line can be divided *ad infinitum* or without limit (ἐπ' ἄπειρον διαιρεῖται). Hence it was argued (φασίν), says Proclus, that the divisibility of magnitudes without limit was admitted and assumed as a geometrical principle. To this he replies, following Geminus, that geometers did indeed assume, by way of a common notion, that a continuous magnitude, i.e. a magnitude consisting of parts connected together (συνημμένων), is divisible (διαιρετόν). But *infinite* divisibility was not assumed by them; it was *proved* by means of the first principles applicable to the case. "For when," he says, "they prove that the incommensurable exists among magnitudes, and that it is not all things that are commensurable with one another, what else will any one say that they prove but that every magnitude can be divided for ever, and that we shall never arrive at the indivisible, that is, the least common measure of the magnitudes? This then is matter of demonstration, whereas it is an *axiom* that everything continuous is divisible, so that a finite continuous line is divisible. The writer of the Elements bisects a finite straight line, starting from the latter notion, and not from any assumption that it is divisible without limit." Proclus adds that the proposition may also serve to refute Xenocrates' theory of indivisible lines (ἄτομοι γραμμαί). The argument given by Proclus to disprove the existence of indivisible lines is substantially that used by Aristotle as regards magnitudes generally (cf. *Physics* VI. 1, 231 a 21 sqq. and especially VI. 2, 233 b 15—32).

PROPOSITION 11.

To draw a straight line at right angles to a given straight line from a given point on it.

Let *AB* be the given straight line, and *C* the given point on it.

Thus it is required to draw from the point *C* a straight line at right angles to the straight line *AB*.



Let a point *D* be taken at random on *AC*;

let *CE* be made equal to *CD*; [I. 3]

on *DE* let the equilateral triangle *FDE* be constructed, [I. 1]

and let *FC* be joined;

I say that the straight line *FC* has been drawn at right angles to the given straight line *AB* from *C* the given point on it.

For, since *DC* is equal to *CE*,

and *CF* is common,

the two sides *DC*, *CF* are equal to the two sides *EC*, *CF* respectively;

and the base *DF* is equal to the base *FE*;

therefore the angle *DCF* is equal to the angle *ECF*; [I. 8]

and they are adjacent angles.

But, when a straight line set up on a straight line makes the adjacent angles equal to one another, each of the equal angles is right; [Def. 10]

therefore each of the angles *DCF*, *FCE* is right.

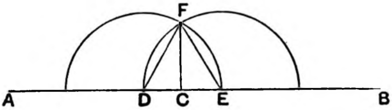
Therefore the straight line *CF* has been drawn at right angles to the given straight line *AB* from the given point *C* on it.

Q. E. F.

10. **let CE be made equal to CD.** The verb is *κείσθω* which, as well as the other parts of *κεῖμαι*, is constantly used for the passive of *τίθημι* "to *place*"; and the latter word is constantly used in the sense of *making*, e.g., one straight line equal to another straight line.

De Morgan remarks that this proposition, which is "to bisect the angle made by a straight line and its continuation" [i.e. a *flat* angle], should be a particular case of I. 9, the constructions being the same. This is certainly worth noting, though I doubt the advantage of rearranging the propositions in consequence.

Apollonius gave a construction for this proposition (see Proclus, p. 282, 8) differing from Euclid's in much the same way as his construction for bisecting a straight line differed from that of I. 10. Instead of assuming an equilateral triangle drawn without repeating the process of I. 1, Apollonius takes *D* and *E* equidistant from *C* as in Euclid, and then draws circles in the manner of I. 1 meeting at *F*. This necessitates proving again that *DF* is equal to *FE*; whereas Euclid's assumption of the construction of I. 1 in the words "let the equilateral triangle *FDE* be constructed" enables him to dispense with the drawing of circles and with the proof that *DF* is equal to *FE* at the same time. While however the substitution of Apollonius' constructions for I. 10 and 11 would show faulty arrangement in a theoretical treatise like Euclid's, they are entirely suitable for what we call *practical* geometry, and such may have been Apollonius' object in these constructions and in his alternative for I. 23.



Proclus gives a construction for drawing a straight line at right angles to another straight line but from *one end* of it, instead of from an intermediate point on it, it being supposed (for the sake of argument) that we are not permitted to *produce* the straight line. In the commentary of an-Nairīzī (ed. Besthorn-Heiberg, pp. 73—4; ed. Curtze, pp. 54—5) this construction is attributed to Heron.

Let it be required to draw from *A* a straight line at right angles to *AB*.

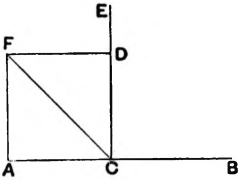
On *AB* take any point *C*, and in the manner of the proposition draw *CE* at right angles to *AB*.

From *CE* cut off *CD* equal to *AC*, bisect the angle *ACE* by the straight line *CF*, [I. 9]
and draw *DF* at right angles to *CE* meeting *CF* in *F*. Join *FA*.

Then the angle *FAC* will be a right angle.

For, since, in the triangles *ACF*, *DCF*, the two sides *AC*, *CF* are equal to the two sides *DC*, *CF* respectively, and the included angles *ACF*, *DCF* are equal,

the triangles are equal in all respects. [I. 4]



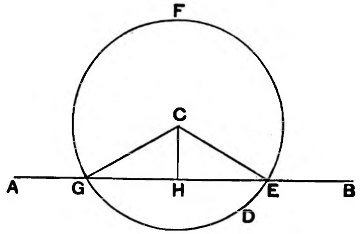
Therefore the angle at *A* is equal to the angle at *D*, and is accordingly a right angle.

PROPOSITION 12.

To a given infinite straight line, from a given point which is not on it, to draw a perpendicular straight line.

Let *AB* be the given infinite straight line, and *C* the given point which is not on it;

thus it is required to draw to the given infinite straight line *AB*, from the given point *C* which is not on it, a perpendicular straight line.



For let a point *D* be taken at random on the other side of the straight line *AB*, and with centre *C* and distance *CD* let the circle *EFG* be described; [Post. 3]

let the straight line *EG* be bisected at *H*, [I. 10]

and let the straight lines *CG*, *CH*, *CE* be joined. [Post. 1]

I say that *CH* has been drawn perpendicular to the given infinite straight line *AB* from the given point *C* which is not on it.

For, since *GH* is equal to *HE*, and *HC* is common,

the two sides *GH*, *HC* are equal to the two sides *EH*, *HC* respectively;

and the base *CG* is equal to the base *CE*;

therefore the angle *CHG* is equal to the angle *EHC*. [I. 8]

And they are adjacent angles.

But, when a straight line set up on a straight line makes the adjacent angles equal to one another, each of the equal angles is right, and the straight line standing on the other is called a perpendicular to that on which it stands. [Def. 10]

Therefore *CH* has been drawn perpendicular to the given infinite straight line *AB* from the given point *C* which is not on it.

Q. E. F.

2. **a perpendicular straight line,** *κάθετον εὐθεῖαν γραμμήν*. This is the full expression for a *perpendicular*, *κάθετος* meaning *let fall* or *let down*, so that the expression corresponds to our *plumb-line*. *ἡ κάθετος* is however constantly used alone for a perpendicular, *γραμμή* being understood.

10. **on the other side of the straight line AB,** literally "towards the other parts of the straight line *AB*," *ἐπὶ τὰ ἕτερα μέρη τῆς* AB. Cf. "on the same side" (*ἐπὶ τὰ αὐτὰ μέρη*) in Post. 5 and "in both directions" (*ἐφ' ἑκάτερα τὰ μέρη*) in Def. 23.

"This problem," says Proclus (p. 283, 7—10), "was first investigated by Oenopides [5th cent. B.C.], who thought it useful for astronomy. He however calls the perpendicular, in the archaic manner, (a line drawn) *gnomon-wise* (κατὰ γνώμονα), because the gnomon is also at right angles to the horizon." In this earlier sense the *gnomon* was a staff placed in a vertical position for the purpose of casting shadows and so serving as a means of measuring time (Cantor, *Geschichte der Mathematik*, I$_3$, p. 161). The later meanings of the word as used in Eucl. Book II. and elsewhere will be explained in the note on Book II. Def. 2.

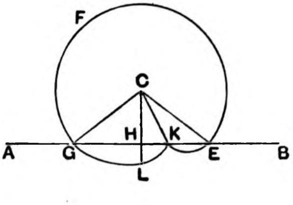
Proclus says that two kinds of perpendicular were distinguished, the "plane" (ἐπίπεδος) and the "solid" (στερεά), the former being the perpendicular dropped on a line in a plane and the latter the perpendicular dropped on a plane. The term "solid perpendicular" is sufficiently curious, but it may perhaps be compared with the Greek term "solid locus" applied to a conic section, apparently on the ground that it has its origin in the section of a solid, namely a cone.

Attention is called by most editors to the assumption in this proposition that, if only *D* be taken on the side of *AB* remote from *C*, the circle described with *CD* as radius must necessarily cut *AB* in two points. To satisfy us of this we need, as in I. 1, some postulate of continuity, e.g. something like that suggested by Killing (see note on the Principle of Continuity above, p. 235): "If a point [here the point describing the circle] moves in a figure which is divided into two parts [by the straight line], and if it belongs at the beginning of the motion to one part and at another stage of the motion to the other part, it must during the motion cut the boundary between the two parts," and this of course applies to the motion in *two* directions from *D*.

But the editors have not, as a rule, noticed a possible *objection* to the Euclidean statement of this problem which is much more difficult to dispose of at this stage, i.e. without employing any proposition later than this in Euclid's order. How do we know, says the supposed critic, that the circle does not cut *AB* in *three* or more points, in which case there would be not *one* perpendicular but *three* or more? Proclus (pp. 286, 12—289, 6) tries to refute this objection, and it is interesting to follow his argument, though it will easily be seen to be inconclusive. He takes in order three possible suppositions.

1. May not the circle meet *AB* in a third point *K* between the middle point of *GE* and either extremity of it, taking the form drawn in the figure appended?

Suppose this possible. Bisect *GE* in *H*. Join *CH*, and produce it to meet the circle in *L*. Join *CG*, *CK*, *CE*.



Then, since *CG* is equal to *CE*, and *CH* is common, while the base *GH* is equal to the base *HE*,

the angles *CHG*, *CHE* are equal and, since they are adjacent, they are both right.

Again, since *CG* is equal to *CE*,

the angles at *G* and *E* are equal.

Lastly, since *CK* is equal to *CG* and also to *CE*, the angles *CGK*, *CKG* are equal, as also are the angles *CKE*, *CEK*.

Since the angles *CGK*, *CEK* are equal, it follows that

the angles *CKG*, *CKE* are equal and therefore both right.

Therefore the angle *CKH* is equal to the angle *CHK*,

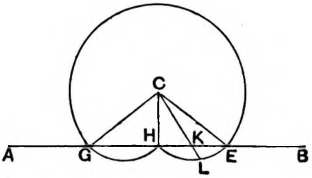
and *CH* is equal to *CK*.

But *CK* is equal to *CL*, by the definition of the circle; therefore *CH* is equal to *CL*: which is impossible.

Thus Proclus; but why should not the circle meet *AB* in *H* as well as *K*?

2. May not the circle meet *AB* in *H* the middle point of *GE* and take the form shown in the second figure?

In that case, says Proclus, join *CG*, *CH*, *CE* as before. Then bisect *HE* at *K*, join *CK* and produce it to meet the circumference at *L*.



Now, since *HK* is equal to *KE*, *CK* is common, and the base *CH* is equal to the base *CE*,

the angles at *K* are equal and therefore both right angles.

Therefore the angle *CHK* is equal to the angle *CKH*, whence *CK* is equal to *CH* and therefore to *CL*: which is impossible.

So Proclus; but why should not the circle meet *AB* in *K* as well as *H*?

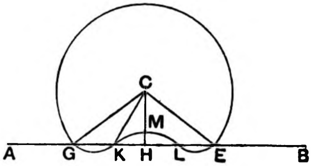
3. May not the circle meet *AB* in *two* points besides *G*, *E* and pass, between those two points, to the side of *AB* towards *C*, as in the next figure?

Here again, by the same method, Proclus proves that, *K*, *L* being the other two points in which the circle cuts *AB*,

CK is equal to *CH*,

and, since the circle cuts *CH* in *M*,

CM is equal to *CK* and therefore to *CH*: which is impossible.



But, again, why should the circle not cut *AB* in the point *H* as well?

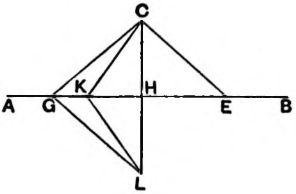
In fact, Proclus' cases are not mutually exclusive, and his method of proof only enables us to show that, if the circle meets *AB* in one more point besides *G*, *E*, it must meet it in more points still. We can always find a new point of intersection by bisecting the distance separating any two points of intersection, and so, applying the method *ad infinitum*, we should have to conclude ultimately that the circle with radius *CH* (or *CG*) coincides with *AB*. It would follow that a circle with centre *C* and radius greater than *CH* would not meet *AB* at all. Also, since all straight lines from *C* to points on *AB* would be equal in length, there would be an infinite number of perpendiculars from *C* on *AB*.

Is this under any circumstances possible? It is not possible in Euclidean space, but it is possible, under the Riemann hypothesis (where a straight line is a "closed series" and returns on itself), in the case where *C* is the pole of the straight line *AB*.

It is natural therefore that, for a proof that in Euclidean space there is only one perpendicular from a point to a straight line, we have to wait until I. 16, the precise proposition which under the Riemann hypothesis is only valid with a certain restriction and not universally. There is no difficulty involved by waiting until I. 16, since I. 12 is not used before that proposition is reached; and we are only in the same position as when, in order to satisfy ourselves of the number of possible solutions of I. 1, we have to wait till I. 7.

But if we wish, after all, to prove the truth of the assumption *without* recourse to any later proposition than I. 12, we can do so by means of this same invaluable I. 7.

If the circle intersects AB as before in G, E, let H be the middle point of GE, and suppose, if possible, that the circle also intersects AB in any other point K on AH.



From H, on the side of AB opposite to C, draw HL at right angles to AB, and make HL equal to HC.

Join CG, LG, CK, LK.

Now, in the triangles CHG, LHG, CH is equal to LH, and HG is common.

Also the angles CHG, LHG, being both right, are equal.

Therefore the base CG is equal to the base LG.

Similarly we prove that CK is equal to LK.

But, by hypothesis, since K is on the circle,

CK is equal to CG.

Therefore CG, CK, LG, LK are all equal.

Now the next proposition, I. 13, will tell us that CH, HL are in a straight line; but we will not assume this. Join CL.

Then on the same base CL and on the same side of it we have two pairs of straight lines drawn from C, L to G and K such that CG is equal to CK and LG to LK.

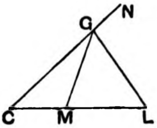
But this is impossible [I. 7].

Therefore the circle cannot cut BA or BA produced in any point other than G on that side of CL on which G is.

Similarly it cannot cut AB or AB produced at any point other than E on the other side of CL.

The only possibility left therefore is that the circle might cut AB in the same point as that in which CL cuts it. But this is shown to be impossible by an adaptation of the proof of I. 7.

For the assumption is that there may be some point M on CL such that CM is equal to CG and LM to LG.



If possible, let this be the case, and produce CG to N.

Then, since CM is equal to CG, the angle NGM is equal to the angle GML [I. 5, part 2].

Therefore the angle GML is greater than the angle MGL.

Again, since LG is equal to LM, the angle GML is equal to the angle MGL.

But it was also greater: which is impossible.

Hence the circle in the original figure cannot cut AB in the point in which CL cuts it.

Therefore the circle cannot cut AB in any point whatever except G and E.

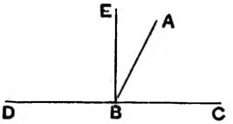
[This proof of course does not prove that CK is *less* than CG, but only that it is not equal to it. The proposition that, of the obliques drawn from C to AB, that is less the foot of which is nearer to H can only be proved later. The proof by I. 7 also fails, under the Riemann hypothesis, if C, L are the poles of the straight line AB, since the broken lines CGL, CKL etc. become equal straight lines, all perpendicular to AB.]

Proclus rightly adds (p. 289, 18 sqq.) that it is not *necessary* to take D on the side of AB away from A if an objector "says that there is no space on that side." If it is not desired to trespass on that side of *AB*, we can take *D* anywhere on *AB* and describe the *arc* of a circle between *D* and the point where it meets *AB* again, drawing the arc on the side of *AB* on which *C* is. If it should happen that the selected point *D* is such that the circle only meets *AB* in *one* point (*D* itself), we have only to describe the circle with *CD* as radius, then, if *E* be a point on this circle, take *F* a point further from *C* than *E* is, and describe with *CF* as radius the circular arc meeting *AB* in two points.

PROPOSITION 13.

If a straight line set up on a straight line make angles, it will make either two right angles or angles equal to two right angles.

For let any straight line *AB* set up on the straight line *CD* make the angles *CBA*, *ABD*;



I say that the angles *CBA*, *ABD* are either two right angles or equal to two right angles.

Now, if the angle *CBA* is equal to the angle *ABD*,

they are two right angles. [Def. 10]

But, if not, let *BE* be drawn from the point *B* at right angles to *CD*; [I. 11]

therefore the angles *CBE*, *EBD* are two right angles.

Then, since the angle *CBE* is equal to the two angles *CBA*, *ABE*,

let the angle *EBD* be added to each;

therefore the angles *CBE*, *EBD* are equal to the three angles *CBA*, *ABE*, *EBD*. [*C. N.* 2]

Again, since the angle *DBA* is equal to the two angles *DBE*, *EBA*,

let the angle *ABC* be added to each;

therefore the angles *DBA*. *ABC* are equal to the three angles *DBE*, *EBA*, *ABC*. [*C. N.* 2]

But the angles *CBE*, *EBD* were also proved equal to the same three angles;

and things which are equal to the same thing are also equal to one another; [*C. N.* 1]

therefore the angles *CBE*, *EBD* are also equal to the angles *DBA*, *ABC*.

But the angles *CBE*, *EBD* are two right angles;
therefore the angles *DBA*, *ABC* are also equal to two right angles.

Therefore etc.

Q. E. D.

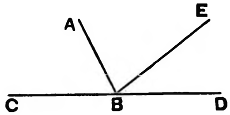
17. **let the angle EBD be added to each,** literally "let the angle *EBD* be added (so as to be) common," κοινὴ προσκείσθω ἡ ὑπὸ ΕΒΔ. Similarly κοινὴ ἀφῃρήσθω is used of subtracting a straight line or angle from each of two others. "Let the common angle *EBD* be added" is clearly an inaccurate translation, for the angle is not common before it is added, i.e. the κοινὴ is proleptic. "Let the common angle be *subtracted*" as a translation of κοινὴ ἀφῃρήσθω would be less unsatisfactory, it is true, but, as it is desirable to use corresponding words when translating the two expressions, it seems hopeless to attempt to keep the word "common," and I have therefore said "to each" and "from each" simply.

PROPOSITION 14.

If with any straight line, and at a point on it, two straight lines not lying on the same side make the adjacent angles equal to two right angles, the two straight lines will be in a straight line with one another.

For with any straight line *AB*, and at the point *B* on it, let the two straight lines *BC*, *BD* not lying on the same side make the adjacent angles *ABC*, *ABD* equal to two right angles;

I say that *BD* is in a straight line with *CB*.



For, if *BD* is not in a straight line with *BC*, let *BE* be in a straight line with *CB*.

Then, since the straight line *AB* stands on the straight line *CBE*,
the angles *ABC*, *ABE* are equal to two right angles. [I. 13]

But the angles *ABC*, *ABD* are also equal to two right angles;
therefore the angles *CBA*, *ABE* are equal to the angles *CBA*, *ABD*. [Post. 4 and *C. N.* 1]

Let the angle *CBA* be subtracted from each;
therefore the remaining angle *ABE* is equal to the remaining angle *ABD*, [*C. N.* 3]
the less to the greater: which is impossible.

Therefore *BE* is not in a straight line with *CB*.

Similarly we can prove that neither is any other straight line except *BD*.

Therefore *CB* is in a straight line with *BD*.

Therefore etc.

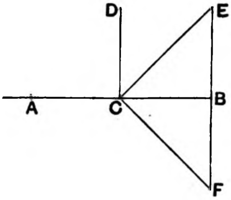
Q. E. D.

1. **If with any straight line....** There is no greater difficulty in translating the works of the Greek geometers than that of accurately giving the force of prepositions. πρός, for instance, is used in all sorts of expressions with various shades of meaning. The present enunciation begins Ἐὰν πρός τινι εὐθείᾳ καὶ τῷ πρὸς αὐτῇ σημείῳ, and it is really necessary in this one sentence to translate πρός by three different words, *with*, *at*, and *on*. The first πρός must be translated by *with* because two straight lines "make" an angle *with* one another. On the other hand, where the similar expression πρὸς τῇ δοθείσῃ εὐθείᾳ occurs in I. 23, but it is a question of "constructing" an angle (συστήσασθαι), we have to say "to construct *on* a given straight line." *Against* would perhaps be the English word coming nearest to expressing all these meanings of πρός, but it would be intolerable as a translation.

17. Todhunter points out that for the inference in this line Post. 4, that all right angles are equal, is necessary as well as the Common Notion that things which are equal to the same thing (or rather, here, to *equal things*) are equal. A similar remark applies to steps in the proofs of I. 15 and I. 28.

24. **we can prove.** The Greek expresses this by the future of the verb, δείξομεν, "we shall prove," which however would perhaps be misleading in English.

Proclus observes (p. 297) that two straight lines on the *same* side of another straight line and meeting it in one and the same point may make with one and the same portion of the straight line terminated at the point two angles which are together equal to two right angles, in which case however the two straight lines would not be in a straight line with one another. And he quotes from Porphyry a construction for two such straight lines in the particular case where they form with the given straight line angles equal respectively to half a right angle and one and a half right angles. There is no particular value in the construction, which will be gathered from the annexed figure where *CE*, *CF* are drawn at the prescribed inclinations to *CD*.



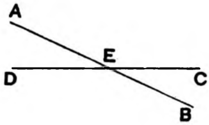
PROPOSITION 15.

If two straight lines cut one another, they make the vertical angles equal to one another.

For let the straight lines *AB*, *CD* cut one another at the point *E*;

I say that the angle *AEC* is equal to the angle *DEB*,

and the angle *CEB* to the angle *AED*.



For, since the straight line *AE* stands on the straight line *CD*, making the angles *CEA*, *AED*, the angles *CEA*, *AED* are equal to two right angles [I. 13]

Again, since the straight line *DE* stands on the straight line *AB*, making the angles *AED*, *DEB*,

the angles *AED*, *DEB* are equal to two right angles. [I. 13]

But the angles *CEA*, *AED* were also proved equal to two right angles;

therefore the angles *CEA*, *AED* are equal to the angles *AED* *DEB*. [Post. 4 and *C. N.* 1]

Let the angle *AED* be subtracted from each;

therefore the remaining angle *CEA* is equal to the remaining angle *BED*. [*C. N.* 3]

Similarly it can be proved that the angles *CEB*, *DEA* are also equal.

Therefore etc. Q. E. D.

[PORISM. From this it is manifest that, if two straight lines cut one another, they will make the angles at the point of section equal to four right angles.]

1. **the vertical angles.** The difference between *adjacent* angles (αἱ ἐφεξῆς γωνίαι) and *vertical* angles (αἱ κατὰ κορυφὴν γωνίαι) is thus explained by Proclus (p. 298, 14—24). The first term describes the angles made by two straight lines when one only is divided by the other, i.e. when one straight line meets another at a point which is not either of its extremities, but is not itself produced beyond the point of meeting. When the first straight line *is* produced, so that the lines cross at the point, they make two pairs of *vertical* angles (which are more clearly described as *vertically opposite* angles), and which are so called because their convergence is from opposite directions to one point (the intersection of the lines) as vertex (κορυφή).

26. **at the point of section,** literally "at the section," πρὸς τῇ τομῇ.

This theorem, according to Eudemus, was first discovered by Thales, but found its scientific demonstration in Euclid (Proclus, p. 299, 3—6).

Proclus gives a converse theorem which may be stated thus. *If a straight line is met at one and the same point intermediate in its length by two other straight lines on different sides of it and such as to make the vertical angles equal, the latter straight lines are in a straight line with one another.* The proof need not be given, since it is almost self-evident, whether (1) it is direct, by means of I. 13, 14, or (2) indirect, by *reductio ad absurdum* depending on I. 15.

The balance of MS. authority seems to be against the genuineness of this *Porism*, but Proclus and Psellus both have it. The word is not here used, as it is in the title of Euclid's lost *Porisms*, to signify a particular class of independent propositions which Proclus describes as being in some sort intermediate between theorems and problems (requiring us, not to bring a thing into existence, but to *find* something which we know to exist). *Porism* has here (and wherever the term is used in the *Elements*) its second meaning; it is what we call a corollary, i.e. an incidental result springing from the proof of a theorem or the solution of a problem, a result not directly sought but appearing as it were by chance without any additional labour, and constituting, as Proclus says, a sort of *windfall* (ἕρμαιον) and *bonus* (κέρδος). These Porisms appear in both the geometrical and arithmetical Books of the *Elements*, and may either result from theorems or problems. Here the Porism is geometrical, and springs out of a theorem; VII. 2 affords an instance of an arithmetical Porism. As an instance of a Porism to a problem Proclus cites "that which is found in the second Book" (τὸ ἐν τῷ δευτέρῳ βιβλίῳ κείμενον); but as to this see notes on II. 4 and IV. 15.

The present Porism, says Proclus, formed the basis of "that paradoxical theorem which proves that only the following three (regular) polygons can fill up the whole space surrounding one point, the equilateral triangle, the square, and the equilateral and equiangular hexagon." We can in fact place round a point in this manner six equilateral triangles, three regular hexagons, or four squares. "But only the angles of these regular figures, to the number specified, can make up four right angles: a theorem due to the Pythagoreans."

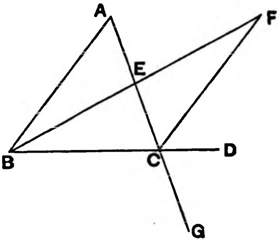
Proclus further adds that it results from the Porism that, if any number of straight lines intersect one another at one point, the sum of all the angles so formed will still be equal to four right angles. This is of course what is generally given in the text-books as Corollary 2.

PROPOSITION 16.

In any triangle, if one of the sides be produced, the exterior angle is greater than either of the interior and opposite angles.

Let *ABC* be a triangle, and let one side of it *BC* be produced to *D*;

I say that the exterior angle *ACD* is greater than either of the interior and opposite angles *CBA*, *BAC*.



Let *AC* be bisected at *E* [I. 10], and let *BE* be joined and produced in a straight line to *F*;

let *EF* be made equal to *BE* [I. 3], let *FC* be joined [Post. 1], and let *AC* be drawn through to *G* [Post. 2].

Then, since *AE* is equal to *EC*, and *BE* to *EF*,

the two sides *AE*, *EB* are equal to the two sides *CE*, *EF* respectively;

and the angle *AEB* is equal to the angle *FEC*,

for they are vertical angles. [I. 15]

Therefore the base *AB* is equal to the base *FC*,

and the triangle *ABE* is equal to the triangle *CFE*,

and the remaining angles are equal to the remaining angles respectively, namely those which the equal sides subtend; [I. 4]

therefore the angle *BAE* is equal to the angle *ECF*.

But the angle ECD is greater than the angle ECF; [*C. N.* 5]

therefore the angle ACD is greater than the angle BAE.

Similarly also, if BC be bisected, the angle BCG, that is, the angle ACD [I. 15], can be proved greater than the angle ABC as well.

Therefore etc. Q. E. D.

1. **the exterior angle,** literally "the outside angle," ἡ ἐκτὸς γωνία.
2. **the interior and opposite angles,** τῶν ἐντὸς καὶ ἀπεναντίον γωνιῶν.
12. **let AC be drawn through to G.** The word is διήχθω, a variation on the more usual ἐκβεβλήσθω, "let it be *produced.*"
21. **CFE,** in the text "*FEC.*"

As is well known, this proposition is not universally true under the Riemann hypothesis of a space endless in extent but not infinite in size. On this hypothesis a straight line is a "closed series" and returns on itself; and two straight lines which have one point of intersection have another point of intersection also, which bisects the whole length of the straight line measured from the first point on it to the same point again; thus the axiom of Euclidean geometry that two straight lines do not enclose a space does not hold. If 4Δ denotes the finite length of a straight line measured from any point once round to the same point again, 2Δ is the distance between the two intersections of two straight lines which meet. Two points A, B do not determine one sole straight line unless the distance between them is different from 2Δ. In order that there may only be one perpendicular from a point C to a straight line AB, C must not be one of the two "poles" of the straight line.

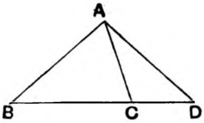
Now, in order that the proof of the present proposition may be universally valid, it is necessary that CF should always fall within the angle ACD so that the angle ACF may be less than the angle ACD. But this will not always be so on the Riemann hypothesis. For, (1) if BE is equal to Δ, so that BF is equal to 2Δ, F will be the second point in which BE and BD intersect; i.e. F will lie on CD, and the angle ACF will be *equal* to the angle ACD. In this case the exterior angle ACD will be *equal* to the interior angle BAC. (2) If BE is greater than Δ and less than 2Δ, so that BF is greater than 2Δ and less than 4Δ, the angle ACF will be *greater* than the angle ACD, and therefore the angle ACD will be *less* than the interior angle BAC. Thus, e.g., in the particular case of a right-angled triangle, the angles other than the right angle may be (1) both acute, (2) one acute and one obtuse, or (3) both obtuse according as the perpendicular sides are (1) both less than Δ, (2) one less and the other greater than Δ, (3) both greater than Δ.

Proclus tells us (p. 307, 1—12) that some combined this theorem with the next in one enunciation thus: *In any triangle, if one side be produced, the exterior angle of the triangle is greater than either of the interior and opposite angles, and any two of the interior angles are less than two right angles*, the combination having been suggested by the similar enunciation of Euclid I. 32, *In any triangle, if one of the sides be produced, the exterior angle is equal to the two interior and opposite angles, and the three interior angles of the triangle are equal to two right angles.*

The present proposition enables Proclus to prove what he did not succeed in establishing conclusively in his note on I. 12, namely that *from one point there cannot be drawn to the same straight line three straight lines equal in length.*

For, if possible, let AB, AC, AD be all equal, B, C, D being in a straight line.

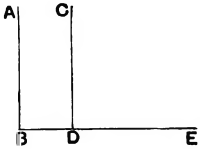
Then, since AB, AC are equal, the angles ABC, ACB are equal.



Similarly, since AB, AD are equal, the angles ABD, ADB are equal.

Therefore the angle ACB is equal to the angle ADC, i.e. the exterior angle to the interior and opposite angle: which is impossible.

Proclus next (p. 308, 14 sqq.) undertakes to prove by means of I. 16 that, *if a straight line falling on two straight lines make the exterior angle equal to the interior and opposite angle, the two straight lines will not form a triangle or meet*, for in that case the same angle would be both greater and equal.



The proof is really equivalent to that of Eucl. I. 27. If BE falls on the two straight lines AB, CD in such a way that the angle CDE is equal to the interior and opposite angle ABD, AB and CD cannot form a triangle or meet. For, if they did, then (by I. 16) the angle CDE would be *greater* than the angle ABD, while by the hypothesis it is at the same time *equal* to it.

Hence, says Proclus, in order that BA, DC may form a triangle it is necessary for them to *approach* one another in the sense of being turned round one pair of corresponding extremities, e.g. B, D, so that the other extremities A, C come nearer. This may be brought about in one of three ways: (1) AB may remain fixed and CD be turned about D so that the angle CDE increases; (2) CD may remain fixed and AB be turned about B so that the angle ABD becomes smaller; (3) both AB and CD may move so as to make the angle ABD smaller and the angle CDE larger at the same time. The *reason*, then, of the straight lines AB, CD coming to form a triangle or to meet is (says Proclus) *the movement of the straight lines.*

Though he does not mention it here, Proclus does in another passage (p. 371, 2—10, quoted on p. 207 above) hint at the possibility that, while I. 16 may remain universally true, either of the straight lines BA, DC (or both together) may be turned through any angle not greater than a certain finite angle and yet may not meet (the Bolyai-Lobachewsky hypothesis).

PROPOSITION 17.

In any triangle two angles taken together in any manner are less than two right angles.

Let ABC be a triangle;

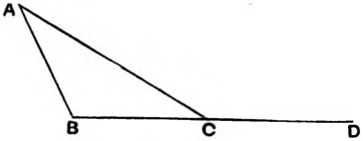
I say that two angles of the triangle ABC taken together in any manner are less than two right angles.

For let BC be produced to D. [Post. 2]

Then, since the angle ACD is an exterior angle of the triangle ABC,

it is greater than the interior and opposite angle ABC.

Let the angle ACB be added to each;
therefore the angles ACD, ACB are greater than the angles ABC, BCA.



But the angles ACD, ACB are equal to two right angles. [I. 13]

Therefore the angles ABC, BCA are less than two right angles.

Similarly we can prove that the angles BAC, ACB are also less than two right angles, and so are the angles CAB, ABC as well.

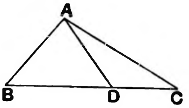
Therefore etc.

Q. E. D.

1. taken together in any manner, πάντῃ μεταλαμβανόμεναι, i.e. any pair added together.

As in his note on the previous proposition, Proclus tries to state the *cause* of the property. He takes the case of two straight lines forming right angles with a transversal and observes that it is the *convergence of the straight lines towards one another* (σύνευσις τῶν εὐθειῶν), the *lessening* of the two right angles, which produces the triangle. He will not have it that the fact of the exterior angle being greater than the interior and opposite angle is the *cause* of the property, for the odd reason that "it is not necessary that a side should be produced, or that there should be any exterior angle constructed...and how can what is not necessary be the cause of what is necessary?" (p. 311, 17—21).

Agreeably to this view, Proclus then sets himself to prove the theorem without producing a side of the triangle.



Let ABC be a triangle. Take any point D on BC, and join AD.

Then the exterior angle ADC of the triangle ABD is greater than the interior and opposite angle ABD.

Similarly the exterior angle ADB of the triangle ADC is greater than the interior and opposite angle ACD.

Therefore, by addition, the angles ADB, ADC are together greater than the angles ABC, ACB.

But the angles ADB, ADC are equal to two right angles; therefore the angles ABC, ACB are less than two right angles.

Lastly, Proclus proves (what is obvious from this proposition) that *there cannot be more than one perpendicular to a straight line from a point without it*. For, if this were possible, two of such perpendiculars would form a triangle in which two angles would be right angles: which is impossible, since any two angles of a triangle are together less than two right angles.

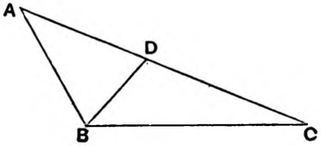
PROPOSITION 18.

In any triangle the greater side subtends the greater angle.

For let ABC be a triangle having the side AC greater than AB;

I say that the angle ABC is also greater than the angle BCA.

For, since AC is greater than AB, let AD be made equal to AB [I. 3], and let BD be joined.



Then, since the angle ADB is an exterior angle of the triangle BCD,

it is greater than the interior and opposite angle DCB. [I. 16]

But the angle ADB is equal to the angle ABD,

since the side AB is equal to AD;

therefore the angle ABD is also greater than the angle ACB;

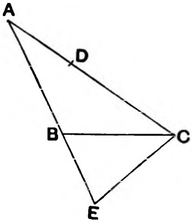
therefore the angle ABC is much greater than the angle ACB.

Therefore etc.

Q. E. D.

In the enunciation of this proposition we have ὑποτείνειν ("subtend") used with the simple accusative instead of the more usual ὑπό with accusative. The latter construction is used in the enunciation of I. 19, which otherwise only differs from that of I. 18 in the order of the words. The point to remember in order to distinguish the two is that the *datum* comes first and the *quaesitum* second, the *datum* being in this proposition the greater *side* and in the next the greater *angle*. Thus the enunciations are (I. 18) παντὸς τριγώνου ἡ μείζων πλευρὰ τὴν μείζονα γωνίαν ὑποτείνει and (I. 19) παντὸς τριγώνου ὑπὸ τὴν μείζονα γωνίαν ἡ μείζων πλευρὰ ὑποτείνει. In order to keep the proper order in English we must use the passive of the verb in I. 19. Aristotle quotes the result of I. 19, using the exact wording, ὑπὸ γὰρ τὴν μείζω γωνίαν ὑποτείνει (*Meteorologica* III. 5, 376 a 12).

"In order to assist the student in remembering which of these two propositions [I. 18, 19] is demonstrated directly and which indirectly, it may be observed that the order is similar to that in I. 5 and I. 6" (Todhunter).



An alternative proof of I. 18 given by Porphyry (see Proclus, pp. 315, 11—316, 13) is interesting. It starts by supposing a length equal to AB cut off from the other end of AC; that is, CD and not AD is made equal to AB.

Produce AB to E so that BE is equal to AD, and join EC.

Then, since AB is equal to CD, and BE to AD,

AE is equal to AC.

Therefore the angle *AEC* is equal to the angle *ACE*.
Now the angle *ABC* is greater than the angle *AEC*, [I. 16]
and therefore greater than the angle *ACE*.
Hence, *a fortiori*, the angle *ABC* is greater than the angle *ACB*.

PROPOSITION 19.

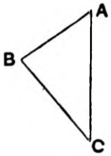
In any triangle the greater angle is subtended by the greater side.

Let *ABC* be a triangle having the angle *ABC* greater than the angle *BCA*;

I say that the side *AC* is also greater than the side *AB*.

For, if not, *AC* is either equal to *AB* or less.

Now *AC* is not equal to *AB*;
for then the angle *ABC* would also have been equal to the angle *ACB*; [I. 5]
but it is not;
therefore *AC* is not equal to *AB*.



Neither is *AC* less than *AB*,
for then the angle *ABC* would also have been less than the angle *ACB*; [I. 18]
but it is not;
therefore *AC* is not less than *AB*.

And it was proved that it is not equal either.
Therefore *AC* is greater than *AB*.

Therefore etc. Q. E. D.

This proposition, like I. 6, can be proved by merely *logical* deduction from I. 5 and I. 18 taken together, as pointed out by De Morgan. The general form of the argument used by De Morgan is given in his *Formal Logic* (1847), p. 25, thus:

"*Hypothesis.* Let there be any number of propositions or assertions—three for instance, *X*, *Y* and *Z*—of which it is the property that one or the other must be true, *and one only*. Let there be three other propositions *P*, *Q* and *R* of which it is also the property that one, and one only, must be true. Let it be a connexion of those assertions that:

when *X* is true, *P* is true,
when *Y* is true, *Q* is true,
when *Z* is true, *R* is true.

Consequence: then it follows that,

when *P* is true, *X* is true,
when *Q* is true, *Y* is true,
when *R* is true, *Z* is true."

To apply this to the case before us, let us denote the sides of the triangle *ABC* by *a*, *b*, *c*, and the angles opposite to these sides by *A*, *B*, *C* respectively, and suppose that *a* is the base.

Then we have the three propositions,

when *b* is equal to *c*, *B* is equal to *C*, [I. 5]
when *b* is greater than *c*, *B* is greater than *C*,
when *b* is less than *c*, *B* is less than *C*, } [I. 18]

and it follows *logically* that,

when *B* is equal to *C*, *b* is equal to *c*, [I. 6]
when *B* is greater than *C*, *b* is greater than *c*,
when *B* is less than *C*, *b* is less than *c*. } [I. 19]

Reductio ad absurdum by exhaustion.

Here, says Proclus (p. 318, 16—23), Euclid proves the impossibility "by means of *division*" (ἐκ διαιρέσεως). This means simply the separation of different hypotheses, each of which is inconsistent with the truth of the theorem to be proved, and which therefore must be successively shown to be impossible. If a straight line is not greater than a straight line, it must be either equal to it or less; thus in a *reductio ad absurdum* intended to prove such a theorem as I. 19 it is necessary to dispose successively of *two* hypotheses inconsistent with the truth of the theorem.

Alternative (direct) proof.

Proclus gives a direct proof (pp. 319—321) which an-Nairīzī also has and attributes to Heron. It requires a lemma and is consequently open to the slight objection of separating a theorem from its converse. But the lemma and proof are worth giving.

Lemma.

If an angle of a triangle be bisected and the straight line bisecting it meet the base and divide it into unequal parts, the sides containing the angle will be unequal, and the greater will be that which meets the greater segment of the base, and the less that which meets the less.

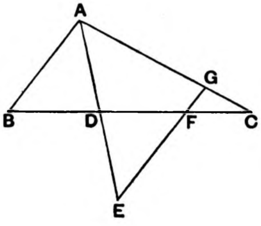
Let *AD*, the bisector of the angle *A* of the triangle *ABC*, meet *BC* in *D*, making *CD* greater than *BD*.

I say that *AC* is greater than *AB*.

Produce *AD* to *E* so that *DE* is equal to *AD*. And, since *DC* is greater than *BD*, cut off *DF* equal to *BD*.

Join *EF* and produce it to *G*.

Then, since the two sides *AD*, *DB* are equal to the two sides *ED*, *DF*, and the vertical angles at *D* are equal,



AB is equal to *EF*,

and the angle *DEF* to the angle *BAD*,

i.e. to the angle *DAG* (by hypothesis).

Therefore *AG* is equal to *EG*,

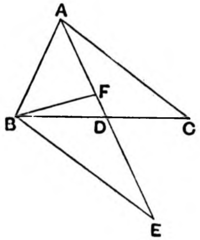
and therefore greater than *EF*, or *AB*.

Hence, *a fortiori*, *AC* is greater than *AB*.

Proof of I. 19.

Let *ABC* be a triangle in which the angle *ABC* is greater than the angle *ACB.*

Bisect *BC* at *D*, join *AD*, and produce it to *E* so that *DE* is equal to *AD*. Join *BE*.



Then the two sides *BD*, *DE* are equal to the two sides *CD*, *DA*, and the vertical angles at *D* are equal;

therefore *BE* is equal to *AC*,

and the angle *DBE* to the angle at *C*.

But the angle at *C* is less than the angle *ABC*;

therefore the angle *DBE* is less than the angle *ABD*.

Hence, if *BF* bisect the angle *ABE*, *BF* meets *AE* between *A* and *D*. Therefore *EF* is greater than *FA*.

It follows, by the lemma, that *BE* is greater than *BA*,

that is, *AC* is greater than *AB*.

PROPOSITION 20.

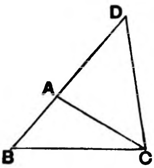
In any triangle two sides taken together in any manner are greater than the remaining one.

For let *ABC* be a triangle;

I say that in the triangle *ABC* two sides taken together in any manner are greater than the remaining one, namely

BA, *AC* greater than *BC*,
AB, *BC* greater than *AC*,
BC, *CA* greater than *AB*.

For let *BA* be drawn through to the point *D*, let *DA* be made equal to *CA*, and let *DC* be joined.



Then, since *DA* is equal to *AC*, the angle *ADC* is also equal to the angle *ACD*; [I. 5]

therefore the angle *BCD* is greater than the angle *ADC*. [C. N. 5]

And, since *DCB* is a triangle having the angle *BCD* greater than the angle *BDC*,

and the greater angle is subtended by the greater side, [I. 19]

therefore *DB* is greater than *BC*.

But DA is equal to AC;

therefore BA, AC are greater than BC.

Similarly we can prove that AB, BC are also greater than CA, and BC, CA than AB.

Therefore etc.

Q. E. D.

It was the habit of the Epicureans, says Proclus (p. 322), to ridicule this theorem as being evident even to an ass and requiring no proof, and their allegation that the theorem was "known" (γνώριμον) even to an ass was based on the fact that, if fodder is placed at one angular point and the ass at another, he does not, in order to get to his food, traverse the two sides of the triangle but only the one side separating them (an argument which makes Savile exclaim that its authors were "digni ipsi, qui cum Asino foenum essent," p. 78). Proclus replies truly that a mere perception of the truth of the theorem is a different thing from a scientific proof of it and a knowledge of the reason *why* it is true. Moreover, as Simson says, the number of axioms should not be increased without necessity.

Alternative Proofs.

Heron and Porphyry, we are told (Proclus, pp. 323—6), proved this theorem in different ways as follows, without producing one of the sides.

First proof.

Let ABC be the triangle, and let it be required to prove that the sides BA, AC are greater than BC.

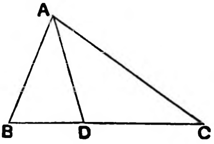
Bisect the angle BAC by AD meeting BC in D.

Then, in the triangle ABD,

the exterior angle ADC is greater than the interior and opposite angle BAD, [I. 16]

that is, greater than the angle DAC.

Therefore the side AC is greater than the side CD. [I. 19]



Similarly we can prove that AB is greater than BD.

Hence, by addition, BA, AC are greater than BC.

Second proof.

This, like the first proof, is direct. There are several cases to be considered.

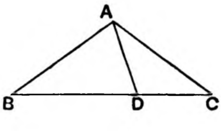
(1) If the triangle is *equilateral*, the truth of the proposition is obvious.

(2) If the triangle is *isosceles*, the proposition needs no proof in the case (*a*) where each of the equal sides is greater than the base.

(*b*) If the base is greater than either of the other sides, we have to prove that the sum of the two equal sides is greater than the base. Let BC be the base in such a triangle.

Cut off from BC a length BD equal to AB, and join AD.

Then, in the triangle ADB, the exterior angle ADC is greater than the interior and opposite angle BAD. [I. 16]



Similarly, in the triangle ADC, the exterior angle ADB is greater than the interior and opposite angle CAD.

By addition, the two angles BDA, ADC are together greater than the two angles BAD, DAC (or the whole angle BAC).

Subtracting the equal angles BDA, BAD, we have the angle ADC greater than the angle CAD.

It follows that AC is greater than CD; [I. 19]
and, adding the equals AB, BD respectively, we have BA, AC together greater than BC.

(3) If the triangle be *scalene*, we can arrange the sides in order of length. Suppose BC is the greatest, AB the intermediate and AC the least side. Then it is obvious that AB, BC are together greater than AC, and BC, CA together greater than AB.

It only remains therefore to prove that CA, AB are together greater than BC.

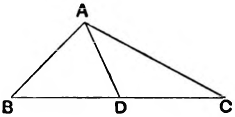
We cut off from BC a length BD equal to the adjacent side, join AD, and proceed exactly as in the above case of the isosceles triangle.

Third proof.

This proof is by *reductio ad absurdum.*

Suppose that BC is the greatest side and, as before, we have to prove that BA, AC are greater than BC.

If they are not, they must be either equal to or less than BC.



(1) Suppose BA, AC are together equal to BC.

From BC cut off BD equal to BA, and join AD.

It follows from the hypothesis that DC is equal to AC.

Then, since BA is equal to BD,
the angle BDA is equal to the angle BAD.

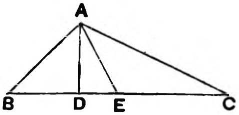
Similarly, since AC is equal to CD,
the angle CDA is equal to the angle CAD.

By addition, the angles BDA, ADC are together equal to the whole angle BAC.

That is, the angle BAC is equal to two right angles: which is impossible.

(2) Suppose BA, AC are together less than BC.

From BC cut off BD equal to BA, and from CB cut off CE equal to CA. Join AD, AE.



In this case, we prove in the same way that the angle BDA is equal to the angle BAD, and the angle CEA to the angle CAE.

By addition, the sum of the angles BDA, AEC is equal to the sum of the angles BAD, CAE.

Now, by I. 16, the angle BDA is greater than the angle DAC, and therefore, *a fortiori*, greater than the angle EAC.

Similarly the angle AEC is greater than the angle BAD.

Hence the sum of the angles BDA, AEC is greater than the sum of the angles BAD, EAC.

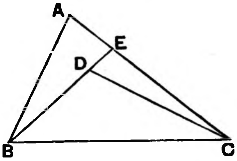
But the former sum was also equal to the latter: which is impossible.

PROPOSITION 21.

If on one of the sides of a triangle, from its extremities, there be constructed two straight lines meeting within the triangle, the straight lines so constructed will be less than the remaining two sides of the triangle, but will contain a greater angle.

On BC, one of the sides of the triangle ABC, from its extremities B, C, let the two straight lines BD, DC be constructed meeting within the triangle;

I say that BD, DC are less than the remaining two sides of the triangle BA, AC, but contain an angle BDC greater than the angle BAC.



For let BD be drawn through to E.

Then, since in any triangle two sides are greater than the remaining one, [I. 20]

therefore, in the triangle ABE, the two sides AB, AE are greater than BE.

Let EC be added to each;

therefore BA, AC are greater than BE, EC.

Again, since, in the triangle CED,

the two sides CE, ED are greater than CD,

let DB be added to each;

therefore CE, EB are greater than CD, DB.

But BA, AC were proved greater than BE, EC;

therefore BA, AC are much greater than BD, DC.

Again, since in any triangle the exterior angle is greater than the interior and opposite angle, [I. 16]

therefore, in the triangle CDE,

the exterior angle BDC is greater than the angle CED.

For the same reason, moreover, in the triangle ABE also,

the exterior angle CEB is greater than the angle BAC.

But the angle BDC was proved greater than the angle CEB;

therefore the angle BDC is much greater than the angle BAC.

Therefore etc. Q. E. D.

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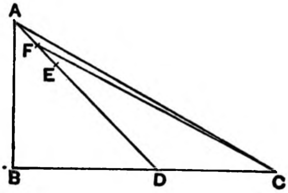
2. **be constructed...meeting within the triangle.** The word "meeting" is not in the Greek, where the words are *ἐντὸς συσταθῶσιν*. *συνίστασθαι* is the word used of constructing two straight lines *to a point* (cf. I. 7) or so as to form a triangle; but it is necessary in English to indicate that they *meet*.

3. **the straight lines so constructed.** Observe the elegant brevity of the Greek *αἱ συσταθεῖσαι*.

The editors generally call attention to the fact that the lines drawn within the triangle in this proposition must be drawn, as the enunciation says, from the *ends* of the side; otherwise it is not necessary that their sum should be less than that of the remaining sides of the triangle. Proclus (p. 327, 12 sqq.) gives a simple illustration.



Let *ABC* be a right-angled triangle. Take any point *D* on *BC*, join *DA*, and cut off from it *DE* equal to *AB*. Bisect *AE* at *F*, and join *FC*.

Then shall *CF*, *FD* be together greater than *CA*, *AB*.

For *CF*, *FE* are equal to *CF*, *FA*,

and therefore greater than *CA*.

Add the equals *ED*, *AB* respectively;

therefore *CF*, *FD* are together greater than *CA*, *AB*.

Pappus gives the same proposition as that just proved, but follows it up by a number of others more elaborate in character, selected apparently from "the so-called paradoxes" of one Erycinus (Pappus, III. p. 106 sqq.). Thus he proves the following:

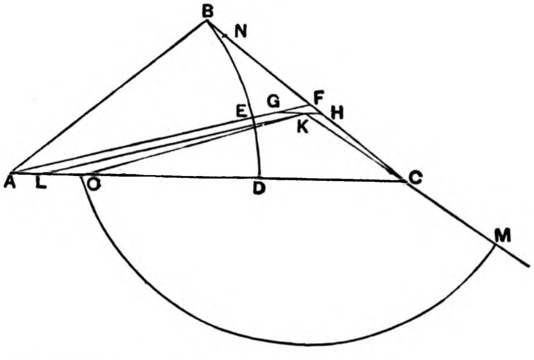
1. In any triangle, except an equilateral triangle or an isosceles triangle with base less than one of the other sides, it is possible to construct on the base and within the triangle two straight lines the sum of which is equal to the sum of the other two sides of the triangle.

2. In any triangle in which it is possible to construct two straight lines on the base which are equal to the sum of the other two sides of the triangle it is also possible to construct two others the sum of which is *greater* than that sum.

3. Under the same conditions, if the base is greater than either of the other two sides, two straight lines can be constructed in the manner described which are *respectively* greater than the other two sides of the triangle; and the lines may be constructed so as to be respectively *equal* to the two sides, if one of those two sides is less than the other and each of them less than the base.

4. The lines may be so constructed that their sum will bear to the sum of the two sides of the triangle any ratio less than 2 : 1.

As a specimen of the proofs we will give that of the proposition which has



been numbered (1) for the case where the triangle is isosceles (Pappus, III. pp. 108—110).

Let ABC be an isosceles triangle in which the base AC is greater than either of the equal sides AB, BC.

With centre A and radius AB describe a circle meeting AC in D.

Draw any radius AEF such that it meets BC in a point F outside the circle.

Take any point G on EF, and through it draw GH parallel to AC. Take any point K on GH, and draw KL parallel to FA meeting AC in L.

From BC cut off BN equal to EG.

Thus AG, or LK, is equal to the sum of AB, BN, and CN is less than LK.

Now GF, FH are together greater than GH,
and CH, HK together greater than CK.

Therefore, by addition,
CF, FG, HK are together greater than CK, HG.

Subtracting HK from each side, we see that
CF, FG are together greater than CK, KG;
therefore, if we add AG to each,
AF, FC are together greater than AG, GK, KC.

And AB, BC are together greater than AF, FC. [I. 21]

Therefore AB, BC are together greater than AG, GK, KC.

But, by construction, AB, BN are together equal to AG;
therefore, by subtraction, NC is greater than GK, KC,
and *a fortiori* greater than KC.

Take on KC produced a point M such that KM is equal to NC;
with centre K and radius KM describe a circle meeting CL in O, and join KO.

Then shall LK, KO be equal to AB, BC.

For, by construction, LK is equal to the sum of AB, BN, and KO is equal to NC;
therefore LK, KO are together equal to AB, BC.

It is after I. 21 that (as remarked by De Morgan) the important proposition about the perpendicular and obliques drawn from a point to a straight line of unlimited length is best introduced:

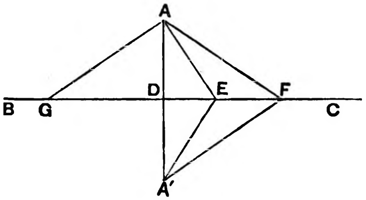
Of all straight lines that can be drawn to a given straight line of unlimited length from a given point without it:

(*a*) *the perpendicular is the shortest;*

(*b*) *of the obliques, that is the greater the foot of which is further from the perpendicular;*

(*c*) *given one oblique, only one other can be found of the same length, namely that the foot of which is equally distant with the foot of the given one from the perpendicular, but on the other side of it.*

Let A be the given point, BC the given straight line; let AD be the perpendicular from A on BC, and AE, AF any two obliques of which AF makes the greater angle with AD.



Produce AD to A', making $A'D$ equal to AD, and join $A'E$, $A'F$.

Then the triangles ADE, $A'DE$ are equal in all respects; and so are the triangles ADF, $A'DF$.

Now (1) in the triangle AEA' the two sides AE, EA' are greater than AA' [I. 20], that is, twice AE is greater than twice AD.

Therefore AE is greater than AD.

(2) Since AE, $A'E$ are drawn to E, a point within the triangle AFA',

AF, FA' are together greater than AE, EA', [I. 21]

or twice AF is greater than twice AE.

Therefore AF is greater than AE.

(3) Along DB measure off DG equal to DF, and join AG.

The triangles AGD, AFD are then equal in all respects, so that the angles GAD, FAD are equal, and AG is equal to AF.

PROPOSITION 22.

Out of three straight lines, which are equal to three given straight lines, to construct a triangle: thus it is necessary that two of the straight lines taken together in any manner should be greater than the remaining one. [I. 20]

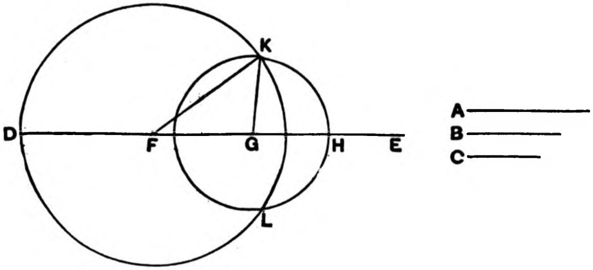
Let the three given straight lines be A, B, C, and of these let two taken together in any manner be greater than the remaining one,

namely A, B greater than C,

A, C greater than B,

and B, C greater than A;

thus it is required to construct a triangle out of straight lines equal to A, B, C.



Let there be set out a straight line DE, terminated at D but of infinite length in the direction of E,

and let DF be made equal to A, FG equal to B, and GH equal to C. [I. 3]

With centre F and distance FD let the circle DKL be described;

again, with centre G and distance GH let the circle KLH be described;

and let KF, KG be joined;

I say that the triangle KFG has been constructed out of three straight lines equal to A, B, C.

For, since the point F is the centre of the circle DKL,

FD is equal to FK.

But FD is equal to A;

therefore KF is also equal to A.

Again, since the point G is the centre of the circle LKH,

GH is equal to GK.

But GH is equal to C;

therefore KG is also equal to C.

And FG is also equal to B;

therefore the three straight lines KF, FG, GK are equal to the three straight lines A, B, C.

Therefore out of the three straight lines KF, FG, GK, which are equal to the three given straight lines A, B, C, the triangle KFG has been constructed.

Q. E. F.

2—4. This is the first case in the *Elements* of a διορισμὸς to a problem in the sense of a statement of the conditions or limits of the possibility of a solution. The criterion is of course supplied by the preceding proposition.

2. **thus it is necessary.** This is usually translated (e.g. by Williamson and Simson) "*But* it is necessary," which is however inaccurate, since the Greek is not δεῖ δέ but δεῖ δή. The words are the same as those used to introduce the διορισμὸς in the other sense of the "definition" or "particular statement" of a construction to be effected. Hence, as in the latter case we say "thus it is required" (e.g. to bisect the finite straight line AB, I. 10), we should here translate "*thus* it is necessary."

4. To this enunciation all the MSS. and Boethius add, after the διορισμὸς, the words "because in any triangle two sides taken together in any manner are greater than the remaining one." But this explanation has the appearance of a gloss, and it is omitted by Proclus and Campanus. Moreover there is no corresponding addition to the διορισμὸς of VI. 28.

It was early observed that Euclid assumes, without giving any reason, that the circles drawn as described will meet if the condition that any two of the straight lines A, B, C are together greater than the third be fulfilled. Proclus (p. 331, 8 sqq.) argues the matter by means of *reductio ad absurdum*, but does not exhaust the possible hypotheses inconsistent with the contention. He says the circles must do one of three things, (1) cut one another, (2) touch one another, (3) stand apart (διεστάναι) from one another. He then considers the hypotheses (*a*) of their touching *externally*, (*b*) of their being separated from one another by a space. He should have considered also the hypothesis (*c*) of one circle touching the other *internally* or lying entirely within the other without touching. These three hypotheses being successively disproved, it follows that the circles must meet (this is the line taken by Camerer and Todhunter).

Simson says: "Some authors blame Euclid because he does not demonstrate that the two circles made use of in the construction of this problem must cut one another: but this is very plain from the determination he has given, namely, that any two of the straight lines DF, FG, GH must be greater than the third. For who is so dull, though only beginning to learn the Elements, as not to perceive that the circle described from the centre F, at the distance FD, must meet FH betwixt F and H, because FD is less than FH; and that, for the like reason, the circle described from the centre *G* at the distance *GH* must meet *DG* betwixt *D* and *G*; and that these circles must meet one another, because *FD* and *GH* are together greater than *FG*."

We have in fact only to satisfy ourselves that one of the circles, e.g. that with centre *G*, has at least one point of its circumference outside the other circle and also at least one point of its circumference inside the same circle; and this is best shown with reference to the points in which the first circle cuts the straight line *DE*. For (1) *FH*, being equal to the sum of *B* and *C*, is greater than *A*, i.e. than the radius of the circle with centre *F*, and therefore *H* is outside that circle. (2) If *GM* be measured along *GF* equal to *GH* or *C*, then, since *GM* is either (*a*) less or (*b*) greater than *GF*, *M* will fall either (*a*) between *G* and *F* or (*b*) beyond *F* towards *D*; in the first case (*a*) the sum of *FM* and *C* is equal to *FG* and therefore less than the sum of *A* and *C*, so that *FM* is less than *A* or *FD*; in the second case (*b*) the sum of *MF* and *FG*, i.e. the sum of *MF* and *B*, is equal to *GM* or *C*, and therefore less than the sum of *A* and *B*, so that *MF* is less than *A* or *FD*; hence in either case *M* falls within the circle with centre *F*.

It being now proved that the circumference of the circle with centre *G* has at least one point outside, and at least one point inside, the circle with centre *F*, we have only to invoke the Principle of Continuity, as we have to do in I. 1 (cf. the note on that proposition, p. 242, where the necessary postulate is stated in the form suggested by Killing).

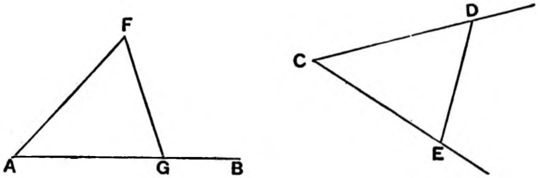
That the construction of the proposition gives only *two* points of intersection between the circles, and therefore only two triangles satisfying the condition, one on each side of *FG*, is clear from I. 7, which, as before pointed out, takes the place, in Book I., of III. 10 proving that two circles cannot intersect in more points than two.

PROPOSITION 23.

On a given straight line and at a point on it to construct a rectilineal angle equal to a given rectilineal angle.

Let *AB* be the given straight line, *A* the point on it, and the angle *DCE* the given rectilineal angle;

thus it is required to construct on the given straight line *AB*, and at the point *A* on it, a rectilineal angle equal to the given rectilineal angle *DCE*.



On the straight lines *CD*, *CE* respectively let the points *D*, *E* be taken at random;

let *DE* be joined,

and out of three straight lines which are equal to the three straight lines *CD*, *DE*, *CE* let the triangle *AFG* be constructed in such a way that *CD* is equal to *AF*, *CE* to *AG*, and further *DE* to *FG*. [I. 22]

Then, since the two sides *DC*, *CE* are equal to the two sides *FA*, *AG* respectively,

and the base *DE* is equal to the base *FG*,

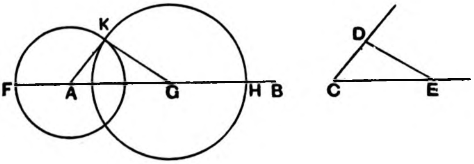
the angle *DCE* is equal to the angle *FAG*. [I. 8]

Therefore on the given straight line *AB*, and at the point *A* on it, the rectilineal angle *FAG* has been constructed equal to the given rectilineal angle *DCE*.

Q. E. F.

This problem was, according to Eudemus (see Proclus, p. 333, 5), "rather the discovery of Oenopides," from which we must apparently infer, not that Oenopides was the first to find any solution of it, but that it was he who discovered the particular solution given by Euclid. (Cf. Bretschneider, p. 65.)

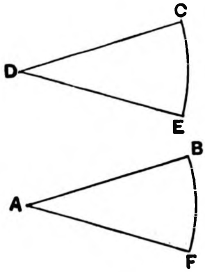
The editors do not seem to have noticed the fact that the construction of the triangle assumed in this proposition is not exactly the construction given in I. 22. We have here to construct a triangle on a certain finite straight line *AG* as base; in I. 22 we have only to construct a triangle with sides of given length without any restriction as to how it is to be placed. Thus in I. 22 we set out any line whatever and measure successively three lengths along it beginning from the given extremity, and what we must regard as the base is the intermediate length, not the length beginning at the given extremity of the straight line arbitrarily set out. Here the base is a given straight line abutting at a given point. Thus the construction has to be modified somewhat from that of the preceding proposition. We must measure *AG* along *AB* so that *AG* is equal to *CE* (or *CD*), and *GH* along *GB* equal to *DE*; and then we must produce *BA*, in the opposite direction, to *F*, so that *AF* is equal to *CD* (or *CE*, if *AG* has been made equal to *CD*).



Then, by drawing circles (1) with centre *A* and radius *AF*, (2) with centre *G* and radius *GH*, we determine *K*, one of their points of intersection, and we prove that the triangle *KAG* is equal in all respects to the triangle *DCE*, and then that the angle at *A* is equal to the angle *DCE*.

I think that Proclus must (though he does not say so) have felt the same difficulty with regard to the use in I. 23 of the result of I. 22, and that this is probably the reason why he gives over again the construction which I have given above, with the remark (p. 334, 6) that "you may obtain the construction of the triangle in a more instructive manner (διδασκαλικώτερον) as follows."

Proclus objects to the procedure of Apollonius in constructing an angle under the same conditions, and certainly, if he quotes Apollonius correctly, the latter's exposition must have been somewhat slipshod.



"He takes an angle *CDE* at random," says Proclus (p. 335, 19 sqq.), "and a straight line *AB*, and with centre *D* and distance *CD* describes the circumference *CE*, and in the same way with centre *A* and distance *AB* the circumference *FB*. Then, cutting off *FB* equal to *CE*, he joins *AF*. And he declares that the angles *A*, *D* standing on equal circumferences are equal."

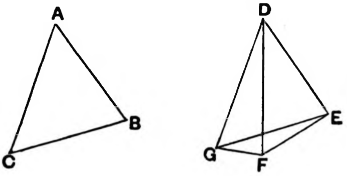
In the first place, as Proclus remarks, it should be premised that *AB* is equal to *CD* in order that the circles may be equal; and the use of Book III. for such an elementary construction is objectionable. The omission to state that *AB* must be taken equal to *CD* was no doubt a slip, if it occurred. And, as regards the equal angles "standing on equal circumferences," it would seem possible that Apollonius said this in *explanation*, for the sake of brevity, rather than by way of proof. It seems to me probable that his construction was only given from the point of view of *practical*, not theoretical, geometry. It really comes to the same thing as Euclid's except that *DC* is taken equal to *DE*. For cutting off the arc *BF* equal to the arc *CE* can only be meant in the sense of measuring the *chord* *CE*, say, with a pair of compasses, and then drawing a circle with centre *B* and radius equal to the chord *CE*. Apollonius' direction was therefore probably intended as a practical short cut, avoiding the actual drawing of the chords *CE*, *BF*, which, as well as a proof of the equality in all respects of the triangles *CDE*, *BAF*, would be required to establish *theoretically* the correctness of the construction.

PROPOSITION 24.

If two triangles have the two sides equal to two sides respectively, but have the one of the angles contained by the equal straight lines greater than the other, they will also have the base greater than the base.

Let *ABC*, *DEF* be two triangles having the two sides *AB*, *AC* equal to the two sides *DE*, *DF* respectively, namely *AB* to *DE*, and *AC* to *DF*, and let the angle at *A* be greater than the angle at *D*;

I say that the base *BC* is also greater than the base *EF*.



For, since the angle *BAC* is greater than the angle *EDF*, let there be constructed, on the straight line *DE*, and at the point *D* on it, the angle *EDG* equal to the angle *BAC*; [I. 23]
let *DG* be made equal to either of the two straight lines *AC*, *DF*, and let *EG*, *FG* be joined.

Then, since *AB* is equal to *DE*, and *AC* to *DG*, the two sides *BA*, *AC* are equal to the two sides *ED*, *DG*, respectively;

and the angle *BAC* is equal to the angle *EDG*;

therefore the base *BC* is equal to the base *EG*. [I. 4]

Again, since *DF* is equal to *DG*,

the angle *DGF* is also equal to the angle *DFG*; [I. 5]

therefore the angle *DFG* is greater than the angle *EGF*.

Therefore the angle *EFG* is much greater than the angle *EGF*.

And, since *EFG* is a triangle having the angle *EFG* greater than the angle *EGF*,

and the greater angle is subtended by the greater side, [I. 19]

the side *EG* is also greater than *EF*.

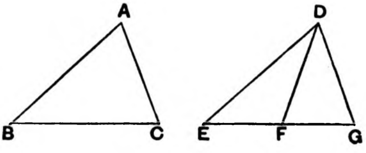
But *EG* is equal to *BC*.

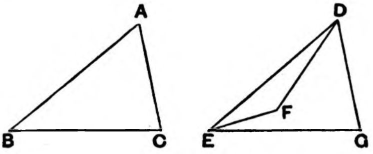
Therefore *BC* is also greater than *EF*.

Therefore etc.

Q. E. D.

10. I have naturally left out the well-known words added by Simson in order to avoid the necessity of considering three cases: "Of the two sides *DE*, *DF* let *DE* be the side which is not greater than the other." I doubt whether Euclid could have been induced to insert the words himself, even if it had been represented to him that their omission meant leaving two possible cases out of consideration. His habit and that of the great Greek geometers was, not to set out all possible cases, but to give as a rule one case, generally the most difficult, as here, and to leave the others to the reader to work out for himself. We have already seen one instance in I. 7.





Proclus of course gives the other two cases which arise if we do not first provide that *DE* is not greater than *DF*.

(1) In the first case *G* may fall on *EF* produced, and it is then obvious that *EG* is greater than *EF*.

(2) In the second case *EG* may fall below *EF*.

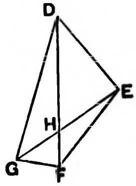
If so, by I. 21, *DF*, *FE* are together less than *DG*, *GE*.

But *DF* is equal to *DG*; therefore *EF* is less than *EG*, i.e. than *BC*.

These two cases are therefore decidedly simpler than the case taken by Euclid as typical, and could well be left to the ingenuity of the learner.

If however after all we prefer to insert Simson's words and avoid the latter two cases, the proof is not complete unless we show that, with his assumption, *F* must, in the figure of the proposition, fall *below EG*.

De Morgan would make the following proposition precede: ***Every straight line drawn from the vertex of a triangle to the base is less than the greater of the two sides, or than either if they are equal***, and he would prove it by means of the proposition relating to perpendicular and obliques given above, p. 291.



But it is easy to prove directly that *F* falls below *EG*, if *DE* is not greater than *DG*, by the method employed by Pfleiderer, Lardner, and Todhunter.

Let *DF*, produced if necessary, meet *EG* in *H*.

Then the angle *DHG* is greater than the angle *DEG*; [I. 16]

and the angle *DEG* is not less than the angle *DGE*; [I. 18]

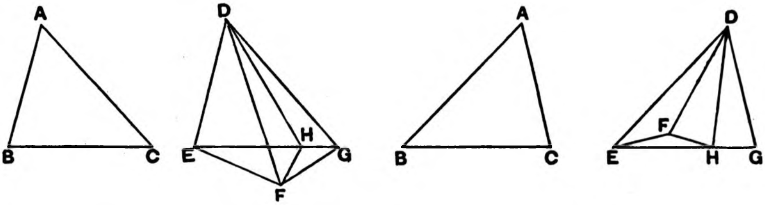
therefore the angle *DHG* is greater than the angle *DGH*.

Hence *DH* is less than *DG*, [I. 19]

and therefore *DH* is less than *DF*.

Alternative proof.

Lastly, the modern alternative proof is worth giving.



Let *DH* bisect the angle *FDG* (after the triangle *DEG* has been made equal in all respects to the triangle *ABC*, as in the proposition), and let *DH* meet *EG* in *H*. Join *HF*.

Then, in the triangles *FDH*, *GDH*,

the two sides *FD*, *DH* are equal to the two sides *GD*, *DH*,

and the included angles *FDH*, *GDH* are equal;

therefore the base *HF* is equal to the base *HG*.

Accordingly *EG* is equal to the sum of *EH*, *HF*;

and *EH*, *HF* are together greater than *EF*; [I. 20]

therefore *EG*, or *BC*, is greater than *EF*.

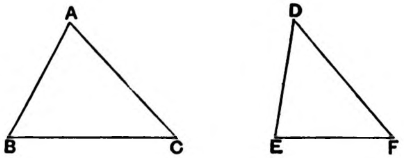
Proclus (p. 339, 11 sqq.) answers by anticipation the possible question that might occur to any one on this proposition, viz. why does Euclid not compare the areas of the triangles as he does in I. 4? He observes that inequality of the areas does not follow from the inequality of the angles contained by the equal sides, and that Euclid leaves out all reference to the question both for this reason and because the areas cannot be compared without the help of the theory of parallels. "But if," says Proclus, "we must anticipate what is to come and make our comparison of the areas at once, we assert that (1) *if the angles A, D*—supposing that our argument proceeds with reference to the figure in the proposition—*are (together) equal to two right angles, the triangles* *are proved equal,* (2) *if greater than two right angles, that triangle which has the greater angle is less, and* (3) *if they are less, greater.*" Proclus then gives the proof, but without any reference to the source from which he quoted the proposition. Now an-Nairīzī adds a similar proposition to I. 38, but definitely attributes it to Heron. I shall accordingly give it in the place where Heron put it

PROPOSITION 25.

If two triangles have the two sides equal to two sides respectively, but have the base greater than the base, they will also have the one of the angles contained by the equal straight lines greater than the other.

Let ABC, DEF be two triangles having the two sides AB, AC equal to the two sides DE, DF respectively, namely AB to DE, and AC to DF; and let the base BC be greater than the base EF;

I say that the angle BAC is also greater than the angle EDF.



For, if not, it is either equal to it or less.

Now the angle BAC is not equal to the angle EDF; for then the base BC would also have been equal to the base EF, [I. 4]

but it is not;

therefore the angle BAC is not equal to the angle EDF.

Neither again is the angle BAC less than the angle EDF; for then the base BC would also have been less than the base EF, [I. 24]

but it is not;

therefore the angle BAC is not less than the angle EDF.

But it was proved that it is not equal either;

therefore the angle BAC is greater than the angle EDF.

Therefore etc.

Q. E. D.

De Morgan points out that this proposition (as also I. 8) is a purely *logical* consequence of I. 4 and I. 24 in the same way as I. 19 and I. 6 are purely *logical* consequences of I. 18 and I. 5. If a, b, c denote the sides, A, B, C the angles opposite to them in a triangle ABC, and a', b', c', A', B', C' the sides and opposite angles respectively in a triangle $A'B'C'$, I. 4 and I. 24 tell us that, b, c being respectively equal to b', c',

(1) if A is equal to A', then a is equal to a',
(2) if A is less than A', then a is less than a',
(3) if A is greater than A', then a is greater than a';

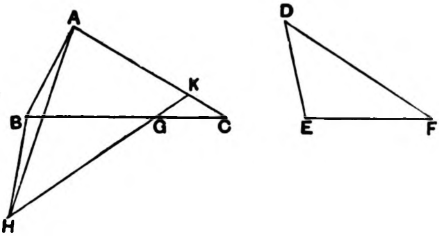
and it follows *logically* that,

(1) if a is equal to a', the angle A is equal to the angle A', [I. 8]
(2) if a is less than a', A is less than A',
(3) if a is greater than a', A is greater than A'. [I. 25]

Two alternative proofs of this theorem are given by Proclus (pp. 345—7), and they are both interesting. Moreover both are *direct*.

I. Proof by Menelaus of Alexandria.

Let ABC, DEF be two triangles having the two sides BA, AC equal to the two sides ED, DF, but the base BC greater than the base EF.



Then shall the angle at A be greater than the angle at D.

From BC cut off BG equal to EF. At B, on the straight line BC, make the angle GBH (on the side of BG remote from A) equal to the angle FED.

Make BH equal to DE; join HG, and produce it to meet AC in K. Join AH.

Then, since the two sides GB, BH are equal to the two sides FE, ED respectively,

and the angles contained by them are equal,

HG is equal to DF or AC,

and the angle BHG is equal to the angle EDF.

Now HK is greater than HG or AC,

and *a fortiori* greater than AK;

therefore the angle KAH is greater than the angle KHA.

And, since AB is equal to BH,

the angle BAH is equal to the angle BHA.

Therefore, by addition,

the whole angle BAC is greater than the whole angle BHG,

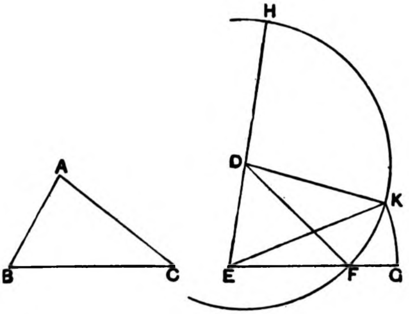
that is, greater than the angle EDF.

II. Heron's proof.

Let the triangles be given as before.

Since BC is greater than EF, produce EF to G so that EG is equal to BC.

Produce ED to H so that DH is equal to DF. The circle with centre D and radius DF will then pass through H. Let it be described, as FKH.



Now, since BA, AC are together greater than BC,

and BA, AC are equal to ED, DH respectively,

while BC is equal to EG,

EH is greater than EG.

Therefore the circle with centre E and radius EG will cut EH, and therefore will cut the circle already drawn. Let it cut that circle in K, and join DK, KE.

Then, since D is the centre of the circle FKH,

DK is equal to DF or AC.

Similarly, since E is the centre of the circle KG,

EK is equal to EG or BC,

And DE is equal to AB.

Therefore the two sides BA, AC are equal to the two sides ED, DK respectively;

and the base BC is equal to the base EK;

therefore the angle BAC is equal to the angle EDK.

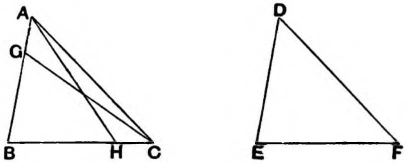
Therefore the angle BAC is greater than the angle EDF.

PROPOSITION 26.

If two triangles have the two angles equal to two angles respectively, and one side equal to one side, namely, either the side adjoining the equal angles, or that subtending one of the equal angles, they will also have the remaining sides equal to the remaining sides and the remaining angle to the remaining angle.

Let ABC, DEF be two triangles having the two angles ABC, BCA equal to the two angles DEF, EFD respectively, namely the angle ABC to the angle DEF, and the angle BCA to the angle EFD; and let them also have one side equal to one side, first that adjoining the equal angles, namely BC to EF;

I say that they will also have the remaining sides equal to the remaining sides respectively, namely AB to DE and AC to DF, and the remaining angle to the remaining angle, namely the angle BAC to the angle EDF.



For, if AB is unequal to DE, one of them is greater.

Let AB be greater, and let BG be made equal to DE; and let GC be joined.

Then, since BG is equal to DE, and BC to EF, the two sides GB, BC are equal to the two sides DE, EF respectively;

and the angle GBC is equal to the angle DEF;

therefore the base GC is equal to the base DF,

and the triangle GBC is equal to the triangle DEF, and the remaining angles will be equal to the remaining angles, namely those which the equal sides subtend; [I. 4]

therefore the angle GCB is equal to the angle DFE.

But the angle DFE is by hypothesis equal to the angle BCA;

therefore the angle BCG is equal to the angle BCA,

the less to the greater: which is impossible.

Therefore AB is not unequal to DE,

and is therefore equal to it.

But BC is also equal to EF;

therefore the two sides AB, BC are equal to the two sides DE, EF respectively,

and the angle ABC is equal to the angle DEF;

therefore the base AC is equal to the base DF,

and the remaining angle BAC is equal to the remaining angle EDF. [I. 4]

Again, let sides subtending equal angles be equal, as *AB* to *DE*;

I say again that the remaining sides will be equal to the remaining sides, namely *AC* to *DF* and *BC* to *EF*, and further the remaining angle *BAC* is equal to the remaining angle *EDF*.

For, if *BC* is unequal to *EF*, one of them is greater.

Let *BC* be greater, if possible, and let *BH* be made equal to *EF*; let *AH* be joined.

Then, since *BH* is equal to *EF*, and *AB* to *DE*, the two sides *AB*, *BH* are equal to the two sides *DE*, *EF* respectively, and they contain equal angles;

therefore the base *AH* is equal to the base *DF*,

and the triangle *ABH* is equal to the triangle *DEF*, and the remaining angles will be equal to the remaining angles, namely those which the equal sides subtend; [I. 4]

therefore the angle *BHA* is equal to the angle *EFD*.

But the angle *EFD* is equal to the angle *BCA*;

therefore, in the triangle *AHC*, the exterior angle *BHA* is equal to the interior and opposite angle *BCA*:

which is impossible. [I. 16]

Therefore *BC* is not unequal to *EF*,

and is therefore equal to it.

But *AB* is also equal to *DE*;

therefore the two sides *AB*, *BC* are equal to the two sides *DE*, *EF* respectively, and they contain equal angles;

therefore the base *AC* is equal to the base *DF*,

the triangle *ABC* equal to the triangle *DEF*,

and the remaining angle *BAC* equal to the remaining angle *EDF*. [I. 4]

Therefore etc.

Q. E. D.

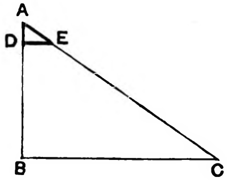
2—3. **the side adjoining the equal angles,** πλευρὰν τὴν πρὸς ταῖς ἴσαις γωνίαις.

29. **is by hypothesis equal.** ὑπόκειται ἴση, according to the elegant Greek idiom. ὑπόκειμαι is used for the passive of ὑποτίθημι, as κεῖμαι is used for the passive of τίθημι, and so with the other compounds. Cf. προσκεῖσθαι, "to be added."

The alternative method of proving this proposition, viz. by applying one triangle to the other, was very early discovered, at least so far as regards the case where the equal sides are adjacent to the equal angles in each. An-Nairīzī gives it for this case, observing that the proof is one which he had found, but of which he did not know the author.

Proclus has the following interesting note (p. 352, 13—18): "Eudemus in his geometrical history refers this theorem to Thales. For he says that, in the method by which they say that Thales proved the distance of ships in the sea, it was necessary to make use of this theorem." As, unfortunately, this information is not sufficient of itself to enable us to determine how Thales solved this problem, there is considerable room for conjecture as to his method.

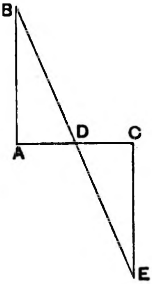
The suggestions of Bretschneider and Cantor agree in the assumption that the necessary observations were probably made from the top of some tower or structure of known height, and that a right-angled triangle was used in which the tower was the perpendicular, and the line connecting the bottom of the tower and the ship was the base, as in the annexed figure, where *AB* is the tower and *C* the ship. Bretschneider (*Die Geometrie und die Geometer vor Eukleides*, § 30) says that it was only necessary for the observer to observe the angle *CAB*, and then the triangle would be completely determined by means of this angle and the known length *AB*. As Bretschneider says that the result would be obtained "in a moment" by this method, it is not clear in what sense he supposes Thales to have "observed" the angle *BAC*. Cantor is more definite (*Gesch. d. Math.* I$_3$, p. 145), for he says that the problem was nearly related to that of finding the *Seqt* from given sides. By the *Seqt* in the Papyrus Rhind is meant the ratio to one another of certain lines in pyramids or obelisks. Eisenlohr and Cantor took the one word to be equivalent, sometimes to the *cosine* of the angle made by the *edge* of the pyramid with the co-terminous diagonal of the base, sometimes to the *tangent* of the angle of slope of the *faces* of the pyramid. It is now certain that it meant one thing, viz. the ratio of half the side of the base to the height of the pyramid, i.e. the *cotangent* of the angle of slope. The calculation of the *Seqt* thus implying a sort of theory of similarity, or even of trigonometry, the suggestion of Cantor is apparently that the *Seqt* in this case would be found from a *small* right-angled triangle *ADE* having a common angle *A* with *ABC* as shown in the figure, and that the ascertained value of the *Seqt* with the length *AB* would determine *BC*. This amounts to the use of the property of similar triangles; and Bretschneider's suggestion must apparently come to the same thing, since, even if Thales measured the *angle* in our sense (e.g. by its ratio to a right angle), he would, in the absence of something corresponding to a table of trigonometrical ratios, have gained nothing and would have had to work out the proportions all the same.



Max C. P. Schmidt also (*Kulturhistorische Beiträge zur Kenntnis des griechischen und römischen Altertums*, 1906, p. 32) similarly supposes Thales to have had a right angle made of wood or bronze with the legs graduated, to have placed it in the position *ADE* (*A* being the position of his eye), and then to have read off the lengths *AD*, *DE* respectively, and worked out the length of *BC* by the rule of three.

How then does the supposed use of similar triangles and their property square with Eudemus' remark about I. 26? As it stands, it asserts the *equality* of *two* triangles which have two angles and one side respectively equal, and the theorem can only be brought into relation with the above explanations by taking it as asserting that, if two angles and one side of *one* triangle are given, the triangle is completely determined. But, if Thales practically used *proportions*, as supposed, I. 26 is surely not at all the theorem which this procedure would naturally suggest as underlying it and being "necessarily used"; the use of proportions or of similar but not equal triangles would surely have taken attention altogether away from I. 26 and fixed it on VI. 4.

For this reason I think Tannery is on the right road when he tries to find a solution using I. 26 as it stands, and withal as primitive as any recorded solution of such a problem. His suggestion (*La Géométrie grecque*, pp. 90—1) is based on the *fluminis varatio* of the Roman agrimensor Marcus Junius Nipsus and is as follows.



To find the distance from a point A to an inaccessible point B. From A measure along a straight line at right angles to AB a length AC and bisect it at D. From C draw CE at right angles to CA on the side of it remote from B, and let E be the point on it which is in a straight line with B and D.

Then, by I. 26, CE is obviously equal to AB.

As regards the equality of angles, it is to be observed that those at D are equal because they are vertically opposite, and, curiously enough, Thales is expressly credited with the discovery of the equality of such angles.

The only objection which I can see to Tannery's solution is that it would require, in the case of the ship, a certain extent of free and level ground for the construction and measurements.

I suggest therefore that the following may have been Thales' method. Assuming that he was on the top of a tower, he had only to use a rough instrument made of a straight stick and a cross-piece fastened to it so as to be capable of turning about the fastening (say a nail) so that it could form any angle with the stick and would remain where it was put. Then the natural thing would be to fix the stick upright (by means of a plumb-line) and direct the cross-piece towards the ship. Next, leaving the cross-piece at the angle so found, the stick could be turned round, still remaining vertical, until the cross-piece pointed to some visible object on the shore, when the object could be mentally noted and the distance from the bottom of the tower to it could be subsequently measured. This would, by I. 26, give the distance from the bottom of the tower to the ship. This solution has the advantage of corresponding better to the simpler and more probable version of Thales' method of measuring the height of the pyramids; Diogenes Laertius says namely (I. 27, p. 6, ed. Cobet) on the authority of Hieronymus of Rhodes (B.C. 290—230), that he waited for this purpose until the moment when *our shadows are of the same length as ourselves*.

Recapitulation of congruence theorems.

Proclus, like other commentators, gives at this point (p. 347, 20 sqq.) a summary of the cases in which the equality of two triangles in all respects can be established. We may, he says, seek the conditions of such equality by successively considering as hypotheses the equality (1) of sides alone, (2) of angles alone, (3) of sides and angles combined. Taking (1) first, we can only establish the equality of the triangles in all respects if all three sides are respectively equal; we cannot establish the equality of the triangles by any hypothesis of class (2), not even the hypothesis that all the three angles are respectively equal; among the hypotheses of class (3), the equality of one side and one angle in each triangle is not enough, the equality (*a*) of one side and all three angles is more than enough, as is also the equality (*b*) of two sides and two or three angles, and (*c*) of three sides and one or two angles.

The only hypotheses therefore to be examined from this point of view are the equality of

(α) three sides [Eucl. I. 8].

(β) two sides and one angle [I. 4 proves one case of this, where the angle is that contained by the sides which are by hypothesis equal].

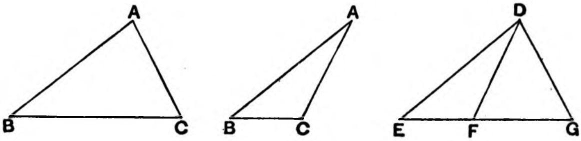
(γ) one side and two angles [I. 26 covers all cases].

It is curious that Proclus makes no allusion to what we call the *ambiguous case*, that case namely of (β) in which it is an angle opposite to one of the two specified sides in one triangle which is equal to the angle opposite to the equal side in the other triangle. Camerer indeed attributes to Proclus the observation that in this case the equality of the triangles cannot, unless some other condition is added, be asserted generally; but it would appear that Camerer was probably misled by a figure (Proclus, p. 351) which looks like a figure of the ambiguous case but is really only used to show that in I. 26 the equal sides must be *corresponding* sides, i.e., they must be either adjacent to the equal angles in each triangle, or opposite to corresponding equal angles, and that, e.g., one of the equal sides must not be adjacent to the two angles in one triangle, while the side equal to it in the other triangle is opposite to one of the two corresponding angles in that triangle.

The ambiguous case.

If two triangles have two sides equal to two sides respectively, and if the angles opposite to one pair of equal sides be also equal, then will the angles opposite the other pair of equal sides be either equal or supplementary; and, in the former case, the triangles will be equal in all respects.

Let *ABC*, *DEF* be two triangles such that *AB* is equal to *DE*, and *AC* to *DF*, while the angle *ABC* is equal to the angle *DEF*;
it is required to prove that the angles *ACB*, *DFE* are either equal or supplementary.



Now (1), if the angle *BAC* be equal to the angle *EDF*, it follows, since the two sides *AB*, *AC* are equal to the two sides *DE*, *DF* respectively, that

the triangles *ABC*, *DEF* are equal in all respects, [I. 4]
and the angles *ACB*, *DFE* are equal.

(2) If the angles *BAC*, *EDF* be not equal, make the angle *EDG* (on the same side of *ED* as the angle *EDF*) equal to the angle *BAC*.

Let *EF*, produced if necessary, meet *DG* in *G*.

Then, in the triangles *ABC*, *DEG*,
the two angles *BAC*, *ABC* are equal to the two angles *EDG*, *DEG* respectively,
and the side *AB* is equal to the side *DE*;

therefore the triangles *ABC*, *DEG* are equal in all respects, [I. 26]

so that the side *AC* is equal to the side *DG*,

and the angle *ACB* is equal to the angle *DGE*.

Again, since *AC* is equal to *DF* as well as to *DG*,

DF is equal to *DG*,

and therefore the angles *DFG*, *DGF* are equal.

But the angle *DFE* is supplementary to the angle *DFG*; and the angle *DGF* was proved equal to the angle *ACB*;

therefore the angle *DFE* is supplementary to the angle *ACB*.

If it is desired to avoid the ambiguity and secure that the triangles may be congruent, we can introduce the necessary conditions into the enunciation, on the analogy of Eucl. VI. 7.

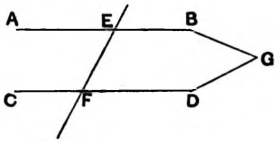
If two triangles have two sides of the one equal to two sides of the other respectively, and the angles opposite to a pair of equal sides equal, then, if the angles opposite to the other pair of equal sides are both acute, or both obtuse, or if one of them is a right angle, the two triangles are equal in all respects.

The proof of the three cases (by *reductio ad absurdum*) was given by Todhunter.

PROPOSITION 27.

If a straight line falling on two straight lines make the alternate angles equal to one another, the straight lines will be parallel to one another.

For let the straight line *EF* falling on the two straight lines *AB*, *CD* make the alternate angles *AEF*, *EFD* equal to one another;



I say that *AB* is parallel to *CD*.

For, if not, *AB*, *CD* when produced will meet either in the direction of *B*, *D* or towards *A*, *C*.

Let them be produced and meet, in the direction of *B*, *D*, at *G*.

Then, in the triangle *GEF*,

the exterior angle *AEF* is equal to the interior and opposite angle *EFG*:

which is impossible. [I. 16]

Therefore *AB*, *CD* when produced will not meet in the direction of *B*, *D*.

Similarly it can be proved that neither will they meet towards *A*, *C*.

But straight lines which do not meet in either direction are parallel; [Def. 23]

therefore *AB* is parallel to *CD*.

Therefore etc.

Q. E. D.

1. **falling on two straight lines,** *εἰς δύο εὐθείας ἐμπίπτουσα*, the phrase being the same as that used in Post. 5, meaning a *transversal*.

2. **the alternate angles,** *αἱ ἐναλλὰξ γωνίαι*. Proclus (p. 357, 9) explains that Euclid uses the word *alternate* (or, more exactly, *alternately*, *ἐναλλάξ*) in two connexions, (1) of a certain transformation of a proportion, as in Book v. and the arithmetical Books, (2) as here, of certain of the angles formed by parallels with a straight line crossing them. *Alternate* angles are, according to Euclid as interpreted by Proclus, those which are not on the same side of the transversal, and are not adjacent, but are separated by the transversal, both being within the parallels but one "above" and the other "below." The meaning is natural enough if we imagine the four internal angles to be taken in cyclic order and *alternate* angles to be any two of them not successive but separated by one angle of the four.

9. **in the direction of B, D or towards A, C,** literally "towards the *parts B, D* or towards *A, C*," *ἐπὶ τὰ* Β, Δ *μέρη ἢ ἐπὶ τὰ* Α, Γ.

With this proposition begins the second section of the first Book. Up to this point Euclid has dealt mainly with triangles, their construction and their properties in the sense of the relation of their parts, the sides and angles, to one another, and the comparison of different triangles in respect of their parts, and of their area in the particular cases where they are congruent.

The second section leads up to the third, in which we pass to relations between the areas of triangles, parallelograms and squares, the special feature being a new conception of *equality* of areas, equality not dependent on *congruence*. This whole subject requires the use of parallels. Consequently the second section beginning at I. 27 establishes the theory of parallels, introduces the cognate matter of the equality of the sum of the angles of a triangle to two right angles (I. 32), and ends with two propositions forming the transition to the third section, namely I. 33, 34, which introduce the parallelogram for the first time.

Aristotle on parallels.

We have already seen reason to believe that Euclid's personal contribution to the subject was nothing less than the formulation of the famous Postulate 5 (see the notes on that Postulate and on Def. 23), since Aristotle indicates that the then current theory of parallels contained a *petitio principii*, and presumably therefore it was Euclid who saw the defect and proposed the remedy.

But it is clear that the propositions I. 27, 28 were contained in earlier text-books. They were familiar to Aristotle, as we may judge from two interesting passages.

(1) In *Anal. Post.* I. 5 he is explaining that a scientific demonstration must not only prove a fact of every individual of a class (*κατὰ παντός*) but must prove it primarily and generally true (*πρῶτον καθόλου*) of the *whole* of the class as one; it will not do to prove it first of one part, then of another part, and so on, until the class is exhausted. He illustrates this (74 a 13—16) by a reference to parallels: "If then one were to show that right (angles) do not meet, the proof of this might be thought to depend on the fact that this is true of all (pairs of actual) right angles. But this is not so, inasmuch as the result does not follow because (the two angles are) equal (to two right

angles) *in the particular way* [i.e. because each is a right angle], but by virtue of their being equal (to two right angles) in any way whatever [i.e. because the *sum* only needs to be equal to two right angles, and the angles themselves may vary as much as we please subject to this]."

(2) The second passage has already been quoted in the note on Def. 23: "there is nothing surprising in different hypotheses leading to the same false conclusion; e.g. the conclusion that parallels meet might equally be drawn from either of the assumptions (*a*) that the interior (angle) is greater than the exterior or (*b*) that the sum of the angles of a triangle is greater than two right angles" (*Anal. Prior.* II. 17, 66 a 11—15).

I do not quite concur in the interpretation which Heiberg places upon these passages (*Mathematisches zu Aristoteles*, pp. 18—19). He says, first, that the allusion to the "interior angle" being "greater than the exterior" in the second passage shows that the reference in the first passage must be to Eucl. I. 28 and not to I. 27, and he therefore takes the words ὅτι ὡδὶ ἴσαι in the first passage (which I have translated "because the two angles are equal to two right angles in the particular way") as meaning "because the angles, viz. the *exterior* and the *interior*, are equal in the particular way." He also takes αἱ ὀρθαὶ οὐ συμπίπτουσι (which I have translated "right angles do not meet," an expression quite in Aristotle's manner) to mean "perpendicular *straight lines* do not meet"; this is very awkward, especially as he is obliged to supply *angles* with ἴσαι in the next sentence.

But I think that the first passage certainly refers to I. 28, although I do not think that the alternative (*a*) in the second passage suggests it. This alternative may, I think, equally with the alternative (*b*) refer to I. 27. That proposition is proved by *reductio ad absurdum* based on the fact that, if the straight lines do meet, they must form a *triangle*, in which case the exterior angle must be greater than the interior (while according to the hypothesis these angles are equal). It is true that Aristotle speaks of the hypothesis that the *interior* angle is greater than the *exterior*; but after all Aristotle had only to state *some* incorrect hypothesis. It is of course only in connexion with straight lines *meeting*, as the hypothesis in I. 27 makes them, that the alternative (*b*) about the sum of the angles of a triangle could come in, and alternative (*a*) implies alternative (*b*).

It seems clear then from Aristotle that I. 27, 28 at least are pre-Euclidean, and that it was only in I. 29 that Euclid made a change by using his Postulate.

De Morgan observes that I. 27 is a *logical* equivalent of I. 16. Thus, if A means "straight lines forming a triangle with a transversal," B "straight lines making angles with a transversal on the same side which are together less than two right angles," we have

$$\text{All } A \text{ is } B,$$

and it follows *logically* that

$$\text{All not-}B \text{ is not-}A.$$

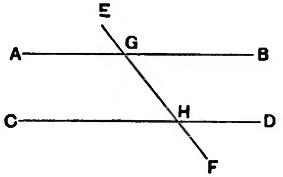
PROPOSITION 28.

If a straight line falling on two straight lines make the exterior angle equal to the interior and opposite angle on the same side, or the interior angles on the same side equal to two right angles, the straight lines will be parallel to one another.

For let the straight line *EF* falling on the two straight lines *AB*, *CD* make the exterior angle *EGB* equal to the interior and opposite angle *GHD*, or the interior angles on the same side, namely *BGH*, *GHD*, equal to two right angles;

I say that *AB* is parallel to *CD*.

For, since the angle *EGB* is equal to the angle *GHD*, while the angle *EGB* is equal to the angle *AGH*, [I. 15] the angle *AGH* is also equal to the angle *GHD*; and they are alternate;



therefore *AB* is parallel to *CD*. [I. 27]

Again, since the angles *BGH*, *GHD* are equal to two right angles, and the angles *AGH*, *BGH* are also equal to two right angles, [I. 13] the angles *AGH*, *BGH* are equal to the angles *BGH*, *GHD*.

Let the angle *BGH* be subtracted from each;

therefore the remaining angle *AGH* is equal to the remaining angle *GHD*;

and they are alternate;

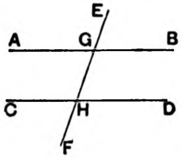
therefore *AB* is parallel to *CD*. [I. 27]

Therefore etc.

Q. E. D.

One criterion of parallelism, the equality of alternate angles, is given in I. 27; here we have two more, each of which is easily reducible, and is actually reduced, to the other.

Proclus observes (pp. 358—9) that Euclid could have stated six criteria as well as three, by using, in addition, other pairs of angles in the figure (not adjacent) of which it could be predicated that the two angles are equal or that their sum is equal to two right angles. A natural division is to consider, first the pairs which are on the same side of the transversal, and secondly the pairs which are on different sides of it.



Taking (1) the possible pairs on the *same* side, we may have a pair consisting of

(*a*) two internal angles, viz. the pairs (*BGH*, *GHD*) and (*AGH*, *GHC*);

(*b*) two external angles, viz. the pairs (*EGB*, *DHF*) and (*EGA*, *CHF*);

(*c*) one external and one internal angle, viz. the pairs (*EGB*, *GHD*), (*FHD*, *HGB*), (*EGA*, *GHC*) and (*FHC*, *HGA*).

And (2) the possible pairs on *different* sides of the transversal may consist respectively of

(*a*) two internal angles, viz. the pairs (*AGH*, *GHD*) and (*CHG*, *HGB*);

(*b*) two external angles, viz. the pairs (*AGE*, *DHF*) and (*EGB*, *CHF*);

(*c*) one external and one internal, viz. the pairs (*AGE*, *GHD*), (*EGB*, *GHC*), (*FHC*, *HGB*) and (*FHD*, *HGA*).

The angles are equal in the pairs (1) (*c*), (2) (*a*) and (2) (*b*), and the sum is equal to two right angles in the case of the pairs (1) (*a*), (1) (*b*) and (2) (*c*). For his criteria Euclid selects the cases (2) (*a*) [I. 27] and (1) (*c*), (1) (*a*) [I. 28], leaving out the other three, which are of course equivalent but are not quite so easily expressed.

From Proclus' note on I. 28 (p. 361) we learn that one Aigeias (? Aineias) of Hierapolis wrote an epitome or abridgment of the *Elements*. This seems to be the only mention of this editor and his work; and they are only mentioned as having combined Eucl. I. 27, 28 into one proposition. To do this, or to make the three hypotheses the subject of *three* separate theorems, would, Proclus thinks, have been more natural than to deal with them, as Euclid does, in two propositions. Proclus has no suggestion for explaining Euclid's arrangement unless the ground were that I. 27 deals with angles on different sides, I. 28 with angles on one and the same side, of the transversal. But may not the reason have been one of convenience, namely that the criterion of I. 27 is that actually used to prove parallelism, and is moreover the basis of the construction of parallels in I. 31, while I. 28 only reduces the other two hypotheses to that of I. 27, so that precision of reference, as well as clearness of exposition, is better secured by the arrangement adopted?

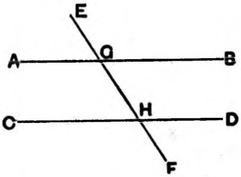
PROPOSITION 29.

A straight line falling on parallel straight lines makes the alternate angles equal to one another, the exterior angle equal to the interior and opposite angle, and the interior angles on the same side equal to two right angles.

For let the straight line *EF* fall on the parallel straight lines *AB*, *CD*;

I say that it makes the alternate angles *AGH*, *GHD* equal, the exterior angle *EGB* equal to the interior and opposite angle *GHD*, and the interior angles on the same side, namely *BGH*, *GHD*, equal to two right angles.

For, if the angle *AGH* is unequal to the angle *GHD*, one of them is greater.



Let the angle *AGH* be greater.

Let the angle *BGH* be added to each;

therefore the angles *AGH*, *BGH* are greater than the angles *BGH*, *GHD*.

But the angles *AGH*, *BGH* are equal to two right angles; [I. 13]

therefore the angles *BGH*, *GHD* are less than two right angles.

But straight lines produced indefinitely from angles less than two right angles meet; [Post. 5]

therefore *AB*, *CD*, if produced indefinitely, will meet; but they do not meet, because they are by hypothesis parallel.

Therefore the angle *AGH* is not unequal to the angle *GHD*,

and is therefore equal to it.

Again, the angle *AGH* is equal to the angle *EGB*; [I. 15]

therefore the angle *EGB* is also equal to the angle *GHD*. [*C. N.* 1]

Let the angle *BGH* be added to each;

therefore the angles *EGB*, *BGH* are equal to the angles *BGH*, *GHD*. [*C. N.* 2]

But the angles *EGB*, *BGH* are equal to two right angles; [I. 13]

therefore the angles *BGH*, *GHD* are also equal to two right angles.

Therefore etc. Q. E. D.

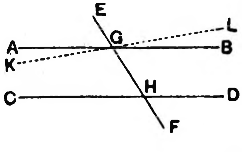
23. **straight lines produced indefinitely from angles less than two right angles**, *αἱ δὲ ἀπ᾽ ἐλασσόνων ἢ δύο ὀρθῶν ἐκβαλλόμεναι εἰς ἄπειρον συμπίπτουσιν*, a variation from the more explicit language of Postulate 5. A good deal is left to be understood, namely that the straight lines begin from points at which they meet a transversal, and make with it internal angles on the same side the sum of which is less than two right angles.

26. **because they are by hypothesis parallel**, literally "because they are supposed parallel," *διὰ τὸ παραλλήλους αὐτὰς ὑποκεῖσθαι*.

Proof by "Playfair's" axiom.

If, instead of Postulate 5, it is preferred to use "Playfair's" axiom in the proof of this proposition, we proceed thus.

To prove that the alternate angles *AGH*, *GHD* are equal.



If they are not equal, draw another straight line *KL* through *G* making the angle *KGH* equal to the angle *GHD*.

Then, since the angles *KGH*, *GHD* are equal,

KL is parallel to *CD*. [I. 27]

Therefore *two straight lines* KL, AB *intersecting at* G *are both parallel to the straight line* CD:

which is impossible (by the axiom).

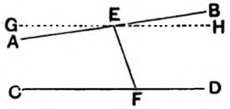
Therefore the angle *AGH* cannot but be equal to the angle *GHD*.

The rest of the proposition follows as in Euclid.

Proof of Euclid's Postulate 5 from "Playfair's" axiom.

Let *AB*, *CD* make with the transversal *EF* the angles *AEF*, *EFC* together less than two right angles.

To prove that *AB*, *CD* meet towards *A*, *C*.



Through *E* draw *GH* making with *EF* the angle *GEF* equal (and alternate) to the angle *EFD*.

Thus *GH* is parallel to *CD*. [I. 27]

Then (1) *AB* must meet *CD* in one direction or the other.

For, if it does not, *AB* must be parallel to *CD*; hence we have two straight lines *AB*, *GH* intersecting at *E* and both parallel to *CD*: which is impossible.

Therefore *AB*, *CD* must meet.

(2) Since *AB*, *CD* meet, they must form a triangle with *EF*.

But in any triangle any two angles are together less than two right angles.

Therefore the angles *AEF*, *EFC* (which are less than two right angles), and not the angles *BEF*, *EFD* (which are together greater than two right angles, by I. 13), are the angles of the triangle;

that is, *EA*, *FC* meet in the direction of *A*, *C*, or on the side of *EF* on which are the angles together less than two right angles.

The usual course in modern text-books which use "Playfair's" axiom in lieu of Euclid's Postulate is apparently to prove I. 29 by means of the axiom, and then Euclid's Postulate by means of I. 29.

De Morgan would introduce the proof of Postulate 5 by means of "Playfair's" axiom *before* I. 29, and would therefore apparently prove I. 29 as Euclid does, without any change.

As between Euclid's Postulate 5 and "Playfair's" axiom, it would appear that the tendency in modern text-books is rather in favour of the latter. Thus, to take a few noteworthy foreign writers, we find that Rausenberger stands almost alone in using Euclid's Postulate, while Hilbert, Henrici and Treutlein, Rouché and De Comberousse, Enriques and Amaldi all use "Playfair's" axiom.

Yet the case for preferring Euclid's Postulate is argued with some force by Dodgson (*Euclid and his modern Rivals*, pp. 44—6). He maintains (1) that "Playfair's" axiom in fact involves Euclid's Postulate, but at the same time involves *more* than the latter, so that, to that extent, it is a needless strain on the faith of the learner. This is shown as follows.

Given *AB*, *CD* making with *EF* the angles *AEF*, *EFC* together less than two right angles, draw *GH* through *E* so that the angles *GEF*, *EFC* are together *equal* to two right angles.

Then, by I. 28, *GH*, *CD* are "separational."

We see then that any lines which have the property (α) that they make with a transversal angles less than two right angles have also the property (β) that one of them intersects a straight line which is "separational" from the other.

Now Playfair's axiom asserts that the lines which have property (β) meet if produced: for, if they did not, we should have two intersecting straight lines both "separational" from a third, which is impossible.

We then argue that lines having property (α) meet because lines having property (α) are lines having property (β). But we do not know, until we have proved I. 29, that all pairs of lines having property (β) have also property (α). For anything we know to the contrary, class (β) *may* be greater than class (α). Hence, if you assert anything of class (β), the logical effect is more extensive than if you assert it of class (α); for you assert it, not only of that portion of class (β) which is known to be included in class (α), but also of the unknown (but possibly existing) portion which is *not* so included.

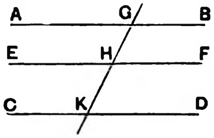
(2) Euclid's Postulate puts before the beginner clear and *positive* conceptions, a pair of straight lines, a transversal, and two angles together less than two right angles, whereas "Playfair's" axiom requires him to realise a pair of straight lines which never meet though produced to infinity: a *negative* conception which does not convey to the mind any clear notion of the relative position of the lines. And (p. 68) Euclid's Postulate gives a direct criterion for judging that two straight lines meet, a criterion which is constantly required, e.g. in I. 44. It is true that the Postulate can be *deduced* from "Playfair's" axiom, but editors frequently omit to deduce it, and then tacitly assume it afterwards: which is the least justifiable course of all.

PROPOSITION 30.

Straight lines parallel to the same straight line are also parallel to one another.

Let each of the straight lines *AB*, *CD* be parallel to *EF*; I say that *AB* is also parallel to *CD*.

For let the straight line *GK* fall upon them.



Then, since the straight line *GK* has fallen on the parallel straight lines *AB*, *EF*,

the angle *AGK* is equal to the angle *GHF*. [I. 29]

Again, since the straight line *GK* has fallen on the parallel straight lines *EF*, *CD*,

the angle *GHF* is equal to the angle *GKD*. [I. 29]

But the angle *AGK* was also proved equal to the angle *GHF*;

therefore the angle *AGK* is also equal to the angle *GKD*; [*C. N.* 1]

and they are alternate.

Therefore *AB* is parallel to *CD*.

Q. E. D.

20. The usual *conclusion* in general terms ("Therefore etc.") repeating the enunciation is, curiously enough, wanting at the end of this proposition.

The proposition is, as De Morgan points out, the *logical* equivalent of "Playfair's" axiom. Thus, if *X* denote "pairs of straight lines intersecting one another," *Y* "pairs of straight lines parallel to one and the same straight line," we have

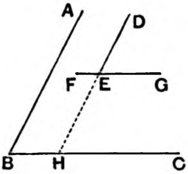
No *X* is *Y*,

and it follows logically that

No *Y* is *X*.

De Morgan adds that a proposition is much wanted about parallels (or perpendiculars) to two straight lines respectively making the same angles with one another as the latter do. The proposition may be enunciated thus:

If the sides of one angle be respectively (1) *parallel or* (2) *perpendicular to the sides of another angle, the two angles are either equal or supplementary.*



(1) Let *DE* be parallel to *AB* and *GEF* parallel to *BC*.

To prove that the angles *ABC*, *DEG* are equal and the angles *ABC*, *DEF* supplementary.

Produce *DE* to meet *BC* in *H*.

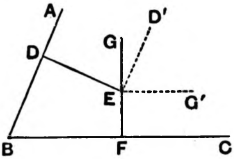
Then [I. 29] the angle *DEG* is equal to the angle *DHC*,

and the angle *ABC* is equal to the angle *DHC*.

Therefore the angle *DEG* is equal to the angle *ABC*; whence also the angle *DEF* is supplementary to the angle *ABC*.

(2) Let *ED* be perpendicular to *AB*, and *GEF* perpendicular to *BC*.

To prove that the angles *ABC*, *DEG* are equal, and the angles *ABC*, *DEF* supplementary.



Draw *ED'* at right angles to *ED* on the side of it opposite to *B*, and draw *EG'* at right angles to *EF* on the side of it opposite to *B*.

Then, since the angles *BDE*, *DED'*, being right angles, are equal,

ED' is parallel to *BA*. [I. 27]

Similarly *EG'* is parallel to *BC*.

Therefore [Part (1)] the angle *D'EG'* is equal to the angle *ABC*.

But, the right angle *DED'* being equal to the right angle *GEG'*, if the common angle *GED'* be subtracted,

the angle *DEG* is equal to the angle *D'EG'*.

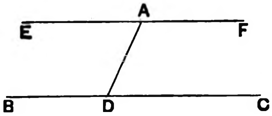
Therefore the angle *DEG* is equal to the angle *ABC*; and hence the angle *DEF* is supplementary to the angle *ABC*.

PROPOSITION 31.

Through a given point to draw a straight line parallel to a given straight line.

Let *A* be the given point, and *BC* the given straight line;

thus it is required to draw through the point *A* a straight line parallel to the straight line *BC*.



Let a point D be taken at random on BC, and let AD be joined; on the straight line DA, and at the point A on it, let the angle DAE be constructed equal to the angle ADC [I. 23]; and let the straight line AF be produced in a straight line with EA.

Then, since the straight line AD falling on the two straight lines BC, EF has made the alternate angles EAD, ADC equal to one another,

therefore EAF is parallel to BC. [I. 27]

Therefore through the given point A the straight line EAF has been drawn parallel to the given straight line BC.

Q. E. F.

Proclus rightly remarks (p. 376, 14—20) that, as it is implied in I. 12 that only one perpendicular can be drawn to a straight line from an external point, so here it is implied that only one straight line can be drawn through a point parallel to a given straight line. The construction, be it observed, depends only upon I. 27, and might therefore have come directly after that proposition. Why then did Euclid postpone it until after I. 29 and I. 30? Presumably because he considered it necessary, before giving the construction, to place beyond all doubt the fact that only one such parallel can be drawn. Proclus infers this fact from I. 30; for, he says, if two straight lines could be drawn through one and the same point parallel to the same straight line, the two straight lines would be *parallel*, though intersecting at the given point: which is impossible. I think it is a fair inference that Euclid would have considered it necessary to justify the assumption that only one parallel can be drawn by some such argument, and that he deliberately determined that his own assumption was more appropriate to be made the subject of a Postulate than the assumption of the uniqueness of the parallel.

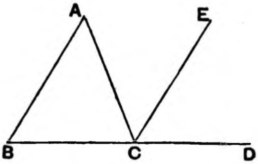
PROPOSITION 32.

In any triangle, if one of the sides be produced, the exterior angle is equal to the two interior and opposite angles, and the three interior angles of the triangle are equal to two right angles.

Let ABC be a triangle, and let one side of it BC be produced to D;

I say that the exterior angle ACD is equal to the two interior and opposite angles CAB, ABC, and the three interior angles of the triangle ABC, BCA, CAB are equal to two right angles.

For let *CE* be drawn through the point *C* parallel to the straight line *AB*. [I. 31]



Then, since *AB* is parallel to *CE*,

and *AC* has fallen upon them,

the alternate angles *BAC*, *ACE* are equal to one another. [I. 29]

Again, since *AB* is parallel to *CE*,

and the straight line *BD* has fallen upon them,

the exterior angle *ECD* is equal to the interior and opposite angle *ABC*. [I. 29]

But the angle *ACE* was also proved equal to the angle *BAC*;

therefore the whole angle *ACD* is equal to the two interior and opposite angles *BAC*, *ABC*.

Let the angle *ACB* be added to each;

therefore the angles *ACD*, *ACB* are equal to the three angles *ABC*, *BCA*, *CAB*.

But the angles *ACD*, *ACB* are equal to two right angles; [I. 13]

therefore the angles *ABC*, *BCA*, *CAB* are also equal to two right angles.

Therefore etc.

Q. E. D.

This theorem was discovered in the very early stages of Greek geometry. What we know of the history of it is gathered from three allusions found in Eutocius, Proclus and Diogenes Laertius respectively.

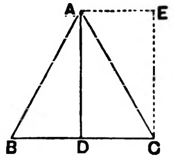
1. Eutocius at the beginning of his commentary on the *Conics* of Apollonius (ed. Heiberg, Vol. II. p. 170) quotes Geminus as saying that "the ancients (οἱ ἀρχαῖοι) investigated the theorem of the two right angles in each individual species of triangle, first in the equilateral, again in the isosceles, and afterwards in the scalene triangle, and later geometers demonstrated the general theorem to the effect that in *any* triangle the three interior angles are equal to two right angles."

2. Now, according to Proclus (p. 379, 2—5), Eudemus the Peripatetic refers the discovery of this theorem to the Pythagoreans and gives what he affirms to be their demonstration of it. This demonstration will be given below, but it should be remarked that it is general, and therefore that the "later geometers" spoken of by Geminus were presumably the Pythagoreans, whence it appears that the "ancients" contrasted with them must have belonged to the time of Thales, if they were not his Egyptian instructors.

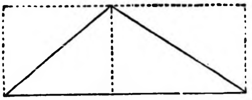
3. That the truth of the theorem was known to Thales might also be inferred from the statement of Pamphile (quoted by Diogenes Laertius, I. 24—5, p. 6, ed. Cobet) that "he, having learnt geometry from the

Egyptians, was the first to inscribe a right-angled triangle in a circle and sacrificed an ox" (on the strength of it); in other words, he discovered that the angle in a semicircle is a right angle. No doubt, when this fact was once discovered (*empirically*, say), the consideration of the two isosceles triangles having the centre for vertex and the sides of the right angle for bases respectively, with the help of the theorem of Eucl. I. 5, also known to Thales, would easily lead to the conclusion that the sum of the angles of a *right-angled* triangle is equal to two right angles, and it could be readily inferred that the angles of *any* triangle were likewise equal to two right angles (by resolving it into two right-angled triangles). But it is not easy to see how the property of the angle in a semicircle could be *proved* except (in the reverse order) by means of the equality of the sum of the angles of a *right-angled* triangle to two right angles; and hence it is most natural to suppose, with Cantor, that Thales proved it (if he did prove it) practically as Euclid does in III. 31, i.e. by means of I. 32 as applied to *right-angled* triangles at all events.

If the theorem of I. 32 was proved before Thales' time, or by Thales himself, by the stages indicated in the note of Geminus, we may be satisfied that the reconstruction of the argument of the older proof by Hankel (pp. 96—7) and Cantor (I_3, pp. 143—4) is not far wrong. First, it must have been observed that six angles equal to an angle of an equilateral triangle would, if placed adjacent to one another round a common vertex, fill up the whole space round that vertex. It is true that Proclus attributes to the Pythagoreans the general theorem that only three kinds of regular polygons, the equilateral triangle, the square and the regular hexagon, can fill up the entire space round a point, but the practical knowledge that equilateral triangles have this property could hardly have escaped the Egyptians, whether they made floors with tiles in the form of equilateral triangles or regular hexagons (Allman, *Greek Geometry from Thales to Euclid*, p. 12) or joined the ends of adjacent radii of a figure like the six-spoked wheel, which was their common form of wheel from the time of Ramses II. of the nineteenth Dynasty, say 1300 B.C. (Cantor, I_3, p. 109). It would then be clear that six angles equal to an angle of an equilateral triangle are equal to four right angles, and therefore that the three angles of an equilateral triangle are equal to two right angles. (It would be as clear or clearer, from observation of a square divided into two triangles by a diagonal, that an isosceles right-angled triangle has each of its equal angles equal to half a right angle, so that an isosceles right-angled triangle must have the sum of its angles equal to two right angles.) Next, with regard to the equilateral triangle, it could not fail to be observed that, if *AD* were drawn from the vertex *A* perpendicular to the base *BC*, each of the two right-angled triangles so formed would have the sum of its angles equal to two right angles; and this would be confirmed by completing the rectangle *ADCE*, when it would be seen that the rectangle (with its angles equal to four right angles) was divided by its diagonal into two equal triangles, each of which had the sum of its angles equal to two right angles. Next it would be inferred, as the result of drawing the diagonal of *any* rectangle and observing the equality of the triangles forming the two halves, that the sum of the angles of *any* right-angled triangle is equal to two right angles, and hence (the two congruent right-angled triangles being then placed so as to form one isosceles triangle) that the same is true of *any isosceles* triangle. Only the last step remained, namely that of observing that *any* triangle could be regarded as the half of a rectangle (drawn as indicated in the next figure), or



simply that any triangle could be divided into two right-angled triangles, whence it would be inferred that in general the sum of the angles of any triangle is equal to two right angles.



Such would be the probabilities if we could absolutely rely upon the statements attributed to Pamphile and Geminus respectively. But in fact there is considerable ground for doubt in both cases.

1. Pamphile's story of the sacrifice of an ox by Thales for joy at his discovery that the angle in a semicircle is a right angle is too suspiciously like the similar story told with reference to Pythagoras and his discovery of the theorem of Eucl. I. 47 (Proclus, p. 426, 6—9). And, as if this were not enough, Diogenes Laertius immediately adds that "others, among whom is Apollodorus the calculator (ὁ λογιστικός), say it was Pythagoras" (sc. who "inscribed the right-angled triangle in a circle"). Now Pamphile lived in the reign of Nero (A.D. 54—68) and therefore some 700 years after the birth of Thales (about 640 B.C.). I do not know on what Max Schmidt bases his statement (*Kulturhistorische Beiträge zur Kenntnis des griechischen und römischen Altertums*, 1906, p. 31) that "other, *much older*, sources name Pythagoras as the discoverer of the said proposition," because nothing more seems to be known of Apollodorus than what is stated here by Diogenes Laertius. But it would at least appear that Apollodorus was only one of several authorities who attributed the proposition to Pythagoras, while Pamphile is alone mentioned as referring it to Thales. Again, the connexion of Pythagoras with the investigation of the right-angled triangle makes it *a priori* more likely that it would be he who would discover its relation to a semicircle. On the whole, therefore, the attribution to Thales would seem to be more than doubtful.

2. As regards Geminus' account of the three stages through which the proof of the theorem of I. 32 passed, we note, first, that it is certainly not confirmed by Eudemus, who referred to the Pythagoreans the *discovery* of the theorem that the sum of the angles of *any* triangle is equal to two right angles and says nothing about any gradual stages by which it was proved. Secondly, it must be admitted, I think, that in the evolution of the proof as reconstructed by Hankel the middle stage is rather artificial and unnecessary, since, once it is proved that *any right-angled* triangle has the sum of its angles equal to two right angles, it is just as easy to pass at once to any *scalene* triangle (which is decomposable into two *unequal* right-angled triangles) as to the isosceles triangle made up of two congruent right-angled triangles. Thirdly, as Heiberg has recently pointed out (*Mathematisches zu Aristoteles*, p. 20), it is quite possible that the statement of Geminus from beginning to end is simply due to a misapprehension of a passage of Aristotle (*Anal. Post.* I. 5, 74 a 25). Aristotle is illustrating his contention that a property is not scientifically proved to belong to a class of things unless it is proved to belong *primarily* (πρῶτον) and *generally* (καθόλου) to the *whole* of the class. His first illustration relates to parallels making with a transversal angles on the same side together equal to two right angles, and has been quoted above in the note on I. 27 (pp. 308—9). His second illustration refers to the transformation of a proportion *alternando*, which (he says) "used at one time to be proved separately" for numbers, lines, solids, and times, although it admits of being proved of all at once by one demonstration. The third illustration is: "For the same reason, even *if one should prove* (οὐδ᾽ ἄν τις δείξῃ) with reference to

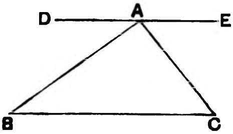
each (sort of) triangle, the equilateral, scalene and isosceles, separately, that each has its angles equal to two right angles, either by one proof or by different proofs, he does not yet know that *the triangle*, i.e. the triangle *in general*, has its angles equal to two right angles, except in a sophistical sense, even though there exists no triangle other than triangles of the kinds mentioned. For he knows it, not *quâ* triangle, nor of *every* triangle, except in a numerical sense (κατ' ἀριθμόν); he does not know it *notionally* (κατ' εἶδος) of every triangle, even though there be actually no triangle which he does not know."

The difference between the phrase "used at one time to be proved" in the second illustration and "if any one should prove" in the third appears to indicate that, while the former referred to a historical fact, the latter does not; the reference to a person who should prove the theorem of I. 32 for the three kinds of triangle separately, and then claim that he had proved it generally, states a purely hypothetical case, a mere illustration. Yet, coming after the historical fact stated in the preceding illustration, it might not unnaturally give the impression, at first sight, that it was historical too.

On the whole, therefore, it would seem that we cannot safely go behind the dictum of Eudemus that the discovery and proof of the theorem of I. 32 in all its generality were Pythagorean. This does not however preclude its having been discovered by stages such as those above set out after Hankel and Cantor. Nor need it be doubted that Thales and even his Egyptian instructors had advanced some way on the same road, so far at all events as to see that in an equilateral triangle, and in an isosceles right-angled triangle, the sum of the angles is equal to two right angles.

The Pythagorean proof.

This proof, handed down by Eudemus (Proclus, p. 379, 2—15), is no less elegant than that given by Euclid, and is a natural development from the last figure in the reconstructed argument of Hankel. It would be seen, after the theory of parallels was added to geometry, that the actual drawing of the perpendicular and the complete rectangle on *BC* as base was unnecessary, and that the parallel to *BC* through *A* was all that was required.



Let *ABC* be a triangle, and through *A* draw *DE* parallel to *BC*. [I. 31]

Then, since *BC*, *DE* are parallel,

the alternate angles *DAB*, *ABC* are equal, [I. 29]

and so are the alternate angles *EAC*, *ACB* also.

Therefore the angles *ABC*, *ACB* are together equal to the angles *DAB*, *EAC*.

Add to each the angle *BAC*;

therefore the sum of the angles *ABC*, *ACB*, *BAC* is equal to the sum of the angles *DAB*, *BAC*, *CAE*, that is, to two right angles.

Euclid's proof pre-Euclidean.

The theorem of I. 32 is Aristotle's favourite illustration when he wishes to refer to some truth generally acknowledged, and so often does it occur that it is often indicated by two or three words in themselves hardly intelligible, e.g. τὸ δυσὶν ὀρθαῖς (*Anal. Post.* I. 24, 85 b 5) and ὑπάρχει παντὶ τριγώνῳ τὸ δύο (*ibid.* 85 b 11).

One passage (*Metaph.* 1051 a 24) makes it clear, as Heiberg (*op. cit.*

p. 19) acutely observes, that in the proof as Aristotle knew it Euclid's construction was used. "Why does the triangle make up two right angles? Because *the angles about one point* are equal to two right angles. If then the parallel to the side had been *drawn up* (ἀνῆκτο), the fact would at once have been clear from merely looking at the figure." The words "the angles about one point" would equally fit the Pythagorean construction, but "drawn *upwards*" applied to the parallel to a side can only indicate Euclid's.

Attempts at proof independently of parallels.

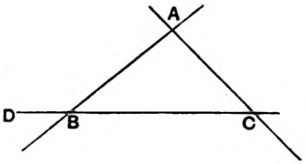
The most indefatigable worker on these lines was Legendre, and a sketch of his work has been given in the note on Postulate 5 above.

One other attempted proof needs to be mentioned here because it has found much favour. I allude to

Thibaut's method.

This appeared in Thibaut's *Grundriss der reinen Mathematik*, Göttingen (2 ed. 1809, 3 ed. 1818), and is to the following effect.

Suppose CB produced to D, and let BD (produced to any necessary extent either way) revolve in one direction (say clockwise) first about B into the position BA, then about A into the position of AC produced both ways, and lastly about C into the position CB produced both ways.



The argument then is that the straight line BD has revolved through the sum of the three exterior angles of the triangle. But, since it has at the end of the revolution assumed a position in the same straight line with its original position, it must have revolved *through four right angles.*

Therefore the sum of the three exterior angles is equal to four right angles;

from which it follows that the sum of the three angles of the triangle is equal to two right angles.

But it is to be observed that the straight line BD revolves about *different points in it*, so that there is *translation* combined with *rotatory* motion, and it is necessary to assume as an axiom that the two motions are independent, and therefore that the *translation* may be neglected.

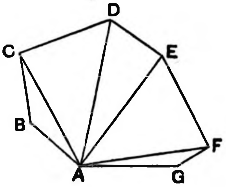
Schumacher (letter to Gauss of 3 May, 1831) tried to represent the rotatory motion graphically in a second figure as mere motion round a point; but Gauss (letter of 17 May, 1831) pointed out in reply that he really assumed, without proving it, a proposition to the effect that "If two straight lines (1) and (2) which cut one another make angles A , A'' with a straight line (3) cutting both of them, and if a straight line (4) in the same plane is likewise cut by (1) at an angle A', then (4) will be cut by (2) at the angle A''. But this proposition not only needs proof, but we may say that it is, in essence, the very proposition to be proved" (see Engel and Stäckel, *Die Theorie der Parallellinien von Euklid bis auf Gauss*, 1895, p. 230).

How easy it is to be deluded in this way is plainly shown by Proclus' attempt on the same lines. He says (p. 384, 13—21) that the truth of the theorem is borne in upon us by the help of "common notions" only. "For, if we conceive a straight line with two perpendiculars drawn to it at its extremities, and if we then suppose the perpendiculars to (revolve about their feet and) approach one another, so as to form a triangle, we see that, to the extent to which they converge, they diminish the right angles which they made with the straight line, so that the amount taken from the right angles is also the amount added to the vertical angle of the triangle, and the three angles are necessarily made equal to two right angles." But a moment's reflection shows that, so far from being founded on mere "common notions," the supposed proof assumes, to begin with, that, if the perpendiculars approach one another ever so little, they will then form a triangle immediately, i.e., it assumes Postulate 5 itself; and the fact about the vertical angle can only be seen by means of the equality of the alternate angles exhibited by drawing a perpendicular from the vertex of the triangle to the base, i.e. a *parallel* to either of the original perpendiculars.

Extension to polygons.

The two important corollaries added to I. 32 in Simson's edition are given by Proclus; but Proclus' proof of the first is different from, and perhaps somewhat simpler than, Simson's.

1. *The sum of the interior angles of a convex rectilineal figure is equal to twice as many right angles as the figure has sides, less four.*



For let one angular point *A* be joined to all the other angular points with which it is not connected already.

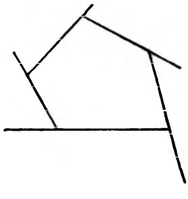
The figure is then divided into triangles, and mere inspection shows

(1) that the number of triangles is two less than the number of sides in the figure,

(2) that the sum of the angles of all the triangles is equal to the sum of all the interior angles of the figure.

Since then the sum of the angles of each triangle is equal to two right angles the sum of the interior angles of the figure is equal to $2(n-2)$ right angles, i.e. $(2n-4)$ right angles, where n is the number of sides in the figure.

2. *The exterior angles of any convex rectilineal figure are together equal to four right angles.*



For the interior and exterior angles together are equal to $2n$ right angles, where n is the number of sides.

And the interior angles are together equal to $(2n-4)$ right angles.

Therefore the exterior angles are together equal to four right angles.

This last property is already quoted by Aristotle as true of all rectilineal figures in two passages (*Anal. Post.* I. 24, 85 b 38 and II. 17, 99 a 19).

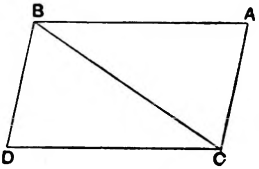
PROPOSITION 33.

The straight lines joining equal and parallel straight lines (at the extremities which are) in the same directions (respectively) are themselves also equal and parallel.

Let *AB*, *CD* be equal and parallel, and let the straight lines *AC*, *BD* join them (at the extremities which are) in the same directions (respectively);

I say that AC, BD are also equal and parallel.

Let BC be joined.



Then, since AB is parallel to CD, and BC has fallen upon them, the alternate angles ABC, BCD are equal to one another. [I. 29]

And, since AB is equal to CD, and BC is common, the two sides AB, BC are equal to the two sides DC, CB; and the angle ABC is equal to the angle BCD;

therefore the base AC is equal to the base BD, and the triangle ABC is equal to the triangle DCB, and the remaining angles will be equal to the remaining angles respectively, namely those which the equal sides subtend; [I. 4]

therefore the angle ACB is equal to the angle CBD.

And, since the straight line BC falling on the two straight lines AC, BD has made the alternate angles equal to one another,

AC is parallel to BD. [I. 27]

And it was also proved equal to it.

Therefore etc.

Q. E. D.

1. **joining...(at the extremities which are) in the same directions (respectively).** I have for clearness' sake inserted the words in brackets though they are not in the original Greek, which has "joining...in the same directions" or "on the same sides," ἐπὶ τὰ αὐτὰ μέρη ἐπιζευγνύουσαι. The expression "towards the same parts," though usage has sanctioned it, is perhaps not quite satisfactory.

15. **DC, CB** and 18. **DCB.** The Greek has "*BC, CD*" and "*BCD*" in these places respectively. Euclid is not always careful to write in corresponding order the letters denoting corresponding points in congruent figures. On the contrary, he evidently prefers the alphabetical order, and seems to disdain to alter it for the sake of beginners or others who might be confused by it. In the case of angles alteration is perhaps unnecessary; but in the case of triangles and pairs of corresponding sides I have ventured to alter the order to that which the mathematician of to-day expects.

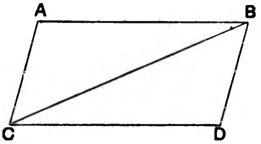
This proposition is, as Proclus says (p. 385, 5), the connecting link between the exposition of the theory of parallels and the investigation of parallelograms. For, while it only speaks of equal and parallel straight lines connecting those ends of equal and parallel straight lines which are in the same directions, it gives, without expressing the fact, the construction or origin of the parallelogram, so that in the next proposition Euclid is able to speak of "parallelogrammic areas" without any further explanation.

PROPOSITION 34.

In parallelogrammic areas the opposite sides and angles are equal to one another, and the diameter bisects the areas.

Let $ACDB$ be a parallelogrammic area, and BC its diameter;

I say that the opposite sides and angles of the parallelogram *ACDB* are equal to one another, and the diameter *BC* bisects it.



For, since *AB* is parallel to *CD*, and the straight line *BC* has fallen upon them,

the alternate angles *ABC*, *BCD* are equal to one another. [I. 29]

Again, since *AC* is parallel to *BD*, and *BC* has fallen upon them,

the alternate angles *ACB*, *CBD* are equal to one another. [I. 29]

Therefore *ABC*, *DCB* are two triangles having the two angles *ABC*, *BCA* equal to the two angles *DCB*, *CBD* respectively, and one side equal to one side, namely that adjoining the equal angles and common to both of them, *BC*;

therefore they will also have the remaining sides equal to the remaining sides respectively, and the remaining angle to the remaining angle; [I. 26]

therefore the side *AB* is equal to *CD*,

and *AC* to *BD*,

and further the angle *BAC* is equal to the angle *CDB*.

And, since the angle *ABC* is equal to the angle *BCD*,

and the angle *CBD* to the angle *ACB*,

the whole angle *ABD* is equal to the whole angle *ACD*. [*C. N.* 2]

And the angle *BAC* was also proved equal to the angle *CDB*.

Therefore in parallelogrammic areas the opposite sides and angles are equal to one another.

I say, next, that the diameter also bisects the areas.

For, since *AB* is equal to *CD*, and *BC* is common,

the two sides *AB*, *BC* are equal to the two sides *DC*, *CB* respectively;

and the angle *ABC* is equal to the angle *BCD*;

therefore the base *AC* is also equal to *DB*,

and the triangle *ABC* is equal to the triangle *DCB*. [I. 4]

Therefore the diameter *BC* bisects the parallelogram *ACDB*. Q. E. D.

1. It is to be observed that, when parallelograms have to be mentioned for the first time, Euclid calls them "**parallelogrammic areas**" or, more exactly, "parallelogram" areas (παραλληλόγραμμα χωρία). The meaning is simply areas bounded by parallel straight lines with the further limitation placed upon the term by Euclid that only *four-sided* figures are so called, although of course there are certain regular polygons which have opposite sides parallel, and which therefore might be said to be areas bounded by parallel straight lines. We gather from Proclus (p. 393) that the word "parallelogram" was first introduced by Euclid, that its use was suggested by I. 33, and that the formation of the word παραλληλόγραμμος (parallel-lined) was analogous to that of εὐθύγραμμος (straight-lined or rectilineal).

17, 18, 40. **DCB** and 36. **DC, CB.** The Greek has in these places "*BCD*" and "*CD, BC*" respectively. Cf. note on I. 33, lines 15, 18.

After specifying the particular kinds of parallelograms (squares and rhombi) in which the diagonals bisect the angles which they join, as well as the areas, and those (rectangles and rhomboids) in which the diagonals do not bisect the angles, Proclus proceeds (pp. 390 sqq.) to analyse this proposition with reference to the distinction in Aristotle's *Anal. Post.* (I. 4, 5, 73 a 21—74 b 4) between attributes which are only predicable of every individual thing (κατὰ παντός) in a class and those which are true of it *primarily* (τούτου πρώτου) and *generally* (καθόλου). We are apt, says Aristotle, to mistake a proof κατὰ παντός for a proof τούτου πρώτου καθόλου because it is either impossible to find a higher generality to comprehend all the particulars of which the predicate is true, or to find a name for it. (Part of this passage of Aristotle has been quoted above in the note on I. 32, pp. 319—320.)

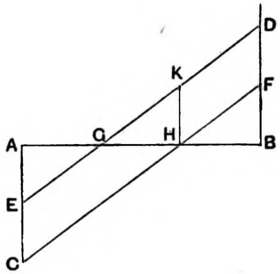
Now, says Proclus, adapting Aristotle's distinction to *theorems*, the present proposition exhibits the distinction between theorems which are *general* and theorems which are *not general.* According to Proclus, the first part of the proposition stating that the opposite sides and angles of a parallelogram are equal is *general* because the property is only true of parallelograms; but the second part which asserts that the diameter bisects the area is *not general* because it does not include all the figures of which this property is true, e.g. circles and ellipses. Indeed, says Proclus, the first attempts upon problems seem usually to have been of this partial character (μερικώτεραι), and generality was only attained by degrees. Thus "the ancients, after investigating the fact that the diameter bisects an ellipse, a circle, and a parallelogram respectively, proceeded to investigate what was common to these cases," though "it is difficult to show what is common to an ellipse, a circle and a parallelogram."

I doubt whether the supposed distinction between the two parts of the proposition, in point of "generality," can be sustained. Proclus himself admits that it is presupposed that the subject of the proposition is a *quadrilateral*, because there are other figures (e.g. regular polygons of an even number of sides) besides parallelograms which have their opposite sides and angles equal; therefore the second part of the theorem is, in this respect, no more *general* than the other, and, if we are entitled to the tacit limitation of the theorem to quadrilaterals in one part, we are equally entitled to it in the other.

It would almost appear as though Proclus had drawn the distinction for the mere purpose of alluding to investigations by Greek geometers on the general subject of *diameters* of all sorts of figures; and it may have been these which brought the subject to the point at which Apollonius could say in the first definitions at the beginning of his *Conics* that "In *any bent line*, such as is in one plane, I give the name *diameter* to any straight line which, being drawn from the bent line, bisects all the straight lines (chords) drawn in the line parallel to any straight line." The term *bent line* (καμπύλη γραμμή) includes, e.g. in Archimedes, not only curves, but any composite line made up of straight lines and curves joined together in any manner. It is of course clear that either diagonal of a parallelogram bisects all lines drawn within the parallelogram parallel to the other diagonal.

An-Nairīzī gives after I. 31 a neat construction for dividing a straight line into any number of equal parts (ed. Curtze, p. 74, ed. Besthorn-Heiberg, pp. 141—3) which requires only one measurement repeated, together with the properties of parallel lines including I. 33, 34. As I. 33, 34 are assumed, I place the problem here. The particular case taken is the problem of dividing a straight line into *three* equal parts.

Let *AB* be the given straight line. Draw *AC*, *BD* at right angles to it on opposite sides.



An-Nairīzī takes *AC*, *BD* of the same length and then bisects *AC* at *E* and *BD* at *F*. But of course it is even simpler to measure *AE*, *EC* along one perpendicular equal and of any length, and *BF*, *FD* along the other also equal and of the same length.

Join *ED*, *CF* meeting *AB* in *G*, *H* respectively.

Then shall *AG*, *GH*, *HB* all be equal.

Draw *HK* parallel to *AC*, or at right angles to *AB*.

Since now *EC*, *FD* are equal and parallel, *ED*, *CF* are equal and parallel. [I. 33]

And *HK* was drawn parallel to *AC*.

Therefore *ECHK* is a parallelogram; whence *KH* is equal as well as parallel to *EC*, and therefore to *EA*.

The triangles *EAG*, *KHG* have now two angles respectively equal and the sides *AE*, *HK* equal.

Thus the triangles are equal in all respects, and

AG is equal to *GH*.

Similarly the triangles *KHG*, *FBH* are equal in all respects, and

GH is equal to *HB*.

If now we wish to extend the problem to the case where *AB* is to be divided into *n* parts, we have only to measure $(n-1)$ successive equal lengths along *AC* and $(n-1)$ successive lengths, each equal to the others, along *BD*. Then join the first point arrived at on *AC* to the last point on *BD*, the second on *AC* to the last but one on *BD*, and so on; and the joining lines cut *AB* in points dividing it into *n* equal parts.

PROPOSITION 35.

Parallelograms which are on the same base and in the same parallels are equal to one another.

Let *ABCD*, *EBCF* be parallelograms on the same base *BC* and in the same parallels *AF*, *BC*;

I say that *ABCD* is equal to the parallelogram *EBCF*.

For, since *ABCD* is a parallelogram,

AD is equal to *BC*. [I. 34]

For the same reason also

EF is equal to *BC*,

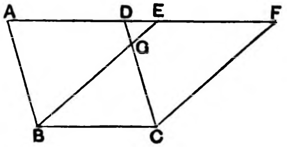
so that *AD* is also equal to *EF*; [*C. N.* 1]

and *DE* is common;

therefore the whole *AE* is equal to the whole *DF*. [*C. N.* 2]

But *AB* is also equal to *DC*; [I. 34]

therefore the two sides *EA*, *AB* are equal to the two sides *FD*, *DC* respectively,



and the angle *FDC* is equal to the angle *EAB*, the exterior to the interior; [I. 29]

therefore the base *EB* is equal to the base *FC*,

and the triangle *EAB* will be equal to the triangle *FDC*. [I. 4]

Let *DGE* be subtracted from each;

therefore the trapezium *ABGD* which remains is equal to the trapezium *EGCF* which remains. [*C. N.* 3]

Let the triangle *GBC* be added to each;

therefore the whole parallelogram *ABCD* is equal to the whole parallelogram *EBCF*. [*C. N.* 2]

Therefore etc.

Q. E. D.

21. **FDC.** The text has "*DFC*."

22. **Let DGE be subtracted.** Euclid speaks of the triangle *DGE* without any explanation that, in the case which he takes (where *AD*, *EF* have no point in common), *BE*, *CD* must meet at a point *G* between the two parallels. He allows this to appear from the figure simply.

Equality in a new sense.

It is important to observe that we are in this proposition introduced for the first time to a new conception of equality between figures. Hitherto we have had equality in the sense of *congruence* only, as applied to straight lines, angles, and even triangles (cf. I. 4). Now, without any explicit reference to any change in the meaning of the term, figures are inferred to be *equal* which are equal in *area* or in *content* but need not be of the same *form*. No *definition* of equality is anywhere given by Euclid; we are left to infer its meaning from the few *axioms* about "equal things." It will be observed that in the above proof the "equality" of two parallelograms on the same base and between the same parallels is inferred by the successive steps (1) of subtracting one and the same area (the triangle *DGE*) from two areas equal in the sense of *congruence* (the triangles *AEB*, *DFC*), and inferring that the remainders (the trapezia *ABGD*, *EGCF*) are "equal"; (2) of adding one and the same area (the triangle GBC) to each of the latter "equal" trapezia, and inferring the equality of the respective sums (the two given parallelograms).

As is well known, Simson (after Clairaut) slightly altered the proof in order to make it applicable to all the three possible cases. The alteration substituted *one* step of subtracting congruent areas (the triangles AEB, DFC) from one and the same area (the trapezium $ABCF$) for the *two* steps above shown of first subtracting and then adding a certain area.

While, in either case, nothing more is explicitly used than the axioms that, *if equals be added to equals, the wholes are equal* and that, *if equals be subtracted from equals, the remainders are equal*, there is the further *tacit* assumption that it is indifferent to *what part* or from *what part* of the same or equal areas the same or equal areas are added or subtracted. De Morgan observes that the postulate "an area taken from an area leaves the same area from whatever part it may be taken" is particularly important as the key to equality of non-rectilineal areas which could not be cut into coincidence geometrically.

Legendre introduced the word *equivalent* to express this wider sense of equality, restricting the term *equal* to things equal in the sense of congruent; and this distinction has been found convenient.

I do not think it necessary, nor have I the space, to give any account of the recent developments of the theory of equivalence on new lines represented by the researches of W. Bolyai, Duhamel, De Zolt, Stolz, Schur, Veronese, Hilbert and others, and must refer the reader to Ugo Amaldi's article *Sulla teoria dell' equivalenza* in *Questioni riguardanti le matematiche elementari*, I. (Bologna, 1912), pp. 145—198, and to Max Simon, *Über die Entwicklung der Elementar-geometrie im XIX. Jahrhundert* (Leipzig, 1906), pp. 115—120, with their full references to the literature of the subject. I may however refer to the suggestive distinction of phraseology used by Hilbert (*Grundlagen der Geometrie*, pp. 39, 40):

(1) "Two polygons are called *divisibly-equal* (*zerlegungsgleich*) if they can be divided into a *finite* number of triangles which are congruent two and two."

(2) "Two polygons are called *equal in content* (*inhaltsgleich*) or *of equal content* if it is possible to add *divisibly-equal* polygons to them in such a way that the two combined polygons are *divisibly-equal*."

(Amaldi suggests as alternatives for the terms in (1) and (2) the expressions *equivalent by sum* and *equivalent by difference* respectively.)

From these definitions it follows that "by combining *divisibly-equal* polygons we again arrive at *divisibly-equal* polygons; and, if we subtract *divisibly-equal* polygons from *divisibly-equal* polygons, the polygons remaining are *equal in content*."

The proposition also follows without difficulty that, "if two polygons are *divisibly-equal* to a third polygon, they are also *divisibly-equal* to one another; and, if two polygons are *equal in content* to a third polygon, they are *equal in content* to one another."

The different cases.

As usual, Proclus (pp. 399—400), observing that Euclid has given only the most difficult of the three possible cases, adds the other two with separate proofs. In the case where E in the figure of the proposition falls between A and D, he *adds* the congruent triangles ABE, DCF respectively to the smaller trapezium $EBCD$, instead of subtracting them (as Simson does) from the larger trapezium $ABCF$.

An ancient "Budget of Paradoxes."

Proclus observes (p. 396, 12 sqq.) that the present theorem and the similar one relating to triangles are among the so-called paradoxical theorems of mathematics, since the uninstructed might well regard it as impossible that the area of the parallelograms should remain the same while the length of the sides other than the base and the side opposite to it may increase indefinitely. He adds that mathematicians had made a collection of such paradoxes, the so-called *treasury of paradoxes* (ὁ παράδοξος τόπος)—cf. the similar expressions τόπος ἀναλυόμενος (treasury of analysis) and τόπος ἀστρονομούμενος—in the same way as the Stoics with their *illustrations* (ὥσπερ οἱ ἀπὸ τῆς Στοᾶς ἐπὶ τῶν δειγμάτων). It may be that this *treasury of paradoxes* was the work of Erycinus quoted by Pappus (III. p. 107, 8) and mentioned above (note on I. 21, p. 290).

Locus-theorems and loci in Greek geometry.

The proposition I. 35 is, says Proclus (pp. 394—6), the first *locus-theorem* (τοπικὸν θεώρημα) given by Euclid. Accordingly it is in his note on this proposition that Proclus gives us his view of the nature of a locus-theorem and of the meaning of the word *locus* (τόπος); and great importance attaches to his words because he is one of the three writers (Pappus and Eutocius being the two others) upon whom we have to rely for all that is known of the Greek conception of geometrical loci.

Proclus' explanation (pp. 394, 15—395, 2) is as follows. "I call those (theorems) *locus-theorems* (τοπικά) in which the same property is found to exist on the whole of some locus (πρὸς ὅλῳ τινὶ τόπῳ), and (I call) a locus a position of a line or a surface producing one and the same property (γραμμῆς ἢ ἐπιφανείας θέσιν ποιοῦσαν ἓν καὶ ταὐτὸν σύμπτωμα). For, of locus-theorems, some are constructed on lines and others on surfaces (τῶν γὰρ τοπικῶν τὰ μέν ἐστι πρὸς γραμμαῖς συνιστάμενα, τὰ δὲ πρὸς ἐπιφανείαις). And, since some lines are plane (ἐπίπεδοι) and others solid (στερεαί)—those being plane which are simply conceived of in a plane (ὧν ἐν ἐπιπέδῳ ἁπλῆ ἡ νόησις), and those solid the origin of which is revealed from some section of a solid figure, as the cylindrical helix and the conic lines (ὡς τῆς κυλινδρικῆς ἕλικος καὶ τῶν κωνικῶν γραμμῶν)—I should say (φαίην ἄν) further that, of locus-theorems on lines, some give a plane locus and others a solid locus."

Leaving out of sight for the moment the class of *loci on surfaces*, we find that the distinction between *plane* and *solid loci*, or *plane* and *solid lines*, was similarly understood by Eutocius, who says (Apollonius, ed. Heiberg, II. p. 184) that "*solid loci* have obtained their name from the fact that the lines used in the solution of problems regarding them have their origin in the section of solids, for example the sections of the cone and several others." Similarly we gather from Pappus that *plane loci* were straight lines and circles, and *solid loci* were conics. Thus he tells us (VII. p. 672, 20) that Aristaeus wrote five books of *Solid Loci* "supplementary to (literally, continuous with) the conics"; and, though Hultsch brackets the passage (VII. p. 662, 10—15) which says plainly that *plane loci* are straight lines and circles, while *solid loci* are sections of cones, i.e. parabolas, ellipses and hyperbolas, we have the exactly corresponding distinction drawn by Pappus (III. p. 54, 7—16) between *plane* and *solid problems*, plane problems being those solved by means of straight lines and circumferences of circles, and solid problems those solved by means of one or more of the sections of the cone. But, whereas Proclus and Eutocius speak of other *solid loci* besides conics, there is nothing in Pappus to support the wider application of the term. According to Pappus (III. p. 54, 16—21) problems which could not be solved by means of straight lines, circles, or conics were *linear* (γραμμικά) because they used for their construction lines having a more complicated and unnatural origin than those mentioned, namely such curves as *quadratrices*, conchoids and cissoids. Similarly, in the passage supposed to be interpolated, *linear loci* are distinguished as those which are neither straight lines nor circles nor any of the conic sections (VII. p. 662, 13—15). Thus the classification given by Proclus and Eutocius is less precise than that which we find in Pappus; and the inclusion by Proclus of the cylindrical helix among solid loci, on the ground that it arises from a section of a solid figure, would seem to be, in any case, due to some misapprehension.

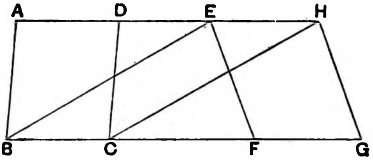
Comparing these passages and the hints in Pappus about *loci on surfaces* (τόποι πρὸς ἐπιφανείᾳ) with special reference to Euclid's two books under that title, Heiberg concludes that *loci on lines* and *loci on surfaces* in Proclus' explanation are loci which *are* lines and loci which *are* surfaces respectively. But some qualification is necessary as regards Proclus' conception of *loci on lines*, because he goes on to say (p. 395, 5), with reference to this proposition, that, while the locus is a *locus on lines* and moreover *plane*, it is "the whole space between the parallels" which is the locus of the various parallelograms on the same base proved to be equal in area. Similarly, when he quotes III. 21 about the equality of the angles in the same segment and III. 31 about the right angle in a semicircle as cases where a circumference of a circle takes the place of a straight line in a *plane* locus-theorem, he appears to imply that it is the segment or semicircle as an *area* which is regarded as the locus of an infinite number of *triangles* with the same base and equal vertical angles, rather than that it is the *circumference* which is the locus of the angular *points*. Likewise he gives the equality of parallelograms inscribed in "the asymptotes and the hyperbola" as an example of a *solid* locus-theorem, as if the area included between the curve and its asymptotes was regarded as the *locus* of the equal parallelograms. However this may be, it is clear that the locus in the present proposition can only be either (1) a *line*-locus of a *line*, not a point, or (2) an *area*-locus of an *area*, not a point or a line; and we seem to be thus brought to another and different classification of loci corresponding to that quoted by Pappus (VII. p. 660, 18 sqq.) from the preliminary exposition given by Apollonius in his *Plane Loci*. According to this, loci in general are of three kinds: (1) ἐφεκτικοί, *holding-in*, in which sense the locus of a point is a point, of a line a line, of a surface a surface, and of a solid a solid, (2) διεξοδικοί, *moving along*, a line being in this sense a locus of a point, a surface of a line and a solid of a surface, (3) ἀναστροφικοί, where a surface is a locus of a point and a solid of a line. Thus the locus in this proposition, whether it is the space between the two parallels regarded as the locus of the equal parallelograms, or the line parallel to the base regarded as the locus of the sides opposite to the base, would seem to be of the first class (ἐφεκτικός); and, as Proclus takes the former view of it, a *locus on lines* is apparently not merely a locus which *is* a line but a locus *bounded by lines* also, the locus being *plane* in the particular case because it is bounded by straight lines, or, in the case of III. 21, 31, by straight lines and circles, but not by any higher curves.

Proclus notes lastly (p. 395, 13—21) that, according to Geminus, "Chrysippus likened locus-theorems to the *ideas*. For, as the ideas confine the genesis of unlimited (particulars) within defined limits, so in such theorems the unlimited (particular figures) are confined within defined *places* or *loci* (τόποι). And it is this boundary which is the cause of the equality; for the height of the parallels, which remains the same, while an infinite number of parallelograms are conceived on the same base, is what makes them all equal to one another."

PROPOSITION 36.

Parallelograms which are on equal bases and in the same parallels are equal to one another.

Let *ABCD*, *EFGH* be parallelograms which are on equal bases *BC*, *FG* and in the same parallels *AH*, *BG*;



I say that the parallelogram *ABCD* is equal to *EFGH*.

For let *BE*, *CH* be joined.

Then, since *BC* is equal to *FG*
while *FG* is equal to *EH*,
BC is also equal to *EH*. [*C. N.* 1]

But they are also parallel.

And *EB*, *HC* join them;

but straight lines joining equal and parallel straight lines (at the extremities which are) in the same directions (respectively) are equal and parallel. [I. 33]

Therefore *EBCH* is a parallelogram. [I. 34]

And it is equal to *ABCD*;

for it has the same base *BC* with it, and is in the same parallels *BC*, *AH* with it. [I. 35]

For the same reason also *EFGH* is equal to the same *EBCH*; [I. 35]

so that the parallelogram *ABCD* is also equal to *EFGH*. [*C. N.* 1]

Therefore etc.

Q. E. D.

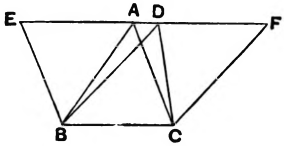
PROPOSITION 37.

Triangles which are on the same base and in the same parallels are equal to one another.

Let ABC, DBC be triangles on the same base BC and in the same parallels AD, BC;

I say that the triangle ABC is equal to the triangle DBC.

Let AD be produced in both directions to E, F;



through B let BE be drawn parallel to CA, [I. 31]

and through C let CF be drawn parallel to BD. [I. 31]

Then each of the figures $EBCA$, $DBCF$ is a parallelogram; and they are equal,

for they are on the same base BC and in the same parallels BC, EF. [I. 35]

Moreover the triangle ABC is half of the parallelogram $EBCA$; for the diameter AB bisects it. [I. 34]

And the triangle DBC is half of the parallelogram $DBCF$; for the diameter DC bisects it. [I. 34]

[But the halves of equal things are equal to one another.]

Therefore the triangle ABC is equal to the triangle DBC.

Therefore etc.

Q. E. D.

21. Here and in the next proposition Heiberg brackets the words "But the halves of equal things are equal to one another" on the ground that, since the *Common Notion* which asserted this fact was interpolated at a very early date (before the time of Theon), it is probable that the words here were interpolated at the same time. Cf. note above (p. 224) on the interpolated *Common Notion.*

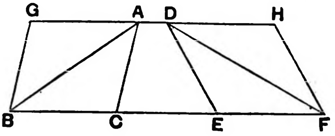
There is a lacuna in the text of Proclus' notes to I. 36 and I. 37. Apparently the end of the former and the beginning of the latter are missing, the MSS. and the *editio princeps* showing no separate note for I. 37 and no lacuna, but going straight on without regard to sense. Proclus had evidently remarked again in the missing passage that, in the case of both parallelograms and triangles between the same parallels, the two sides which stretch from one parallel to the other may increase in length to any extent, while the area remains the same. Thus the *perimeter* in parallelograms or triangles is of itself no criterion as to their area. Misconception on this subject was rife among non-mathematicians; and Proclus (p. 403, 5 sqq.) tells us (1) of describers of countries (χωρογράφοι) who drew conclusions regarding the size of cities from their perimeters, and (2) of certain members of communistic societies in his own time who cheated their fellow members by giving them land of greater perimeter but less area than they took themselves, so that, on the one hand, they got a reputation for greater honesty while, on the other, they took more than their share of produce. Cantor (*Gesch. d. Math.* I_3, p. 172) quotes several remarks of ancient authors which show the prevalence of the same misconception. Thus Thucydides estimates the size of Sicily according to the time required for circumnavigating it. About 130 B.C. Polybius said that there were people who could not understand that camps of the same periphery might have different capacities. Quintilian has a similar remark, and Cantor thinks he may have had in his mind the calculations of Pliny, who compares the size of different parts of the earth by adding their length to their breadth.

The comparison however of the areas of different figures of equal contour had not been neglected by mathematicians. Theon of Alexandria, in his commentary on Book I. of Ptolemy's *Syntaxis*, has preserved a number of propositions on the subject taken from a treatise by Zenodorus *περὶ ἰσομέτρων σχημάτων* (reproduced in Latin on pp. 1190—1211 of Hultsch's edition of Pappus) which was written at some date between, say, 200 B.C. and 90 A.D., and probably not long after the former date. Pappus too has at the beginning of Book v. of his *Collection* (pp. 308 sqq.) the same propositions, in which he appears to have followed Zenodorus pretty closely while making some changes in detail. The propositions proved by Zenodorus and Pappus include the following: (1) that, *of all polygons of the same number of sides and equal perimeter, the equilateral and equiangular polygon is the greatest in area*, (2) that, *of regular polygons of equal perimeter, that is the greatest in area which has the most angles*, (3) that *a circle is greater than any regular polygon of equal contour*, (4) that, *of all circular segments in which the arcs are equal in length, the semicircle is the greatest.* The treatise of Zenodorus was not confined to propositions about plane figures, but gave also the theorem that, *of all solid figures the surfaces of which are equal, the sphere is the greatest in volume.*

PROPOSITION 38.

Triangles which are on equal bases and in the same parallels are equal to one another.

Let *ABC*, *DEF* be triangles on equal bases *BC*, *EF* and in the same parallels *BF*, *AD*;



I say that the triangle *ABC* is equal to the triangle *DEF*.

For let *AD* be produced in both directions to *G*, *H*; through *B* let *BG* be drawn parallel to *CA*, [I. 31]

and through *F* let *FH* be drawn parallel to *DE*.

Then each of the figures *GBCA*, *DEFH* is a parallelogram;

and *GBCA* is equal to *DEFH*;

for they are on equal bases BC, EF and in the same parallels BF, GH. [I. 36]

Moreover the triangle ABC is half of the parallelogram $GBCA$; for the diameter AB bisects it. [I. 34]

And the triangle FED is half of the parallelogram $DEFH$; for the diameter DF bisects it. [I. 34]

[But the halves of equal things are equal to one another.]

Therefore the triangle ABC is equal to the triangle DEF.

Therefore etc.

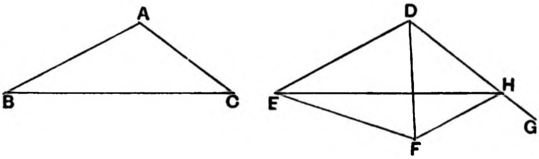
Q. E. D.

On this proposition Proclus remarks (pp. 405—6) that Euclid seems to him to have given in VI. 1 one proof including all the four theorems from I. 35 to I. 38, and that most people had failed to notice this. When Euclid, he says, proves that triangles and parallelograms of the same altitude have to one another the same ratio as their bases, he simply proves all these propositions more generally by the use of proportion; for of course to be of the same altitude is equivalent to being in the same parallels. It is true that VI. 1 generalises these propositions, but it must be observed that it does not prove the propositions themselves, as Proclus seems to imply; they are in fact assumed in order to prove VI. 1.

Comparison of areas of triangles of I. 24.

The theorem already mentioned as given by Proclus on I. 24 (pp. 340—4) is placed here by Heron, who also enunciates it more clearly (an-Nairīzī, ed. Besthorn-Heiberg, pp. 155—161, ed. Curtze, pp. 75—8).

If in two triangles two sides of the one be equal to two sides of the other respectively, and the angle of the one be greater than the angle of the other, namely the angles contained by the equal sides, then, (1) *if the sum of the two angles contained by the equal sides is equal to two right angles, the two triangles are equal to one another;* (2) *if less than two right angles, the triangle which has the greater angle is also itself greater than the other;* (3) *if greater than two right angles, the triangle which has the less angle is greater than the other triangle.*



Let two triangles ABC, DEF have the sides AB, AC respectively equal to DE, DF.

(1) First, suppose that the angles at A and D in the triangles ABC, DEF are together equal to two right angles.

Heron's construction is now as follows.

Make the angle EDG equal to the angle BAC.

Draw FH parallel to ED meeting DG in H.

Join EH.

Then, since the angles *BAC*, *EDF* are equal to two right angles, the angles *EDH*, *EDF* are equal to two right angles.

But so are the angles *EDH*, *DHF*.

Therefore the angles *EDF*, *DHF* are equal.

And the alternate angles *EDF*, *DFH* are equal. [I. 29]

Therefore the angles *DHF*, *DFH* are equal,

and *DF* is equal to *DH*. [I. 6]

Hence the two sides *ED*, *DH* are equal to the two sides *BA*, *AC*; and the included angles are equal.

Therefore the triangles *ABC*, *DEH* are equal in all respects.

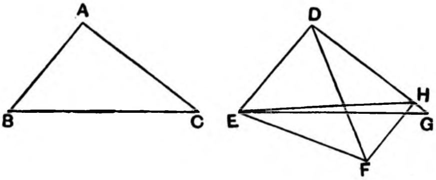
And the triangles *DEF*, *DEH* between the same parallels are equal. [I. 37]

Therefore the triangles *ABC*, *DEF* are equal.

[Proclus takes the construction of Eucl. I. 24, i.e., he makes *DH* equal to *DF* and then proves that *ED*, *FH* are parallel.]

(2) Suppose the angles *BAC*, *EDF* together *less* than two right angles.

As before, make the angle *EDG* equal to the angle *BAC*, draw *FH* parallel to *ED*, and join *EH*.



In this case the angles *EDH*, *EDF* are together less than two right angles, while the angles *EDH*, *DHF* are equal to two right angles. [I. 29]

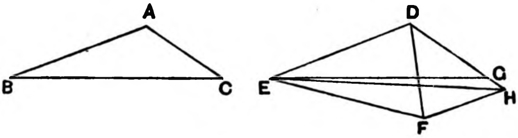
Hence the angle *EDF*, and therefore the angle *DFH*, is less than the angle *DHF*.

Therefore *DH* is less than *DF*. [I. 19]

Produce *DH* to *G* so that *DG* is equal to *DF* or *AC*, and join *EG*.

Then the triangle *DEG*, which is equal to the triangle *ABC*, is greater than the triangle *DEH*, and therefore greater than the triangle *DEF*.

(3) Suppose the angles *BAC*, *EDF* together greater than two right angles.



We make the same construction in this case, and we prove in like manner that the angle *DHF* is less than the angle *DFH*,

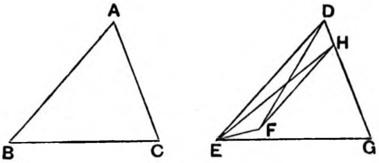
whence *DH* is greater than *DF* or *AC*.

Make *DG* equal to *AC*, and join *EG*.

It then follows that the triangle *DEF* is greater than the triangle *ABC*.

[In the second and third cases again Proclus starts from the construction in I. 24, and proves, in the second case, that the parallel, *FH*, to *ED* cuts *DG* and, in the third case, that it cuts *DG* produced.]

There is no necessity for Heron to take account of the position of *F* in relation to the side opposite *D*. For in the first and third cases *F must* fall



in the position in which Euclid draws it in I. 24, whatever be the relative lengths of *AB*, *AC*. In the second case the figure *may* be as annexed, but the proof is the same, or rather the case needs no proof at all.

PROPOSITION 39.

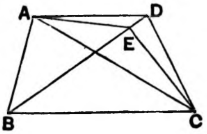
Equal triangles which are on the same base and on the same side are also in the same parallels.

Let *ABC*, *DBC* be equal triangles which are on the same base *BC* and on the same side of it;

[I say that they are also in the same parallels.]

And [For] let *AD* be joined;

I say that *AD* is parallel to *BC*.



For, if not, let *AE* be drawn through the point *A* parallel to the straight line *BC*, [I. 31]

and let *EC* be joined.

Therefore the triangle *ABC* is equal to the triangle *EBC*;

for it is on the same base *BC* with it and in the same parallels. [I. 37]

But *ABC* is equal to *DBC*;

therefore *DBC* is also equal to *EBC*, [C. N. 1]

the greater to the less: which is impossible.

Therefore *AE* is not parallel to *BC*.

Similarly we can prove that neither is any other straight line except *AD*;

therefore *AD* is parallel to *BC*.

Therefore etc.

Q. E. D.

5. **[I say that they are also in the same parallels.]** Heiberg has proved (*Hermes*, XXXVIII., 1903, p. 50) from a recently discovered papyrus-fragment (*Fayûm towns and their papyri*, p. 96, No. IX.) that these words are an interpolation by some one who did not observe that the words "And let *AD* be joined" are part of the *setting-out* (ἔκθεσις), but took them as belonging to the *construction* (κατασκευή) and consequently thought that a διορισμός or "definition" (of the thing to be proved) should precede. The interpolator then altered "And" into "For" in the next sentence.

This theorem is of course the *partial* converse of I. 37. In I. 37 we have triangles which are (1) on the same base, (2) in the same parallels, and the theorem proves (3) that the triangles are equal. Here the hypothesis (1) and the conclusion (3) are combined as hypotheses, and the conclusion is the hypothesis (2) of I. 37, that the triangles are in the same parallels. The additional qualification in this proposition that the triangles must be *on the same side* of the base is necessary because it is not, as in I. 37, involved in the other hypotheses.

Proclus (p. 407, 4—17) remarks that Euclid only converts I. 37 and I. 38 relative to triangles, and omits the converses of I. 35, 36 about parallelograms as unnecessary because it is easy to see that the method would be the same, and therefore the reader may properly be left to prove them for himself.

The proof is, as Proclus points out (p. 408, 5—21), equally easy on the supposition that the assumed parallel *AE* meets *BD* or *CD* produced beyond *D*.

[PROPOSITION 40.

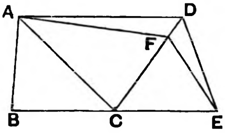
Equal triangles which are on equal bases and on the same side are also in the same parallels.

Let *ABC*, *CDE* be equal triangles on equal bases *BC*, *CE* and on the same side.

I say that they are also in the same parallels.

For let *AD* be joined;

I say that *AD* is parallel to *BE*.



For, if not, let *AF* be drawn through *A* parallel to *BE* [I. 31], and let *FE* be joined.

Therefore the triangle *ABC* is equal to the triangle *FCE*;

for they are on equal bases *BC*, *CE* and in the same parallels *BE*, *AF*. [I. 38]

But the triangle *ABC* is equal to the triangle *DCE*;

therefore the triangle *DCE* is also equal to the triangle *FCE*, [*C. N.* 1]

the greater to the less: which is impossible.

Therefore *AF* is not parallel to *BE*.

Similarly we can prove that neither is any other straight line except AD;

therefore AD is parallel to BE.

Therefore etc.

Q. E. D.]

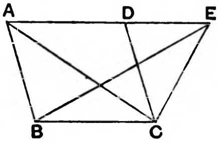
Heiberg has proved by means of the papyrus-fragment mentioned in the last note that this proposition is an interpolation by some one who thought that there should be a proposition following I. 39 and related to it in the same way as I. 38 is related to I. 37, and I. 36 to I. 35.

PROPOSITION 41.

If a parallelogram have the same base with a triangle and be in the same parallels, the parallelogram is double of the triangle.

For let the parallelogram $ABCD$ have the same base BC with the triangle EBC, and let it be in the same parallels BC, AE;

I say that the parallelogram $ABCD$ is double of the triangle BEC.



For let AC be joined.

Then the triangle ABC is equal to the triangle EBC;

for it is on the same base BC with it and in the same parallels BC, AE. [I. 37]

But the parallelogram $ABCD$ is double of the triangle ABC;

for the diameter AC bisects it; [I. 34]

so that the parallelogram $ABCD$ is also double of the triangle EBC.

Therefore etc.

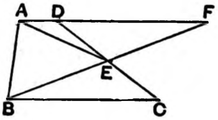
Q. E. D.

On this proposition Proclus (pp. 414, 15—415, 16), "by way of practice" (γυμνασίας ἕνεκα), considers the area of a *trapezium* (a quadrilateral with only one pair of opposite sides parallel) in comparison with that of the triangles in the same parallels and having the greater and less of the parallel sides of the trapezium for bases respectively, and proves that the trapezium is less than double of the former triangle and more than double of the latter.

He next (pp. 415, 22—416, 14) proves the proposition that,

If a triangle be formed by joining the middle point of either of the non-parallel sides to the extremities of the opposite side, the area of the trapezium is always double of that of the triangle.

Let *ABCD* be a trapezium in which *AD*, *BC* are the parallel sides, and *E* the middle point of one of the non-parallel sides, say *DC*.



Join *EA*, *EB* and produce *BE* to meet *AD* produced in *F*.

Then the triangles *BEC*, *FED* have two angles equal respectively, and one side *CE* equal to one side *DE*;

therefore the triangles are equal in all respects. [I. 26]

Add to each the quadrilateral *ABED*;

therefore the trapezium *ABCD* is equal to the triangle *ABF*,

that is, to twice the triangle *AEB*, since *BE* is equal to *EF*. [I. 38]

The three properties proved by Proclus may be combined in one enunciation thus:

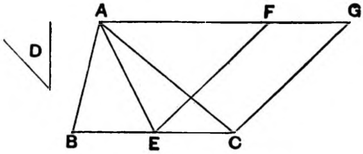
If a triangle be formed by joining the middle point of one side of a trapezium to the extremities of the opposite side, the area of the trapezium is (1) *greater than*, (2) *equal to*, *or* (3) *less than*, *double the area of the triangle according as the side the middle point of which is taken is* (1) *the greater of the parallel sides*, (2) *either of the non-parallel sides*, *or* (3) *the lesser of the parallel sides.*

Proposition 42.

To construct, in a given rectilineal angle, a parallelogram equal to a given triangle.

Let *ABC* be the given triangle, and *D* the given rectilineal angle;

thus it is required to construct in the rectilineal angle *D* a parallelogram equal to the triangle *ABC*.



Let *BC* be bisected at *E*, and let *AE* be joined;

on the straight line *EC*, and at the point *E* on it, let the angle *CEF* be constructed equal to the angle *D*; [I. 23]

through *A* let *AG* be drawn parallel to *EC*, and [I. 31]

through *C* let *CG* be drawn parallel to *EF*.

Then *FECG* is a parallelogram.

And, since *BE* is equal to *EC*,

the triangle *ABE* is also equal to the triangle *AEC*,

for they are on equal bases *BE*, *EC* and in the same parallels *BC*, *AG*; [I. 38]

therefore the triangle *ABC* is double of the triangle *AEC*.

But the parallelogram *FECG* is also double of the triangle *AEC*, for it has the same base with it and is in the same parallels with it; [I. 41]

therefore the parallelogram *FECG* is equal to the triangle *ABC*.

And it has the angle *CEF* equal to the given angle *D*.

Therefore the parallelogram *FECG* has been constructed equal to the given triangle *ABC*, in the angle *CEF* which is equal to *D*. Q. E. F.

PROPOSITION 43.

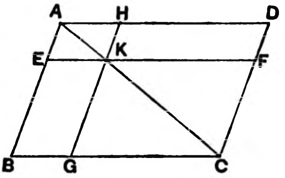
In any parallelogram the complements of the parallelograms about the diameter are equal to one another.

Let *ABCD* be a parallelogram, and *AC* its diameter; and about *AC* let *EH*, *FG* be parallelograms, and *BK*, *KD* the so-called complements;

I say that the complement *BK* is equal to the complement *KD*.

For, since *ABCD* is a parallelogram, and *AC* its diameter,

the triangle *ABC* is equal to the triangle *ACD*. [I. 34]



Again, since *EH* is a parallelogram, and *AK* is its diameter,

the triangle *AEK* is equal to the triangle *AHK*.

For the same reason

the triangle *KFC* is also equal to *KGC*.

Now, since the triangle *AEK* is equal to the triangle *AHK*,

and *KFC* to *KGC*,

the triangle *AEK* together with *KGC* is equal to the triangle *AHK* together with *KFC*. [*C. N.* 2]

And the whole triangle *ABC* is also equal to the whole *ADC*;

therefore the complement *BK* which remains is equal to the complement *KD* which remains. [*C. N.* 3]

Therefore etc.

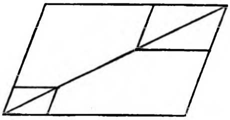
Q. E. D.

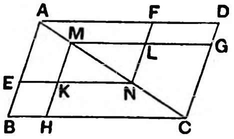
1. **complements**, παραπληρώματα, the figures put in to fill up (interstices).

4. **and about AC....** Euclid's phraseology here and in the next proposition implies that the complements as well as the other parallelograms are "about" the diagonal. The words are here περὶ δὲ τὴν ΑΓ παραλληλόγραμμα μὲν ἔστω τὰ ΕΘ, ΖΗ, τὰ δὲ λεγόμενα παραπληρώματα τὰ ΒΚ, ΚΔ. The expression "the so-called complements" indicates that this technical use of παραπληρώματα was not new, though it might not be universally known.

In the text of Proclus' commentary as we have it, the end of the note on I. 41, the whole of that on I. 42, and the beginning of that on I. 43 are missing.

Proclus remarks (p. 418, 15—20) that Euclid did not need to give a formal definition of *complement* because the name was simply suggested by the facts; when once we have the two "parallelograms about the diameter," the *complements* are necessarily the areas remaining over on each side of the diameter, which fill up the complete parallelogram. Thus (p. 417, 1 sqq.) the complements need not be parallelograms. They are so if the two "parallelograms about the diameter" are formed by straight lines drawn through *one point* of the diameter parallel to the sides of the original parallelogram, but not otherwise. If, as in the first of the accompanying figures, the parallelograms have no common point, the complements are five-sided figures as shown. When the parallelograms overlap, as in the second figure, Proclus regards the complements as being the small parallelograms *FG*, *EH*. But, if complements are strictly the areas required to fill up the original parallelogram, Proclus is inaccurate in describing *FG*, *EH* as the complements. The complements are really (1) the parallelogram *FG minus* the triangle *LMN*, and (2) the parallelogram *EH minus* the triangle *KMN*, respectively; the possibility that the respective differences may be negative merely means the possibility that the sum of the two parallelograms about the diameter may be together greater than the original parallelogram.





In all the cases it is easy to show, as Proclus does, that the complements are still equal.

PROPOSITION 44.

To a given straight line to apply, in a given rectilineal angle, a parallelogram equal to a given triangle.

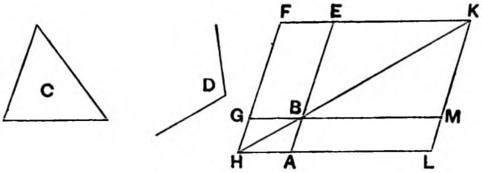
Let *AB* be the given straight line, *C* the given triangle and *D* the given rectilineal angle;

thus it is required to apply to the given straight line *AB*, in an angle equal to the angle *D*, a parallelogram equal to the given triangle *C*.

Let the parallelogram *BEFG* be constructed equal to the triangle *C*, in the angle *EBG* which is equal to *D* [I. 42]; let it be placed so that *BE* is in a straight line with *AB*; let

FG be drawn through to *H*, and let *AH* be drawn through *A* parallel to either *BG* or *EF*. [I. 31]

Let *HB* be joined.



Then, since the straight line *HF* falls upon the parallels *AH*, *EF*,

the angles *AHF*, *HFE* are equal to two right angles. [I. 29]

Therefore the angles *BHG*, *GFE* are less than two right angles;

and straight lines produced indefinitely from angles less than two right angles meet; [Post. 5]

therefore *HB*, *FE*, when produced, will meet.

Let them be produced and meet at *K*; through the point *K* let *KL* be drawn parallel to either *EA* or *FH*, [I. 31]

and let *HA*, *GB* be produced to the points *L*, *M*.

Then *HLKF* is a parallelogram, *HK* is its diameter, and *AG*, *ME* are parallelograms. and *LB*, *BF* the so-called complements, about *HK*;

therefore *LB* is equal to *BF*. [I. 43]

But *BF* is equal to the triangle *C*;

therefore *LB* is also equal to *C*. [*C. N.* 1]

And, since the angle *GBE* is equal to the angle *ABM*, [I. 15]

while the angle *GBE* is equal to *D*,

the angle *ABM* is also equal to the angle *D*.

Therefore the parallelogram *LB* equal to the given triangle *C* has been applied to the given straight line *AB*, in the angle *ABM* which is equal to *D*.

Q. E. F.

14. **since the straight line HF falls....** The verb is in the aorist (ἐνέπεσεν) here and in similar expressions in the following propositions.

This proposition will always remain one of the most impressive in all geometry when account is taken (1) of the great importance of the result obtained, the transformation of a parallelogram of any shape into another with the same angle and of equal area but with one side of any given length, e.g. a *unit* length, and (2) of the simplicity of the means employed, namely the mere application of the property that the complements of the "parallelograms about the diameter" of a parallelogram are equal. The marvellous ingenuity of the solution is indeed worthy of the "godlike men of old," as Proclus calls the discoverers of the method of "application of areas"; and there would seem to be no reason to doubt that the particular solution, like the whole theory, was Pythagorean, and not a new solution due to Euclid himself.

Application of areas.

On this proposition Proclus gives (pp. 419, 15—420, 23) a valuable note on the method of "application of areas" here introduced, which was one of the most powerful methods on which Greek geometry relied. The note runs as follows:

"These things, says Eudemus (οἱ περὶ τὸν Εὔδημον), are ancient and are discoveries of the Muse of the Pythagoreans, I mean the *application of areas* (παραβολὴ τῶν χωρίων), their *exceeding* (ὑπερβολή) and their *falling-short* (ἔλλειψις). It was from the Pythagoreans that later geometers [i.e. Apollonius] took the names, which they again transferred to the so-called *conic* lines, designating one of these a *parabola* (application), another a *hyperbola* (exceeding) and another an *ellipse* (falling-short), whereas those godlike men of old saw the things signified by these names in the construction, in a plane, of areas upon a finite straight line. For, when you have a straight line set out and lay the given area exactly alongside the whole of the straight line, then they say that you *apply* (παραβάλλειν) the said area; when however you make the length of the area greater than the straight line itself, it is said to *exceed* (ὑπερβάλλειν), and when you make it less, in which case, after the area has been drawn, there is some part of the straight line extending beyond it, it is said to *fall short* (ἐλλείπειν). Euclid too, in the sixth book, speaks in this way both of *exceeding* and *falling-short*; but in this place he needed the *application* simply, as he sought to apply to a given straight line an area equal to a given triangle in order that we might have in our power, not only the *construction* (σύστασις) of a parallelogram equal to a given triangle, but also the *application* of it to a finite straight line. For example, given a triangle with an area of 12 feet, and a straight line set out the length of which is 4 feet, we apply to the straight line the area equal to the triangle if we take the whole length of 4 feet and find how many feet the breadth must be in order that the parallelogram may be equal to the triangle. In the particular case, if we find a breadth of 3 feet and multiply the length into the breadth, supposing that the angle set out is a right angle, we shall have the area. Such then is the *application* handed down from early times by the Pythagoreans."

Other passages to a similar effect are quoted from Plutarch. (1) "Pythagoras sacrificed an ox on the strength of his proposition (διάγραμμα) as Apollodotus (?-rus) says...whether it was the theorem of the hypotenuse, viz. that the square on it is equal to the squares on the sides containing the right angle, or the problem about the *application of an area*." (*Non posse suaviter vivi secundum Epicurum*, c. 11.) (2) "Among the most geometrical theorems, or rather problems, is the following: given two figures, to *apply* a third equal to the one and similar to the other, on the strength of which discovery they say moreover that Pythagoras sacrificed. This is indeed unquestionably more subtle and more scientific than the theorem which demonstrated that the square on the hypotenuse is equal to the squares on the sides about the right angle" (*Symp.* VIII. 2, 4).

The story of the sacrifice must (as noted by Bretschneider and Hankel) be given up as inconsistent with Pythagorean ritual, which forbade such sacrifices; but there is no reason to doubt that the first distinct formulation and introduction into Greek geometry of the method of *application of areas* was due to the Pythagoreans. The complete exposition of the *application* of areas, their *exceeding* and their *falling-short*, and of the construction of a rectilineal figure equal to one given figure and similar to another, takes us into the sixth Book of Euclid; but it will be convenient to note here the general features of the theory of *application*, *exceeding* and *falling-short.*

The simple *application* of a parallelogram of given area to a given straight line as one of its sides is what we have in I. 44 and 45; the general form of the problem with regard to *exceeding* and *falling-short* may be stated thus:

"To apply to a given straight line a rectangle (or, more generally, a parallelogram) equal to a given rectilineal figure and (1) *exceeding* or (2) *falling-short* by a square (or, in the more general case, a parallelogram similar to a given parallelogram)."

What is meant by saying that the applied parallelogram (1) *exceeds* or (2) *falls short* is that, while its base coincides and is coterminous *at one end* with the straight line, the said base (1) overlaps or (2) falls short of the straight line *at the other end*, and the portion by which the applied parallelogram exceeds a parallelogram of the same angle and height on the given straight line (exactly) as base is a parallelogram similar to a given parallelogram (or, in particular cases, a square). In the case where the parallelogram is to *fall short*, a διορισμός is necessary to express the condition of possibility of solution.

We shall have occasion to see, when we come to the relative propositions in the second and sixth Books, that the general problem here stated is equivalent to that of solving geometrically a mixed quadratic equation. We shall see that, even by means of II. 5 and 6, we can solve geometrically the equations

$$ax \pm x^2 = b^2,$$

$$x^2 - ax = b^2;$$

but in VI. 28, 29 Euclid gives the equivalent of the solution of the general equations

$$ax \pm \frac{b}{c} x^2 = \frac{C}{m}.$$

We are now in a position to understand the application of the terms *parabola* (application), *hyperbola* (exceeding) and *ellipse* (falling-short) to conic sections. These names were first so applied by Apollonius as expressing in each case the fundamental property of the curves as stated by him. This fundamental property is the geometrical equivalent of the Cartesian equation referred to any diameter of the conic and the tangent at its extremity as (in general, oblique) axes. If the *parameter* of the ordinates from the several points of the conic drawn to the given diameter be denoted by p (p being accordingly, in the case of the hyperbola and ellipse, equal to $\frac{d'^2}{d}$, where d is the length of the given diameter and d' that of its conjugate), Apollonius gives the properties of the three conics in the following form.

(1) For the *parabola*, the square on the ordinate at any point is equal to a rectangle applied to p as base with altitude equal to the corresponding abscissa. That is to say, with the usual notation,

$$y^2 = px.$$

(2) For the *hyperbola* and *ellipse*, the square on the ordinate is equal to the rectangle applied to p having as its width the abscissa and *exceeding* (for the hyperbola) or *falling-short* (for the ellipse) by a figure similar and similarly situated to the rectangle contained by the given diameter and p.

That is, in the *hyperbola* $$y^2 = px + \frac{x^2}{d^2} pd,$$

or $$y^2 = px + \frac{p}{d} x^2;$$

and in the *ellipse* $$y^2 = px - \frac{p}{d} x^2.$$

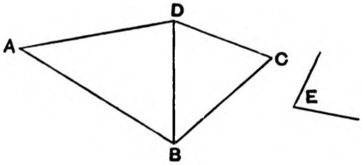
The form of these equations will be seen to be exactly the same as that of the general equations above given, and thus Apollonius' nomenclature followed exactly the traditional theory of *application*, *exceeding*, and *falling-short*.

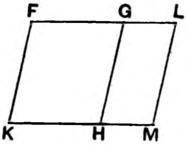
PROPOSITION 45.

To construct, in a given rectilineal angle, a parallelogram equal to a given rectilineal figure.

Let *ABCD* be the given rectilineal figure and *E* the given rectilineal angle;

thus it is required to construct, in the given angle *E*, a parallelogram equal to the rectilineal figure *ABCD*.





Let *DB* be joined, and let the parallelogram *FH* be constructed equal to the triangle *ABD*, in the angle *HKF* which is equal to *E*; [I. 42]

let the parallelogram *GM* equal to the triangle *DBC* be applied to the straight line *GH*, in the angle *GHM* which is equal to *E*. [I. 44]

Then, since the angle *E* is equal to each of the angles *HKF*, *GHM*,

the angle *HKF* is also equal to the angle *GHM*. [*C. N.* 1]

Let the angle *KHG* be added to each;
therefore the angles *FKH*, *KHG* are equal to the angles *KHG*, *GHM*.

But the angles *FKH*, *KHG* are equal to two right angles; [I. 29]
therefore the angles *KHG*, *GHM* are also equal to two right angles.

Thus, with a straight line *GH*, and at the point *H* on it, two straight lines *KH*, *HM* not lying on the same side make the adjacent angles equal to two right angles;

therefore *KH* is in a straight line with *HM*. [I. 14]

And, since the straight line *HG* falls upon the parallels *KM*, *FG*, the alternate angles *MHG*, *HGF* are equal to one another. [I. 29]

Let the angle *HGL* be added to each;
therefore the angles *MHG*, *HGL* are equal to the angles *HGF*, *HGL*. [*C. N.* 2]

But the angles *MHG*, *HGL* are equal to two right angles; [I. 29]
therefore the angles *HGF*, *HGL* are also equal to two right angles. [*C. N.* 1]

Therefore *FG* is in a straight line with *GL*. [I. 14]

And, since *FK* is equal and parallel to *HG*, [I. 34]
and *HG* to *ML* also,
KF is also equal and parallel to *ML*; [*C. N.* 1; I. 30]
and the straight lines *KM*, *FL* join them (at their extremities);
therefore *KM*, *FL* are also equal and parallel. [I. 33]

Therefore *KFLM* is a parallelogram.

And, since the triangle *ABD* is equal to the parallelogram *FH*,
and *DBC* to *GM*,
the whole rectilineal figure *ABCD* is equal to the whole parallelogram *KFLM*.

Therefore the parallelogram *KFLM* has been constructed equal to the given rectilineal figure *ABCD*, in the angle *FKM* which is equal to the given angle *E*. Q. E. F.

2, 3, 6, 45, 48. **rectilineal figure**, in the Greek "rectilineal" simply, without "figure," *εὐθύγραμμον* being here used as a substantive, like the similarly formed *παραλληλόγραμμον*.

Transformation of areas.

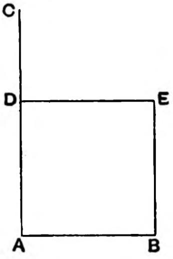
We can now take stock of how far the propositions I. 43—45 bring us in the matter of *transformation of areas*, which constitutes so important a part of what has been fitly called the *geometrical algebra* of the Greeks. We have now learnt how to represent any rectilineal area, which can of course be resolved into triangles, by a single parallelogram having one side equal to any given straight line and one angle equal to any given rectilineal angle. Most important of all such parallelograms is the rectangle, which is one of the simplest forms in which an area can be shown. Since a rectangle corresponds to the product of two magnitudes in algebra, we see that *application* to a given straight line of a rectangle equal to a given area is the geometrical equivalent of algebraical *division* of the product of two quantities by a third. Further than this, it enables us to *add* or *subtract* any rectilineal areas and to represent the sum or difference by *one* rectangle with one side of any given length, the process being the equivalent of obtaining a common factor. But one step still remains, the finding of a *square* equal to a given rectangle, i.e. to a given rectilineal figure; and this step is not taken till II. 14. In general, the transformation of combinations of rectangles and squares into other combinations of rectangles and squares is the subject-matter of Book II., with the exception of the expression of the sum of two squares as a single square which appears earlier in the other Pythagorean theorem I. 47. Thus the transformation of rectilineal areas is made complete *in one direction*, i.e. in the direction of their simplest expression in terms of rectangles and squares, by the end of Book II. The reverse process of transforming the simpler rectangular area into an equal area which shall be similar to any rectilineal figure requires, of course, the use of proportions, and therefore does not appear till VI. 25.

Proclus adds to his note on this proposition the remark (pp. 422, 24—423, 6): "I conceive that it was in consequence of this problem that the ancient geometers were led to investigate the squaring of the circle as well. For, if a parallelogram can be found equal to any rectilineal figure, it is worth inquiring whether it be not also possible to prove rectilineal figures equal to circular. And Archimedes actually proved that any circle is equal to the right-angled triangle which has one of its sides about the right angle [the perpendicular] equal to the radius of the circle and its base equal to the perimeter of the circle. But of this elsewhere."

Proposition 46.

On a given straight line to describe a square.

Let AB be the given straight line; thus it is required to describe a square on the straight line AB.



Let AC be drawn at right angles to the straight line AB from the point A on it [I. 11], and let AD be made equal to AB;

through the point D let DE be drawn parallel to AB,

and through the point B let BE be drawn parallel to AD.

[I. 31]

Therefore *ADEB* is a parallelogram;

therefore *AB* is equal to *DE*, and *AD* to *BE*. [I. 34]

But *AB* is equal to *AD*;

therefore the four straight lines *BA*, *AD*, *DE*, *EB* are equal to one another;

therefore the parallelogram *ADEB* is equilateral.

I say next that it is also right-angled.

For, since the straight line *AD* falls upon the parallels *AB*, *DE*,

the angles *BAD*, *ADE* are equal to two right angles. [I. 29]

But the angle *BAD* is right;

therefore the angle *ADE* is also right.

And in parallelogrammic areas the opposite sides and angles are equal to one another; [I. 34]

therefore each of the opposite angles *ABE*, *BED* is also right.

Therefore *ADEB* is right-angled.

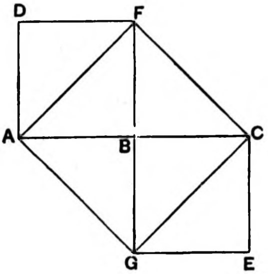
And it was also proved equilateral.

Therefore it is a square; and it is described on the straight line *AB*.

Q. E. F.

1, 3, 30. Proclus (p. 423, 18 sqq.) notes the difference between the word *construct* (συστήσασθαι) applied by Euclid to the construction of a *triangle* (and, he might have added, of an *angle*) and the words *describe on* (ἀναγράφειν ἀπό) used of drawing a square on a given straight line as one side. The *triangle* (or *angle*) is, so to say, pieced together, while the describing of a square on a given straight line is the making of a figure "from" *one* side, and corresponds to the multiplication of the number representing the side by itself.

Proclus (pp. 424—5) proves that, *if squares are described on equal straight lines, the squares are equal*; and, *conversely*, that, *if two squares are equal, the straight lines are equal on which they are described*. The first proposition is immediately obvious if we divide the squares into two triangles by drawing a diagonal in each. The converse is proved as follows.



Place the two equal squares *AF*, *CG* so that *AB*, *BC* are in a straight line. Then, since the angles are right, *FB*, *BG* will also be in a straight line. Join *AF*, *FC*, *CG*, *GA*.

Now, since the squares are equal, the triangles *ABF*, *CBG* are equal.

Add to each the triangle *FBC*; therefore the triangles *AFC*, *GFC* are equal, and hence they must be in the same parallels.

Therefore *AG*, *CF* are parallel.

Also, since each of the alternate angles *AFG*, *FGC* is half a right angle,

AF, *CG* are parallel.

Hence *AFCG* is a parallelogram; and *AF*, *CG* are equal.

Thus the triangles *ABF*, *CBG* have two angles and one side respectively equal;

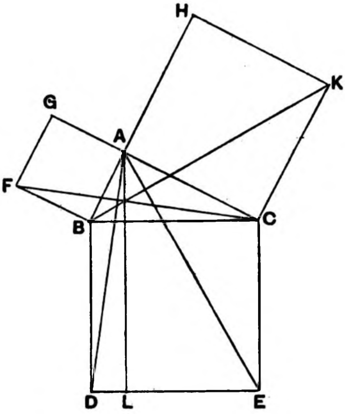
therefore *AB* is equal to *BC*, and *BF* to *BG*.

PROPOSITION 47.

In right-angled triangles the square on the side subtending the right angle is equal to the squares on the sides containing the right angle.

Let *ABC* be a right-angled triangle having the angle *BAC* right;

I say that the square on *BC* is equal to the squares on *BA*, *AC*.



For let there be described on *BC* the square *BDEC*, and on *BA*, *AC* the squares *GB*, *HC*; [I. 46]

through *A* let *AL* be drawn parallel to either *BD* or *CE*, and let *AD*, *FC* be joined.

Then, since each of the angles *BAC*, *BAG* is right, it follows that with a straight line *BA*, and at the point *A* on it, the two straight lines *AC*, *AG* not lying on the same side make the adjacent angles equal to two right angles;

therefore *CA* is in a straight line with *AG*. [I. 14]

For the same reason

BA is also in a straight line with *AH*.

And, since the angle *DBC* is equal to the angle *FBA*: for each is right:

let the angle *ABC* be added to each;

therefore the whole angle *DBA* is equal to the whole angle *FBC*. [*C. N.* 2]

And, since *DB* is equal to *BC*, and *FB* to *BA*,
the two sides *AB*, *BD* are equal to the two sides *FB*, *BC* respectively,
and the angle *ABD* is equal to the angle *FBC*;
therefore the base *AD* is equal to the base *FC*,
and the triangle *ABD* is equal to the triangle *FBC*. [I. 4]

Now the parallelogram *BL* is double of the triangle *ABD*, for they have the same base *BD* and are in the same parallels *BD*, *AL*. [I. 41]

And the square *GB* is double of the triangle *FBC*, for they again have the same base *FB* and are in the same parallels *FB*, *GC*. [I. 41]

[But the doubles of equals are equal to one another.]

Therefore the parallelogram *BL* is also equal to the square *GB*.

Similarly, if *AE*, *BK* be joined,
the parallelogram *CL* can also be proved equal to the square *HC*;
therefore the whole square *BDEC* is equal to the two squares *GB*, *HC*. [*C. N.* 2]

And the square *BDEC* is described on *BC*,
and the squares *GB*, *HC* on *BA*, *AC*.

Therefore the square on the side *BC* is equal to the squares on the sides *BA*, *AC*.

Therefore etc. Q. E. D.

1. **the square on,** *τὸ ἀπὸ...τετράγωνον*, the word *ἀναγραφέν* or *ἀναγεγραμμένον* being understood.

subtending the right angle. Here *ὑποτεινούσης*, "subtending," is used with the simple accusative (*τὴν ὀρθὴν γωνίαν*) instead of being followed by *ὑπό* and the accusative, which seems to be the original and more orthodox construction. Cf. I. 18, note.

33. **the two sides AB, BD....** Euclid actually writes "*DB*, *BA*," and therefore the equal sides in the two triangles are not mentioned in corresponding order, though he adheres to the words *ἑκατέρα ἑκατέρα* "respectively." Here *DB* is equal to *BC* and *BA* to *FB*.

44. **[But the doubles of equals are equal to one another.]** Heiberg brackets these words as an interpolation, since it quotes a *Common Notion* which is itself interpolated. Cf. notes on I. 37, p. 332, and on interpolated *Common Notions*, pp. 223—4.

"If we listen," says Proclus (p. 426, 6 sqq.), "to those who wish to recount ancient history, we may find some of them referring this theorem to Pythagoras and saying that he sacrificed an ox in honour of his discovery. But for my part, while I admire those who first observed the truth of this theorem, I marvel more at the writer of the Elements, not only because he made it fast (*κατεδήσατο*) by a most lucid demonstration, but because he compelled assent to the still more general theorem by the irrefragable arguments of science in the sixth Book. For in that Book he proves generally that, in right-angled triangles, the figure on the side subtending the right angle is equal to the similar and similarly situated figures described on the sides about the right angle."

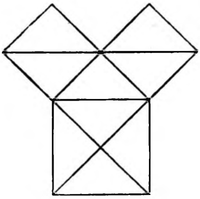
In addition, Plutarch (in the passages quoted above in the note on I. 44), Diogenes Laertius (VIII. 12) and Athenaeus (X. 13) agree in attributing this proposition to Pythagoras. It is easy to point out, as does G. Junge ("Wann haben die Griechen das Irrationale entdeckt?" in *Novae Symbolae Joachimicae*, Halle a. S., 1907, pp. 221—264), that these are late witnesses, and that the Greek literature which we possess belonging to the first five centuries after Pythagoras contains no statement specifying this or any other particular great geometrical discovery as due to him. Yet the distich of Apollodorus the "calculator," whose date (though it cannot be fixed) is at least earlier than that of Plutarch and presumably of Cicero, is quite definite as to the existence of *one* "famous proposition" discovered by Pythagoras, whatever it was. Nor does Cicero, in commenting apparently on the verses (*De nat. deor.* III. c. 36, § 88), seem to dispute the fact of the geometrical discovery, but only the story of the sacrifice. Junge naturally emphasises the apparent uncertainty in the statements of Plutarch and Proclus. But, as I read the passages of Plutarch, I see nothing in them inconsistent with the supposition that Plutarch unhesitatingly accepted as discoveries of Pythagoras *both* the theorem of the square of the hypotenuse and the problem of the application of an area, and the only doubt he felt was as to which of the two discoveries was the more appropriate occasion for the supposed sacrifice. There is also other evidence not without bearing on the question. The theorem is closely connected with the whole of the matter of Eucl. Book II., in which one of the most prominent features is the use of the *gnomon*. Now the gnomon was a well-understood term with the Pythagoreans (cf. the fragment of Philolaus quoted on p. 141 of Boeckh's *Philolaos des Pythagoreers Lehren*, 1819). Aristotle also (*Physics* III. 4, 203 a 10—15) clearly attributes to the Pythagoreans the placing of odd numbers as *gnomons* round successive squares beginning with 1, thereby forming new squares, while in another place (*Categ.* 14, 15 a 30) the word *gnomon* occurs in the same (obviously familiar) sense: "e.g. a square, when a gnomon is placed round it, is increased in size but is not altered in form." The inference must therefore be that practically the whole doctrine of Book II. is Pythagorean. Again Heron (? 3rd cent. A.D.), like Proclus, credits Pythagoras with a general rule for forming right-angled triangles with rational whole numbers for sides. Lastly, the "summary" of Proclus appears to credit Pythagoras with the discovery of the theory, or study, of irrationals (τὴν τῶν ἀλόγων πραγματείαν). But it is now more or less agreed that the reading here should be, not τῶν ἀλόγων, but τῶν ἀναλόγων, or rather τῶν ἀνὰ λόγον ("of proportionals"), and that the author intended to attribute to Pythagoras a theory of *proportion*, i.e. the (arithmetical) theory of proportion applicable only to commensurable magnitudes, as distinct from the theory of Eucl. Book V., which was due to Eudoxus. It is not however disputed that the *Pythagoreans* discovered the irrational (cf. the scholium No. 1 to Book X.). Now everything goes to show that this discovery of the irrational was made with reference to $\sqrt{2}$, the ratio of the diagonal of a square to its side. It is clear that this presupposes the knowledge that I. 47 is true of an isosceles right-angled triangle; and the fact that some triangles of which it had been discovered to be true were *rational* right-angled triangles was doubtless what suggested the inquiry whether the ratio between the lengths of the diagonal and the side of a square could also be expressed in whole numbers. On the whole, therefore, I see no sufficient reason to question the tradition that, *so far as Greek geometry is concerned* (the possible priority of the discovery of the same proposition in India will be considered later), Pythagoras was the first to introduce the theorem of I. 47 and to give a general proof of it.

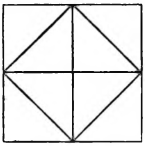
On this assumption, how was Pythagoras led to this discovery? It has been suggested and commonly assumed that the Egyptians were aware that a triangle with its sides in the ratio 3, 4, 5 was right-angled. Cantor inferred this from the fact that this was precisely the triangle with which Pythagoras began, if we may accept the testimony of Vitruvius (IX. 2) that Pythagoras taught how to make a right angle by means of three lengths measured by the numbers 3, 4, 5. If then he took from the Egyptians the triangle 3, 4, 5, he presumably learnt its property from them also. Now the Egyptians must certainly be credited from a period at least as far back as 2000 B.C. with the knowledge that $4^2 + 3^2 = 5^2$. Cantor finds proof of this in a fragment of papyrus belonging to the time of the 12th Dynasty newly discovered at Kahun. In this papyrus we have extractions of square roots: e.g. that of 16 is 4, that of $1\frac{9}{16}$ is $1\frac{1}{4}$, that of $6\frac{1}{4}$ is $2\frac{1}{2}$, and the following equations can be traced:

$$
\begin{aligned}
1^2 + (\tfrac{3}{4})^2 &= (1\tfrac{1}{4})^2\\
8^2 + 6^2 &= 10^2\\
2^2 + (1\tfrac{1}{2})^2 &= (2\tfrac{1}{2})^2\\
16^2 + 12^2 &= 20^2.
\end{aligned}
$$

It will be seen that $4^2 + 3^2 = 5^2$ can be derived from each of these by multiplying, or dividing out, by one and the same factor. We may therefore admit that the Egyptians knew that $3^2 + 4^2 = 5^2$. But there seems to be no evidence that they knew that the triangle (3, 4, 5) is *right-angled*; indeed, according to the latest authority (T. Eric Peet, *The Rhind Mathematical Papyrus*, 1923), nothing in Egyptian mathematics suggests that the Egyptians were acquainted with this or any special cases of the Pythagorean theorem.

How then did Pythagoras discover the general theorem? Observing that 3, 4, 5 was a right-angled triangle, while $3^2 + 4^2 = 5^2$, he was probably led to





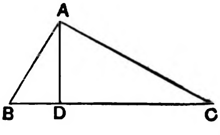
consider whether a similar relation was true of the sides of right-angled triangles other than the particular one. The simplest case (geometrically) to investigate was that of the *isosceles* right-angled triangle; and the truth of the theorem in this particular case would easily appear from the mere construction of a figure. Cantor (I_3, p. 185) and Allman (*Greek Geometry from Thales to Euclid*, p. 29) illustrate by a figure in which the squares are drawn outwards, as in I. 47, and divided by diagonals into equal triangles; but I think that the truth was more likely to be first observed from a figure of the kind suggested by Bürk (*Das Āpastamba-Śulba-Sūtra* in *Zeitschrift der deuts hen morgenländ. Gesellschaft*, LV., 1901, p. 557) to explain how the Indians arrived at the same thing. The two figures are as shown above. When the geometrical consideration of the figure had shown that the isosceles right-angled triangle had the property in question, the investigation of the same fact from the arithmetical point of view would ultimately lead to the other momentous discovery of the irrationality of the length of the diagonal of a square expressed in terms of its side.

The *irrational* will come up for discussion later; and our next question is: Assuming that Pythagoras had observed the geometrical truth of the theorem in the case of the two particular triangles, and doubtless of other rational right-angled triangles, how did he establish it generally?

There is no positive evidence on this point. Two possible lines are however marked out. (1) Tannery says (*La Géométrie grecque*, p. 105) that the geometry of Pythagoras was sufficiently advanced to make it possible for him to prove the theorem by *similar triangles.* He does not say in what particular manner similar triangles would be used, but their use must apparently have involved the use of *proportions*, and, in order that the proof should be conclusive, of the theory of proportions in its complete form applicable to incommensurable as well as commensurable magnitudes. Now Eudoxus was the first to make the theory of proportion independent of the hypothesis of commensurability; and as, before Eudoxus' time, this had not been done, any proof of the general theorem by means of proportions given by Pythagoras must at least have been inconclusive. But this does not constitute any objection to the supposition that the truth of the general theorem may have been discovered in such a manner; on the contrary, the supposition that Pythagoras proved it by means of an imperfect theory of proportions would better than anything else account for the fact that Euclid had to devise an entirely new proof, as Proclus says he did in I. 47. This proof had to be independent of the theory of proportion even in its rigorous form, because the plan of the *Elements* postponed that theory to Books V. and VI., while the Pythagorean theorem was required as early as Book II. On the other hand, if the Pythagorean proof had been based on the doctrine of Books I. and II. only, it would scarcely have been necessary for Euclid to supply a new proof.

The possible proofs by means of proportion would seem to be practically limited to two.

(*a*) One method is to prove, from the similarity of the triangles *ABC*, *DBA*, that the rectangle *CB*, *BD* is equal to the square on *BA*, and, from the similarity of the triangles *ABC*, *DAC*, that the rectangle *BC*, *CD* is equal to the square on *CA*; whence the result follows by addition.

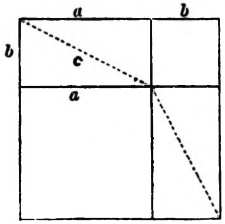


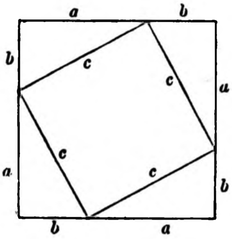
It will be observed that this proof is *in substance* identical with that of Euclid, the only difference being that the equality of the two smaller squares to the respective rectangles is inferred by the method of Book VI. instead of from the relation between the areas of parallelograms and triangles on the same base and between the same parallels established in Book I. It occurred to me whether, if Pythagoras' proof had come, even in substance, so near to Euclid's, Proclus would have emphasised so much as he does the originality of Euclid's, or would have gone so far as to say that he marvelled more at that proof than at the original discovery of the theorem. But on the whole I see no difficulty; for there can be little doubt that the proof by proportion is what suggested to Euclid the method of I. 47, and the transformation of the method of proportions into one based on Book I. only, effected by a construction and proof so extraordinarily ingenious, is a veritable *tour de force* which compels admiration, notwithstanding the ignorant strictures of Schopenhauer, who wanted something as obvious as the second figure in the case of the isosceles right-angled triangle (p. 352), and accordingly (*Sämmtliche Werke*, III. § 39 and I. § 15) calls Euclid's proof "a mouse-trap proof" and "a proof walking on stilts, nay, a mean, underhand, proof" ("Des Eukleides stelzbeiniger, ja, hinterlistiger Beweis").

(*b*) The other possible method is this. As it would be seen that the triangles into which the original triangle is divided by the perpendicular from the right angle on the hypotenuse are similar to one another and to the whole triangle, while in these three triangles the two sides about the right angle in the original triangle, and the hypotenuse of the original triangle, are corresponding sides, and that the sum of the two former similar triangles is identically equal to the similar triangle on the hypotenuse, it might be inferred that the same would also be true of *squares* described on the corresponding three sides respectively, because squares as well as similar triangles are to one another in the duplicate ratio of corresponding sides. But the same thing is equally true of any similar rectilineal figures, so that this proof would practically establish the extended theorem of Eucl. VI. 31, which theorem, however, Proclus appears to regard as being entirely Euclid's discovery.

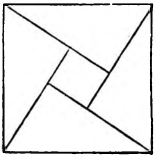
On the whole, the most probable supposition seems to me to be that Pythagoras used the first method (*a*) of proof by means of the theory of proportion as he knew it, i.e. in the defective form which was in use up to the date of Eudoxus.

(2) I have pointed out the difficulty in the way of the supposition that Pythagoras' proof depended upon the principles of Eucl. Books I. and II. only.



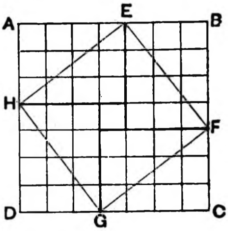


Were it not for this difficulty, the conjecture of Bretschneider (p. 82), followed by Hankel (p. 98), would be the most tempting hypothesis. According to this suggestion, we are to suppose a figure like that of Eucl. II. 4 in which *a*, *b* are the sides of the two inner squares respectively, and $a+b$ is the side of the complete square. Then, if the two complements, which are equal, are divided by their two diagonals into four equal triangles of sides *a*, *b*, *c*, we can place these triangles round another square of the same size as the whole square, in the manner shown in the second figure, so that the sides *a*, *b* of *successive* triangles make up one of the sides of the square and are arranged in cyclic order. It readily follows that the remainder of the square when the four triangles are deducted is, in the one case, a square whose side is *c*, and in the other the sum of two squares whose sides are *a*, *b* respectively. Therefore the square on *c* is equal to the sum of the squares on *a*, *b*. All that can be said against this conjectural proof is that it has no specifically Greek colouring but rather recalls the Indian method. Thus Bhāskara (born 1114 A.D.; see Cantor, I$_3$, p. 656) simply draws four right-angled triangles equal to the original one inwards, one on each side of the square on the hypotenuse, and says "see!", without even adding that inspection shows that



$$c^2 = 4\frac{ab}{2} + (a-b)^2 = a^2 + b^2.$$

Though, for the reason given, there is difficulty in supposing that Pythagoras used a general proof of this kind, which applies of course to right-angled triangles with sides incommensurable as well as commensurable, there is no objection, I think, to supposing that the truth of the proposition in the case of the first *rational* right-angled triangles discovered, e.g. 3, 4, 5, was proved by a method of this sort. Where the sides are commensurable in this way, the squares can be divided up into small (unit) squares, which would much facilitate the comparison between them. That this subdivision was in fact resorted to in adding and subtracting squares is made probable by Aristotle's allusion to odd numbers as *gnomons* placed round unity to form successive squares in *Physics* III. 4; this must mean that the squares were represented by dots arranged in the form of a square and a gnomon formed of dots put round, or that (if the given square was drawn in the usual way) the gnomon was divided up into unit squares. Zeuthen has shown ("*Théorème de Pythagore*," *Origine de la Géométrie scientifique* in *Comptes rendus du II*^{me} *Congrès international de Philosophie*, Genève, 1904), how easily the proposition could be proved by a method of this kind for the triangle 3, 4, 5. To admit of the two smaller squares being shown side by side, take a square on a line containing 7 units of length (4 + 3), and divide it up into 49 small squares. It would be obvious that the whole square could be exhibited as containing four rectangles of sides 4, 3 cyclically arranged round the figure with one unit square in the middle. (This same figure is given by Cantor, I$_3$, p. 680. to illustrate the method given in the Chinese "Chóu-peï".) It would be seen that



(i) the whole square (7^2) is made up of two squares 3^2 and 4^2, and two rectangles 3, 4;

(ii) the same square is made up of the square *EFGH* and the halves of four of the same rectangles 3, 4, whence the square *EFGH*, being equal to the sum of the squares 3^2 and 4^2, must contain 25 unit squares and its side, or the diagonal of one of the rectangles, must contain 5 units of length.

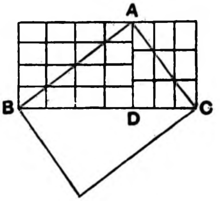
Or the result might equally be seen by observing that

(i) the square *EFGH* on the diagonal of one of the rectangles is made up of the halves of four rectangles and the unit square in the middle, while

(ii) the squares 3^2 and 4^2 placed at adjacent corners of the large square make up two rectangles 3, 4 with the unit square in the middle.

The procedure would be equally easy for any *rational* right-angled triangle, and would be a natural method of trying to *prove* the property when it had once been *empirically* observed that triangles like 3, 4, 5 did in fact contain a right angle.

Zeuthen has, in the same paper, shown in a most ingenious way how the property of the triangle 3, 4, 5 could be verified by a sort of combination of the second possible method by similar triangles, (*b*) on p. 354 above, with subdivision of *rectangles* into similar small *rectangles*. I give the method on account of its interest, although it is no doubt too advanced to have been used by those who first proved the property of the particular triangle.



Let ABC be a triangle right-angled at A, and such that the lengths of the sides AB, AC are 4 and 3 units respectively.

Draw the perpendicular AD, divide up AB, AC into unit lengths, complete the rectangle on BC as base and with AD as altitude, and subdivide this rectangle into small rectangles by drawing parallels to BC, AD through the points of division of AB, AC.

Now, since the diagonals of the small rectangles are all equal, each being of unit length, it follows by similar triangles that the small rectangles are all equal. And the rectangle with AB for diagonal contains 16 of the small rectangles, while the rectangle with diagonal AC contains 9 of them.

But the sum of the triangles ABD, ADC is equal to the triangle ABC.

Hence the rectangle with BC as diagonal contains $9 + 16$ or 25 of the small rectangles;

and therefore $BC = 5$.

Rational right-angled triangles from the arithmetical standpoint.

Pythagoras investigated the *arithmetical* problem of finding rational numbers which could be made the sides of right-angled triangles, or of finding square numbers which are the sum of two squares; and herein we find the beginning of the *indeterminate analysis* which reached so high a stage of development in Diophantus. Fortunately Proclus has preserved Pythagoras' method of solution in the following passage (pp. 428, 7—429, 8). "Certain methods for the discovery of triangles of this kind are handed down, one of which they refer to Plato, and another to Pythagoras. [The latter] starts from odd numbers. For it makes the odd number the smaller of the sides about the right angle; then it takes the square of it, subtracts unity, and makes half the difference the greater of the sides about the right angle; lastly it adds unity to this and so forms the remaining side, the hypotenuse. For example, taking 3, squaring it, and subtracting unity from the 9, the method takes half of the 8, namely 4; then, adding unity to it again, it makes 5, and a right-angled triangle has been found with one side 3, another 4 and another 5. But the method of Plato argues from even numbers. For it takes the given even number and makes it one of the sides about the right angle; then, bisecting this number and squaring the half, it adds unity to the square to form the hypotenuse, and subtracts unity from the square to form the other side about the right angle. For example, taking 4, the method squares half of this, or 2, and so makes 4; then, subtracting unity, it produces 3, and adding unity it produces 5. Thus it has formed the same triangle as that which was obtained by the other method."

The formula of Pythagoras amounts, if m be an odd number, to

$$m^2 + \left(\frac{m^2 - 1}{2}\right)^2 = \left(\frac{m^2 + 1}{2}\right)^2,$$

the sides of the right-angled triangle being m, $\frac{m^2 - 1}{2}$, $\frac{m^2 + 1}{2}$. Cantor (I_3, pp. 185—6), taking up an idea of Röth (*Geschichte der abendländischen Philosophie*, II. 527), gives the following as a possible explanation of the way in which Pythagoras arrived at his formula. If $c^2 = a^2 + b^2$, it follows that

$$a^2 = c^2 - b^2 = (c + b)(c - b).$$

Numbers can be found satisfying the first equation if (1) $c + b$ and $c - b$ are either both even or both odd, and if further (2) $c + b$ and $c - b$ are such numbers as, when multiplied together, produce a square number. The first condition is necessary because, in order that c and b may both be whole numbers, the sum and difference of $c + b$ and $c - b$ must both be even. The second condition is satisfied if $c + b$ and $c - b$ are what were called ***similar numbers*** (ὅμοιοι ἀριθμοί); and that such numbers were most probably known in the time before Plato may be inferred from their appearing in Theon of Smyrna (*Expositio rerum mathematicarum ad legendum Platonem utilium*, ed. Hiller, p. 36, 12), who says that similar plane numbers are, first, all square numbers and, secondly, such oblong numbers as have the sides which contain them proportional. Thus 6 is an oblong number with length 3 and breadth 2; 24 is another with length 6 and breadth 4. Since therefore 6 is to 3 as 4 is to 2, the numbers 6 and 24 are similar.

Now the simplest case of two similar numbers is that of 1 and a^2, and, since 1 is odd, the condition (1) requires that a^2, and therefore a, is also odd. That is, we may take 1 and $(2n + 1)^2$ and equate them respectively to $c - b$ and $c + b$, whence we have

$$b = \frac{(2n + 1)^2 - 1}{2},$$

$$c = \frac{(2n + 1)^2 - 1}{2} + 1,$$

while

$$a = 2n + 1.$$

As Cantor remarks, the form in which c and b appear correspond sufficiently closely to the description in the text of Proclus.

Another obvious possibility would be, instead of equating $c - b$ to unity, to put $c - b = 2$, in which case the similar number $c + b$ must be equated to double of some square, i.e. to a number of the form $2n^2$, or to the half of an even square number, say $\frac{(2n)^2}{2}$. This would give

$$a = 2n,$$

$$b = n^2 - 1,$$

$$c = n^2 + 1,$$

which is Plato's solution, as given by Proclus.

The two solutions supplement each other. It is interesting to observe that the method suggested by Röth and Cantor is very like that of Eucl. X. (Lemma 1 following Prop. 28). We shall come to this later, but it may be mentioned here that the problem is *to find two square numbers such that their* *sum is also a square.* Euclid there uses the property of II. 6 to the effect that, if AB is bisected at C and produced to D,

$$AD.DB + BC^2 = CD^2.$$

We may write this $$uv = c^2 - b^2,$$
where $$u = c + b, \quad v = c - b.$$
In order that uv may be a square, Euclid points out that u and v must be similar numbers, and further that u and v must be either both odd or both even in order that b may be a whole number. We may then put for the similar numbers, say, $\alpha\beta^2$ and $\alpha\gamma^2$, whence (if $\alpha\beta^2$, $\alpha\gamma^2$ are either both odd or both even) we obtain the solution

$$\alpha\beta^2 . \alpha\gamma^2 + \left(\frac{\alpha\beta^2 - \alpha\gamma^2}{2}\right)^2 = \left(\frac{\alpha\beta^2 + \alpha\gamma^2}{2}\right)^2.$$

But I think a serious, and even fatal, objection to the conjecture of Cantor and Röth is the very fact that the method enables both the Pythagorean and the Platonic series of triangles to be deduced with equal ease. If this had been the case with the method used by Pythagoras, it would not, I think, have been left to Plato to discover the second series of such triangles. It seems to me therefore that Pythagoras must have used some method which would produce his rule *only*; and further it would be some less recondite method, suggested by direct *observation* rather than by argument from general principles.

One solution satisfying these conditions is that of Bretschneider (p. 83), who suggests the following simple method. Pythagoras was certainly aware that the successive odd numbers are *gnomons*, or the differences between successive square numbers. It was then a simple matter to write down in three rows (*a*) the natural numbers, (*b*) their squares, (*c*) the successive odd numbers constituting the differences between the successive squares in (*b*), thus:

1 2 3 4 5 6 7 8 9 10 11 12 13 14
1 4 9 16 25 36 49 64 81 100 121 144 169 196
1 3 5 7 9 11 13 15 17 19 21 23 25 27

Pythagoras had then only to pick out the numbers in the third row which are squares, and his rule would be obtained by finding the formula connecting the square in the third line with the two adjacent squares in the second line. But even this would require some little argument; and I think a still better suggestion, because making pure observation play a greater part, is that of P. Treutlein (*Zeitschrift für Mathematik und Physik*, XXVIII., 1883, Hist.-litt. Abtheilung, pp. 209 sqq.).

We have the best evidence (e.g. in Theon of Smyrna) of the practice of representing square numbers and other figured numbers, e.g. oblong, triangular, hexagonal, by dots or signs arranged in the shape of the particular figure. (Cf. Aristotle, *Metaph.* 1092 b 12). Thus, says Treutlein, it would be easily seen that any square number can be turned into the next higher square by putting a single row of dots round two adjacent sides, in the form of a gnomon (see figures on next page).

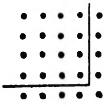
If a is the side of a particular square, the gnomon round it is shown by simple inspection to contain $2a + 1$ dots or units. Now, in order that $2a + 1$ may itself be a square, let us suppose

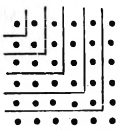
$$2a + 1 = n^2,$$
whence $$a = \tfrac{1}{2}(n^2 - 1),$$
and $$a + 1 = \tfrac{1}{2}(n^2 + 1).$$

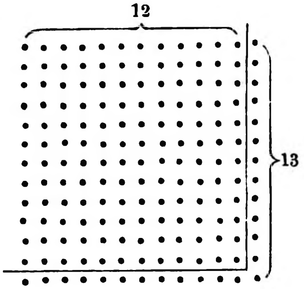
In order that a and $a + 1$ may be integral, n must be odd, and we have at once the Pythagorean formula

$$n^2 + \left(\frac{n^2 - 1}{2}\right)^2 = \left(\frac{n^2 + 1}{2}\right)^2.$$

I think Treutlein's hypothesis is shown to be the correct one by the passage in Aristotle's *Physics* already quoted, where the reference is undoubtedly to the Pythagoreans, and odd numbers are clearly identified with *gnomons* "placed round 1." But the ancient commentaries on the passage make the matter clearer still. Philoponus says: "As a proof...the Pythagoreans refer to what happens with the addition of numbers; for when the odd numbers are successively added to a square number they keep it square and equilateral.... Odd numbers are accordingly called *gnomons* because, when added to what are already squares, they preserve the square form....Alexander has excellently said in explanation that the phrase 'when gnomons are placed round' means *making a figure* with the odd numbers (τὴν κατὰ τοὺς περιττοὺς ἀριθμοὺς σχηματογραφίαν)...for it is the practice with the Pythagoreans to *represent things in figures* (σχηματογραφεῖν)."





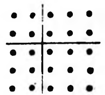


The next question is: assuming this explanation of the Pythagorean formula, what are we to say of the origin of Plato's? It could of course be obtained as a particular case of the general formula of Eucl. x. already referred to; but there are two simple alternative explanations in this case also. (1) Bretschneider observes that, to obtain Plato's formula, we have only to double the sides of the squares in the Pythagorean formula,

for $$(2n)^2 + (n^2 - 1)^2 = (n^2 + 1)^2,$$

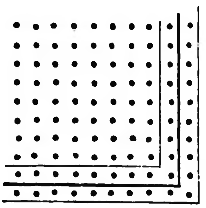
where however n is not necessarily odd.

(2) Treutlein would explain by means of an extension of the gnomon idea. As, he says, the Pythagorean formula was obtained by placing a gnomon consisting of a single row of dots round two adjacent sides of a square, it would be natural to try whether another solution could not be found by placing round the square a gnomon consisting of a *double* row of dots. Such a gnomon would equally turn the square into a larger square; and the question would be whether the double-row gnomon itself could be a square. If the side of the original square was a, it would easily be seen that the number of units in the double-row gnomon would be $4a + 4$, and we have only to put



$$4a + 4 = 4n^2,$$

whence
$$a = n^2 - 1,$$
$$a + 2 = n^2 + 1,$$
and we have the Platonic formula
$$(2n)^2 + (n^2 - 1)^2 = (n^2 + 1)^2.$$
I think this is, in substance, the right explanation, but, in form, not quite correct. The Greeks would not, I think, have treated the *double* row as a gnomon. Their comparison would have been between (1) a certain square *plus* a single-row gnomon and (2) the same square *minus* a single-row gnomon. As the application of Eucl. II. 4 to the case where the segments of the side of the square are a, 1 enables the Pythagorean formula to be obtained as Treutlein obtains it, so I think that Eucl. II. 8 confirms the idea that the Platonic formula was obtained by comparing a square *plus* a gnomon with the same square *minus* a gnomon. For II. 8 proves that



$$4ab + (a - b)^2 = (a + b)^2,$$
whence, substituting 1 for b, we have
$$4a + (a - 1)^2 = (a + 1)^2,$$
and we have only to put $a = n^2$ to obtain Plato's formula.

The "theorem of Pythagoras" in India.

This question has been discussed anew in the last few years as the result of the publication of two important papers by Albert Bürk on *Das Āpastamba-Sulba-Sūtra* in the *Zeitschrift der deutschen morgenländischen Gesellschaft* (LV., 1901, pp. 543—591, and LVI., 1902, pp. 327—391). The first of the two papers contains the introduction and the text, the second the translation with notes. A selection of the most important parts of the material was made and issued by G. Thibaut in the *Journal of the Asiatic Society of Bengal*, XLIV., 1875, Part I. (reprinted also at Calcutta, 1875, as *The Śulvasūtras*, by G. Thibaut). Thibaut in this work gave a most valuable comparison of extracts from the three Śulvasūtras by Bāudhāyana, Āpastamba and Kātyāyana respectively, with a running commentary and an estimate of the date and originality of the geometry of the Indians. Bürk has however done good service by making the Āpastamba-Ś.-S. accessible in its entirety and investigating the whole subject afresh. With the natural enthusiasm of an editor for the work he is editing, he roundly maintains, not only that the Pythagorean theorem was known and proved in all its generality by the Indians long before the date of Pythagoras (about 580—500 B.C.), but that they had also discovered the irrational; and further that, so far from Indian geometry being indebted to the Greek, the much-travelled Pythagoras probably obtained his theory from India (*loc. cit.* LV., p. 575 note). Three important notices and criticisms of Bürk's work have followed, by H. G. Zeuthen ("*Théorème de Pythagore,*" *Origine de la Géométrie scientifique*, 1904, already quoted), by Moritz Cantor (*Über die älteste indische Mathematik* in the *Archiv der Mathematik und Physik*, VIII., 1905, pp. 63—72) and by Heinrich Vogt (*Haben die alten Inder den Pythagoreischen Lehrsatz und das Irrationale gekannt?* in the *Bibliotheca Mathematica*, VII$_3$, 1906, pp. 6—23. See also Cantor's *Geschichte der Mathematik*, I$_3$, pp. 635—645.

The general effect of the criticisms is, I think, to show the necessity for the greatest caution, to say the least, in accepting Bürk's conclusions.

I proceed to give a short summary of the portions of the contents of the Āpastamba-Ś.-S. which are important in the present connexion. It may be premised that the general object of the book is to show how to construct altars of certain shapes, and to vary the dimensions of altars without altering the form. It is a collection of *rules* for carrying out certain constructions. There are no proofs, the nearest approach to a proof being in the rule for obtaining the area of an isosceles trapezium, which is done by drawing a perpendicular from one extremity of the smaller of the two parallel sides to the greater, and then taking away the triangle so cut off and placing it, the other side up, adjacent to the other equal side of the trapezium, thereby transforming the trapezium into a rectangle. It should also be observed that Āpastamba does not speak of *right-angled triangles*, but of two adjacent sides and the diagonal of a *rectangle*. For brevity, I shall use the expression "rational rectangle" to denote a rectangle the two sides and the diagonal of which can be expressed in terms of rational numbers. The references in brackets are to the chapters and numbers of Āpastamba's work.

(1) Constructions of right angles by means of cords of the following relative lengths respectively:

$$\left\{\begin{array}{l} 3,\ 4,\ 5 \quad (\text{I. } 3,\ \text{V. } 3) \\ 12,\ 16,\ 20 \quad (\text{V. } 3) \\ 15,\ 20,\ 25 \quad (\text{V. } 3) \end{array}\right.$$

$$\left\{\begin{array}{l} 5,\ 12,\ 13 \quad (\text{V. } 4) \\ 15,\ 36,\ 39 \quad (\text{I. } 2,\ \text{V. } 2,\ 4) \end{array}\right.$$

8, 15, 17 (V. 5)
12, 35, 37 (V. 5)

(2) A general enunciation of the Pythagorean theorem thus: "The diagonal of a rectangle produces [i.e. the square on the diagonal is equal to] the sum of what the longer and shorter sides separately produce [i.e. the squares on the two sides]." (I. 4)

(3) The application of the Pythagorean theorem to a *square* instead of a rectangle [i.e. to an *isosceles* right-angled triangle]: "The diagonal of a square produces an area double [of the original square]." (I. 5)

(4) An approximation to the value of $\sqrt{2}$; the diagonal of a square is $\left(1 + \frac{1}{3} + \frac{1}{3 \cdot 4} - \frac{1}{3 \cdot 4 \cdot 34}\right)$ times the side. (I. 6)

(5) Application of this approximate value to the construction of a square with side of any length. (II. 1)

(6) The construction of $a\sqrt{3}$, by means of the Pythagorean theorem, as the diagonal of a rectangle with sides a and $a\sqrt{2}$. (II. 2)

(7) Remarks equivalent to the following:

(*a*) $a\sqrt{\frac{1}{3}}$ is the side of $\frac{1}{9}(a\sqrt{3})^2$, or $a\sqrt{\frac{1}{3}} = \frac{1}{3}a\sqrt{3}$. (II. 3)

(*b*) A square on length of 1 unit gives 1 unit square (III. 4)
" " 2 units gives 4 unit squares (III. 6)
" " 3 " 9 " (III. 6)
" " $1\frac{1}{2}$ " $2\frac{1}{4}$ " (III. 8)

A square on length of $2\frac{1}{2}$ units gives $6\frac{1}{4}$ unit squares (III. 8)
" " $\frac{1}{2}$ unit gives $\frac{1}{4}$ unit square (III. 10)
" " $\frac{1}{3}$ " $\frac{1}{9}$ " (III. 10)

(*c*) Generally, the square on any length contains as many rows (of small, unit, squares) as the length contains units. (III. 7)

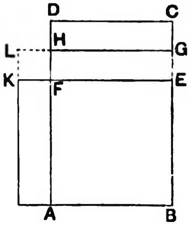
(8) Constructions, by means of the Pythagorean theorem, of

(*a*) the *sum* of two squares as one square, (II. 4)

(*b*) the *difference* of two squares as one square. (II. 5)

(9) A transformation of a rectangle into a square. (II. 7)

[This is not directly done as by Euclid in II. 14, but the rectangle is first transformed into a gnomon, i.e. into the difference between two squares, which difference is then transformed into one square by the preceding rule. If *ABCD* be the given rectangle of which *BC* is the longer side, cut off the square *ABEF*, bisect the rectangle *DE* left over by *HG* parallel to *FE*, move the upper half *DG* and place it on *AF* as base in the position *AK*. Then the rectangle *ABCD* is equal to the gnomon which is the difference between the square *LB* and the square *LF*. In other words, Āpastamba transforms the rectangle *ab* into the difference between the squares $\left(\frac{a+b}{2}\right)^2$ and $\left(\frac{a-b}{2}\right)^2$.]



(10) An attempt at a transformation of a square (a^2) into a rectangle which shall have one side of given length (b). (III. 1)

[This shows no sign of such a procedure as that of Eucl. I. 44, and indeed does no more than say that we must subtract ab from a^2 and then adapt the remainder $a^2 - ab$ so that it may "fit on" to the rectangle ab. The problem is therefore only reduced to another of the same kind, and presumably it was only solved *arithmetically* in the case where a, b are given numerically. The Indian was therefore far from the general, geometrical, solution.]

(11) Increase of a given square into a larger square. (III. 9)

[This amounts to saying that you must add two rectangles (a, b) and another square (b^2) in order to transform a square a^2 into a square $(a+b)^2$. The formula is therefore that of Eucl. II. 4, $a^2 + 2ab + b^2 = (a+b)^2$.]

The first important question in relation to the above is that of date. Bürk assigns to the *Āpastamba-Śulba-Sūtra* a date at least as early as the 5th or 4th century B.C. He observes however (what is likely enough) that the matter of it must have been much older than the book itself. Further, as regards one of the constructions for right angles, that by means of cords of lengths 15, 36, 39, he shows that it was known at the time of the *Tāittirīya-Saṃhitā* and the *Śatapatha-Brāhmaṇa*, still older works belonging to the 8th century B.C. at latest. It may be that (as Bürk maintains) the discovery that triangles with sides (a, b, c) in rational numbers such that $a^2 + b^2 = c^2$ are right-angled was nowhere made so early as in India. We find however in two ancient Chinese treatises (1) a statement that the diagonal of the rectangle (3, 4) is 5 and (2) a rule for finding the hypotenuse of a "right triangle" from the sides, while tradition connects both works with the name of Chou Kung who died 1105 B.C. (D. E. Smith, *History of Mathematics*, I. pp. 30—33, II. p. 288).

As regards the various "rational rectangles" used by Apastamba, it is to be observed that two of the seven, viz. 8, 15, 17 and 12, 35, 37, do not belong to the Pythagorean series, the others consist of two which belong to it, viz. 3, 4, 5 and 5, 12, 13, and multiples of these. It is true, as remarked by Zeuthen (*op. cit.* p. 842), that the rules of II. 7 and III. 9, numbered (9) and (11) above respectively, would furnish the means of finding any number of "rational rectangles." But it would not appear that the Indians had been able to formulate any general rule; otherwise their list of such rectangles would hardly have been so meagre. Āpastamba mentions seven only, really reducible to four (though one other, 7, 24, 25, appears in the Bāudhāyana-Ś.-S., supposed to be older than Āpastamba). These are all that Āpastamba knew of, for he adds (v. 6): "So many *recognisable* (erkennbare) constructions are there," implying that he knew of no other "rational rectangles" that could be employed. But the words also imply that the theorem of the square on the diagonal is also true of other rectangles not of the "recognisable" kind, i.e. rectangles in which the sides and the diagonal are not in the ratio of integers; this is indeed implied by the constructions for $\sqrt{2}$, $\sqrt{3}$ etc. up to $\sqrt{6}$ (cf. II. 2, VIII. 5). This is all that can be said. The theorem is, it is true, enunciated as a general proposition, but there is no sign of anything like a general proof; there is nothing to show that the assumption of its universal truth was founded on anything better than an imperfect induction from a certain number of cases, discovered empirically, of triangles with sides in the ratio of whole numbers in which the property (1) that the square on the longest side is equal to the sum of the squares on the other two sides was found to be always accompanied by the property (2) that the latter two sides include a right angle.

It remains to consider Bürk's claim that the Indians had discovered the *irrational*. This is based upon the approximate value of $\sqrt{2}$ given by Āpastamba in his rule I. 6 numbered (4) above. There is nothing to show how this was arrived at, but Thibaut's suggestion certainly seems the best and most natural. The Indians may have observed that $17^2 = 289$ is nearly double of $12^2 = 144$. If so, the next question which would naturally occur to them would be, by how much the side 17 must be diminished in order that the square on it may be 288 *exactly*. If, in accordance with the Indian fashion, a gnomon with unit area were to be subtracted from a square with 17 as side, this would approximately be secured by giving the gnomon the breadth $\frac{1}{34}$, for $2 \times 17 \times \frac{1}{34} = 1$. The side of the smaller square thus arrived at would be $17 - \frac{1}{34} = 12 + 4 + 1 - \frac{1}{34}$, whence, dividing out by 12, we have

$$\sqrt{2} \doteq 1 + \frac{1}{3} + \frac{1}{3 \cdot 4} - \frac{1}{3 \cdot 4 \cdot 34}, \text{ approximately.}$$

But it is a far cry from this calculation of an approximate value to the discovery of the *irrational*. First, we ask, is there any sign that this value was known to be inexact? It comes directly after the statement (I. 6) that the square on the diagonal of a square is double of that square, and the rule is quite boldly stated without any qualification: "lengthen the unit by one-third and the latter by one-quarter of itself less one-thirty-fourth of this part." Further, the approximate value is actually used for the purpose of constructing a square when the side is given (II. 1). So familiar was the formula that it was apparently made the basis of a sub-division of measures of length.

Thibaut observes (*Journal of the Asiatic Society of Bengal*, XLIX., p. 241) that, according to Bāudhāyana, the unit of length was divided into 12 *fingerbreadths*, and that one of two divisions of the *fingerbreadth* was into 34 *sesame-corns*, and he adds that he has no doubt that this division, which he has not elsewhere met, owes its origin to the formula for $\sqrt{2}$. The result of using this subdivision would be that, in a square with side equal to 12 *fingerbreadths*, the diagonal would be 17 *fingerbreadths* less 1 *sesame-corn*. Is it conceivable that a sub-division of a measure of length would be based on an evaluation known to be inexact? No doubt the first discoverer would be aware that the area of a gnomon with breadth $\frac{1}{34}$ and outer side 17 is not exactly equal to 1 but less than it by the square of $\frac{1}{34}$ or by $\frac{1}{1156}$, and therefore that, in taking that gnomon as the proper area to be subtracted from 17^2, he was leaving out of account the small fraction $\frac{1}{1156}$; as, however, the object of the whole proceeding was purely practical, he would, without hesitation, ignore this as being of no practical importance, and, thereafter, the formula would be handed down and taken as a matter of course without arousing suspicion as to its accuracy. This supposition is confirmed by reference to the sort of rules which the Indians allowed themselves to regard as accurate. Thus Āpastamba himself gives a construction for a circle equal in area to a given square, which is equivalent to taking $\pi = 3\cdot09$, and yet observes that it gives the required circle "*exactly*" (III. 2), while his construction of a square equal to a circle, which he equally calls "exact," makes the side of the square equal to $\frac{13}{15}$ths of the diameter of the circle (III. 3), and is equivalent to taking $\pi = 3\cdot004$. But, even if some who used the approximation for $\sqrt{2}$ were conscious that it was not quite accurate (of which there is no evidence), there is an immeasurable difference between arrival at this consciousness and the discovery of the irrational. As Vogt says, three stages had to be passed through before the irrationality of the diagonal of a square was discovered in any real sense. (1) All values found by direct measurement or calculations based thereon have to be recognised as being inaccurate. Next (2) must supervene the conviction that it is *impossible* to arrive at an accurate arithmetical expression of the value. And lastly (3) the impossibility must be proved. Now there is no real evidence that the Indians, at the date in question, had even reached the first stage, still less the second or third.

The net results then of Bürk's papers and of the criticisms to which they have given rise appear to be these. (1) It must be admitted that Indian geometry had reached the stage at which we find it in Āpastamba quite independently of Greek influence. But (2) the old Indian geometry was purely empirical and practical, far removed from abstractions such as the irrational. The Indians had indeed, by trial in particular cases, persuaded themselves of the truth of the Pythagorean theorem and enunciated it in all its generality; but they had not established it by scientific proof.

Alternative proofs.

I. The well-known proof of I. 47 obtained by putting two squares side by side, with their bases continuous, and cutting off right-angled triangles which can then be put on again in different positions, is attributed by an-Nairīzī to Thābit b. Qurra (826—901 A.D.).

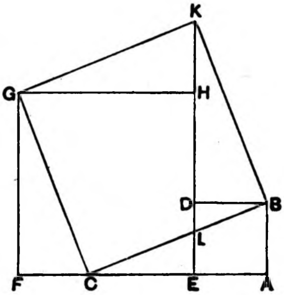
His actual construction proceeds thus.

Let ABC be the given triangle right-angled at A.

Construct on AB the square AD;

produce AC to F so that EF may be equal to AC.

Construct on *EF* the square *EG*, and produce *DH* to *K* so that *DK* may be equal to *AC*.



It is then proved that, in the triangles *BAC*, *CFG*, *KHG*, *BDK*,

the sides *BA*, *CF*, *KH*, *BD* are all equal, and

the sides *AC*, *FG*, *HG*, *DK* are all equal.

The angles included by the equal sides are all right angles; hence the four triangles are equal in all respects. [I. 4]

Hence *BC*, *CG*, *GK*, *KB* are all equal.

Further the angles *DBK*, *ABC* are equal; hence, if we add to each the angle *DBC*, the angle *KBC* is equal to the angle *ABD* and is therefore a right angle.

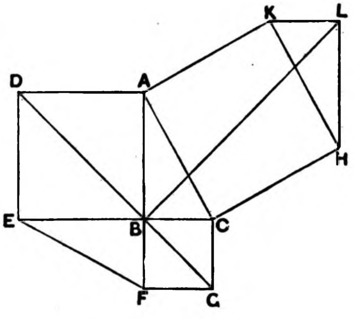
In the same way the angle *CGK* is right; therefore *BCGK* is a square, i.e. the square on *BC*.

Now the sum of the quadrilateral *GCLH* and the triangle *LDB* together with two of the equal triangles make the squares on *AB*, *AC*, and together with the other two make the square on *BC*.

Therefore etc.

II. Another proof is easily arrived at by taking the particular case of Pappus' more general proposition given below in which the given triangle is right-angled and the parallelograms on the sides containing the right angles are squares. If the figure is drawn, it will be seen that, with no more than one additional line inserted, it contains Thābit's figure, so that Thābit's proof may have been practically derived from that of Pappus.

III. The most interesting of the remaining proofs seems to be that shown in the accompanying figure. It is given by J. W. Müller, *Systematische Zusammenstellung der wichtigsten bisher bekannten Beweise des Pythag. Lehrsatzes* (Nürnberg, 1819), and in the second edition (Mainz, 1821) of Ign. Hoffmann, *Der Pythag. Lehrsatz mit* 32 *theils bekannten theils neuen Beweisen* [3 more in second edition]. It appears to come from one of the scientific papers of Lionardo da Vinci (1452—1519).



The triangle *HKL* is constructed on the base *KH* with the side *KL* equal to *BC* and the side *LH* equal to *AB*.

Then the triangle *HLK* is equal in all respects to the triangle *ABC*, and to the triangle *EBF*.

Now *DB*, *BG*, which bisect the angles *ABE*, *CBF* respectively, are in a straight line. Join *BL*.

It is easily proved that the four quadrilaterals *ADGC*, *EDGF*, *ABLK*, *HLBC* are all equal.

Hence the hexagons *ADEFGC*, *ABCHLK* are equal.

Subtracting from the former the two triangles *ABC*, *EBF*, and from the latter the two equal triangles *ABC*, *HLK*, we prove that

the square *CK* is equal to the sum of the squares *AE*, *CF*.

Pappus' extension of I. 47.

In this elegant extension the triangle may be *any* triangle (not necessarily right-angled), and *any* parallelograms take the place of squares on two of the sides.

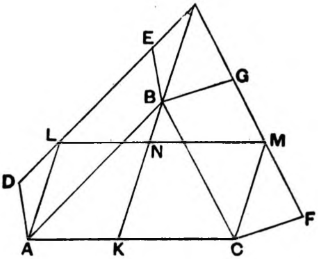
Pappus (IV. p. 177) enunciates the theorem as follows:

If ABC *be a triangle, and any parallelograms whatever* ABED, BCFG *be described on* AB, BC, *and if* DE, FG *be produced to* H, *and* HB *be joined, the parallelograms* ABED, BCFG *are equal to the parallelogram contained by* AC, HB *in an angle which is equal to the sum of the angles* BAC, DHB.

Produce *HB* to *K*; through *A*, *C* draw *AL*, *CM* parallel to *HK*, and join *LM*.

Then, since *ALHB* is a parallelogram, *AL*, *HB* are equal and parallel. Similarly *MC*, *HB* are equal and parallel.

Therefore *AL*, *MC* are equal and parallel;



whence *LM*, *AC* are also equal and parallel,

and *ALMC* is a parallelogram.

Further, the angle *LAC* of this parallelogram is equal to the sum of the angles *BAC*, *DHB*, since the angle *DHB* is equal to the angle *LAB*.

Now, since the parallelogram *DABE* is equal to the parallelogram *LABH* (for they are on the same base *AB* and in the same parallels *AB*, *DH*), and likewise *LABH* is equal to *LAKN* (for they are on the same base *LA* and in the same parallels *LA*, *HK*),

the parallelogram *DABE* is equal to the parallelogram *LAKN*.

For the same reason,

the parallelogram *BGFC* is equal to the parallelogram *NKCM*.

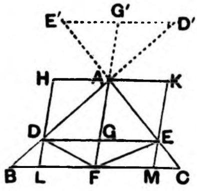
Therefore the sum of the parallelograms *DABE*, *BGFC* is equal to the parallelogram *LACM*, that is, to the parallelogram which is contained by *AC*, *HB* in an angle *LAC* which is equal to the sum of the angles *BAC*, *BHD*.

"And this is far more general than what is proved in the Elements about squares in the case of right-angled (triangles)."

Heron's proof that AL, BK, CF in Euclid's figure meet in a point.

The final words of Proclus' note on I. 47 (p. 429, 9—15) are historically interesting. He says: "The demonstration by the writer of the Elements being clear, I consider that it is unnecessary to add anything further, and that we may be satisfied with what has been written, since in fact those who have added anything more, like Pappus and Heron, were obliged to draw upon what is proved in the sixth Book, for no really useful object." These words cannot of course refer to the extension of I. 47 given by Pappus; but the key to them, so far as Heron is concerned, is to be found in the commentary of an-Nairīzī (pp. 175—185, ed. Besthorn-Heiberg; pp. 78—84, ed. Curtze) on I. 47, wherein he gives Heron's proof that the lines *AL*, *FC*, *BK* in Euclid's figure meet in a point. Heron proved this by means of three lemmas which would most naturally be proved from the principle of similitude as laid down in Book VI., but which Heron, as a *tour de force*, proved on the principles of Book I. only. The *first* lemma is to the following effect.

If, in a triangle ABC, DE *be drawn parallel to the base* BC, *and if* AF *be drawn from the vertex* A *to the middle point* F *of* BC, *then* AF *will also bisect* DE.



This is proved by drawing *HK* through *A* parallel to *DE* or *BC*, and *HDL*, *KEM* through *D*, *E* respectively parallel to *AGF*, and lastly joining *DF*, *EF*.

Then the triangles *ABF*, *AFC* are equal (being on equal bases), and the triangles *DBF*, *EFC* are also equal (being on equal bases and between the same parallels).

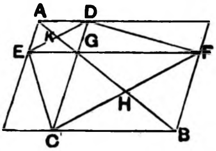
Therefore, by subtraction, the triangles *ADF*, *AEF* are equal, and hence the parallelograms *AL*, *AM* are equal.

These parallelograms are between the same parallels *LM*, *HK*; therefore *LF*, *FM* are equal, whence *DG*, *GE* are also equal.

The *second* lemma is an extension of this to the case where *DE* meets *BA*, *CA* produced beyond *A*.

The *third* lemma proves the converse of Euclid I. 43, that, *If a parallelogram* AB *is cut into four others* ADGE, DF, FGCB, CE, *so that* DF, CE *are equal, the common vertex* G *will be on the diagonal* AB.

Heron produces *AG* till it meets *CF* in *H*. Then, if we join *HB*, we have to prove that *AHB* is one straight line. The proof is as follows. Since the areas *DF*, *EC* are equal, the triangles *DGF*, *ECG* are equal.



If we add to each the triangle *GCF*,

the triangles *ECF*, *DCF* are equal;

therefore *ED*, *CF* are parallel.

Now it follows from I. 34, 29 and 26 that the triangles *AKE*, *GKD* are equal in all respects;

therefore *EK* is equal to *KD*.

Hence, by the second lemma,

CH is equal to *HF*.

Therefore, in the triangles *FHB*, *CHG*,

the two sides *BF*, *FH* are equal to the two sides *GC*, *CH*,

and the angle *BFH* is equal to the angle *GCH*;

hence the triangles are equal in all respects,

and the angle *BHF* is equal to the angle *GHC*.

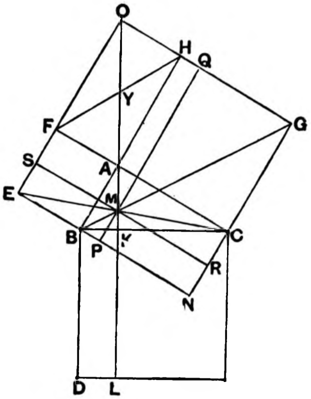
Adding to each the angle *GHF*, we find that the angles *BHF*, *FHG* are equal to the angles *CHG*, *GHF*,

and therefore to two right angles.

Therefore *AHB* is a straight line.

Heron now proceeds to prove the proposition that, in the accompanying figure, if AKL perpendicular to BC meet EC in M, and if BM, MG be joined,

BM, MG are in one straight line.



Parallelograms are completed as shown in the figure, and the diagonals OA, FH of the parallelogram FH are drawn.

Then the triangles FAH, BAC are clearly equal in all respects;

therefore the angle HFA is equal to the angle ABC, and therefore to the angle CAK (since AK is perpendicular to BC).

But, the diagonals of the rectangle FH cutting one another in Y,

FY is equal to YA,

and the angle HFA is equal to the angle OAF.

Therefore the angles OAF, CAK are equal, and accordingly

OA, AK are in a straight line.

Hence OM is the diagonal of SQ;

therefore AS is equal to AQ,

and, if we add AM to each,

FM is equal to MH.

But, since EC is the diagonal of the parallelogram FN,

FM is equal to MN.

Therefore MH is equal to MN;

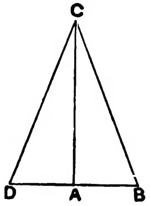
and, by the third lemma, BM, MG are in a straight line.

PROPOSITION 48.

If in a triangle the square on one of the sides be equal to the squares on the remaining two sides of the triangle, the angle contained by the remaining two sides of the triangle is right.

For in the triangle ABC let the square on one side BC be equal to the squares on the sides BA, AC;

I say that the angle BAC is right.



For let AD be drawn from the point A at right angles to the straight line AC, let AD be made equal to BA, and let DC be joined.

Since DA is equal to AB,

the square on DA is also equal to the square on AB.

Let the square on AC be added to each;

therefore the squares on *DA*, *AC* are equal to the squares on *BA*, *AC*.

But the square on *DC* is equal to the squares on *DA*, *AC*, for the angle *DAC* is right; [I. 47]

and the square on *BC* is equal to the squares on *BA*, *AC*, for this is the hypothesis;

therefore the square on *DC* is equal to the square on *BC*,

so that the side *DC* is also equal to *BC*.

And, since *DA* is equal to *AB*,

and *AC* is common,

the two sides *DA*, *AC* are equal to the two sides *BA*, *AC*;

and the base *DC* is equal to the base *BC*;

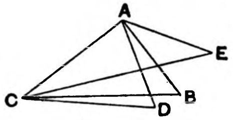
therefore the angle *DAC* is equal to the angle *BAC*. [I. 8]

But the angle *DAC* is right;

therefore the angle *BAC* is also right.

Therefore etc. Q. E. D.

Proclus' note (p. 430) on this proposition, though it does not mention Heron's name, gives an alternative proof, which is the same as that definitely attributed by an-Nairīzī to Heron, the only difference being that Proclus demonstrates two cases in full, while Heron dismisses the second with a "similarly." The alternative proof is another instance of the use of I. 7 as a means of answering objections. If, says Proclus, it be not admitted that the perpendicular *AD* may be drawn on the opposite side of *AC* from *B*, we may draw it on the same side as *AB*, in which case it is impossible that it should not coincide with *AB*. Proclus takes two cases, first supposing that the perpendicular falls, as *AD*, within the angle *CAB*, and secondly that it falls, as *AE*, outside that angle. In either case the absurdity results that, on the same straight line *AC* and on the same side of it, *AD*, *DC* must be respectively equal to *AB*, *BC*, which contradicts I. 7.



Much to the same effect is the note of De Morgan that there is here "an appearance of avoiding indirect demonstration by drawing the triangles on different sides of the base and appealing to I. 8, because drawing them on the same side would make the appeal to I. 7 (on which, however, I. 8 is founded)."

BOOK II.

DEFINITIONS.

1. Any rectangular parallelogram is said to be **contained** by the two straight lines containing the right angle.

2. And in any parallelogrammic area let any one whatever of the parallelograms about its diameter with the two complements be called a **gnomon**.

DEFINITION 1.

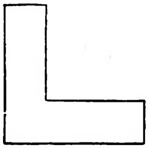
Πᾶν παραλληλόγραμμον ὀρθογώνιον περιέχεσθαι λέγεται ὑπὸ δύο τῶν την ὀρθὴν γωνίαν περιεχουσῶν εὐθειῶν.

As the full expression in Greek for "the angle *BAC*" is "the angle contained by the (straight lines) *BA*, *AC*," ἡ ὑπὸ τῶν ΒΑ, ΑΓ περιεχομένη γωνία, so the full expression for "the rectangle contained by *BA*, *AC*" is τὸ ὑπὸ τῶν ΒΑ, ΑΓ περιεχόμενον ὀρθογώνιον. In this case too ΒΑ, ΑΓ is commonly abbreviated by the Greek geometers into ΒΑΓ. Thus in Archimedes and Apollonius τὸ ὑπὸ ΒΑΓ or τὸ ὑπὸ τῶν ΒΑΓ means *the rectangle BA, AC*, just as ἡ ὑπὸ ΒΑΓ means *the angle BAC*; the gender of the article shows which is meant in each case. In the early Books Euclid uses the full expression τὸ ὑπὸ τῶν ΒΑ, ΑΓ; but the shorter form τὸ ὑπὸ τῶν ΒΑΓ is found from Book x. onwards. Cf. XII. 11, where τὰ (τμήματα) ἐπὶ τῶν ΘΟΕ, ΕΠΖ, ΖΡΗ, ΗΣΘ means the segments on the eight straight lines ΘΟ, ΟΕ, ΕΠ, ΠΖ, ΖΡ, ΡΗ, ΗΣ, ΣΘ.

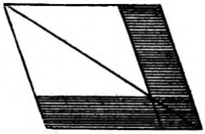
DEFINITION 2.

Παντὸς δὲ παραλληλογράμμου χωρίου τῶν περὶ τὴν διάμετρον αὐτοῦ παραλληλογράμμων ἓν ὁποιονοῦν σὺν τοῖς δυσὶ παραπληρώμασι γνώμων καλείσθω.

Meaning literally a thing enabling something to be *known*, *observed* or *verified*, a *teller* or *marker*, as we might say, the word *gnomon* (γνώμων) was first used in the sense (1) in which it appears in a passage of Herodotus (II. 109) stating that "the Greeks learnt the πόλος, the *gnomon* and the twelve parts of the day from the Babylonians." According to Suidas, it was Anaximander (611—545 B.C.) who introduced the *gnomon* into Greece. Whatever may be the details of the construction of the two instruments called the πόλος and the *gnomon*, so much is certain, that the gnomon had to do with the measurement of time by shadows thrown by the sun, and that the word signified the placing of a staff perpendicular to the horizon. This is borne out by the statement of Proclus that Oenopides of Chios, who first investigated the problem (Eucl. I. 12) of drawing a perpendicular from an external point to a given straight line, called the perpendicular a straight line drawn "*gnomon-wise*" (κατὰ γνώμονα). Then (2) we find the term used of a mechanical instrument for drawing right angles, as shown in the figure annexed. This seems to be the meaning in Theognis 805, where it is said that the envoy sent to consult the oracle at Delphi should be "straighter (ἰθύτερος) than the τόρνος, the στάθμη and the *gnomon*," and all three words evidently denote appliances, the τόρνος being an instrument for drawing a circle (probably a string stretched between a fixed and a moving point), and the στάθμη a plumb-line. Next (3) it was natural that the *gnomon*, owing to its shape, should become the figure which remained of a square when a smaller square was cut out of one corner (or the figure, as Aristotle says, which when *added* to a square increases its size but does not alter its form). We have seen (note on I. 47, p. 351) that the Pythagoreans used the term in this sense, and further applied it, by analogy, to the series of odd numbers as having the same property in relation to square *numbers*. The earliest evidence for this is the fragment of Philolaus (*c.* 460 B.C.) already mentioned (see Boeckh, *Philolaos des Pythagoreers Lehren*, p. 141) where he says that "number makes all things knowable and mutually agreeing (ποτάγορα ἀλλάλοις) in the way characteristic of the gnomon" (κατὰ γνώμονος φύσιν). As Boeckh says (p. 144), it would appear from the fragment that the connexion between the gnomon and the square to which it is added was regarded as symbolical of union and agreement, and that Philolaus used the idea to explain the knowledge of things, making the *knowing* embrace and grasp the *known* as the gnomon does the square. Cf. Scholium II. No. 11 (Euclid, ed. Heiberg, Vol. v. p. 225), which says "It is to be noted that the gnomon was discovered by geometers with a view to brevity, while the name came from its incidental property, namely that from it the whole is known, whether of the whole area or of the remainder, when it is either placed round or taken away. In sundials too its sole function is to make the actual time of day known."



The geometrical meaning of the word is extended in the definition of *gnomon* given by Euclid, where (4) the gnomon has the same relation to *any parallelogram* as it before had to a *square*. From the fact that Euclid says "*let*" the figure described "*be called* a gnomon" we may infer that he was using the word in the wider sense for the first time. Later still (5) we find Heron of Alexandria defining a *gnomon in general* as any figure which, when added to any figure whatever, makes the whole figure similar to that to which it is added. In this definition of Heron (Def. 58) Hultsch brackets the words which make it apply to any *number* as well; but Theon of Smyrna, who explains that plane, triangular, square, solid and other kinds of numbers are so called after the likeness of the areas which they measure, does make the term in its most general sense apply to numbers. "All the successive numbers which [by being successively added] produce triangles or squares or polygons are called gnomons" (p. 37, 11—13, ed. Hiller). Thus the successive odd numbers added



together make square numbers; the gnomons in the case of triangular numbers are the successive numbers 1, 2, 3, 4...; those for pentagonal numbers are the series 1, 4, 7, 10...(the common difference being 3), and so on. In general, the successive *gnomonic* numbers for any polygonal number, say of n sides, have $n-2$ for their common difference (Theon of Smyrna, p. 34, 13—15).

GEOMETRICAL ALGEBRA.

We have already seen (cf. part of the note on I. 47 and the above note on the *gnomon*) how the Pythagoreans and later Greek mathematicians exhibited different kinds of numbers as forming different geometrical figures. Thus, says Theon of Smyrna (p. 36, 6—11), "plane numbers, triangular, square and solid numbers, and the rest, are not so called independently (κυρίως) but in virtue of their similarity to the areas which they measure; for 4, since it measures a square area, is called square by adaptation from it, and 6 is called oblong for the same reason." A "plane number" is similarly described as a number obtained by multiplying two numbers together, which two numbers are sometimes spoken of as "sides," sometimes as the "length" and "breadth" respectively, of the number which is their product.

The *product* of two numbers was thus represented geometrically by the *rectangle* contained by the straight lines representing the two numbers respectively. It only needed the discovery of incommensurable or irrational straight lines in order to represent geometrically by a rectangle the product of any two quantities whatever, rational or irrational; and it was possible to advance from a geometrical arithmetic to a geometrical *algebra*, which indeed by Euclid's time (and probably long before) had reached such a stage of development that it could solve the same problems as our algebra so far as they do not involve the manipulation of expressions of a degree higher than the second. In order to make the geometrical algebra so generally effective, the theory of proportions was essential. Thus, suppose that x, y, z etc. are quantities which can be represented by straight lines, while α, β, γ etc. are coefficients which can be expressed by ratios between straight lines. We can then by means of Book VI. find a single straight line d such that

$$\alpha x + \beta y + \gamma z + \ldots = d.$$

To solve the simple equation in its general form

$$\alpha x + a = b,$$

where α represents any ratio between straight lines also requires recourse to the sixth Book, though, e.g., if α is $\frac{1}{2}$ or $\frac{1}{3}$ or any submultiple of unity, or if α is 2, 4 or any power of 2, we should not require anything beyond Book I. for solving the equation. Similarly the general form of a quadratic equation requires Book VI. for its geometrical solution, though particular quadratic equations may be so solved by means of Book II. alone.

Besides enabling us to solve geometrically these particular quadratic equations, Book II. gives the geometrical proofs of a number of algebraical formulae. Thus the first ten propositions give the equivalent of the several identities

1. $a(b + c + d + \ldots) = ab + ac + ad + \ldots,$
2. $(a + b)a + (a + b)b = (a + b)^2,$
3. $(a + b)a = ab + a^2,$
4. $(a + b)^2 = a^2 + b^2 + 2ab.$

5. $ab + \left(\frac{a+b}{2} - b\right)^2 = \left(\frac{a+b}{2}\right)^2$,
   or $(\alpha + \beta)(\alpha - \beta) + \beta^2 = \alpha^2$,
6. $(2a + b)b + a^2 = (a + b)^2$,
   or $(\alpha + \beta)(\beta - \alpha) + \alpha^2 = \beta^2$,
7. $(a + b)^2 + a^2 = 2(a + b)a + b^2$,
   or $\alpha^2 + \beta^2 = 2\alpha\beta + (\alpha - \beta)^2$,
8. $4(a + b)a + b^2 = \{(a + b) + a\}^2$,
   or $4\alpha\beta + (\alpha - \beta)^2 = (\alpha + \beta)^2$,
9. $a^2 + b^2 = 2\left\{\left(\frac{a+b}{2}\right)^2 + \left(\frac{a+b}{2} - b\right)^2\right\}$,
   or $(\alpha + \beta)^2 + (\alpha - \beta)^2 = 2(\alpha^2 + \beta^2)$,
10. $(2a + b)^2 + b^2 = 2\{a^2 + (a + b)^2\}$,
    or $(\alpha + \beta)^2 + (\beta - \alpha)^2 = 2(\alpha^2 + \beta^2)$.

The form of these identities may of course be varied according to the different symbols which we may use to denote particular portions of the lines given in Euclid's figures. They are, for the most part, simple identities, but there is no reason to suppose that these were the only applications of the geometrical algebra that Euclid and his predecessors had been able to make. We may infer the very contrary from the fact that Apollonius in his *Conics* frequently states without proof much more complicated propositions of the kind.

It is important however to bear in mind that the whole procedure of Book II. is *geometrical*; rectangles and squares are shown in the figures, and the equality of certain combinations to other combinations is proved by those figures. We gather that this was the classical or standard method of proving such propositions, and that the *algebraical* method of proving them, with no figure except a line with points marked thereon, was a later introduction. Accordingly Eutocius' method of proving certain lemmas assumed by Apollonius (*Conics*, II. 23 and III. 29) probably represents more nearly than Pappus' proof of the same the point of view from which Apollonius regarded them.

It would appear that Heron was the first to adopt the *algebraical* method of demonstrating the propositions of Book II., beginning from the second, without figures, as consequences of the first proposition corresponding to

$$a(b + c + d) = ab + ac + ad.$$

According to an-Nairīzī (ed. Curtze, p. 89), Heron explains that it is not possible to prove II. 1 without drawing a number of lines (i.e. without actually drawing the rectangles), but that the following propositions up to II. 10 inclusive can be proved by merely drawing one line. He distinguishes two varieties of the method, one by *dissolutio*, the other by *compositio*, by which he seems to mean *splitting-up* of rectangles and squares, and *combination* of them into others. But in his proofs he sometimes combines the two varieties.

When he comes to II. 11, he says that it is not possible to do without a figure because the proposition is a problem, which accordingly requires an *operation* and therefore the drawing of a figure.

The algebraical method has been preferred to Euclid's by some English editors; but it should not find favour with those who wish to preserve the essential features of Greek geometry as presented by its greatest exponents, or to appreciate their point of view.

It may not be out of place to add a word with reference to the geometrical equivalent of the algebraical operations. The addition and subtraction of quantities represented in the geometrical algebra by lines is of course effected by producing the line to the required extent or cutting off a portion of it. The equivalent of multiplication is the construction of the rectangle of which the given lines are adjacent sides. The equivalent of the division of one quantity represented by a line by another quantity represented by a line is simply the statement of a *ratio* between lines on the principles of Books v. and vi. The division of a product of two quantities by a third is represented in the geometrical algebra by the finding of a rectangle with one side of a given length and equal to a given rectangle or square. This is the problem of *application of areas* solved in i. 44, 45. The addition and subtraction of products is, in the geometrical algebra, the addition and subtraction of rectangles or squares; the sum or difference can be transformed into a single rectangle by means of the *application of areas* to any line of given length, corresponding to the algebraical process of finding a common measure. Lastly, the extraction of the square root is, in the geometrical algebra, the finding of a square equal to a given rectangle, which is done in ii. 14 with the help of i. 47.

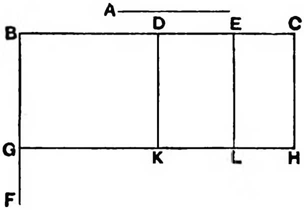
BOOK II. PROPOSITIONS.

PROPOSITION 1.

If there be two straight lines, and one of them be cut into any number of segments whatever, the rectangle contained by the two straight lines is equal to the rectangles contained by the uncut straight line and each of the segments.

Let A, BC be two straight lines, and let BC be cut at random at the points D, E;

I say that the rectangle contained by A, BC is equal to the rectangle contained by A, BD, that contained by A, DE and that contained by A, EC.



For let BF be drawn from B at right angles to BC; [I. 11]
let BG be made equal to A, [I. 3]
through G let GH be drawn parallel to BC, [I. 31]
and through D, E, C let DK, EL, CH be drawn parallel to BG.

Then BH is equal to BK, DL, EH.

Now BH is the rectangle A, BC, for it is contained by GB, BC, and BG is equal to A;

BK is the rectangle A, BD, for it is contained by GB, BD, and BG is equal to A;

and DL is the rectangle A, DE, for DK, that is BG [I. 34], is equal to A.

Similarly also EH is the rectangle A, EC.

Therefore the rectangle A, BC is equal to the rectangle A, BD, the rectangle A, DE and the rectangle A, EC.

Therefore etc.

Q. E. D.

20. **the rectangle A, BC.** From this point onward I shall translate thus in cases where Euclid leaves out the word *contained* (περιεχόμενον). Though the word "rectangle" is also omitted in the Greek (the neuter article being sufficient to show that the rectangle is meant), it cannot be dispensed with in English. De Morgan advises the use of the expression "the rectangle *under* two lines." This does not seem to me a very good expression, and, if used in a translation from the Greek, it might suggest that ὑπό in τὸ ὑπό meant *under*, which it does not.

This proposition, the geometrical equivalent of the algebraical formula

$$a(b+c+d+\dots)=ab+ac+ad+\dots,$$

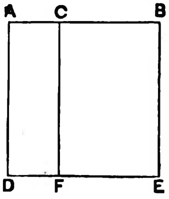
can, of course, easily be extended so as to correspond to the more general algebraical proposition that the product of an expression consisting of any number of terms added together and another expression also consisting of any number of terms added together is equal to the sum of all the products obtained by multiplying each term of one expression by all the terms of the other expression, one after another. The geometrical proof of the more general proposition would be effected by means of a figure showing all the rectangles corresponding to the partial products, in the same way as they are shown in the simpler case of II. 1; the difference would be that a series of parallels to BC would have to be drawn as well as the series of parallels to BF.

PROPOSITION 2.

If a straight line be cut at random, the rectangle contained by the whole and both of the segments is equal to the square on the whole.

For let the straight line AB be cut at random at the point C;

I say that the rectangle contained by AB, BC together with the rectangle contained by BA, AC is equal to the square on AB.



For let the square $ADEB$ be described on AB [I. 46], and let CF be drawn through C parallel to either AD or BE. [I. 31]

Then AE is equal to AF, CE.

Now AE is the square on AB;

AF is the rectangle contained by BA, AC, for it is contained by DA, AC, and AD is equal to AB;

and CE is the rectangle AB, BC, for BE is equal to AB.

Therefore the rectangle BA, AC together with the rectangle AB, BC is equal to the square on AB.

Therefore etc.

Q. E. D.

The *fact* asserted in the enunciation of this proposition has already been used in the proof of I. 47; but there was no occasion in that proof to observe that the two rectangles *BL*, *CL* making up the square on *BC* are the rectangles contained by *BC* and the two parts, respectively, into which it is divided by the perpendicular from *A* on *BC*. It is this fact which it is necessary to state in this proposition, in accordance with the plan of Book II.

The second and third propositions are of course particular cases of the first. They were no doubt separately enunciated by Euclid in order that they might be immediately available for use hereafter, instead of having to be deduced for the particular occasion from II. 1. For, if they had not been thus separately stated, it would scarcely have been practicable to quote them later without explaining at the same time that they are included in II. 1 as particular cases. And, though the propositions are not used by Euclid in the later propositions of Book II., they are used afterwards in XIII. 10 and IX. 15 respectively; and they are of extreme importance for geometry generally, being constantly used by Pappus, for example, who frequently quotes the third proposition by the Book and number.

Attention has been called to the fact that II. 1 is never used by Euclid; and this may seem no less remarkable than the fact that II. 2, 3 are not again used in Book II. But it is important, I think, to observe that the proofs of all the first ten propositions of Book II. are practically independent of each other, though the results are really so interwoven that they can often be deduced from each other in a variety of ways. What then was Euclid's intention, first in inserting some propositions not immediately required, and secondly in making the proofs of the first ten practically independent of each other? Surely the object was to show the power of the *method* of geometrical algebra as much as to arrive at results. From the point of view of illustrating the *method*, there can be no doubt that Euclid's procedure is far more instructive than the semi-algebraical substitutes which seem to find a good deal of favour; practically it means that, instead of relying on our memory of a few standard formulae, we can use the machinery given us by Euclid's method to prove immediately *ab initio* any of the propositions taken at random.

Let us contrast with Euclid's plan the semi-algebraical alternative. One editor, for example, thinks that, as II. 1 is not used by Euclid afterwards, it seems more logical to deduce from it those of the subsequent propositions which can be readily so deduced. Putting this idea into practice, he proves II. 2 and 3 by quoting II. 1, then proves II. 4 by means of II. 1 and 3, II. 5 and 6 by means of II. 1, 3 and 4, and so on. The result is ultimately to deduce the whole of the first ten propositions from II. 1, which Euclid does not use at all; and this is to give an importance to II. 1 which is altogether disproportionate and, by starting with such a narrow foundation, to make the whole structure of Book II. top-heavy.

Editors have of course been much influenced by a desire to make the proofs of the propositions of Book II. easier, as they think, for schoolboys. But, even from this point of view, is it an improvement to deduce II. 2 and 3 from II. 1 as corollaries? I doubt it. For, in the first place, Euclid's figures *visualise* the results and so make it easier to grasp their meaning; the truth of the propositions is made clear even to the eye. Then, in the matter of brevity, to which such an exaggerated importance is attached, Euclid's proof positively has the advantage. Counting a capital letter or a collocation of such as one word, I find, e.g., that Mr H. M. Taylor's proof of II. 2 contains

120 words, of which 8 represent the construction. Euclid's as above translated has 126 words, of which 22 are descriptive of the construction; therefore the actual *proof* by Euclid has 8 words fewer than Mr Taylor's, and the extra words due to the construction in Euclid are much more than atoned for by the advantage of picturing the result in the figure.

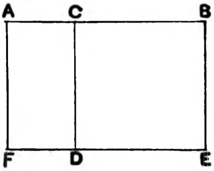
The advantages then which Euclid's method may claim are, I think, these: in the case of II. 2, 3 it produces the result more easily and clearly than does the alternative proof by means of II. 1, and, in its general application, it is more powerful in that it makes us independent of any recollection of results.

PROPOSITION 3.

If a straight line be cut at random, the rectangle contained by the whole and one of the segments is equal to the rectangle contained by the segments and the square on the aforesaid segment.

For let the straight line *AB* be cut at random at *C*;

I say that the rectangle contained by *AB*, *BC* is equal to the rectangle contained by *AC*, *CB* together with the square on *BC*.



For let the square *CDEB* be described on *CB*; [I. 46]

let *ED* be drawn through to *F*, and through *A* let *AF* be drawn parallel to either *CD* or *BE*. [I. 31]

Then *AE* is equal to *AD*, *CE*.

Now *AE* is the rectangle contained by *AB*, *BC*, for it is contained by *AB*, *BE*, and *BE* is equal to *BC*;

AD is the rectangle *AC*, *CB*, for *DC* is equal to *CB*;

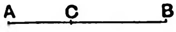
and *DB* is the square on *CB*.

Therefore the rectangle contained by *AB*, *BC* is equal to the rectangle contained by *AC*, *CB* together with the square on *BC*.

Therefore etc.

Q. E. D.

If we leave out of account the contents of Book II. itself and merely look to the applicability of propositions to general use, this proposition and the preceding are, as already indicated, of great importance, and particularly so to the semi-algebraical method just described, which seems to have found its first exponents in Heron and Pappus. Thus the proposition that *the difference of the squares on two straight lines is equal to the rectangle contained by the sum* *and the difference of the straight lines*, which is generally given as equivalent to II. 5, 6, can be proved by means of II. 1, 2, 3, as shown by Lardner. For suppose the given straight lines are *AB*, *BC*, the latter being measured along *BA*.



Then, by II. 2, the square on *AB* is equal to the sum of the rectangles *AB*, *BC* and *AB*, *AC*.

By II. 3, the rectangle *AB*, *BC* is equal to the sum of the square on *BC* and the rectangle *AC*, *CB*.

Therefore the square on *AB* is equal to the square *BC* together with the sum of the rectangles *AC*, *AB* and *AC*, *CB*.

But, by II. 1, the sum of the latter rectangles is equal to the rectangle contained by *AC* and the sum of *AB*, *BC*, i.e. the rectangle contained by the sum and difference of *AB*, *BC*.

Hence the square on *AB* is equal to the square on *BC* and the rectangle contained by the sum and difference of *AB*, *BC*:

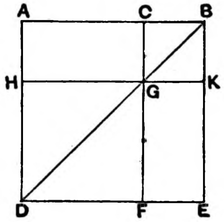
that is, the difference of the squares on *AB*, *BC* is equal to the rectangle contained by the sum and difference of *AB*, *BC*.

PROPOSITION 4.

If a straight line be cut at random, the square on the whole is equal to the squares on the segments and twice the rectangle contained by the segments.

For let the straight line *AB* be cut at random at *C*;

I say that the square on *AB* is equal to the squares on *AC*, *CB* and twice the rectangle contained by *AC*, *CB*.



For let the square *ADEB* be described on *AB*, [I. 46]

let *BD* be joined;

through *C* let *CF* be drawn parallel to either *AD* or *EB*,

and through *G* let *HK* be drawn parallel to either *AB* or *DE*. [I. 31]

Then, since *CF* is parallel to *AD*, and *BD* has fallen on them, the exterior angle *CGB* is equal to the interior and opposite angle *ADB*. [I. 29]

But the angle *ADB* is equal to the angle *ABD*,

since the side *BA* is also equal to *AD*; [I. 5]

therefore the angle *CGB* is also equal to the angle *GBC*, so that the side *BC* is also equal to the side *CG*. [I. 6]

But *CB* is equal to *GK*, and *CG* to *KB*; [I. 34]
therefore *GK* is also equal to *KB*;
therefore *CGKB* is equilateral.

I say next that it is also right-angled.

For, since *CG* is parallel to *BK*,
the angles *KBC*, *GCB* are equal to two right angles. [I. 29]

But the angle *KBC* is right;
therefore the angle *BCG* is also right,
so that the opposite angles *CGK*, *GKB* are also right. [I. 34]

Therefore *CGKB* is right-angled;
and it was also proved equilateral;
therefore it is a square;
and it is described on *CB*.

For the same reason
HF is also a square;
and it is described on *HG*, that is *AC*. [I. 34]

Therefore the squares *HF*, *KC* are the squares on *AC*, *CB*.

Now, since *AG* is equal to *GE*,
and *AG* is the rectangle *AC*, *CB*, for *GC* is equal to *CB*,
therefore *GE* is also equal to the rectangle *AC*, *CB*.

Therefore *AG*, *GE* are equal to twice the rectangle *AC*, *CB*.

But the squares *HF*, *CK* are also the squares on *AC*, *CB*;
therefore the four areas *HF*, *CK*, *AG*, *GE* are equal to the squares on *AC*, *CB* and twice the rectangle contained by *AC*, *CB*.

But *HF*, *CK*, *AG*, *GE* are the whole *ADEB*,
which is the square on *AB*.

Therefore the square on *AB* is equal to the squares on *AC*, *CB* and twice the rectangle contained by *AC*, *CB*.

Therefore etc. Q. E. D.

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2. **twice the rectangle contained by the segments.** By a curious idiom this is in Greek "the rectangle *twice contained* by the segments." Similarly "twice the rectangle contained by *AC*, *CB*" is expressed as "the rectangle *twice contained* by *AC*, *CB*" (τὸ δὶς ὑπὸ τῶν ΑΓ, ΓΒ περιεχόμενον ὀρθογώνιον).

35, 38. **described.** 39, 45. **the squares** (before "on"). These words are not in the Greek, which simply says that the squares "are on" (εἰσὶν ἀπό) their respective sides.

46. **areas.** It is necessary to supply some substantive (the Greek leaves it to be understood); and I prefer "areas" to "figures."

The editions of the Greek text which preceded that of E. F. August (Berlin, 1826—9) give a second proof of this proposition introduced by the usual word ἄλλως or "otherwise thus." Heiberg follows August in omitting this proof, which is attributed to Theon, and which is indeed not worth reproducing, since it only differs from the genuine proof in that portion of it which proves that $CGKB$ is a square. The proof that $CGKB$ is equilateral is rather longer than Euclid's, and the only interesting point to notice is that, whereas Euclid still, as in I. 46, seems to regard it as necessary to prove that *all* the angles of $CGKB$ are right angles before he concludes that it is *right-angled*, Theon says simply "And it also has the angle CBK right; therefore CK is a square." The shorter form indicates a legitimate abbreviation of the genuine proof; because there can be no need to repeat exactly that part of the proof of I. 46 which shows that *all* the angles of the figure there constructed are right when one is.

There is also in the Greek text a Porism which is undoubtedly interpolated: "From this it is manifest that in square areas the parallelograms about the diameter are squares." Heiberg doubted its genuineness when preparing his edition, and conjectured that it too may have been added by Theon; but the matter is placed beyond doubt by a papyrus-fragment referred to already (see Heiberg, *Paralipomena zu Euklid*, in *Hermes* XXXVIII., 1903, p. 48) in which the Porism was evidently wanting. It is the only Porism in Book II., but does not correspond to Proclus' remark (p. 304, 2) that "the Porism found in the second book belongs to a *problem*." Heiberg regards these words as referring to the Porism to IV. 15, the correct reading having probably been not δευτέρῳ but δ′, i.e. τετάρτῳ.

The semi-algebraical proof of this proposition is very easy, and is of course old enough, being found in Clavius and in most later editions. It proceeds thus.

By II. 2, the square on AB is equal to the sum of the rectangles AB, AC and AB, CB.

But, by II. 3, the rectangle AB, AC is equal to the sum of the square on AC and the rectangle AC, CB;

while, by II. 3, the rectangle AB, CB is equal to the sum of the square on BC and the rectangle AC, CB.

Therefore the square on AB is equal to the sum of the squares on AC, CB and twice the rectangle AC, CB.

The figure of the proposition also helps to visualise, in the orthodox manner, the proof of the theorem deduced above from II. 1—3, viz. that *the difference of the squares on two given straight lines is equal to the rectangle contained by the sum and the difference of the lines.*

For, if the lines be AB, BC respectively, the shorter of the lines being measured along BA, the figure shows that

the square AE is equal to the sum of the square CK and the rectangles AF, FK;

that is, the square on AB is equal to the sum of the square on BC and the rectangles AB, AC and AC, BC.

But the rectangles AB, AC and BC, AC are, by II. 1, together equal to the rectangle contained by AC and the sum of AB, BC,

i.e to the rectangle contained by the sum and difference of AB, BC.

Whence the result follows as before.

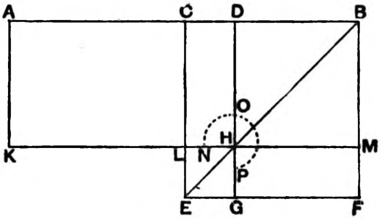
The proposition II. 4 can also be extended to the case where a straight line is divided into any number of segments; for the figure will show in like manner that the square on the whole line is equal to the sum of the squares on all the parts together with twice the rectangles contained by every pair of the parts.

PROPOSITION 5.

If a straight line be cut into equal and unequal segments, the rectangle contained by the unequal segments of the whole together with the square on the straight line between the points of section is equal to the square on the half.

For let a straight line *AB* be cut into equal segments at *C* and into unequal segments at *D*;

I say that the rectangle contained by *AD*, *DB* together with the square on *CD* is equal to the square on *CB*.



For let the square *CEFB* be described on *CB*, [I. 46]
and let *BE* be joined;
through *D* let *DG* be drawn parallel to either *CE* or *BF*,
through *H* again let *KM* be drawn parallel to either *AB* or *EF*,
and again through *A* let *AK* be drawn parallel to either *CL* or *BM*. [I. 31]

Then, since the complement *CH* is equal to the complement *HF*, [I. 43]
let *DM* be added to each;

therefore the whole *CM* is equal to the whole *DF*.

But *CM* is equal to *AL*,

since *AC* is also equal to *CB*; [I. 36]

therefore *AL* is also equal to *DF*.

Let *CH* be added to each;

therefore the whole *AH* is equal to the gnomon *NOP*.

But *AH* is the rectangle *AD*, *DB*, for *DH* is equal to *DB*,

therefore the gnomon *NOP* is also equal to the rectangle *AD*, *DB*.

Let *LG*, which is equal to the square on *CD*, be added to each;

therefore the gnomon *NOP* and *LG* are equal to the rectangle contained by *AD*, *DB* and the square on *CD*.

But the gnomon *NOP* and *LG* are the whole square *CEFB*, which is described on *CB*;

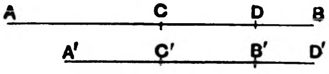
therefore the rectangle contained by *AD*, *DB* together with the square on *CD* is equal to the square on *CB*.

Therefore etc. Q. E. D.

3. **between the points of section**, literally "between the *sections*," the word being the same (τομή) as that used of a conic *section*.

It will be observed that the gnomon is indicated in the figure by three separate letters and a dotted curve. This is no doubt a clearer way of showing what exactly the gnomon is than the method usual in our text-books. In this particular case the figure of the MSS. has *two* M's in it, the gnomon being MNΞ. I have corrected the lettering to avoid confusion.

It is easily seen that this proposition and the next give exactly the theorem already alluded to under the last propositions, namely that *the difference of the squares on two straight lines is equal to the rectangle contained by their sum and difference.* The two given lines are, in II. 5, the lines *CB* and *CD*, and their sum and difference are respectively equal to *AD* and *DB*. To show that II. 6 gives the same theorem we have only to make *CD* the greater line and *CB* the less, i.e. to draw *C′D′* equal to *CB*, measure *C′B′* along it equal to *CD*, and then produce *B′C′* to *A′*, making *A′C′* equal to *B′C′*, whence it is immediately clear that *A′D′* on the second line is equal to *AD* on the first, while *D′B′* is also equal to *DB*, so that the rectangles *AD*, *DB* and *A′D′*, *D′B′* are equal, while the difference of the squares on *CB*, *CD* is equal to the difference of the squares on *C′D′*, *C′B′*.



Perhaps the most important fact about II. 5, 6 is however their bearing on the

Geometrical solution of a quadratic equation.

Suppose, in the figure of II. 5, that $AB = a$, $DB = x$;

then $ax - x^2 =$ the rectangle AH

$=$ the gnomon NOP.

Thus, if the area of the gnomon is given ($= b^2$, say), and if a is given ($= AB$), the problem of solving the equation

$$ax - x^2 = b^2$$

is, in the language of geometry, *To a given straight line* (a) *to apply a rectangle which shall be equal to a given square* (b^2) *and shall fall short by a square figure*, i.e. to construct the rectangle *AH* or the gnomon *NOP*.

Now we are told by Proclus (on I. 44) that "these propositions are ancient and the discoveries of the Muse of the Pythagoreans, the application of areas, their exceeding and their falling-short." We can therefore hardly avoid crediting the Pythagoreans with the geometrical solution, based upon II. 5, 6, of the problems corresponding to the quadratic equations which are directly obtainable from them. It is certain that the Pythagoreans solved the problem in II. 11, which corresponds to the quadratic equation

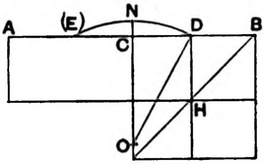
$$a(a-x)=x^2,$$

and Simson has suggested the following easy solution of the equation now in question,

$$ax-x^2=b^2,$$

on exactly similar lines.

Draw CO perpendicular to AB and equal to b; produce OC to N so that $ON=CB$ (or $\frac{1}{2}a$); and with O as centre and radius ON describe a circle cutting CB in D.



Then DB (or x) is found, and therefore the required rectangle AH.

For the rectangle AD, DB together with the square on CD is equal to the square on CB, [II. 5]

i.e. to the square on OD,

i.e. to the squares on OC, CD; [I. 47]

whence the rectangle AD, DB is equal to the square on OC,

or $$ax-x^2=b^2.$$

It is of course a necessary condition of the possibility of a real solution that b^2 must not be greater that $(\frac{1}{2}a)^2$. This condition itself can easily be obtained from Euclid's proposition; for, since the sum of the rectangle AD, DB and the square on CD is equal to the square on CB, which is constant, it follows that, as CD diminishes, i.e. as D moves nearer to C, the rectangle AD, DB increases and, when D actually coincides with C, so that CD vanishes, the rectangle AD, DB becomes the rectangle AC, CB, i.e. the square on CB, and is a maximum. It will be seen also that the geometrical solution of the quadratic equation derived from Euclid does not differ from our practice of solving a quadratic by completing the square on the side containing the terms in x^2 and x.

But, while in this case there are two geometrically real solutions (because the circle described with ON as radius will not only cut CB in D but will also cut AC in another point E), Euclid's figure corresponds to one only of the two solutions. Not that there is any doubt that Euclid was aware that the method of solving the quadratic gives two solutions; he could not fail to see that $x=BE$ satisfies the equation as well as $x=BD$. If however he had actually given us the solution of the equation, he would probably have omitted to specify the solution $x=BE$ because the rectangle found by means of it, which would be a rectangle on the base AE (equal to BD) and with altitude EB (equal to AD), is really an equal rectangle to that corresponding to the other solution $x=BD$; there is therefore no real object in distinguishing two solutions. This is easily understood when we regard the equation as a statement of the problem of finding two magnitudes when their sum (a) and product (b^2) are given, i.e. as equivalent to the simultaneous equations

$$\begin{aligned} x+y&=a,\\ xy&=b^2. \end{aligned}$$

These symmetrical equations have really only one solution, as the two apparent solutions are simply the result of interchanging the values of x and y. This form of the problem was known to Euclid, as appears from the *Data*, Prop. 85, which states that, *If two straight lines contain a parallelogram given in magnitude in a given angle, and if the sum of them be given, then shall each of them be given.*

This proposition then enables us to solve the problem of finding a rectangle the area and perimeter of which are both given; and it also enables us to infer that, of all rectangles of given perimeter, the square has the greatest area, while, the more unequal the sides are, the less is the area.

If in the figure of II. 5 we suppose that $AD=a$, $BD=b$, we find that $CB=(a+b)/2$ and $CD=(a-b)/2$, and we may state the result of the proposition in the following algebraical form

$$\left(\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2 = ab.$$

This way of stating it (which could hardly have escaped the Pythagoreans) gives a ready means of obtaining the two rules, respectively attributed to the Pythagoreans and Plato, for finding integral square numbers which are the sum of two other integral square numbers. We have only to make ab a perfect square in the above formula. The simplest way in which this can be done is to put $a=n^2$, $b=1$, whence we have

$$\left(\frac{n^2+1}{2}\right)^2 - \left(\frac{n^2-1}{2}\right)^2 = n^2,$$

and in order that the first two squares may be integral n^2, and therefore n, must be odd. Hence the Pythagorean rule.

Suppose next that $a=2n^2$, $b=2$, and we have

$$(n^2+1)^2-(n^2-1)^2=4n^2,$$

whence Plato's rule starting from an *even* number $2n$.

PROPOSITION 6.

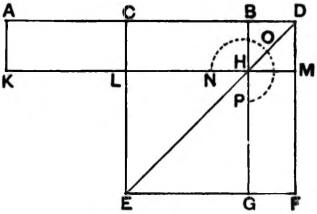
If a straight line be bisected and a straight line be added to it in a straight line, the rectangle contained by the whole with the added straight line and the added straight line together with the square on the half is equal to the square on the straight line made up of the half and the added straight line.

For let a straight line AB be bisected at the point C, and let a straight line BD be added to it in a straight line;

I say that the rectangle contained by AD, DB together with the square on CB is equal to the square on CD.

For let the square $CEFD$ be described on CD, [I. 46] and let DE be joined;

through the point B let BG be drawn parallel to either EC or DF, through the point *H* let *KM* be drawn parallel to either *AB* or *EF*,



and further through *A* let *AK* be drawn parallel to either *CL* or *DM*. [I. 31]

Then, since *AC* is equal to *CB*,

AL is also equal to *CH*. [I. 36]

But *CH* is equal to *HF*. [I. 43]

Therefore *AL* is also equal to *HF*.

Let *CM* be added to each;

therefore the whole *AM* is equal to the gnomon *NOP*.

But *AM* is the rectangle *AD*, *DB*,

for *DM* is equal to *DB*;

therefore the gnomon *NOP* is also equal to the rectangle *AD*, *DB*.

Let *LG*, which is equal to the square on *BC*, be added to each;

therefore the rectangle contained by *AD*, *DB* together with the square on *CB* is equal to the gnomon *NOP* and *LG*.

But the gnomon *NOP* and *LG* are the whole square *CEFD*, which is described on *CD*;

therefore the rectangle contained by *AD*, *DB* together with the square on *CB* is equal to the square on *CD*.

Therefore etc.

Q. E. D.

In this case the rectangle *AD*, *DB* is "a rectangle applied to a given straight line (*AB*) but *exceeding* by a square (the side of which is equal to *BD*)"; and the problem suggested by II. 6 is to find a rectangle of this description equal to a given area, which we will, for convenience, suppose to be a square; i.e., in the language of geometry, *to apply to a given straight line a rectangle which shall be equal to a given square and shall exceed by a square figure.*

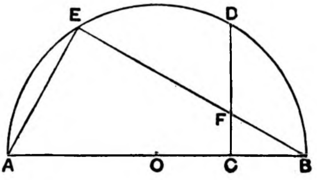
We suppose that in Euclid's figure $AB = a$, $BD = x$; then, if the given square be b^2, the problem is to solve geometrically the equation

$$ax + x^2 = b^2.$$

The solution of a problem theoretically equivalent to the solution of a quadratic equation of this kind is presupposed in the fragment of Hippocrates' *Quadrature of lunes* preserved in a quotation by Simplicius (*Comment. in*

Aristot. Phys. pp. 61—68, ed. Diels) from Eudemus' *History of Geometry*. In this fragment Hippocrates (5th cent. B.C.) assumes the following construction.

AB being the diameter and O the centre of a semicircle, and C being the middle point of OB and CD at right angles to AB, a straight line of length such that its square is $1\frac{1}{2}$ times the square on the radius (i.e. of length $a\sqrt{\frac{3}{2}}$, where a is the radius) is to be so placed, as EF, between CD and the circumference AD that it "verges towards B," that is, EF when produced passes through B.



Now the right-angled triangles BFC, BAE are similar, so that

$$BF : BC = BA : BE,$$

and therefore the rectangle BE, BF = rect. BA, BC

= sq. on BO.

In other words, EF $(= a\sqrt{\frac{3}{2}})$ being given in length, BF $(= x$, say) has to be found such that

$$(\sqrt{\tfrac{3}{2}}\,a + x)\,x = a^2;$$

or the quadratic equation

$$\sqrt{\tfrac{3}{2}}\,ax + x^2 = a^2$$

has to be solved.

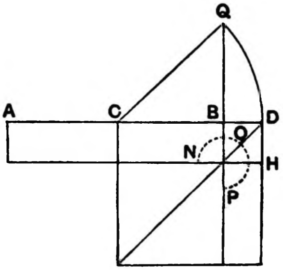
A straight line of length $a\sqrt{\frac{3}{2}}$ would easily be constructed, for, in the figure, $CD^2 = AC \cdot CB = \frac{3}{4}a^2$, or $CD = \frac{1}{2}a\sqrt{3}$, and $a\sqrt{\frac{3}{2}}$ is the diagonal of a square of which CD is the side.

There is no doubt that Hippocrates could have solved the equation by the geometrical construction given below, but he may have contemplated, on this occasion, the merely *mechanical* process of *placing* the straight line of the length required between CD and the circumference AD and *moving* it until E, F, B were in a straight line. Zeuthen (*Die Lehre von den Kegelschnitten im Altertum*, pp. 270, 271) thinks this probable because, curiously enough, the fragment speaks immediately afterwards of "joining B to F."

To solve the equation

$$ax + x^2 = b^2$$

we have to find the rectangle AH, or the gnomon NOP, which is equal in area to b^2 and has one of the sides containing the inner right angle equal to CB or $\frac{1}{2}a$. Thus we know $(\frac{1}{2}a)^2$ and b^2, and we have to find, by I. 47, a square equal to the sum of two given squares.



To do this Simson draws BQ at right angles to AB and equal to b, joins CQ and, with centre C and radius CQ, describes a circle cutting AB produced in D. Thus BD, or x, is found.

Now the rectangle AD, DB together with the square on CB

is equal to the square on CD,

i.e. to the square on CQ,

i.e. to the squares on CB, BQ.

Therefore the rectangle AD, DB is equal to the square on BQ, that is,

$$ax + x^2 = b^2.$$

From Euclid's point of view there would only be one solution in this case.

This proposition enables us also to solve the equation

$$x^2 - ax = b^2$$

in a similar manner.

We have only to suppose that $AB = a$, and AD (instead of BD) $= x$; then

$$x^2 - ax = \text{the gnomon}.$$

To find the gnomon we have its area (b^2) and the area, CB^2 or $(\tfrac{1}{2}a)^2$, by which the gnomon differs from CD^2. Thus we can find D (and therefore AD or x) by the same construction as that just given.

Converse propositions to II. 5, 6 are given by Pappus (VII. pp. 948—950) among his lemmas to the *Conics* of Apollonius to the effect that,

(1) if D be a point dividing AB unequally, and C another point on AB such that the rectangle AD, DB together with the square on CD is equal to the square on AC, then

AC is equal to CB;

(2) if D be a point on AB produced, and C a point on AB such that the rectangle AD, DB together with the square on CB is equal to the square on CD, then

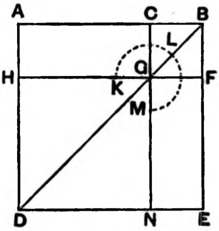
AC is equal to CB.

PROPOSITION 7.

If a straight line be cut at random, the square on the whole and that on one of the segments both together are equal to twice the rectangle contained by the whole and the said segment and the square on the remaining segment.

For let a straight line AB be cut at random at the point C;

I say that the squares on AB, BC are equal to twice the rectangle contained by AB, BC and the square on CA.



For let the square $ADEB$ be described on AB, [I. 46]

and let the figure be drawn.

Then, since AG is equal to GE, [I. 43]

let CF be added to each;

therefore the whole AF is equal to the whole CE.

Therefore AF, CE are double of AF.

But AF, CE are the gnomon KLM and the square CF; therefore the gnomon KLM and the square CF are double of AF.

But twice the rectangle *AB*, *BC* is also double of *AF*; for *BF* is equal to *BC*;
therefore the gnomon *KLM* and the square *CF* are equal to twice the rectangle *AB*, *BC*.

Let *DG*, which is the square on *AC*, be added to each;
therefore the gnomon *KLM* and the squares *BG*, *GD* are equal to twice the rectangle contained by *AB*, *BC* and the square on *AC*.

But the gnomon *KLM* and the squares *BG*, *GD* are the whole *ADEB* and *CF*,

which are squares described on *AB*, *BC*;

therefore the squares on *AB*, *BC* are equal to twice the rectangle contained by *AB*, *BC* together with the square on *AC*.

Therefore etc.

Q. E. D.

An interesting variation of the form of this proposition may be obtained by regarding *AB*, *BC* as two given straight lines of which *AB* is the greater, and *AC* as the *difference* between the two straight lines. Thus the proposition shows that the squares on two straight lines are together equal to twice the rectangle contained by them and the square on their difference. That is, *the square on the difference of two straight lines is equal to the sum of the squares on the straight lines diminished by twice the rectangle contained by them.* In other words, just as II. 4 is the geometrical equivalent of the identity

$$(a+b)^2 = a^2 + b^2 + 2ab,$$

so II. 7 proves that

$$(a-b)^2 = a^2 + b^2 - 2ab.$$

The addition and subtraction of these formulae give the algebraical equivalent of the propositions II. 9, 10 and II. 8 respectively; and we have accordingly a suggestion of alternative methods of proving those propositions.

PROPOSITION 8.

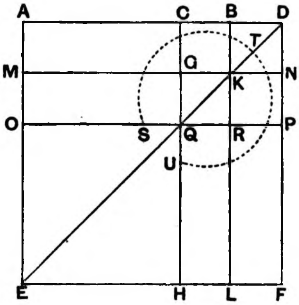
If a straight line be cut at random, four times the rectangle contained by the whole and one of the segments together with the square on the remaining segment is equal to the square described on the whole and the aforesaid segment as on one straight line.

For let a straight line *AB* be cut at random at the point *C*;

I say that four times the rectangle contained by *AB*, *BC* together with the square on *AC* is equal to the square described on *AB*, *BC* as on one straight line.

For let [the straight line] *BD* be produced in a straight line [with *AB*], and let *BD* be made equal to *CB*;

let the square *AEFD* be described on *AD*, and let the figure be drawn double.



Then, since *CB* is equal to *BD*, while *CB* is equal to *GK*, and *BD* to *KN*,

therefore *GK* is also equal to *KN*.

For the same reason

QR is also equal to *RP*.

And, since *BC* is equal to *BD*, and *GK* to *KN*,

therefore *CK* is also equal to *KD*, and *GR* to *RN*. [I. 36]

But *CK* is equal to *RN*, for they are complements of the parallelogram *CP*; [I. 43]

therefore *KD* is also equal to *GR*;

therefore the four areas *DK*, *CK*, *GR*, *RN* are equal to one another.

Therefore the four are quadruple of *CK*.

Again, since *CB* is equal to *BD*,

while *BD* is equal to *BK*, that is *CG*,

and *CB* is equal to *GK*, that is *GQ*,

therefore *CG* is also equal to *GQ*.

And, since *CG* is equal to *GQ*, and *QR* to *RP*,

AG is also equal to *MQ*, and *QL* to *RF*. [I. 36]

But *MQ* is equal to *QL*, for they are complements of the parallelogram *ML*; [I. 43]

therefore *AG* is also equal to *RF*;

therefore the four areas *AG*, *MQ*, *QL*, *RF* are equal to one another.

Therefore the four are quadruple of *AG*.

But the four areas *CK*, *KD*, *GR*, *RN* were proved to be quadruple of *CK*;

therefore the eight areas, which contain the gnomon *STU*, are quadruple of *AK*.

Now, since *AK* is the rectangle *AB*, *BD*, for *BK* is equal to *BD*, therefore four times the rectangle *AB*, *BD* is quadruple of *AK*.

But the gnomon *STU* was also proved to be quadruple of *AK*;

therefore four times the rectangle *AB*, *BD* is equal to the gnomon *STU*.

Let *OH*, which is equal to the square on *AC*, be added to each;

therefore four times the rectangle *AB*, *BD* together with the square on *AC* is equal to the gnomon *STU* and *OH*.

But the gnomon *STU* and *OH* are the whole square *AEFD*,

which is described on *AD*·

therefore four times the rectangle *AB*, *BD* together with the square on *AC* is equal to the square on *AD*

But *BD* is equal to *BC*;

therefore four times the rectangle contained by *AB*, *BC* together with the square on *AC* is equal to the square on *AD*, that is to the square described on *AB* and *BC* as on one straight line.

Therefore etc.

Q. E. D.

This proposition is quoted by Pappus (p. 428, ed. Hultsch) and is used also by Euclid himself in the *Data*, Prop. 86. Further, it is of decided use in proving the fundamental property of a parabola.

Two alternative proofs are worth giving.

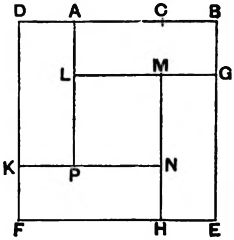
The first is that suggested by the consideration mentioned in the last note, though the proof is old enough, being given by Clavius and others. It is of the semi-algebraical type.

Produce *AB* to *D* (in the figure of the proposition), so that *BD* is equal to *BC*.

By II. 4, the square on *AD* is equal to the squares on *AB*, *BD* and twice the rectangle *AB*, *BD*, i.e. to the squares on *AB*, *BC* and twice the rectangle *AB*, *BC*.

By II. 7, the squares on *AB*, *BC* are equal to twice the rectangle *AB*, *BC* together with the square on *AC*.

Therefore the square on *AD* is equal to four times the rectangle *AB*, *BC* together with the square on *AC*.



The second proof is after the manner of Euclid but with a difference. Produce *BA* to *D* so that *AD* is equal to *BC*. On *BD* construct the square *BEFD*.

Take *BG*, *EH*, *FK* each equal to *BC* or *AD*, and draw *ALP*, *HNM* parallel to *BE* and *GML*, *KPN* parallel to *BD*.

Then it can be shown that each of the rectangles *BL*, *AK*, *FN*, *EM* is equal to the rectangle *AB*, *BC*, and that *PM* is equal to the square on *AC*.

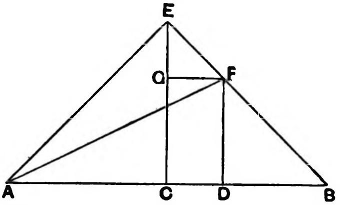
Therefore the square on *BD* is equal to four times the rectangle *AB*, *BC* together with the square on *AC*.

PROPOSITION 9.

If a straight line be cut into equal and unequal segments, the squares on the unequal segments of the whole are double of the square on the half and of the square on the straight line between the points of section.

For let a straight line *AB* be cut into equal segments at *C*, and into unequal segments at *D*;

I say that the squares on *AD*, *DB* are double of the squares on *AC*, *CD*.



For let *CE* be drawn from *C* at right angles to *AB*, and let it be made equal to either *AC* or *CB*;

let *EA*, *EB* be joined,

let *DF* be drawn through *D* parallel to *EC*,

and *FG* through *F* parallel to *AB*,

and let *AF* be joined.

Then, since *AC* is equal to *CE*,

the angle *EAC* is also equal to the angle *AEC*.

And, since the angle at *C* is right,

the remaining angles *EAC*, *AEC* are equal to one right angle. [I. 32]

And they are equal;

therefore each of the angles *CEA*, *CAE* is half a right angle.

For the same reason

each of the angles *CEB*, *EBC* is also half a right angle;

therefore the whole angle *AEB* is right.

And, since the angle *GEF* is half a right angle, and the angle *EGF* is right, for it is equal to the interior and opposite angle *ECB*, [I. 29]

the remaining angle *EFG* is half a right angle; [I. 32]

therefore the angle *GEF* is equal to the angle *EFG*,

so that the side *EG* is also equal to *GF*. [I. 6]

Again, since the angle at *B* is half a right angle, and the angle *FDB* is right, for it is again equal to the interior and opposite angle *ECB*, [I. 29]

the remaining angle *BFD* is half a right angle; [I. 32]

therefore the angle at *B* is equal to the angle *DFB*,

so that the side *FD* is also equal to the side *DB*. [I. 6]

Now, since *AC* is equal to *CE*,

the square on *AC* is also equal to the square on *CE*;

therefore the squares on *AC*, *CE* are double of the square on *AC*.

But the square on *EA* is equal to the squares on *AC*, *CE*, for the angle *ACE* is right; [I. 47]

therefore the square on *EA* is double of the square on *AC*.

Again, since *EG* is equal to *GF*,

the square on *EG* is also equal to the square on *GF*;

therefore the squares on *EG*, *GF* are double of the square on *GF*.

But the square on *EF* is equal to the squares on *EG*, *GF*;

therefore the square on *EF* is double of the square on *GF*.

But *GF* is equal to *CD*; [I. 34]

therefore the square on *EF* is double of the square on *CD*.

But the square on *EA* is also double of the square on *AC*;

therefore the squares on *AE*, *EF* are double of the squares on *AC*, *CD*.

And the square on *AF* is equal to the squares on *AE*, *EF*, for the angle *AEF* is right; [I. 47]

therefore the square on *AF* is double of the squares on *AC*, *CD*.

But the squares on *AD*, *DF* are equal to the square on *AF*, for the angle at *D* is right; [I. 47]

therefore the squares on *AD*, *DF* are double of the squares on *AC*, *CD*.

And *DF* is equal to *DB*;
therefore the squares on *AD*, *DB* are double of the squares on *AC*, *CD*.

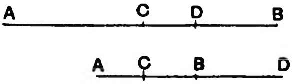
Therefore etc.

Q. E. D.

It is noteworthy that, while the first eight propositions of Book II. are proved independently of the Pythagorean theorem I. 47, all the remaining propositions beginning with the 9th are proved by means of it. Also the 9th and 10th propositions mark a new departure in another respect; the method of demonstration by showing in the figures the various rectangles and squares to which the theorems relate is here abandoned.

The 9th and 10th propositions are related to one another in the same way as the 5th and 6th; they really prove the same result which can, as in the earlier case, be comprised in a single enunciation thus: *The sum of the squares on the sum and difference of two given straight lines is equal to twice the sum of the squares on the lines.*

The semi-algebraical proof of Prop. 9 is that suggested by the remark on the algebraical formulae given at the end of the note on II. 7. It applies with a very slight modification to both II. 9 and II. 10. We will put in brackets the variations belonging to II. 10. The first of the annexed lines is the figure for II. 9 and the second for II. 10.



By II. 4, the square on *AD* is equal to the squares on *AC*, *CD* and twice the rectangle *AC*, *CD*.

By II. 7, the squares on *CB*, *CD* (*CD*, *CB*) are equal to
twice the rectangle *CB*, *CD* together with the square on *BD*.

By addition of these equals crosswise,
the squares on *AD*, *DB* together with twice the rectangle *CB*, *CD* are equal to the squares on *AC*, *CD*, *CB*, *CD* together with twice the rectangle *AC*, *CD*.

But *AC*, *CB* are equal, and therefore the rectangles *AC*, *CD* and *CB*, *CD* are equal.

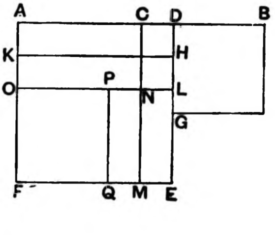
Taking away the equals, we see that
the squares on *AD*, *DB* are equal to the squares on *AC*, *CD*, *CB*, *CD*,
i.e. to twice the squares on *AC*, *CD*.

To show also that the method of geometrical algebra illustrated by II. 1—8 is still effective for the purpose of proving II. 9, 10, we will now prove II. 9 in that manner.

Draw squares on *AD*, *DB* respectively as shown in the figure. Measure *DH* along *DE* equal to *CD*, and *HL* along *HE* also equal to *CD*.

Draw *HK*, *LNO* parallel to *EF*, and *CNM* parallel to *DE*.

Measure *NP* along *NO* equal to *CD*, and draw *PQ* parallel to *DE*.



Now, since AD, CD are respectively equal to DE, DH,

HE is equal to AC or CB;

and, since HL is equal to CD, LE is equal to DB.

Similarly, since each of the segments EM, MQ is equal to CD,

FQ is equal to EL or BD.

Therefore OQ is equal to the square on DB.

We have to prove that the squares on AD, DB are equal to twice the squares on AC, CD.

Now the square on AD includes KM (the square on AC) and CH, HN (that is, twice the square on CD).

Therefore we have to prove that what is left over of the square on AD together with the square on DB is equal to the square on AC.

The parts left over are the rectangles CK and NE, which are equal to KN, PM respectively.

But the latter with the square on DB are equal to the rectangles KN, PM and the square OQ,

i.e. to the square KM, or the square on AC.

Hence the required result follows.

PROPOSITION 10.

If a straight line be bisected, and a straight line be added to it in a straight line, the square on the whole with the added straight line and the square on the added straight line both together are double of the square on the half and of the square described on the straight line made up of the half and the added straight line as on one straight line.

For let a straight line AB be bisected at C, and let a straight line BD be added to it in a straight line;

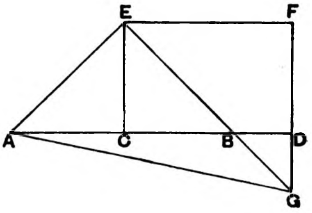
I say that the squares on AD, DB are double of the squares on AC, CD.

For let CE be drawn from the point C at right angles to AB [I. 11], and let it be made equal to either AC or CB [I. 3];

let EA, EB be joined;

through E let EF be drawn parallel to AD,

and through D let FD be drawn parallel to CE. [I. 31]



Then, since a straight line EF falls on the parallel straight lines EC, FD,

the angles *CEF*, *EFD* are equal to two right angles; [I. 29]
therefore the angles *FEB*, *EFD* are less than two right angles.

But straight lines produced from angles less than two right angles meet; [I. Post. 5]

therefore *EB*, *FD*, if produced in the direction *B*, *D*, will meet.

Let them be produced and meet at *G*,
and let *AG* be joined.

Then, since *AC* is equal to *CE*,
the angle *EAC* is also equal to the angle *AEC*; [I. 5]
and the angle at *C* is right;

therefore each of the angles *EAC*, *AEC* is half a right angle. [I. 32]

For the same reason

each of the angles *CEB*, *EBC* is also half a right angle;

therefore the angle *AEB* is right.

And, since the angle *EBC* is half a right angle,
the angle *DBG* is also half a right angle. [I. 15]

But the angle *BDG* is also right,
for it is equal to the angle *DCE*, they being alternate; [I. 29]

therefore the remaining angle *DGB* is half a right angle; [I. 32]
therefore the angle *DGB* is equal to the angle *DBG*,

so that the side *BD* is also equal to the side *GD*. [I. 6]

Again, since the angle *EGF* is half a right angle,
and the angle at *F* is right, for it is equal to the opposite angle, the angle at *C*, [I. 34]

the remaining angle *FEG* is half a right angle; [I. 32]
therefore the angle *EGF* is equal to the angle *FEG*,

so that the side *GF* is also equal to the side *EF*. [I. 6]

Now, since the square on *EC* is equal to the square on *CA*,
the squares on *EC*, *CA* are double of the square on *CA*.

But the square on *EA* is equal to the squares on *EC*, *CA*; [I. 47]
therefore the square on *EA* is double of the square on *AC*. [*C. N.* I]

Again, since *FG* is equal to *EF*,
the square on *FG* is also equal to the square on *FE*;
therefore the squares on *GF*, *FE* are double of the square on *EF*.

But the square on *EG* is equal to the squares on *GF*, *FE*; [I. 47]
therefore the square on *EG* is double of the square on *EF*.

And *EF* is equal to *CD*; [I. 34]

therefore the square on *EG* is double of the square on *CD*.
But the square on *EA* was also proved double of the square on *AC*;
therefore the squares on *AE*, *EG* are double of the squares on *AC*, *CD*.

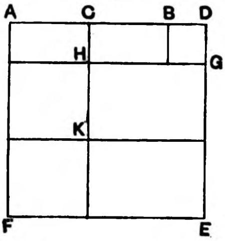
And the square on *AG* is equal to the squares on *AE*, *EG*; [I. 47]
therefore the square on *AG* is double of the squares on *AC*, *CD*.
But the squares on *AD*, *DG* are equal to the square on *AG*; [I. 47]
therefore the squares on *AD*, *DG* are double of the squares on *AC*, *CD*.

And *DG* is equal to *DB*;
therefore the squares on *AD*, *DB* are double of the squares on *AC*, *CD*.

Therefore etc.

Q. E. D.

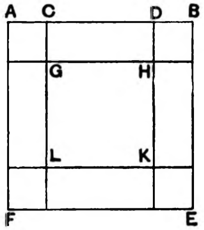
The alternative proof of this proposition by means of the principles exhibited in II. 1—8 follows the lines of that which I have given for the preceding proposition.



It is at once obvious from the figure that the square on *AD* includes within it twice the square on *AC* together with once the square on *CD*. What is left over is the sum of the rectangles *AH*, *KE*. These, which are equivalent to *BH*, *GK*, make up the square on *CD* *less* the square on *BD*. Adding therefore the square *BG* to each side, we have the required result.

Another alternative proof of the theorem which includes both II. 9 and 10 is worth giving. The theorem states that *the sum of the squares on the sum and difference of two given straight lines is equal to twice the sum of the squares on the lines.*

Let AD, DB be the two given straight lines (of which AD is the greater), placed so as to be in one straight line. Make AC equal to DB and complete the figure as shown, each of the segments CG and DH being equal to AC or DB.



Now, AD, DB being the given straight lines, AB is their sum and CD is equal to their difference.

Also AD is equal to BC.

And AE is the square on AB, GK is equal to the square on CD, AK or FH is the square on AD, and BL the square on CB, while each of the small squares AG, BH, EK, FL is equal to the square on AC or DB.

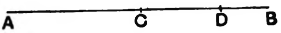
We have to prove that twice the squares on AD, DB are equal to the squares on AB, CD.

Now twice the square on AD is the sum of the squares on AD, CB, which is equal to the sum of the squares BL, FH; and the figure shows these to be equal to twice the inner square GK and once the remainder of the large square AE excluding the two squares AG, KE, which latter squares are equal to twice the square on AC or DB.

Therefore twice the squares on AD, DB are equal to twice the inner square GK together with once the remainder of the large square AE, that is, to the sum of the squares AE, GK, which are the squares on AB, CD.

"Side" and "diagonal" numbers giving successive approximations to $\sqrt{2}$.

Zeuthen pointed out (*Die Lehre von den Kegelschnitten im Altertum*, 1886, pp. 27, 28) that II. 9, 10 have great interest in connexion with a problem of indeterminate analysis which received much attention from the ancient Greeks. If we take the straight line AB divided at C and D as in II. 9, and if we put $CD = x$, $DB = y$, the result obtained by Euclid, namely:



$$AD^2 + DB^2 = 2AC^2 + 2CD^2,$$

or $$AD^2 - 2AC^2 = 2CD^2 - DB^2,$$

becomes the formula

$$(2x+y)^2 - 2(x+y)^2 = 2x^2 - y^2.$$

If therefore x, y be numbers which satisfy one of the two equations

$$2x^2 - y^2 = \pm 1,$$

the formula gives us two higher numbers, $x+y$ and $2x+y$, which satisfy the other of the two equations.

Euclid's propositions thus give a general proof of the very formula used for the formation of the succession of what were called "*side*" and "*diagonal* numbers."

As is well known, Theon of Smyrna (pp. 43, 44, ed. Hiller) describes this system of numbers. The unit, being the beginning of all things, must be potentially both a *side* and a *diameter*. Consequently we begin with two units, the one being the first *side* and the other the first *diameter*, and (*a*) from the sum of them, (*b*) from the sum of twice the first unit and once the second, we form two new numbers

$$1 \cdot 1 + 1 = 2, \quad 2 \cdot 1 + 1 = 3.$$

Of these new numbers the first is a *side-* and the second a *diagonal*-number, or (as we may say)

$$a_2 = 2, \qquad d_2 = 3.$$

In the same way as these numbers were formed from $a_1 = 1$, $d_1 = 1$, successive pairs of numbers are formed from a_2, d_2, and so on, according to the formula

$$a_{n+1} = a_n + d_n, \quad d_{n+1} = 2a_n + d_n.$$

Thus

$$a_3 = 2 + 3 = 5, \quad d_3 = 2 \cdot 2 + 3 = 7,$$
$$a_4 = 5 + 7 = 12, \quad d_4 = 2 \cdot 5 + 7 = 17,$$

and so on.

Theon states, with reference to these numbers, the general proposition that

$$d_n^2 = 2a_n^2 \pm 1,$$

and he observes (1) that the signs alternate as successive *d*'s and *a*'s are taken, $d_1^2 - 2a_1^2$ being equal to -1, $d_2^2 - 2a_2^2$ equal to $+1$, $d_3^2 - 2a_3^2$ equal to 1, and so on, (2) that the sum of the squares of *all* the *d*'s will be double of the sum of the squares of *all* the *a*'s. [If the number of successive terms in each series is *finite*, it is of course necessary that the number should be even.] The proof, no doubt omitted because it was well known, may be put algebraically thus

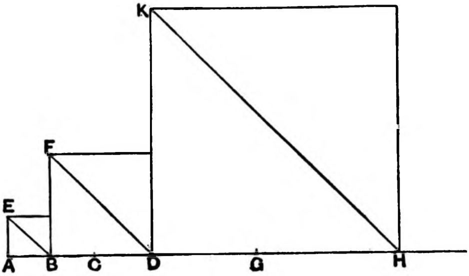
$$\begin{aligned} d_n^2 - 2a_n^2 &= (2a_{n-1} + d_{n-1})^2 - 2(a_{n-1} + d_{n-1})^2 \\ &= 2a_{n-1}^2 - d_{n-1}^2 \\ &= -(d_{n-1}^2 - 2a_{n-1}^2) \\ &= +(d_{n-2}^2 - 2a_{n-2}^2), \text{ in like manner,} \end{aligned}$$

and so on, while $d_1^2 - 2a_1^2 = -1$. Thus the theorem is established.

Euclid's propositions enable us to establish the theorem geometrically; and this fact might well be thought to confirm the conjecture that the investigation of the indeterminate equation $2x^2 - y^2 = \pm 1$ in the manner explained by Theon was no new thing but began at a period long before Euclid's time. No one familiar with the truth of the proposition stated by Theon could have failed to observe that, as the corresponding *side-* and *diagonal*-numbers were successively formed, the value of d_n^2/a_n^2 would approach more and more nearly to 2, and consequently that the successive fractions d_n/a_n would give nearer and nearer approximations to the value of $\sqrt{2}$, viz. $\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \ldots$.

It is fairly clear that in the famous passage of Plato's *Republic* (546 C) about the "geometrical number" some such system of approximations is hinted at. Plato there contrasts the "*rational diameter* of five" (ῥητὴ διάμετρος τῆς πεμπάδος) with the "irrational" (diameter). This was certainly taken from the Pythagorean theory of numbers (cf. the expression immediately preceding, 546 B, C πάντα προσήγορα καὶ ῥητὰ πρὸς ἄλληλα ἀπέφηναν, with the phrase πάντα γνωστὰ καὶ ποτάγορα ἀλλάλοις ἀπεργάζεται in the fragment of Philolaus). The reference of Plato is to the following consideration. If the square of side 5 be taken, the diagonal is $\sqrt{2 \cdot 25}$ or $\sqrt{50}$. This is the Pythagorean "*irrational* diameter" of 5; and the "rational diameter" was the approximation $\sqrt{50 - 1}$, or 7.

But the conjecture of Zeuthen, and the attribution of the whole theory of *side-* and *diagonal*-numbers to the Pythagoreans, have now been fully confirmed by the publication of Kroll's edition of *Procli Diadochi in Platonis rempublicam commentarii* (Teubner), Vol. II., 1901. The passages (cc. 23 and 27, pp. 24, 25 and 27—29) which there saw the light for the first time describe the same system of forming *side-* and *diagonal-*numbers and definitely attribute it, as well as the distinction between the "rational" and "irrational diameter," to the Pythagoreans. Proclus further says (p. 27, 16—22) that the property of the *side-* and *diagonal-*numbers "is proved graphically (γραμμικῶς) in the second book of the Elements by '*him*' (ἀπ' ἐκείνου). For, *if a straight line be bisected and a straight line be added to it, the square on the whole line including the added straight line and the square on the latter alone are double of the square on the half of the original straight line and of the square on the straight line made up of the half and of the added straight line.*" And this is simply Eucl. II. 10. Proclus then goes on to show specifically how this proposition was used to prove that, with the notation above used, the *diameter* corresponding to the *side* $a + d$ is $2a + d$. Let AB be a *side* and BC equal to it, while CD is the *diameter* corresponding to AB, i.e. a straight line such that the square on it is double of the square on AB. (I use the figure supplied by Hultsch on p. 397 of Kroll's Vol. II.)



Then, by the theorem of Eucl. II. 10, the squares on AD, DC are double of the squares on AB, BD.

But the square on DC (i.e. BE) is double of the square on AB; therefore, by subtraction, the square on AD is double of the square on BD.

And the square on DF, the *diagonal* corresponding to the *side* BD, is double the square of BD.

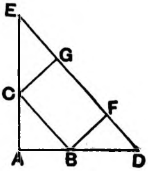
Therefore the square on DF is equal to the square on AD, so that DF is equal to AD.

That is, while the *side* BD is, with our notation, $a + d$, the corresponding *diagonal*, being equal to AD, is $2a + d$.

In the above reference by Proclus to II. 10 ἀπ' ἐκείνου "by *him*" must apparently mean ὑπ' Εὐκλείδου, "by Euclid," although Euclid's name has not been mentioned in the chapter; the phrase would be equivalent to saying "in the second Book of the famous Elements." But, when Proclus says "this *is proved* in the second Book of the Elements," he does not imply that it had not been proved before; on the contrary, it is clear that the theorem had been proved by the Pythagoreans, and we have therefore here a confirmation of the inference from the part played by the *gnomon* and by I. 47 in Book II. that the whole of the substance of that Book was Pythagorean. For further detailed explanation of the passages of Proclus reference should be made to Hultsch's note in Kroll's Vol. II. pp. 393—400, and to the separate article, also by Hultsch, in the *Bibliotheca Mathematica* 1_3, 1900, pp. 8—12.

P. Bergh has an ingenious suggestion (see *Zeitschrift für Math. u. Physik*

xxxi. Hist.-litt. Abt. p. 135, and Cantor, *Geschichte der Mathematik*, I_3, p. 437) as to the way in which the formation of the successive *side-* and *diagonal*-numbers may have been discovered, namely by observation from a very simple geometrical figure. Let ABC be an isosceles triangle, right-angled at A, with sides a_{n-1}, a_{n-1}, d_{n-1} respectively. If now the two sides AB, AC about the right angle be lengthened by adding d_{n-1} to each, and the extremities D, E be joined, it is easily seen by means of the figure (in which BF, CG are perpendicular to DE) that the new *diagonal* d_n is equal to $2a_{n-1} + d_{n-1}$, while the equal *sides* a_n are, by construction, equal to $a_{n-1} + d_{n-1}$.

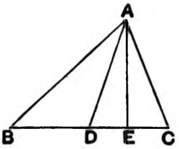


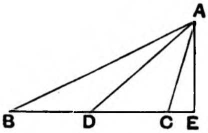
Important deductions from II. 9, 10.

I. Pappus (VII. pp. 856—8) uses II. 9, 10 for proving tne well-known theorem that

The sum of the squares on two sides of a triangle is equal to twice the square on half the base together with twice the square on the straight line joining the middle point of the base to the opposite vertex.

Let ABC be the given triangle and D the middle point of the base BC. Join AD, and draw AE perpendicular to BC (produced if necessary).





Now, by II. 9, 10,

the squares on BE, EC are equal to twice the squares on BD, DE.

Add to each twice the square on AE.

Then, remembering that

the squares on BE, EA are equal to the square on BA,

the squares on AE, EC are equal to the square on AC,

and the squares on AE, ED are equal to the square on AD,

we find that

the squares on BA, AC are equal to twice the squares on AD, BD.

The proposition is generally proved by means of II. 12, 13, but not, I think, so conveniently as by the method of Pappus.

II. The inference was early made by Gregory of St. Vincent (1584–1667) and Viviani (1622–1703) that *In any parallelogram the squares on the diagonals are together equal to the squares on the sides, or to twice the squares on adjacent sides.*

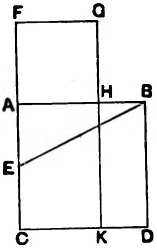
III. It appears that Leonhard Euler (1707–83) was the first to discover the corresponding theorem with reference to any quadrilateral, namely that *In any quadrilateral the sum of the squares on the sides is equal to the sum of the squares on the diagonals and four times the square on the line joining the middle* *points of the diagonals.* Euler seems however to have proved the property from the corresponding theorem for parallelograms just quoted (cf. Camerer's Euclid, Vol. I. pp. 468, 469) and not from the property of the triangle, though the latter brings out the result more easily.

PROPOSITION 11.

To cut a given straight line so that the rectangle contained by the whole and one of the segments is equal to the square on the remaining segment.

Let *AB* be the given straight line;

thus it is required to cut *AB* so that the rectangle contained by the whole and one of the segments is equal to the square on the remaining segment.



For let the square *ABDC* be described on *AB*; [I. 46]

let *AC* be bisected at the point *E*, and let *BE* be joined;

let *CA* be drawn through to *F*, and let *EF* be made equal to *BE*;

let the square *FH* be described on *AF*, and let *GH* be drawn through to *K*.

I say that *AB* has been cut at *H* so as to make the rectangle contained by *AB*, *BH* equal to the square on *AH*.

For, since the straight line *AC* has been bisected at *E*, and *FA* is added to it,

the rectangle contained by *CF*, *FA* together with the square on *AE* is equal to the square on *EF*. [II. 6]

But *EF* is equal to *EB*;

therefore the rectangle *CF*, *FA* together with the square on *AE* is equal to the square on *EB*.

But the squares on *BA*, *AE* are equal to the square on *EB*, for the angle at *A* is right: [I. 47]

therefore the rectangle *CF*, *FA* together with the square on *AE* is equal to the squares on *BA*, *AE*.

Let the square on *AE* be subtracted from each;

therefore the rectangle *CF*, *FA* which remains is equal to the square on *AB*.

Now the rectangle *CF, FA* is *FK*, for *AF* is equal to *FG*;

and the square on *AB* is *AD*;

therefore *FK* is equal to *AD*.

Let *AK* be subtracted from each;

therefore *FH* which remains is equal to *HD*.

And *HD* is the rectangle *AB, BH*, for *AB* is equal to *BD*;

and *FH* is the square on *AH*;

therefore the rectangle contained by *AB, BH* is equal to the square on *HA*.

therefore the given straight line *AB* has been cut at *H* so as to make the rectangle contained by *AB, BH* equal to the square on *HA*.

Q. E. F.

As the solution of this problem is necessary to that of inscribing a regular pentagon in a circle (Eucl. IV. 10, 11), we must necessarily conclude that it was solved by the Pythagoreans, or, in other words, that they discovered the geometrical solution of the quadratic equation

$$a(a-x)=x^2,$$

or

$$x^2+ax=a^2.$$

The solution in II. 11, too, exactly corresponds to the solution of the more general equation

$$x^2+ax=b^2,$$

which, as shown above (pp. 387—8), Simson based upon II. 6. Only Simson's solution, if applied here, gives us the point *F* on *CA* produced and does not directly find the point *H*. It takes *E* the middle point of *CA*, draws *AB* at right angles to *CA* and of length equal to *CA*, and then describes a circle with *EB* as radius cutting *EA* produced in *F*. The only difference between the solution in this case and in the more general case is that *AB* is here equal to *CA* instead of being equal to another given straight line *b*.

As in the more general case, there is, from Euclid's point of view, only one solution.

The construction shows that *CF* is also divided at *A* in the manner described in the enunciation, since the rectangle *CF, FA* is equal to the square on *CA*.

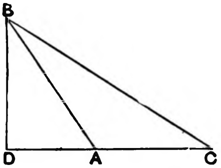
The problem in II. 11 reappears in VI. 30 in the form of *cutting a given straight line in extreme and mean ratio.*

PROPOSITION 12.

In obtuse-angled triangles the square on the side subtending the obtuse angle is greater than the squares on the sides containing the obtuse angle by twice the rectangle contained by one of the sides about the obtuse angle, namely that on which the *perpendicular falls, and the straight line cut off outside by the perpendicular towards the obtuse angle.*

Let *ABC* be an obtuse-angled triangle having the angle *BAC* obtuse, and let *BD* be drawn from the point *B* perpendicular to *CA* produced;

I say that the square on *BC* is greater than the squares on *BA*, *AC* by twice the rectangle contained by *CA*, *AD*.



For, since the straight line *CD* has been cut at random at the point *A*, the square on *DC* is equal to the squares on *CA*, *AD* and twice the rectangle contained by *CA*, *AD*. [II. 4]

Let the square on *DB* be added to each;

therefore the squares on *CD*, *DB* are equal to the squares on *CA*, *AD*, *DB* and twice the rectangle *CA*, *AD*.

But the square on *CB* is equal to the squares on *CD*, *DB*, for the angle at *D* is right; [I. 47]

and the square on *AB* is equal to the squares on *AD*, *DB*; [I. 47]

therefore the square on *CB* is equal to the squares on *CA*, *AB* and twice the rectangle contained by *CA*, *AD*;

so that the square on *CB* is greater than the squares on *CA*, *AB* by twice the rectangle contained by *CA*, *AD*.

Therefore etc. Q. E. D.

Since in this proposition and the next we have to do with the squares on the sides of triangles, the particular form of graphic representation of areas which we have had in Book II. up to this point does not help us to visualise the results of the propositions in the same way, and only two lines of proof are possible, (1) by means of the *results* of certain earlier propositions in Book II. combined with the *result* of I. 47 and (2) by means of the procedure in Euclid's proof of I. 47 itself. The alternative proofs of II. 12, 13 after the manner of Euclid's proof of I. 47 are therefore alone worth giving.

These proofs appear in certain modern text-books (e.g. Mehler, Henrici and Treutlein, H. M. Taylor, Smith and Bryant). Smith and Bryant are not correct in saying (p. 142) that they cannot be traced further back than Lardner's Euclid (1828); they are to be found in Gregory of St. Vincent's work (published in 1647) *Opus geometricum quadraturae circuli et sectionum coni*, Book I. Pt. 2, Props. 44, 45 (pp. 31, 32).

To prove II. 12, take an obtuse-angled triangle *ABC* in which the angle at *A* is the obtuse angle

Describe squares on *BC*, *CA*, *AB*, as *BCED*, *CAGF*, *ABKH*.

Draw *AL*, *BM*, *CN*, perpendicular to *BC*, *CA*, *AB* (produced if necessary), and produce them to meet the further sides of the squares on them in *P*, *Q*, *R* respectively.

Join *AD*, *CK*.

Then, as in I. 47, the triangles *KBC*, *ABD* are equal in all respects;

therefore their doubles, the parallelograms in the same parallels respectively, are equal;

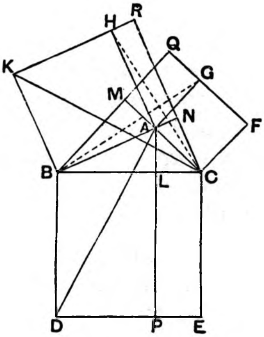
that is, the rectangle *BP* is equal to the rectangle *BR*.

Similarly the rectangle *CP* is equal to the rectangle *CQ*.

Also, if *BG*, *CH* be joined, we see that

the triangles *BAG*, *HAC* are equal in all respects;

therefore their doubles, the rectangles *AQ*, *AR*, are equal.



Now the square on *BC* is equal to the sum of the rectangles *BP*, *CP*,

i.e. to the sum of the rectangles *BR*, *CQ*,

i.e. to the sum of the squares *BH*, *CG* and the rectangles *AR*, *AQ*.

But the rectangles *AR*, *AQ* are equal, and they are respectively the rectangle contained by *BA*, *AN* and the rectangle contained by *CA*, *AM*.

Therefore the square on *BC* is equal to the squares on *BA*, *AC* together with twice the rectangle *BA*, *AN* or *CA*, *AM*.

Incidentally this proof shows that the rectangle *BA*, *AN* is equal to the rectangle *CA*, *AM*: a result which will be seen later on to be a particular case of the theorem in III. 35.

Heron (in an-Nairīzī, ed. Curtze, p. 109) gives a "converse" of II. 12 related to it as I. 48 is related to I. 47.

In any triangle, if the square on one of the sides is greater than the squares on the other two sides, the angle contained by the latter is obtuse.

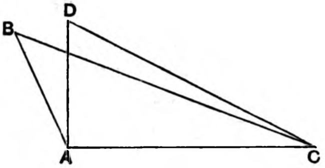
Let *ABC* be a triangle such that the square on *BC* is greater than the squares on *BA*, *AC*.

Draw *AD* at right angles to *AC* and of length equal to *AB*.

Join *DC*.

Then, since *DAC* is a right angle, the square on *DC* is equal to the squares on *DA*, *AC*, [I. 47]

i.e. to the squares on *BA*, *AC*.



But the square on *BC* is greater than the squares on *BA*, *AC*; therefore the square on *BC* is greater than the square on *DC*.

Therefore *BC* is greater than *DC*.

Thus, in the triangles *BAC*, *DAC*,

the two sides *BA*, *AC* are equal to the two sides *DA*, *AC* respectively, but the base *BC* is greater than the base *DC*.

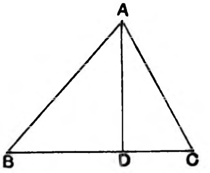
Therefore the angle BAC is greater than the angle DAC; [I. 25]
that is, the angle BAC is *obtuse*.

PROPOSITION 13.

In acute-angled triangles the square on the side subtending the acute angle is less than the squares on the sides containing the acute angle by twice the rectangle contained by one of the sides about the acute angle, namely that on which the perpendicular falls, and the straight line cut off within by the perpendicular towards the acute angle.

Let ABC be an acute-angled triangle having the angle at B acute, and let AD be drawn from the point A perpendicular to BC;

I say that the square on AC is less than the squares on CB, BA by twice the rectangle contained by CB, BD.



For, since the straight line CB has been cut at random at D,

the squares on CB, BD are equal to twice the rectangle contained by CB, BD and the square on DC. [II. 7]

Let the square on DA be added to each;

therefore the squares on CB, BD, DA are equal to twice the rectangle contained by CB, BD and the squares on AD, DC.

But the square on AB is equal to the squares on BD, DA, for the angle at D is right; [I. 47]

and the square on AC is equal to the squares on AD, DC;

therefore the squares on CB, BA are equal to the square on AC and twice the rectangle CB, BD,

so that the square on AC alone is less than the squares on CB, BA by twice the rectangle contained by CB, BD.

Therefore etc.

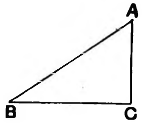
Q. E. D.

As the text stands, this proposition is unequivocally enunciated of *acute-angled* triangles; and, as if to obviate any doubt as to whether the restriction was fully intended, the enunciation speaks of the rectangle contained by one of the sides containing the acute angle and the straight line intercepted *within* by the perpendicular towards the acute angle. On the other hand, it is curious that it speaks of the square on the side subtending *the* acute angle; and again the *setting-out* begins "let ABC be an acute-angled triangle *having the angle at* B *acute*," though the last words have no point if all the angles of the triangle are necessarily acute.

It was however very early noticed, not only by Isaacus Monachus, Campanus, Peletarius, Clavius, Commandinus and the rest, but by the Greek scholiast (Heiberg, Vol. v. p. 253), that the relation between the sides of a triangle established by this theorem is true of the side opposite to, and the sides about, an acute angle respectively in any sort of triangle whether acute-angled, right-angled or obtuse-angled. The scholiast tries to explain away the word "acute-angled" in the enunciation: "Since in the definitions he calls acute-angled the triangle which has three acute angles, you must know that he does not mean that here, but calls all triangles acute-angled because all have an acute angle, one at least, if not all. The enunciation therefore is: 'In any triangle the square on the side subtending the acute angle is less than the squares on the sides containing the acute angle by twice the rectangle, etc.'"

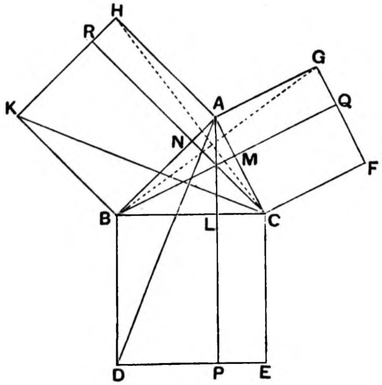
We may judge too by Heron's enunciation of his "converse" of the proposition that he would have left the word "acute-angled" out of the enunciation. His converse is: *In any triangle in which the square on one of the sides is less than the squares on the other two sides, the angle contained by the latter sides is acute.*

If the triangle that we take is a right-angled triangle, and the perpendicular is drawn, not from the right angle, but from the acute angle *not* referred to in the enunciation, the proposition reduces to I. 47, and this case need not detain us.



The other cases can be proved, like II. 12, after the manner of I. 47.

Let us take first the case where all the angles of the triangle are acute.



As before, if we draw ALP, BMQ, CNR perpendicular to BC, CA, AB and meeting the further sides of the squares on BC, CA, AB in P, Q, R, and if we join KC, AD, we have

the triangles KBC, ABD equal in all respects,

and consequently the rectangles BP, BR equal to one another.

Similarly the rectangles CP, CQ are equal to one another.

Next, by joining *BG*, *CH*, we prove in like manner that the rectangles *AR*, *AQ* are equal.

Now the square on *BC* is equal to the sum of the rectangles *BP*, *CP*,

i.e. to the sum of the rectangles *BR*, *CQ*,

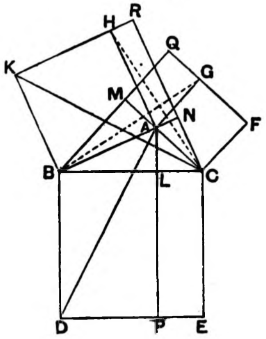
i.e. to the sum of the squares *BH*, *CG* diminished by the rectangles *AR*, *AQ*.

But the rectangles *AR*, *AQ* are equal, and they are respectively the rectangles contained by *BA*, *AN* and by *CA*, *AM*.

Therefore the square on *BC* is less than the squares on *BA*, *AC* by twice the rectangle *BA*, *AN* or *CA*, *AM*.

Next suppose that we have to prove the theorem in the case where the triangle has an obtuse angle at *A*.

Take *B* as the acute angle under consideration, so that *AC* is the side opposite to it.



Now the square on *CA* is equal to the difference of the rectangles *CQ*, *AQ*,

i.e. to the difference between *CP* and *AQ*,

i.e. to the difference between the square *BE* and the sum of the rectangles *BP*, *AQ*,

i.e. to the difference between the square *BE* and the sum of the rectangles *BP*, *AR*,

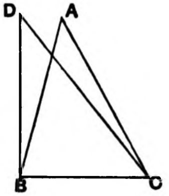
i.e. to the difference between the sum of the squares *BE*, *BH* and the sum of the rectangles *BP*, *BR*

(since *AR* is the difference between *BR* and *BH*).

But *BP*, *BR* are equal, and they are respectively the rectangles *CB*, *BL* and *AB*, *BN*.

Therefore the square on *CA* is less than the squares on *AB*, *BC* by twice the rectangle *CB*, *BL* or *AB*, *BN*.

Heron's proof of his converse proposition (an-Nairīzī, ed. Curtze, p. 110), which is also given by the Greek scholiast above quoted, is of course simple. For let *ABC* be a triangle in which the square on *AC* is less than the squares on *AB*, *BC*.



Draw *BD* at right angles to *BC* and of length equal to *BA*.

Join *DC*.

Then, since the angle *CBD* is right, the square on *DC* is equal to the squares on *DB*, *BC*, i.e. to the squares on *AB*, *BC*. [I. 47]

But the square on *AC* is less than the squares on *AB*, *BC*.

Therefore the square on *AC* is less than the square on *DC*.

Therefore *AC* is less than *DC*.

Hence in the two triangles *DBC*, *ABC* the sides about the angles *DBC*, *ABC* are respectively equal, but the base *DC* is greater than the base *AC*.

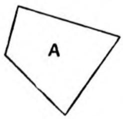
Therefore the angle DBC (a right angle) is greater than the angle ABC [I. 25], which latter is therefore *acute*.

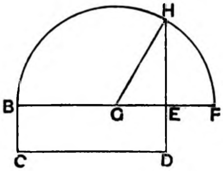
It may be noted, lastly, that II. 12, 13 are supplementary to I. 47 and complete the theory of the relations between the squares on the sides of *any* triangle, whether right-angled or not.

PROPOSITION 14.

To construct a square equal to a given rectilineal figure.

Let A be the given rectilineal figure;
thus it is required to construct a square equal to the rectilineal figure A.





For let there be constructed the rectangular parallelogram BD equal to the rectilineal figure A. [I. 45]

Then, if BE is equal to ED, that which was enjoined will have been done; for a square BD has been constructed equal to the rectilineal figure A.

But, if not, one of the straight lines BE, ED is greater.

Let BE be greater, and let it be produced to F;
let EF be made equal to ED, and let BF be bisected at G.

With centre G and distance one of the straight lines GB, GF let the semicircle BHF be described; let DE be produced to H, and let GH be joined.

Then, since the straight line BF has been cut into equal segments at G, and into unequal segments at E,

the rectangle contained by BE, EF together with the square on EG is equal to the square on GF. [II. 5]

But GF is equal to GH;
therefore the rectangle BE, EF together with the square on GE is equal to the square on GH.

But the squares on HE, EG are equal to the square on GH; [I. 47]
therefore the rectangle BE, EF together with the square on GE is equal to the squares on HE, EG.

Let the square on GE be subtracted from each;

therefore the rectangle contained by *BE*, *EF* which remains is equal to the square on *EH*.

But the rectangle *BE*, *EF* is *BD*, for *EF* is equal to *ED*;

therefore the parallelogram *BD* is equal to the square on *HE*.

And *BD* is equal to the rectilineal figure *A*.

Therefore the rectilineal figure *A* is also equal to the square which can be described on *EH*.

Therefore a square, namely that which can be described on *EH*, has been constructed equal to the given rectilineal figure *A*. Q. E. F.

7. **that which was enjoined will have been done**, literally "would have been done," γεγονὸς ἂν εἴη τὸ ἐπιταχθέν.

35, 36. **which can be described**, expressed by the future passive participle, ἀναγραφησομένῳ, ἀναγραφησόμενον.

Heiberg (*Mathematisches zu Aristoteles*, p. 20) quotes as bearing on this proposition Aristotle's remark (*De anima* II. 2, 413 a 19: cf. *Metaph.* 996 b 21) that "squaring" (τετραγωνισμός) is better defined as the "finding of the mean (proportional)" than as "the making of an equilateral rectangle equal to a given oblong," because the former definition states the *cause*, the latter the *conclusion* only. This, Heiberg thinks, implies that in the text-books which were in Aristotle's hands the problem of II. 14 was solved by means of proportions. As a matter of fact, the actual *construction* is the same in II. 14 as in VI. 13; and the change made by Euclid must have been confined to substituting in the *proof* of the correctness of the construction an argument based on the principles of Books I. and II. instead of Book VI.

As II. 12, 13 are supplementary to I. 47, so II. 14 completes the theory of *transformation of areas* so far as it can be carried without the use of proportions. As we have seen, the propositions I. 42, 44, 45 enable us to construct a parallelogram having a given side and angle, and equal to any given rectilineal figure. The parallelogram can also be transformed into an equal triangle with the same given side and angle by making the other side about the angle twice the length. Thus we can, as a particular case, construct a rectangle on a given base (or a right-angled triangle with one of the sides about the right angle of given length) equal to a given square. Further, I. 47 enables us to make a square equal to the sum of any number of squares or to the difference between any two squares. The problem still remaining unsolved is to transform any rectangle (as representing an area equal to that of any rectilineal figure) into a square of equal area. The solution of this problem, given in II. 14, is of course the equivalent of the extraction of the square root, or of the solution of the pure quadratic equation

$$x^2 = ab.$$

Simson pointed out that, in the construction given by Euclid in this case, it was not necessary to put in the words "*Let* BE *be greater*," since the construction is not affected by the question whether *BE* or *ED* is the greater. This is true, but after all the words do little harm, and perhaps Euclid may have regarded it as conducive to clearness to have the points *B*, *G*, *E*, *F* in the same relative positions as the corresponding points *A*, *C*, *D*, *B* in the figure of II. 5 which he quotes in the proof.

EXCURSUS I.

PYTHAGORAS AND THE PYTHAGOREANS.

The problem of determining how much of the Pythagorean discoveries in mathematics can be attributed to Pythagoras himself is not only difficult; it may be said to be insoluble. Tradition on the subject is very meagre and uncertain, and further doubt is thrown upon it by the well-known tendency of the later Pythagoreans to ascribe everything to the Master himself (αὐτὸς ἔφα, *Ipse dixit*). Pythagoras himself left no written exposition of his doctrines, nor did any of his immediate successors, not even Hippasus, about whom the different stories ran (1) that he was expelled from the school because he published doctrines of Pythagoras, and (2) that he was drowned at sea for revealing the construction of the dodecahedron in the sphere and claiming it as his own, or (as others have it) for making known the discovery of the irrational or incommensurable. Nor is the absence of any written record of Pythagorean doctrines down to the time of Philolaus to be put down to a pledge of secrecy binding the school; at all events this did not apply to their mathematics or their physics; and it may be that the supposed secrecy was invented to account for the absence of documents. The fact seems to be that oral communication was the tradition of the school, while their doctrines would in the main be too abstruse to be understood by the generality of people outside. Even Aristotle felt the difficulty; he evidently knew nothing for certain about any ethical or physical doctrines going back to Pythagoras himself; when he speaks of the Pythagorean system, he always refers it to "the Pythagoreans." sometimes even to "the so-called Pythagoreans."

Since my note on Eucl. I. 47 was originally written the part of Pythagoras in the Pythagorean mathematical discoveries has been further discussed and every scrap of evidence closely, and even meticulously, examined in two long articles by Heinrich Vogt, "Die Geometrie des Pythagoras" (*Bibliotheca Mathematica* IX_3, 1908/9, pp. 15—54) and "Die Entstehungsgeschichte des Irrationalen nach Plato und anderen Quellen des 4. Jahrhunderts" (*Bibliotheca Mathematica* X_3, 1910, pp. 97—155). These papers would not indeed have enabled me to modify greatly what I have written regarding the supposed discoveries of Pythagoras and the early Pythagoreans, because I have throughout been careful to give the traditions on the subject for what they are worth and no more, and not to build too much upon them. It is right however to give, in a separate note, a few details of Vogt's arguments.

G. Junge had, in his paper "Wann haben die Griechen das Irrationale entdeckt?" mentioned above (p. 351), tried to prove that Pythagoras himself could not have discovered the irrational; and the object of Vogt's papers is to go further on the same lines and to show (1) that it was only the later Pythagoreans who (before 410 B.C.) recognised the incommensurability of the diagonal with the side of a square, (2) that the *theory* of the irrational was first discovered by Theodorus, to whom Plato refers (*Theaetetus* 147 D), and (3) that Pythagoras himself could not have been the discoverer of any one of the things specifically attributed to him, namely (*a*) the theorem of Eucl. I. 47, (*b*) the construction of the five regular solids in the sense in which they are respectively constructed in Eucl. XIII, (*c*) the application of an area in its widest sense, equivalent to the solution of a quadratic equation in its most general form.

Vogt's main argument as regards (*a*) the theorem of I. 47 is based on a new translation which he gives of the well-known passage of Proclus' note on the proposition (p. 426, 6—9), Τῶν μὲν ἱστορεῖν τὰ ἀρχαῖα βουλομένων ἀκούοντας τὸ θεώρημα τοῦτο εἰς Πυθαγόραν ἀναπεμπόντων ἐστὶν εὑρεῖν καὶ βουθύτην λεγόντων αὐτὸν ἐπὶ τῇ εὑρέσει. Vogt translates this as follows: "Unter denen, welche das Altertum erforschen wollen, kann man einige finden, welche denen Gehör geben, die dieses Theorem auf Pythagoras zurückführen und ihn als Stieropferer bei dieser Gelegenheit bezeichnen," "Among those who have a taste for research into antiquity, we can find some who give ear to those who refer this theorem to Pythagoras and describe him as sacrificing an ox on the strength of the discovery." According to this version the words τῶν... βουλομένων and the words ἀναπεμπόντων...καὶ...λεγόντων refer respectively to two different sets of persons, in fact two different generations; the latter are older authorities who are supposed to be cited by the former; the former are a later generation, perhaps contemporaries of Proclus, some of whom accepted the view of the older authorities while others did not. But this would have required the article τῶν before ἀναπεμπόντων, or some such expression as ἄλλων τινῶν οἳ ἀναπέμπουσι instead of ἀναπεμπόντων. Vogt's interpretation is therefore quite inadmissible. The persons denoted by ἀναπεμπόντων are *some of* the persons denoted by τῶν βουλομένων; hence Tannery's translation, to which mine (p. 350 above) is equivalent, is the only possible one, namely "Si l'on écoute ceux qui veulent raconter l'histoire des anciens temps, on peut en trouver qui attribuent ce théorème à Pythagore et lui font sacrifier un bœuf après sa découverte" (*La Géométrie grecque*, p. 103). ἀκούοντας agrees with the assumed *subject* of εὑρεῖν; ἀναπεμπόντων and λεγόντων should, strictly speaking, have been ἀναπέμποντας and λέγοντας agreeing with τινὰς (the direct *object* of εὑρεῖν) understood, but are simply attracted into the case of βουλομένων; the construction is quite intelligible. I agree with Vogt that Eudemus' history contained nothing attributing the theorem to Pythagoras. The words of Proclus imply this; but I do not think that they imply (as Vogt maintains) any pronouncement by Proclus himself *against* such attribution. In my opinion, Proclus is simply determined not to commit himself to any view; his way of evading a decision is the sentence following, ἐγὼ δὲ θαυμάζω μὲν καὶ τοὺς πρώτους ἐπιστάντας τῇ τοῦδε τοῦ θεωρήματος ἀληθείᾳ, μειζόνως δὲ ἄγαμαι τὸν στοιχειωτήν...; the plural τοὺς πρώτους ἐπιστάντας is, I hold, used for the very purpose of making the statement as vague as possible; he will not even allow it to be inferred that he attributed the discovery to any single person. Returning to ἡ τῶν ἀλόγων πραγματεία (Proclus, p. 65, 19), we may now concede (following Diels) that we should read τῶν ἀνὰ λόγον ("proportionals") instead of τῶν ἀλόγων ("irrationals") and that the author intended to attribute to Pythagoras a theory of *proportion* (the arithmetical theory applicable to commensurable magnitudes only) rather than the theory of irrationals. But I do not agree in Vogt's contention that the theory of the irrational was first discovered by Theodorus. It seems to me that we have evidence to the contrary in the very passage of Plato referred to. Plato (*Theaetetus* 147 D) mentions $\sqrt{3}$, $\sqrt{5}$, ... up to $\sqrt{17}$ as dealt with by Theodorus, but *omits* $\sqrt{2}$. This fact, along with Plato's allusions elsewhere to the irrationality of $\sqrt{2}$, and to approximations to it, in the expressions *ἄρρητος* and *ῥητὴ διάμετρος τῆς πεμπάδος*, as if those expressions had a well-known signification, implies that the discovery of the irrationality of $\sqrt{2}$ had been made before the time of Theodorus. The words *ἡ τῶν ἀλόγων πραγματεία* might well be used even if the reference is only to $\sqrt{2}$, because the first step would be the most difficult, and *πραγματεία* need not mean the establishment of a complete theory or anything more than "investigation" of a subject.

Junge and Vogt hold that the theory of the irrational was not discovered by the early Pythagoreans any more than Pythagoras because, if it had been so discovered, an impossibly long period would intervene between the investigation of the particular case of $\sqrt{2}$ and the extension of the theory by Theodorus to the cases of $\sqrt{3}$, $\sqrt{5}$ etc. But might not this well be due to the fact that in the meantime the minds of geometers were engrossed by other problems of importance, namely the quadrature of the circle (Hippocrates of Chios and his quadratures of lunes), the trisection of any angle (Hippias of Elis and his curve, afterwards known as the *quadratrix*), and the doubling of the cube (reduced by Hippocrates to the problem of finding two mean proportionals in continued proportion between two given straight lines), the last of which problems, which meant finding geometrically the equivalent of $\sqrt[3]{2}$, would naturally follow the investigation of $\sqrt{2}$? Now Hippias was probably born about 460 B.C., while Hippocrates seems to have been in Athens during a considerable portion of the second half of the fifth century, perhaps from about 450 to 430 B.C. Moreover Vogt has to get over the fact that Democritus (born 470/469 B.C.) wrote a book *περὶ ἀλόγων γραμμῶν καὶ ναστῶν*, *On irrational lines and solids* (or *atoms*). This difficulty he seeks to overcome by maintaining that *ἀλόγων* does not here mean "irrational" at all, but "without ratio" ("verhältnislos"), in the sense that any two straight lines are "without ratio" because they both contain an infinite number of the indivisible (or atomic) lines, and therefore their ratio, being of the form ∞/∞, is indeterminate. But, if these were so, *all* lines (including commensurable lines) would be "without a ratio" to one another, whereas the title of Democritus' work clearly implies that *ἄλογοι γραμμαί* are a class or classes of lines distinguished from other lines. The fact is that Democritus was too good a mathematician to have anything to do with "indivisible lines." This is confirmed by a scholium to Aristotle's *De caelo* (p. 469 b 14, Brandis) which implicitly denies to Democritus any theory of indivisible lines: "of those who have maintained the existence of indivisibles, some, as for example Leucippus and Democritus, believe in indivisible bodies, others, like Xenocrates, in indivisible lines." Moreover Simplicius tells us that, according to Democritus himself, even the atoms were, in a mathematical sense, divisible further and in fact *ad infinitum*.

Coming now to (*b*) the construction of the cosmic figures, *ἡ τῶν κοσμικῶν σχημάτων σύστασις* (Proclus, p. 65, 20), I agree with Vogt to the following extent. It is unlikely that Pythagoras or even the early Pythagoreans "constructed" the five regular solids in the sense of a complete theoretical construction such as we find, say, in Eucl. XIII.; and it is possible that Theaetetus was the first to give these constructions, whether *ἔγραψε* in Suidas' notice, *πρῶτος δὲ τὰ πέντε καλούμενα στερεὰ ἔγραψε*, means "constructed" or "wrote upon." But *σύστασις* in the above phrase of Proclus may well mean something less than the theoretical constructions and proofs of Eucl. XIII.; it may mean, as Vogt says, simply the "putting together" of the figures in the same way as Plato puts them together in the *Timaeus*, i.e. by bringing a certain number of angles of equilateral triangles and of regular pentagons together at one point. There is no reason why the early Pythagoreans should not have "constructed" the five regular solids in this sense; in fact the supposition that they did so agrees well with what we know of their having put angles of certain regular figures together round a point (in connexion with the theorem of Eucl. I. 32) and shown that only three kinds of such angles would fill up the space *in one plane* round the point. But I do not agree in the apparent refusal of Vogt to credit the Pythagoreans with the knowledge of the theoretical construction of the regular pentagon as we find it in Eucl. IV. 10, 11. I do not know of any reason for rejecting the evidence of the Scholia IV. Nos. 2 and 4 which say categorically that "this Book" (Book IV.) and "the whole of the theorems" in it (including therefore Props. 10, 11) are discoveries of the Pythagoreans. And the division of a straight line in extreme and mean ratio, on which the construction of the regular pentagon depends, comes in Eucl. Book II. (Prop. 11), while we have sufficient grounds for regarding the whole of the substance of Book II. as Pythagorean.

I will permit myself one more criticism on Vogt's first paper. I think he bases too much on the fact that it was left for Oenopides (in the period from, say, 470 to 450 B.C.) to discover two elementary constructions (with ruler and compasses only), namely that of a perpendicular to a straight line from an external point (Eucl. I. 12), and that of an angle equal to a given rectilineal angle (Eucl. I. 23). Vogt infers that geometry must have been in a very rudimentary condition at the time. I do not think this follows; the explanation would seem to be rather that, the restriction of the instruments used in constructions to the ruler and compasses not having been definitely established before the time when Oenopides wrote, it had not previously occurred to anyone to substitute new constructions based on that principle for others previously in vogue. In the case of the perpendicular, for example, the construction would no doubt, in earlier days, have been made by means of a set square.

EXCURSUS II.

POPULAR NAMES FOR EUCLIDEAN PROPOSITIONS.

Although some of these time-honoured names are familiar to most educated people, it seems to be impossible to trace them to their original sources, or to say who applied them for the first time respectively. It may be that they were handed down by oral tradition for long periods in each case before they found their way into written documents.

We begin with

I. 5.

1. This proposition is in this country universally known as the *Pons Asinorum*, "Asses' Bridge." Even in this case opinion is not unanimous as to the exact implication of the term. Perhaps the more general view is that taken in the *Stanford Dictionary of Anglicised Words and Phrases* (by C. A. M. Fennell) where the description is: "Name of the fifth proposition of the first Book of Euclid, suggested by the figure and the difficulties which poor geometricians find in mastering it." This is certainly the equivalent of what I gathered, in my early days at school, from a former Fellow of St John's, the Reverend Anthony Bower, who was a high Wrangler in 1846 and a friend of Todhunter's. The "ass" on this interpretation is a synonym for "fool." But there is another view (as I have learnt lately) which is more complimentary to the ass. It is that, the figure of the proposition being like that of a trestle-bridge, with a ramp at each end which is the more practicable the flatter the figure is drawn, the bridge is such that, while a horse could not surmount the ramp, an ass could; in other words, the term is meant to refer to the surefootedness of the ass rather than to any want of intelligence on his part. (I may perhaps mention that Sir George Greenhill is a strong supporter of this view.)

An epigram of 1780 is the earliest reference to the term in Murray's English Dictionary:

> "If this be rightly called the bridge of asses,
> He's not the fool that sticks but he that passes."

The writer's own view is not too clear. He seems to imply that, while the inventor of the name meant that only the fool finds the bridge difficult to pass, the more proper view would be that, since the ass can get over, and "ass" is synonymous with "fool," therefore it must be the fool who *can* get over; in other words, he seems to object to the phrase as being a contradiction in terms.

But we have also to take account of the fact that the French apply the term to I. 47. Now in Euclid's figure for I. 47 there is no suggestion of a bridge, and the reference can only be to the nature of the theorem, its difficulty or otherwise. It is curious that the French dictionaries give two different explanations of *Pont aux ânes*. Littré makes it "ce que personne ne doit ni ne peut ignorer; ce qui est si facile que tout le monde doit y réussir." Now no intelligent person could have applied the name to Eucl. I. 47 for this reason, namely that it was so easy that even a fool could not help knowing it. Larousse is better informed; there we find "*Pont aux ânes*, certaine difficulté, certaine question qui n'arrête que les ignorants, et qui sert de critérium pour juger l'intelligence de quelqu'un, et particulièrement d'un écolier. C'est ainsi que, dans les classes de mathématiques, on ne manque jamais de dire que le carré de l'hypoténuse est le *pont aux ânes* de la géométrie. La plupart des dictionnaires entendent par ce mot une chose si simple, si facile, que personne ne doit l'ignorer : c'est une erreur évidente." Larousse is clearly right. But it will be observed that, so far as it goes, Larousse's interpretation rather supports the first of the two alternative explanations of the meaning of "Asses' Bridge" as applied to I. 5, namely that it is difficult for the fool (= "ass") to master.

In the *Stanford Dictionary* it is added that "in logic the term was in the 16 c. applied to the conversion of propositions by the aid of a difficult diagram for finding middle terms"; and if the mathematicians borrowed the term from logic, this again would be rather in favour of the first explanation of its use for I. 5.

If it is permitted *desipere in loco*, I would add for the benefit of future generations (in the hope that they will still be able to appreciate the joke or, in the alternative, will be tempted to discuss learnedly what could possibly have been meant) a very topical allusion in a recent *Punch* (14 Oct. 1925): "When they film Euclid, as is suggested, we shall no doubt see a very thrilling rescue over the burning Pons Asinorum."—And yet it is safe to prophesy that the "Asses' Bridge" will outlive the "film"!

2. *Elefuga.*

This name for Eucl. I. 5 is mentioned by Roger Bacon (about 1250), who also gives an explanation of it (*Opus Tertium*, c. vi). He observes that in his day people in general, finding no utility in any science such as geometry, for example, recoiled from the idea of studying it unless they were boys forced to it by the rod, so that they would hardly learn so much as three or four propositions. Hence it is, he says, that the fifth proposition is called "Elefuga, id est, fuga miserorum; elegia enim Graece dicitur, Latine miseria; et elegi sunt miseri." That is, according to Roger Bacon, Elefuga is "flight of the miserable." This explanation no doubt accounts for the verses about *Dulcarnon* in Chaucer's *Troilus and Criseyde*, III, ll. 933–5:

> "Dulcarnon called is 'fleminge of wrecches';
> It seemeth hard, for wrecches wol not lere
> For verray slouthe or othere wilful tecches";

since "fleminge of wrecches," "banishment of the miserable," is a translation of "fuga miserorum." Only Dulcarnon is there wrongly taken to be the same proposition as Elefuga, i.e. I. 5, whereas, as we shall see, Dulcarnon was really the name for the Pythagorean theorem I. 47.

Etymologically, Roger Bacon's explanation leaves something to be desired. The word would really seem to be an attempt to compound the two Greek words ἔλεος, pity (or the object of pity), and φυγή, flight (cf. note *ad loc.* in Skeat's edition of Chaucer). Notwithstanding the confusion of tongues, the object seems to be a play upon the two words *Elementa* and ἔλεος, which both begin with the same three letters, and the implication is that "escape from the Elements" (which normally came when Prop. 5 was reached) was equivalent to "escape from misery" or "trouble." A better form for the word would perhaps be Eleufuga; and this form actually occurs in Alanus' *Anticlaudianus*, III, c. 6 (cited by Du Cange, *Glossarium*, s.v.). The word also occurs, according to Skeat's note, in Richard of Bury's *Philobiblon*, c. xiii, where it was somewhat oddly translated by J. B. Inglis in 1832 "How many scholars has the Helleflight of Euclid repelled!"

I. 47.

The Pythagorean proposition about the square on the hypotenuse has taken even a deeper hold of the minds of men, and has been known by a number of names.

1. *The Theorem of the Bride* (θεώρημα τῆς νύμφης).

This name is found in a MS. of Georgius Pachymeres (1242–1310) in the Bibliothèque Nationale at Paris; there is a note to this effect by Tannery (*La Géométrie grecque*, p. 105), but, as he says nothing more, it is probable that the passage gives the mere name without any explanation of it. We have, however, much earlier evidence of the supposed connexion of the proposition with marriage. Plutarch (born about 46 A.D.) says (*De Iside et Osiride* 56, p. 373 F) "We may imagine the Egyptians (thinking of) the most beautiful of triangles (and) likening the nature of the All to this triangle most particularly, for it is this same triangle which Plato is thought to have employed in the *Republic*, when he put together the Nuptial Figure (γαμήλιον διάγραμμα)"—διάγραμμα, though literally meaning "diagram" or "figure," was commonly used in the sense of "proposition"—"and in that triangle the perpendicular side is 3, the base 4, and the hypotenuse, the square on which is equal to the sum of the squares on the sides containing (the right angle), 5. We must, then, liken the perpendicular to the male, the base to the female and the hypotenuse to the offspring of both....For 3 is the first odd number and is perfect, 4 is the square on an even side, 2, while the 5 partly resembles the father and partly the mother, being the sum of 3 and 2."

Plato used the three numbers 3, 4, 5 of the Pythagorean triangle in the formation of his famous Geometrical Number; but Plato himself does not call the triangle the Nuptial Triangle nor the number the Nuptial Number. It is later writers, Plutarch, Nicomachus and Iamblichus, who connect the passage about the Geometrical Number with marriage; Nicomachus (*Introd. Ar.*, II, 24, 11) merely alludes to "the passage in the Republic connected with the so-called Marriage," while Iamblichus (*In Nicom.*, p. 82, 20 Pistelli) only speaks of "the Nuptial Number in the Republic."

It would appear, then, that the name "Nuptial Figure" or "Theorem of the Bride" was originally used of one particular right-angled triangle, namely (3, 4, 5). A late Arabian writer Behā-ad-dīn (1547–1622) seems to have applied the term "Figure of the Bride" to the same triangle; the Arabs therefore seemingly followed the Greeks. The idea underlying the use of the term, first for the triangle (3, 4, 5), and then for the general theorem of I. 47, seems to be roughly that of the two parties to a marriage becoming one, just as the two squares on the sides containing the right angle become the one square on the hypotenuse in the said theorem.

2. *The "Bride's Chair."*

The origin of this name is more obscure. It must presumably have been suggested by a supposed resemblance between the figure of the proposition and such a chair. D. E. Smith (*History of Mathematics*, II, pp. 289–90) remarks that the "Bride's Chair" may be so-called "because the Euclid figure is not unlike the chair which a slave carries on his back and in which the Eastern bride is sometimes transported to the ceremony," and he cites a note from Edouard Lucas' *Récréations Mathématiques*, II, p. 130: "La démonstration que nous venons de donner du théorème de Pythagore sur le carré de l'hypoténuse ne diffère pas essentialement de la démonstration hindoue, connue sous le nom de la *Chaise de la petite mariée*, que l'on rencontre dans l'ouvrage de Bhascara (Bija-Ganita, § 146)." The figure of Bhāskara is not that of Euclid but that shown at the top of p. 355 above; I have however not been able to find the name "Bride's Chair" in Colebrooke's translation of the work of Bhāskara.

Notwithstanding the apparent frivolity of the setting, I venture to suggest that light may be thrown on the question by a very modern version of the "Bride's Chair" which appeared during or since the War in *La Vie Parisienne*. The illustration represents Euclid's figure for I. 47 and, drawn over it, as on a frame, a *poilu* in full fighting kit carrying on his back his bride and his household belongings. Roughly speaking, the soldier is standing (or rather walking) in the middle of the large square, his head and shoulders are bending to the right within the contour of one of the small squares, while the lady, with mirror and powder-puff in action, is sitting with her back to him in the right angle between the two smaller squares (*HAG* in the figure on p. 349 above)[1]. I am informed by Sir George Greenhill that there was also an earlier version "showing the chair as it is in use to-day in Cairo and Egypt, the earliest version of a taxi-chair, a pattern as early as Euclid and suggesting the nickname of the proposition." This recalls to my mind the remark of a friend to whom I mentioned the subject and showed the figure of the proposition; he observed at once on seeing it "But I should have said it was more like a sedan chair," the large square suggesting to him the actual chair and the two smaller squares the two bearers.

3. *Dulcarnon*.

This name for I. 47 appears, as above mentioned, in Chaucer's *Troilus and Criseyde*, III, ll. 930–3, where Criseyde says:

> 'I am, til God me bettre minde sende,
> At dulcarnon, right at my wittes ende.'
> Quod Pandarus, 'ye, nece, wol ye here?
> Dulcarnon called is "fleminge of wrecches."'

Billingsley, too, in his edition of Euclid (1570) observes of I. 47 that "it hath bene commonly called of barbarous writers of the latter time Dulcarnon."

Dulcarnon (see Skeat's note *ad loc.*) seems to represent the Persian and Arabic *du 'lkarnayn*, lit. *two-horned*, from Pers. *du*, two, and *karn*, horn. The name was applied to I. 47 because the two smaller squares stick up like two horns and, as the proposition is difficult, the word here takes the sense of "puzzle"; hence Criseyde was "at dulcarnon" because she was perplexed and at her wit's end.

4. *Francisci tunica* = "Franciskaner Kutte," "Franciscan's cowl."

This name is quoted by Weissenborn (*Die Uebersetzungen des Euklid durch Campano und Zamberti*, p. 42) as given in a *Geometrie* by one Kunze. The name is quite appropriate, one of the squares representing the hood thrown back.

III. 7, 8.

I have already mentioned the names "Goose's Foot" (*Pes anseris*) and "Peacock's Tail" (*Cauda pavonis*) applied, suitably enough, to these propositions respectively. They come from Luca Paciuolo's edition of Euclid published in 1509 (*vide* Weissenborn, *ibid.*).

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[1] Old Cambridge men will recall a picture in some respects not unlike, though less artistic than, the cartoon in *La Vie Parisienne*, I mean the painting of "The Man Loaded with Mischief" which used to be over the door of the former inn of that name on the St Neots Road, a short distance from Cambridge.

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THE THIRTEEN BOOKS
OF
EUCLID'S ELEMENTS

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TRANSLATED FROM THE TEXT OF HEIBERG

WITH INTRODUCTION AND COMMENTARY

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BOOK III.

DEFINITIONS.

1. **Equal circles** are those the diameters of which are equal, or the radii of which are equal.

2. A straight line is said to **touch a circle** which, meeting the circle and being produced, does not cut the circle.

3. **Circles** are said to **touch one another** which, meeting one another, do not cut one another.

4. In a circle straight lines are said **to be equally distant from the centre** when the perpendiculars drawn to them from the centre are equal.

5. And that straight line is said to be **at a greater distance** on which the greater perpendicular falls.

6. A **segment of a circle** is the figure contained by a straight line and a circumference of a circle.

7. An **angle of a segment** is that contained by a straight line and a circumference of a circle.

8. An **angle in a segment** is the angle which, when a point is taken on the circumference of the segment and straight lines are joined from it to the extremities of the straight line which is the **base of the segment,** is contained by the straight lines so joined.

9. And, when the straight lines containing the angle cut off a circumference, the angle is said to **stand upon** that circumference.

10. **A sector of a circle** is the figure which, when an angle is constructed at the centre of the circle, is contained by the straight lines containing the angle and the circumference cut off by them.

11. **Similar segments of circles** are those which admit equal angles, or in which the angles are equal to one another.

DEFINITION I.

Ἴσοι κύκλοι εἰσίν, ὧν αἱ διάμετροι ἴσαι εἰσίν, ἢ ὧν αἱ ἐκ τῶν κέντρων ἴσαι εἰσίν.

Many editors have held that this should not have been included among definitions. Some, e.g. Tartaglia, would call it a *postulate*; others, e.g. Borelli and Playfair, would call it an *axiom*; others again, as Billingsley and Clavius, while admitting it as a *definition*, add explanations based on the mode of constructing a circle; Simson and Pfleiderer hold that it is a *theorem*. I think however that Euclid would have maintained that it is a definition in the proper sense of the term; and certainly it satisfies Aristotle's requirement that a "definitional statement" (ὁριστικὸς λόγος) should not only state the *fact* (τὸ ὅτι) but should indicate the *cause* as well (*De anima* II. 2, 413 a 13). The equality of circles with equal radii can of course be proved by superposition, but, as we have seen, Euclid avoided this method wherever he could, and there is nothing technically wrong in saying "By *equal circles* I mean circles with equal radii." No flaw is thereby introduced into the system of the *Elements*; for the definition could only be objected to if it could be proved that the equality predicated of the two circles in the definition was not the same thing as the equality predicated of other equal figures in the *Elements* on the basis of the Congruence-Axiom, and, needless to say, this cannot be proved because it is not true. The *existence* of equal circles (in the sense of the definition) follows from the existence of equal straight lines and I. Post. 3.

The Greeks had no distinct word for *radius*, which is with them, as here, *the (straight line drawn) from the centre* ἡ ἐκ τοῦ κέντρου (εὐθεῖα); and so definitely was the expression appropriated to the radius that ἐκ τοῦ κέντρου was used without the article as a predicate, just as if it were one word. Thus, e.g., in III. 1 ἐκ κέντρου γάρ means "for they are radii": cf. Archimedes, *On the Sphere and Cylinder* II. 2, ἡ BE ἐκ τοῦ κέντρου ἐστὶ τοῦ...κύκλου, *BE is a radius of the circle.*

DEFINITION 2.

Εὐθεῖα κύκλου ἐφάπτεσθαι λέγεται, ἥτις ἁπτομένη τοῦ κύκλου καὶ ἐκβαλλομένη οὐ τέμνει τὸν κύκλον.

Euclid's phraseology here shows the regular distinction between ἅπτεσθαι and its compound ἐφάπτεσθαι, the former meaning "to *meet*" and the latter "to *touch*." The distinction was generally observed by Greek geometers from Euclid onwards. There are however exceptions so far as ἅπτεσθαι is concerned; thus it means "to *touch*" in Eucl. IV. Def. 5 and sometimes in Archimedes. On the other hand, ἐφάπτεσθαι is used by Aristotle in certain cases where the orthodox geometrical term would be ἅπτεσθαι. Thus in *Meteorologica* III. 5 (376 b 9) he says a certain circle *will pass through all the angles* (ἁπασῶν ἐφάψεται τῶν γωνιῶν), and (376 a 6) M *will lie on a given (circular) circumference* (δεδομένης περιφερείας ἐφάψεται τὸ M). We shall find ἅπτεσθαι used in these senses in Book IV. Deff. 2, 6 and Deff. 1, 3 respectively. The latter of the two expressions quoted from Aristotle means that *the locus of* M *is a given circle*, just as in Pappus ἅψεται τὸ σημεῖον θέσει δεδομένης εὐθείας means that *the locus of* the point is a straight line given in position.

DEFINITION 3.

Κύκλοι ἐφάπτεσθαι ἀλλήλων λέγονται οἵτινες ἁπτόμενοι ἀλλήλων οὐ τέμνουσιν ἀλλήλους.

Todhunter remarks that different opinions have been held as to what is, or should be, included in this definition, one opinion being that it only means that the circles do not cut in the neighbourhood of the point of contact, and that it must be shown that they do not cut elsewhere, while another opinion is that the definition means that the circles do not cut at all. Todhunter thinks the latter opinion correct. I do not think this is proved; and I prefer to read the definition as meaning simply that the circles meet at a point but do not cut *at that point*. I think this interpretation preferable for the reason that, although Euclid does practically assume in III. 11—13, without stating, the theorem that circles touching at one point do not intersect anywhere else, he has given us, before reaching that point in the Book, means for proving for ourselves the truth of that statement. In particular, he has given us the propositions III. 7, 8 which, taken as a whole, give us more information as to the general nature of a circle than any other propositions that have preceded, and which can be used, as will be seen in the sequel, to solve any doubts arising out of Euclid's unproved assumptions. Now, as a matter of fact, the propositions are not used in any of the genuine proofs of the theorems in Book III.; III. 8 is required for the second proof of III. 9 which Simson selected in preference to the first proof, but the first proof only is regarded by Heiberg as genuine. Hence it would not be easy to account for the appearance of III. 7, 8 at all unless as affording means of answering possible *objections* (cf. Proclus' explanation of Euclid's reason for inserting the second part of I. 5).

External and *internal* contact are not distinguished in Euclid until III. 11, 12, though the *figure* of III. 6 (not the *enunciation* in the original text) represents the case of internal contact only. But the definition of touching circles here given must be taken to imply so much about *internal* and *external* contact respectively as that (*a*) a circle touching another internally must, immediately before "meeting" it, have passed through points *within* the circle that it touches, and (*b*) a circle touching another externally must, immediately before meeting it, have passed through points *outside* the circle which it touches. These facts must indeed be admitted if *internal* and *external* are to have any meaning at all in this connexion, and they constitute a minimum admission necessary to the proof of III. 6.

DEFINITION 4.

Ἐν κύκλῳ ἴσον ἀπέχειν ἀπὸ τοῦ κέντρου εὐθεῖαι λέγονται, ὅταν αἱ ἀπὸ τοῦ κέντρου ἐπ᾽ αὐτὰς κάθετοι ἀγόμεναι ἴσαι ὦσιν.

DEFINITION 5.

Μεῖζον δὲ ἀπέχειν λέγεται, ἐφ' ἣν ἡ μείζων κάθετος πίπτει.

DEFINITION 6.

Τμῆμα κύκλου ἐστὶ τὸ περιεχόμενον σχῆμα ὑπό τε εὐθείας καὶ κύκλου περιφερείας.

DEFINITION 7.

Τμήματος δὲ γωνία ἐστὶν ἡ περιεχομένη ὑπό τε εὐθείας καὶ κύκλου περιφερείας.

This definition is only interesting historically. The *angle of a segment*, being the "angle" formed by a straight line and a "circumference," is of the kind described by Proclus as "mixed." A particular "angle" of this sort is the "*angle of a semicircle*," which we meet with again in III. 16, along with the so-called "horn-like angle" (κερατοειδής), the supposed "angle" between a tangent to a circle and the circle itself. The "angle of a semicircle" occurs once in Pappus (VII. p. 670, 19), but it there means scarcely more than the *corner* of a semicircle regarded as a point to which a straight line is directed. Heron does not give the definition of the *angle of a segment*, and we may conclude that the mention of it and of the *angle of a semicircle* in Euclid is a survival from earlier text-books rather than an indication that Euclid considered either to be of importance in elementary geometry (cf. the note on III. 16 below).

We have however, in the note on I. 5 above (Vol. I. pp. 252—3), seen evidence that the *angle of a segment* had played some part in geometrical proofs up to Euclid's time. It would appear from the passage of Aristotle there quoted (*Anal. prior.* I. 24, 41 b 13 sqq.) that the theorem of I. 5 was, in the text-books immediately preceding Euclid, proved by means of the equality of the two "*angles of*" any one segment. This latter property must therefore have been regarded as more elementary (for whatever reason) than the theorem of I. 5; indeed the definition as given by Euclid practically implies the same thing, since it speaks of only *one* "angle of a segment," namely "*the* angle contained by a straight line and a circumference of a circle." Euclid abandoned the actual use of the "angle" in question, but no doubt thought it unnecessary to break with tradition so far as to strike the definition out also.

DEFINITION 8.

Ἐν τμήματι δὲ γωνία ἐστίν, ὅταν ἐπὶ τῆς περιφερείας τοῦ τμήματος ληφθῇ τι σημεῖον καὶ ἀπ' αὐτοῦ ἐπὶ τὰ πέρατα τῆς εὐθείας, ἥ ἐστι βάσις τοῦ τμήματος, ἐπιζευχθῶσιν εὐθεῖαι, ἡ περιεχομένη γωνία ὑπὸ τῶν ἐπιζευχθεισῶν εὐθειῶν.

DEFINITION 9.

Ὅταν δὲ αἱ περιέχουσαι τὴν γωνίαν εὐθεῖαι ἀπολαμβάνωσί τινα περιφέρειαν, ἐπ' ἐκείνης λέγεται βεβηκέναι ἡ γωνία.

DEFINITION 10.

Τομεὺς δὲ κύκλου ἐστίν, ὅταν πρὸς τῷ κέντρῳ τοῦ κύκλου συσταθῇ γωνία, τὸ περιεχόμενον σχῆμα ὑπό τε τῶν τὴν γωνίαν περιεχουσῶν εὐθειῶν καὶ τῆς ἀπολαμβανομένης ὑπʼ αὐτῶν περιφερείας.

A scholiast says that it was the *shoemaker's knife*, σκυτοτομικὸς τομεύς, which suggested the name τομεύς for a sector of a circle. The derivation of the name from a resemblance of shape is parallel to the use of ἄρβηλος (also a *shoemaker's knife*) to denote the well known figure of the Book of Lemmas partly attributed to Archimedes.

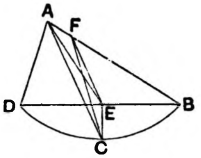
A wider definition of a sector than that given by Euclid is found in a Greek scholiast (Heiberg's Euclid, Vol. v. p. 260) and in an-Nairīzī (ed. Curtze, p. 112). "There are two varieties of sectors; the one kind have the angular vertices at the centres, the other at the circumferences. Those others which have their vertices neither at the circumferences nor at the centres, but at some other points, are for that reason not called sectors but sector-like figures (τομοειδῆ σχήματα)." The exact agreement between the scholiast and an-Nairīzī suggests that Heron was the authority for this explanation.

The *sector-like figure* bounded by an arc of a circle and two lines drawn from its extremities to meet at any point actually appears in Euclid's book *On divisions* (περὶ διαιρέσεων) discovered in an Arabic MS. and edited by Woepcke (cf. Vol. I. pp. 8—10 above). This treatise, alluded to by Proclus, had for its object the division of figures such as triangles, trapezia, quadrilaterals and circles, by means of straight lines, into parts equal or in given ratios. One proposition e.g. is, *To divide a triangle into two equal parts by a straight line passing through a given point on one side.* The proposition (28) in which the *quasi-sector* occurs is, *To divide such a figure by a straight line into two equal parts.* The solution in this case is given by Cantor (*Gesch. d. Math.* I$_3$, pp. 287—8).

If $ABCD$ be the given figure, E the middle point of BD and EC at right angles to BD, the broken line AEC clearly divides the figure into two equal parts.

Join AC, and draw EF parallel to it meeting AB in F.

Join CF, when it is seen that CF divides the figure into two equal parts.



DEFINITION 11.

Ὅμοια τμήματα κύκλων ἐστὶ τὰ δεχόμενα γωνίας ἴσας, ἢ ἐν οἷς αἱ γωνίαι ἴσαι ἀλλήλαις εἰσίν.

De Morgan remarks that the use of the word *similar* in "similar segments" is an anticipation, and that similarity *of form* is meant. He adds that the definition is a theorem, or would be if "similar" had taken its final meaning.

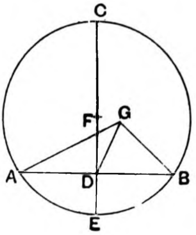
BOOK III. PROPOSITIONS.

PROPOSITION 1.

To find the centre of a given circle.

Let ABC be the given circle;

thus it is required to find the centre of the circle ABC.



Let a straight line AB be drawn through it at random, and let it be bisected at the point D;

from D let DC be drawn at right angles to AB and let it be drawn through to E;

let CE be bisected at F;

I say that F is the centre of the circle ABC.

For suppose it is not, but, if possible, let G be the centre,

and let GA, GD, GB be joined.

Then, since AD is equal to DB,

and DG is common,

the two sides AD, DG are equal to the two sides BD, DG respectively;

and the base GA is equal to the base GB, for they are radii;

therefore the angle ADG is equal to the angle GDB. [I. 8]

But, when a straight line set up on a straight line makes the adjacent angles equal to one another, each of the equal angles is right; [I. Def. 10]

therefore the angle GDB is right.

But the angle FDB is also right; therefore the angle FDB is equal to the angle GDB, the greater to the less: which is impossible.

Therefore G is not the centre of the circle ABC.

Similarly we can prove that neither is any other point except F.

Therefore the point F is the centre of the circle ABC.

PORISM. From this it is manifest that, if in a circle a straight line cut a straight line into two equal parts and at right angles, the centre of the circle is on the cutting straight line.

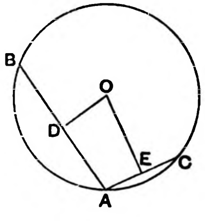
Q. E. F.

12. **For suppose it is not.** This is expressed in the Greek by the two words Μὴ γάρ, but such an elliptical phrase is impossible in English.

17. **the two sides AD, DG are equal to the two sides BD, DG respectively.** As before observed, Euclid is not always careful to put the equals in corresponding order. The text here has "*GD*, *DB*."

Todhunter observes that, when, in the construction, DC is said to be *produced* to E, it is assumed that D is within the circle, a fact which Euclid first demonstrates in III. 2. This is no doubt true, although the word διήχθω, "let it be *drawn through*," is used instead of ἐκβεβλήσθω, "let it be *produced*." And, although it is not necessary to assume that D is within the circle, it is necessary for the success of the construction that the straight line drawn through D at right angles to AB shall meet the circle in two points (and no more): an assumption which we are not entitled to make on the basis of what has gone before only.

Hence there is much to be said for the alternative procedure recommended by De Morgan as preferable to that of Euclid. De Morgan would first prove the fundamental theorem that "the line which bisects a chord perpendicularly must contain the centre," and then make III. 1, III. 25 and IV. 5 immediate corollaries of it. The fundamental theorem is a direct consequence of the theorem that, if P is *any* point equidistant from A and B, then P lies on the straight line bisecting AB perpendicularly. We then take any two chords AB, AC of the given circle and draw DO, EO bisecting them perpendicularly. Unless BA, AC are in one straight line, the straight lines DO, EO must meet in some point O (see note on IV. 5 for possible methods of proving this). And, since both DO, EO must contain the centre, O must be the centre.



This method, which seems now to be generally preferred to Euclid's, has the advantage of showing that, in order to find the centre of a circle, it is sufficient to know three points on the circumference. If therefore two circles have three points in common, they must have the same centre and radius, so that two circles cannot have three points in common without coinciding entirely. Also, as indicated by De Morgan, the same construction enables us (1) to draw the complete circle of which a segment or arc only is given (III. 25), and (2) to circumscribe a circle to any triangle (IV. 5).

But, if the Greeks had used this construction for finding the centre of a circle, they would have considered it necessary to add a proof that no other point than that obtained by the construction can be the centre, as is clear both from the similar *reductio ad absurdum* in III 1 and also from the fact that Euclid thinks it necessary to prove as a separate theorem (III. 9) that, if a point within a circle be such that three straight lines (at least) drawn from it to the circumference are equal, that point must be the centre. In fact, however, the proof amounts to no more than the remark that the two perpendicular bisectors can have no more than one point common.

And even in De Morgan's method there is a yet unproved assumption. In order that *DO*, *EO* may meet, it is necessary that *AB*, *AC* should not be in one straight line or, in other words, that *BC* should not pass through *A*. This results from III. 2, which therefore, strictly speaking, should precede.

To return to Euclid's own proposition III. 1, it will be observed that the demonstration only shows that the centre of the circle cannot lie on either side of *CD*, so that it must lie on *CD* or *CD* produced. It is however taken for granted rather than proved that the centre must be the middle point of *CE*. The proof of this by *reductio ad absurdum* is however so obvious as to be scarcely worth giving. The same consideration which would prove it may be used to show that *a circle cannot have more than one centre*, a proposition which, if thought necessary, may be added to III. 1 as a corollary.

Simson observed that the proof of III. 1 could not but be by *reductio ad absurdum*. At the beginning of Book III. we have nothing more to base the proof upon than the definition of a circle, and this cannot be made use of unless we assume some point to be the centre. We cannot however assume that the point found by the construction is the centre, because that is the thing to be proved. Nothing is therefore left to us but to assume that some other point is the centre and then to prove that, whatever other point is taken, an absurdity results; whence we can infer that the point found is the centre.

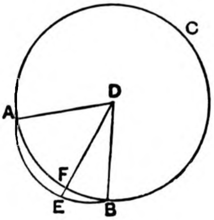
The Porism to III. 1 is inserted, as usual, parenthetically before the words ὅπερ ἔδει ποιῆσαι, which of course refer to the problem itself.

PROPOSITION 2.

If on the circumference of a circle two points be taken at random, the straight line joining the points will fall within the circle.

Let *ABC* be a circle, and let two points *A*, *B* be taken at random on its circumference;

I say that the straight line joined from *A* to *B* will fall within the circle.



For suppose it does not, but, if possible, let it fall outside, as *AEB*; let the centre of the circle *ABC* be taken [III. 1], and let it be *D*; let *DA*, *DB* be joined, and let *DFE* be drawn through.

Then, since *DA* is equal to *DB*,

the angle *DAE* is also equal to the angle *DBE*. [I. 5]

And, since one side *AEB* of the triangle *DAE* is produced,

the angle *DEB* is greater than the angle *DAE*. [I. 16]

But the angle *DAE* is equal to the angle *DBE*;

therefore the angle *DEB* is greater than the angle *DBE*.

And the greater angle is subtended by the greater side; [I. 19]

therefore *DB* is greater than *DE*.

But *DB* is equal to *DF*;

therefore *DF* is greater than *DE*,

the less than the greater: which is impossible.

Therefore the straight line joined from *A* to *B* will not fall outside the circle.

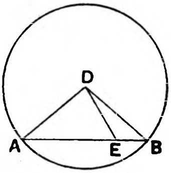
Similarly we can prove that neither will it fall on the circumference itself;

therefore it will fall within.

Therefore etc.

Q. E. D.

The *reductio ad absurdum* form of proof is not really necessary in this case, and it has the additional disadvantage that it requires the destruction of two hypotheses, namely that the chord is (1) outside, (2) on the circle. To prove the proposition directly, we have only to show that, if *E* be any point on the straight line *AB* between *A* and *B*, *DE* is less than the radius of the circle. This may be done by the method shown above, under I. 24, for proving what is assumed in that proposition, namely that, in the figure of the proposition, *F* falls below *EG* if *DE* is not greater than *DF*. The assumption amounts to the following proposition, which De Morgan would make to precede I. 24: "Every straight line drawn from the vertex of a triangle to the base is less than the greater of the two sides, or than either if they be equal." The case here is that in which the two sides are equal; and, since the angle *DAB* is equal to the angle *DBA*, while the exterior angle *DEA* is greater than the interior and opposite angle *DBA*, it follows that the angle *DEA* is greater than the angle *DAE*, whence *DE* must be less than *DA* or *DB*.



Camerer points out that we may add to this proposition the further statement that all points on *AB produced* in either direction are outside the circle. This follows from the proposition (also proved by means of the theorems that the exterior angle of a triangle is greater than either of the interior and opposite angles and that the greater angle is subtended by the greater side) which De Morgan proposes to introduce after I. 21, namely, "The perpendicular is the shortest straight line that can be drawn from a given point to a given straight line, and of others that which is nearer to the perpendicular is less than the more remote, and the converse; also not more than two equal straight lines can be drawn from the point to the line, one on each side of the perpendicular."

The fact that not more than two equal straight lines can be drawn from a given point to a given straight line not passing through it is proved by Proclus on I. 16 (see the note to that proposition) and can alternatively be proved by means of I. 7, as shown above in the note on I. 12. It follows that

A straight line cannot cut a circle in more than two points

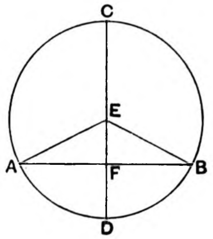
a proposition which De Morgan would introduce here after III. 2. The proof given does not apply to a straight line *passing through the centre*; but that such a line only cuts the circle in two points is self-evident.

PROPOSITION 3.

If in a circle a straight line through the centre bisect a straight line not through the centre, it also cuts it at right angles; and if it cut it at right angles, it also bisects it.

Let *ABC* be a circle, and in it let a straight line *CD* through the centre bisect a straight line *AB* not through the centre at the point *F*;

I say that it also cuts it at right angles.



For let the centre of the circle *ABC* be taken, and let it be *E*; let *EA*, *EB* be joined.

Then, since *AF* is equal to *FB*, and *FE* is common,

two sides are equal to two sides;

and the base *EA* is equal to the base *EB*;

therefore the angle *AFE* is equal to the angle *BFE*. [I. 8]

But, when a straight line set up on a straight line makes the adjacent angles equal to one another, each of the equal angles is right; [I. Def. 10]

therefore each of the angles *AFE*, *BFE* is right.

Therefore *CD*, which is through the centre, and bisects *AB* which is not through the centre, also cuts it at right angles.

Again, let *CD* cut *AB* at right angles;

I say that it also bisects it, that is, that *AF* is equal to *FB*.

For, with the same construction,

since *EA* is equal to *EB*,

the angle *EAF* is also equal to the angle *EBF*. [I. 5]

But the right angle *AFE* is equal to the right angle *BFE*, therefore *EAF*, *EBF* are two triangles having two angles equal to two angles and one side equal to one side, namely *EF*, which is common to them, and subtends one of the equal angles;

therefore they will also have the remaining sides equal to the remaining sides; [I. 26]

therefore *AF* is equal to *FB*.

Therefore etc.

Q. E. D.

26. **with the same construction,** τῶν αὐτῶν κατασκευασθέντων.

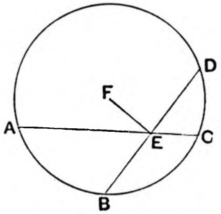
This proposition asserts the two *partial* converses (cf. note on I. 6) of the Porism to III. 1. De Morgan would place it next to III. 1.

PROPOSITION 4.

If in a circle two straight lines cut one another which are not through the centre, they do not bisect one another.

Let *ABCD* be a circle, and in it let the two straight lines *AC*, *BD*, which are not through the centre, cut one another at *E*;

I say that they do not bisect one another.



For, if possible, let them bisect one another, so that *AE* is equal to *EC*, and *BE* to *ED*;

let the centre of the circle *ABCD* be taken [III. 1], and let it be *F*; let *FE* be joined.

Then, since a straight line *FE* through the centre bisects a straight line *AC* not through the centre,

it also cuts it at right angles; [III. 3]

therefore the angle *FEA* is right.

Again, since a straight line *FE* bisects a straight line *BD*,

it also cuts it at right angles; [III. 3]

therefore the angle *FEB* is right.

But the angle FEA was also proved right;

therefore the angle FEA is equal to the angle FEB, the less to the greater: which is impossible.

Therefore AC, BD do not bisect one another.

Therefore etc.

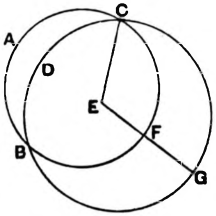
Q. E. D.

PROPOSITION 5.

If two circles cut one another, they will not have the same centre.

For let the circles ABC, CDG cut one another at the points B, C;

I say that they will not have the same centre.



For, if possible, let it be E; let EC be joined, and let EFG be drawn through at random.

Then, since the point E is the centre of the circle ABC,

EC is equal to EF. [I. Def. 15]

Again, since the point E is the centre of the circle CDG,

EC is equal to EG.

But EC was proved equal to EF also;

therefore EF is also equal to EG, the less to the greater: which is impossible.

Therefore the point E is not the centre of the circles ABC, CDG.

Therefore etc.

Q. E. D.

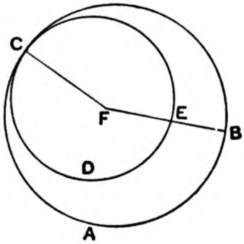
The propositions III. 5, 6 could be combined in one. It makes no difference whether the circles cut, or meet without cutting, so long as they do not coincide altogether; in either case they cannot have the same centre. The two cases are covered by the enunciation: *If the circumferences of two circles meet at a point they cannot have the same centre.* On the other hand, *If two circles have the same centre and one point in their circumferences common, they must coincide altogether.*

PROPOSITION 6.

If two circles touch one another, they will not have the same centre.

For let the two circles *ABC*, *CDE* touch one another at the point *C*;

I say that they will not have the same centre.



For, if possible, let it be *F*; let *FC* be joined, and let *FEB* be drawn through at random.

Then, since the point *F* is the centre of the circle *ABC*,

FC is equal to *FB*.

Again, since the point *F* is the centre of the circle *CDE*,

FC is equal to *FE*.

But *FC* was proved equal to *FB*;

therefore *FE* is also equal to *FB*, the less to the greater: which is impossible.

Therefore *F* is not the centre of the circles *ABC*, *CDE*.

Therefore etc.

Q. E. D.

The English editions enunciate this proposition of circles touching *internally*, but the word (ἐντός) is a mere interpolation, which was no doubt made because Euclid's figure showed only the case of *internal* contact. The fact is that, in his usual manner, he chose for demonstration the more difficult case, and left the other case (that of *external* contact) to the intelligence of the reader. It is indeed sufficiently self-evident that circles touching externally cannot have the same centre; but Euclid's proof can really be used for this case too.

Camerer remarks that the proof of III. 6 seems to assume tacitly that the points *E* and *B* cannot coincide, or that circles which touch internally at *C* cannot meet in any other point, whereas this fact is not proved by Euclid till III. 13. But no such general assumption is necessary here; it is only necessary that *one* line drawn from the assumed common centre should meet the circles in different points; and the very notion of internal contact requires that, before one circle *meets* the other on its inner side, it must have passed through points *within* the latter circle.

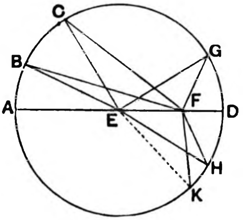
PROPOSITION 7.

If on the diameter of a circle a point be taken which is not the centre of the circle, and from the point straight lines fall upon the circle, that will be greatest on which the centre is, the remainder of the same diameter will be least, and of the rest the nearer to the straight line through the centre is always greater than the more remote, and only two equal straight lines will fall from the point on the circle, one on each side of the least straight line.

Let $ABCD$ be a circle, and let AD be a diameter of it; on AD let a point F be taken which is not the centre of the circle, let E be the centre of the circle,

and from F let straight lines FB, FC, FG fall upon the circle $ABCD$;

I say that FA is greatest, FD is least, and of the rest FB is greater than FC, and FC than FG.



For let BE, CE, GE be joined.

Then, since in any triangle two sides are greater than the remaining one, [I. 20]

EB, EF are greater than BF.

But AE is equal to BE;

therefore AF is greater than BF.

Again, since BE is equal to CE, and FE is common,

the two sides BE, EF are equal to the two sides CE, EF.

But the angle BEF is also greater than the angle CEF; therefore the base BF is greater than the base CF. [I. 24]

For the same reason

CF is also greater than FG.

Again, since GF, FE are greater than EG, and EG is equal to ED,

GF, FE are greater than ED.

Let EF be subtracted from each;

therefore the remainder GF is greater than the remainder FD.

Therefore FA is greatest, FD is least, and FB is greater than FC, and FC than FG.

I say also that from the point *F* only two equal straight lines will fall on the circle *ABCD*, one on each side of the least *FD*.

For on the straight line *EF*, and at the point *E* on it, let the angle *FEH* be constructed equal to the angle *GEF* [I. 23], and let *FH* be joined.

Then, since *GE* is equal to *EH*, and *EF* is common,

the two sides *GE*, *EF* are equal to the two sides *HE*, *EF*; and the angle *GEF* is equal to the angle *HEF*;

therefore the base *FG* is equal to the base *FH*. [I. 4]

I say again that another straight line equal to *FG* will not fall on the circle from the point *F*.

For, if possible, let *FK* so fall.

Then, since *FK* is equal to *FG*, and *FH* to *FG*,

FK is also equal to *FH*,

the nearer to the straight line through the centre being thus equal to the more remote: which is impossible.

Therefore another straight line equal to *GF* will not fall from the point *F* upon the circle;

therefore only one straight line will so fall.

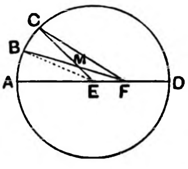
Therefore etc.

Q. E. D.

4. **of the same diameter.** I have inserted these words for clearness' sake. The text has simply ἐλαχίστη δὲ ἡ λοιπή, "and the remaining (straight line) least."

7, 39. **one on each side.** The word "one" is not in the Greek, but is necessary to give the force of ἐφ᾽ ἑκάτερα τῆς ἐλαχίστης, literally "on both sides," or "on each of the two sides, of the least."

De Morgan points out that there is an unproved assumption in this demonstration. We draw straight lines *from F*, as *FB*, *FC*, such that the angle *DFB* is greater than the angle *DFC* and then assume, with respect to the straight lines drawn *from the centre E* to *B*, *C*, that the angle *DEB* is greater than the angle *DEC*. This is most easily proved, I think, by means of the converse of part of the theorem about the lengths of different straight lines drawn to a given straight line from an external point which was mentioned above in the note on III. 2. This converse would be to the effect that, *If two unequal straight lines be drawn from a point to a given straight line which are not perpendicular to the straight line, the greater of the two is the further from the perpendicular from the point to the given straight line.* This can either be proved from its converse by *reductio ad absurdum*, or established directly by means of I. 47. Thus, in the accompanying figure, *FB* must cut *EC* in some point *M*, since the angle *BFE* is less than the angle *CFE*.



Therefore *EM* is less than *EC*, and therefore than *EB*.

Hence the point B in which FB meets the circle is further from the foot of the perpendicular from E on FB than M is;

therefore the angle BEF is greater than the angle CEF.

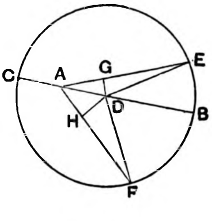
Another way of enunciating the first part of the proposition is that of Mr H. M. Taylor, viz. "Of all straight lines drawn to a circle from an internal point not the centre, the one which passes through the centre is the greatest, and the one which when produced passes through the centre is the least; and of any two others the one which *subtends the greater angle at the centre* is the greater." The substitution of the *angle subtended at the centre* as the criterion no doubt has the effect of avoiding the necessity of dealing with the unproved assumption in Euclid's proof referred to above, and the similar substitution in the enunciation of the first part of III. 8 has the effect of avoiding the necessity for dealing with like unproved assumptions in Euclid's proof, as well as the complication caused by the distinction in Euclid's enunciation between lines falling from an external point on the *convex circumference* and on the *concave circumference* of a circle respectively, terms which are not defined but taken as understood.

Mr Nixon (*Euclid Revised*) similarly substitutes as the criterion the angle subtended at the centre, but gives as his reason that the words "nearer" and "more remote" in Euclid's enunciation are scarcely clear enough without some definition of the sense in which they are used, Smith and Bryant make the substitution in III. 8, but follow Euclid in III. 7.

On the whole, I think that Euclid's plan of taking straight lines drawn from the point which is not the centre direct to the circumference and making greater or less angles *at that point* with the straight line containing it and the centre is the more instructive and useful of the two, since it is such lines drawn in any manner to the circle from the point which are immediately useful in the proofs of later propositions or in resolving difficulties connected with those proofs.

Heron again (an-Nairīzī, ed. Curtze, pp. 114—5) has a note on this proposition which is curious. He first of all says that Euclid proves that lines *nearer the centre* are greater than those more remote *from it*. This is a different view of the question from that taken in Euclid's proposition as we have it, in which the lines are not nearer to and more remote from the *centre* but from *the line through the centre*. Euclid takes lines inclined to the latter line at a greater or less angle; Heron introduces distance *from the centre* in the sense of Deff. 4, 5, i.e. in the sense of *the length of the perpendicular* drawn to the line from the centre, which Euclid does not use till III. 14, 15. Heron then observes that in Euclid's proposition the lines compared are all drawn on one side of the line through the centre, and sets himself to prove the same truth of lines on *opposite* sides which are more or less distant *from the centre*. The new point of view necessitates a quite different line of proof, anticipating the methods of later propositions.

The first case taken by Heron is that of two straight lines such that the perpendiculars from the centre on them fall on the lines themselves and not in either case on the line produced.



Let A be the given point, D the centre, and let AE be nearer the centre than AF, so that the perpendicular DG on AE is less than the perpendicular DH on AF.

Then sqs. on DG, GE = sqs. on DH, HF,

and sqs. on DG, GA = sqs. on DH, HA.

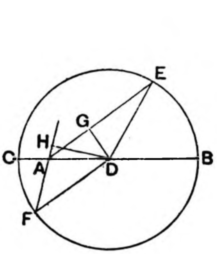
But sq. on DG < sq. on DH.

Therefore sq. on $GE >$ sq. on HF,
and sq. on $GA >$ sq. on HA,
whence $GE > HF$,
$GA > HA$.

Therefore, by addition, $AE > AF$.

The other case taken by Heron is that where one perpendicular falls on the line produced, as in the annexed figure. In this case we prove in like manner that $GE > HF$,
and $GA > AH$.

Thus AE is greater than the sum of HF, AH, whence, *a fortiori*, AE is greater than the difference of HF, AH, i.e. than AF.



Heron does not give the third possible case, that, namely, where *both* perpendiculars fall on the lines produced. The fact is that, in this case, the foregoing method breaks down. Though AE be nearer to the centre than AF in the sense that DG is less than DH,

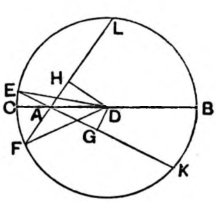
AE is not greater but *less* than AF.

Moreover this cannot be proved by the same method as before.

For, while we can prove that

$$GE > HF,$$
$$GA > AH,$$

we cannot make any inference as to the comparative length of AE, AF.



To judge by Heron's corresponding note to III. 8, he would, to prove this case, practically prove III. 35 first, i.e. prove that, if EA be produced to K and FA to L,

$$\text{rect. } FA, AL = \text{rect. } EA, AK,$$

from which he would infer that, since $AK > AL$ by the first case,

$$AE < AF.$$

An excellent moral can, I think, be drawn from the note of Heron. Having the appearance of supplementing, or giving an alternative for, Euclid's proposition, it cannot be said to do more than confuse the subject. Nor was it necessary to find a new proof for the case where the two lines which are compared are on *opposite* sides of the diameter, since Euclid shows that for each line from the point to the circumference on one side of the diameter there is another of the same length equally inclined to it on the other side.

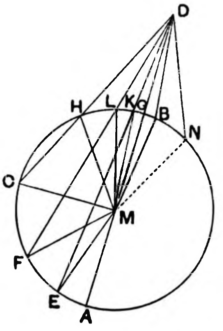
PROPOSITION 8.

If a point be taken outside a circle and from the point straight lines be drawn through to the circle, one of which is through the centre and the others are drawn at random, then, of the straight lines which fall on the concave circumference, that through the centre is greatest, while of the rest *the nearer to that through the centre is always greater than the more remote, but, of the straight lines falling on the convex circumference, that between the point and the diameter is least, while of the rest the nearer to the least is always less than the more remote, and only two equal straight lines will fall on the circle from the point, one on each side of the least.*

Let *ABC* be a circle, and let a point *D* be taken outside *ABC*; let there be drawn through from it straight lines *DA*, *DE*, *DF*, *DC*, and let *DA* be through the centre; I say that, of the straight lines falling on the concave circumference *AEFC*, the straight line *DA* through the centre is greatest,

while *DE* is greater than *DF* and *DF* than *DC*;

but, of the straight lines falling on the convex circumference *HLKG*, the straight line *DG* between the point and the diameter *AG* is least; and the nearer to the least *DG* is always less than the more remote, namely *DK* than *DL*, and *DL* than *DH*.



For let the centre of the circle *ABC* be taken [III. 1], and let it be *M*; let *ME*, *MF*, *MC*, *MK*, *ML*, *MH* be joined.

Then, since *AM* is equal to *EM*,

let *MD* be added to each;

therefore *AD* is equal to *EM*, *MD*.

But *EM*, *MD* are greater than *ED*; [I. 20]

therefore *AD* is also greater than *ED*.

Again, since *ME* is equal to *MF*,

and *MD* is common,

therefore *EM*, *MD* are equal to *FM*, *MD*;

and the angle *EMD* is greater than the angle *FMD*;

therefore the base *ED* is greater than the base *FD*. [I. 24]

Similarly we can prove that *FD* is greater than *CD*; therefore *DA* is greatest, while *DE* is greater than *DF*, and *DF* than *DC*.

Next, since *MK*, *KD* are greater than *MD*, [I. 20]
and *MG* is equal to *MK*,
therefore the remainder *KD* is greater than the remainder *GD*,

so that *GD* is less than *KD*.

And, since on *MD*, one of the sides of the triangle *MLD*, two straight lines *MK*, *KD* were constructed meeting within the triangle,
therefore *MK*, *KD* are less than *ML*, *LD*; [I. 21]
and *MK* is equal to *ML*;

therefore the remainder *DK* is less than the remainder *DL*.

Similarly we can prove that *DL* is also less than *DH*;

therefore *DG* is least, while *DK* is less than *DL*, and *DL* than *DH*.

I say also that only two equal straight lines will fall from the point *D* on the circle, one on each side of the least *DG*.

On the straight line *MD*, and at the point *M* on it,
let the angle *DMB* be constructed equal to the angle *KMD*,
and let *DB* be joined.

Then, since *MK* is equal to *MB*,
and *MD* is common,

the two sides *KM*, *MD* are equal to the two sides *BM*, *MD* respectively;
and the angle *KMD* is equal to the angle *BMD*;

therefore the base *DK* is equal to the base *DB*. [I. 4]

I say that no other straight line equal to the straight line *DK* will fall on the circle from the point *D*.

For, if possible, let a straight line so fall, and let it be *DN*.

Then, since *DK* is equal to *DN*,
while *DK* is equal to *DB*,

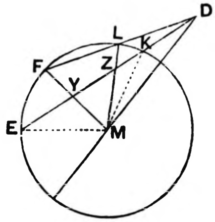
DB is also equal to *DN*,

that is, the nearer to the least *DG* equal to the more remote: which was proved impossible.

Therefore no more than two equal straight lines will fall on the circle *ABC* from the point *D*, one on each side of *DG* the least.

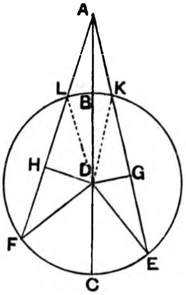
Therefore etc.

As De Morgan points out, there are here two assumptions similar to that tacitly made in the proof of III. 7, namely that K falls within the triangle DLM and E outside the triangle DFM. These facts can be proved in the same way as the assumption in III. 7. Let DE meet FM in Y and LM in Z. Then, as before, MZ is less than ML and therefore than MK. Therefore K lies further than Z from the foot of the perpendicular from M on DE. Similarly E lies further than Y from the foot of the same perpendicular.



Heron deals with lines on *opposite* sides of the diameter through the external point in a manner similar to that adopted in his previous note.

For the case where E, F are the *second* points in which AE, AF meet the circle the method answers well enough.



If AE is *nearer the centre* D than AF is,

sqs. on DG, GE = sqs. on DH, HF

and sqs. on DG, GA = sqs. on DH, HA,

whence, since $DG < DH$,

it follows that $GE > HF$,

and $AG > AH$,

so that, by addition, $AE > AF$.

But, if K, L be the points in which AE, AF *first* meet the circle, the method fails, and Heron is reduced to proving, in the first instance, the property usually deduced from III. 36. He argues thus:

AKD being an obtuse angle,

sq. on AD = sum of sqs. on AK, KD and twice rect. AK, KG. [II. 12]

ALD is also an obtuse angle, and it follows that

sum of sqs. on AK, KD and twice rect. AK, KG is equal to
sum of sqs. on AL, LD and twice rect. AL, LH.

Therefore, the squares on KD, LD being equal,

sq on AK and twice rect. AK, KG = sq. on AL and twice rect. AL, LH,

or sq on AK and rect. AK, KE = sq. on AL and rect. AL, LF,

i.e. rect. AK, AE = rect. AL, AF.

But, by the first part, $AE > AF$.

Therefore $AK < AL$.

III. 7, 8 deal with the lengths of the several lines drawn to the circumference of a circle (1) from a point within it, (2) from a point outside it; but a similar proposition is true of straight lines drawn from a point on the circumference itself: *If any point be taken on the circumference of a circle, then, of all the straight lines which can be drawn from it to the circumference, the greatest is that in which the centre is; of any others that which is nearer to the straight line which passes through the centre is greater than one more remote; and from the same point there can be drawn to the circumference two straight lines, and only two, which are equal to one another, one on each side of the greatest line.*

The converses of III. 7, 8 and of the proposition just given are also true and can easily be proved by *reductio ad absurdum*. They could be employed to throw light on such questions as that of internal contact, and the relative position of the centres of circles so touching. This is clear when part of the converses is stated: thus (1) if from any point in the plane of a circle a number of straight lines be drawn to the circumference of the circle, and one of these is greater than any other, the centre of the circle must lie on that one, (2) if one of them is less than any other, then, (*a*) if the point is within the circle, the centre is on the minimum straight line produced *beyond the point*, (*b*) if the point is outside the circle, the centre is on the minimum straight line produced *beyond the point in which it meets the circle*.

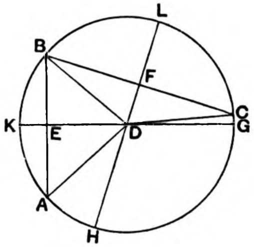
PROPOSITION 9.

If a point be taken within a circle, and more than two equal straight lines fall from the point on the circle, the point taken is the centre of the circle.

Let *ABC* be a circle and *D* a point within it, and from *D* let more than two equal straight lines, namely *DA*, *DB*, *DC*, fall on the circle *ABC*;

I say that the point *D* is the centre of the circle *ABC*.

For let *AB*, *BC* be joined and bisected at the points *E*, *F*, and let *ED*, *FD* be joined and drawn through to the points *G*, *K*, *H*, *L*.



Then, since *AE* is equal to *EB*, and *ED* is common,

the two sides *AE*, *ED* are equal to the two sides *BE*, *ED*; and the base *DA* is equal to the base *DB*;

therefore the angle *AED* is equal to the angle *BED*. [I. 8]

Therefore each of the angles *AED*, *BED* is right; [I. Def. 10]

therefore *GK* cuts *AB* into two equal parts and at right angles.

And since, if in a circle a straight line cut a straight line into two equal parts and at right angles, the centre of the circle is on the cutting straight line, [III. 1, Por.]

the centre of the circle is on *GK*.

For the same reason

the centre of the circle *ABC* is also on *HL*.

And the straight lines *GK*, *HL* have no other point common but the point *D*;

therefore the point *D* is the centre of the circle *ABC*.

Therefore etc. Q. E. D.

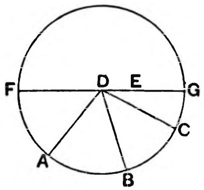
The result of this proposition is quoted by Aristotle, *Meteorologica* III. 3, 373 a 13—16 (cf. note on I. 8).

III. 9 is, as De Morgan remarks, a *logical* equivalent of part of III. 7, where it is proved that every *non*-central point is *not* a point from which three equal straight lines can be drawn to the circle. Thus III. 7 says that every *not-A* is *not-B*, and III. 9 states the equivalent fact that every *B* is *A*. Mr H. M. Taylor does in effect make a *logical* inference of the theorem that, *If from a point three equal straight lines can be drawn to a circle, that point is the centre*, by making it a corollary to his proposition which includes the part of III. 7 referred to. Euclid does not allow himself these logical inferences, as we shall have occasion to observe elsewhere also.

Of the two proofs of this proposition given in earlier texts of Euclid, August and Heiberg regard that translated above as genuine, relegating the other, which Simson gave alone, to a place in an Appendix. Camerer remarks that the genuine proof should also have contemplated the case in which one or other of the straight lines *AB*, *BC* passes through *D*. This would however have been a departure from Euclid's manner of taking the most obscure case for proof and leaving others to the reader.

The other proof, that selected by Simson, is as follows:

"For let a point *D* be taken within the circle *ABC*, and from *D* let more than two equal straight lines, namely *AD*, *DB*, *DC*, fall on the circle *ABC*;



I say that the point *D* so taken is the centre of the circle *ABC*.

For suppose it is not; but, if possible, let it be *E*, and let *DE* be joined and carried through to the points *F*, *G*.

Therefore *FG* is a diameter of the circle *ABC*.

Since, then, on the diameter *FG* of the circle *ABC* a point has been taken which is not the centre of the circle, namely *D*,

DG is greatest, and *DC* is greater than *DB*, and *DB* than *DA*.

But the latter are also equal: which is impossible

Therefore *E* is not the centre of the circle.

Similarly we can prove that neither is any other point except *D*;

therefore the point *D* is the centre of the circle *ABC*.

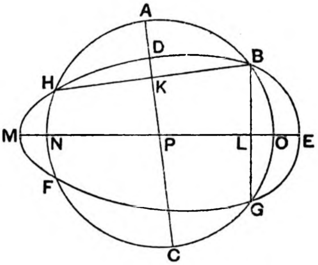
Q. E. D."

On this Todhunter correctly points out that the point *E* might be supposed to fall *within* the angle *ADC*. It cannot then be shown that *DC* is greater than *DB* and *DB* than *DA*, but only that either *DC* or *DA* is less than *DB*; this however is sufficient for establishing the proposition.

PROPOSITION 10.

A circle does not cut a circle at more points than two.

For, if possible, let the circle ABC cut the circle DEF at more points than two, namely B, C, F, H;



let BH, BG be joined and bisected at the points K, L, and from K, L let KC, LM be drawn at right angles to BH, BG and carried through to the points A, E.

Then, since in the circle ABC a straight line AC cuts a straight line BH into two equal parts and at right angles,

the centre of the circle ABC is on AC. [III. 1, Por.]

Again, since in the same circle ABC a straight line NO cuts a straight line BG into two equal parts and at right angles,

the centre of the circle ABC is on NO.

But it was also proved to be on AC, and the straight lines AC, NO meet at no point except at P;

therefore the point P is the centre of the circle ABC.

Similarly we can prove that P is also the centre of the circle DEF;

therefore the two circles ABC, DEF which cut one another have the same centre P: which is impossible. [III. 5]

Therefore etc. Q. E. D.

1. The word circle (κύκλος) is here employed in the unusual sense of the *circumference* (περιφέρεια) of a circle. Cf. note on I. Def. 15.

There is nothing in the demonstration of this proposition which assumes that the circles *cut* one another; it proves that two circles cannot *meet* at more than two points, whether they cut or meet without cutting, i.e. *touch* one another.

Here again, of two demonstrations given in the earlier texts, Simson chose the second, which August and Heiberg relegate to an Appendix and which is as follows:

"For again let the circle ABC cut the circle DEF at more points than two, namely B, G, H, F;

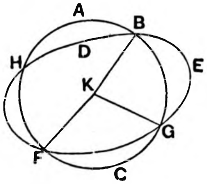
let the centre K of the circle ABC be taken, and let KB, KG, KF be joined.

Since then a point K has been taken within the circle DEF, and from K more than two straight lines, namely KB, KF, KG, have fallen on the circle DEF, the point K is the centre of the circle DEF. [III. 9]

But K is also the centre of the circle ABC.

Therefore two circles cutting one another have the same centre K: which is impossible. [III. 5]

Therefore a circle does not cut a circle at more points than two.



Q. E. D."

This demonstration is claimed by Heron (see an-Nairīzī, ed. Curtze, pp. 120—1). It is incomplete because it assumes that the point K which is taken as the centre of the circle ABC is *within* the circle DEF. It can however be completed by means of III. 8 and the corresponding proposition with reference to a point *on* the circumference of a circle which was enunciated in the note on III. 8. For (1) if the point K is *on* the circumference of the circle DEF, we obtain a contradiction of the latter proposition which asserts that only *two* equal straight lines can be drawn from K to the circumference of the circle DEF; (2) if the point K is *outside* the circle DEF, we obtain a contradiction of the corresponding part of III. 8.

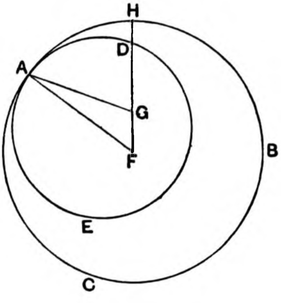
Euclid's proof contains an unproved assumption, namely that the lines bisecting BG, BH at right angles *will* meet in a point P. For a discussion of this assumption see note on IV. 5.

PROPOSITION 11.

If two circles touch one another internally, and their centres be taken, the straight line joining their centres, if it be also produced, will fall on the point of contact of the circles.

For let the two circles ABC, ADE touch one another internally at the point A, and let the centre F of the circle ABC, and the centre G of ADE, be taken;

I say that the straight line joined from G to F and produced will fall on A.



For suppose it does not, but, if possible, let it fall as FGH, and let AF, AG be joined.

Then, since AG, GF are greater than FA, that is, than FH,

let FG be subtracted from each;

therefore the remainder AG is greater than the remainder GH.

But *AG* is equal to *GD*;

therefore *GD* is also greater than *GH*,

the less than the greater: which is impossible.

Therefore the straight line joined from *F* to *G* will not fall outside;

therefore it will fall at *A* on the point of contact.

Therefore etc.

Q. E. D.

2. **the straight line joining their centres**, literally "the straight line joined to their centres" (ἡ ἐπὶ τὰ κέντρα αὐτῶν ἐπιζευγνυμένη εὐθεῖα).

3. **point of contact** is here συναφή, and in the enunciation of the next proposition ἐπαφή.

Again August and Heiberg give in an Appendix the additional or alternative proof, which however shows little or no variation from the genuine proof and can therefore well be dispensed with.

The genuine proof is beset with difficulties in consequence of what it tacitly assumes in the figure, on the ground, probably, of its being obvious to the eye. Camerer has set out these difficulties in a most careful note, the heads of which may be given as follows:

He observes, first, that the straight line joining the centres, when produced, must necessarily (though this is not stated by Euclid) be produced *in the direction of the centre of the circle which touches the other internally*. (For brevity, I shall call this circle the "inner circle," though I shall imply nothing by that term except that it is the circle which touches the other on the inner side of the latter, and therefore that, in accordance with the definition of *touching*, points on it in the immediate neighbourhood of the point of contact are necessarily *within* the circle which it touches.) Camerer then proceeds by the following steps.

1. The two circles, touching at the given point, cannot *intersect* at any point. For, since points on the "inner" in the immediate neighbourhood of the point of contact are within the "outer" circle, the inner circle, if it intersects the other anywhere, must pass outside it and then return. This is only possible (*a*) if it passes out at one point and returns at another point, or (*b*) if it passes out and returns through one and the same point. (*a*) is impossible because it would require two circles to have *three* common points; (*b*) would require that the inner circle should have a *node* at the point where it passes outside the other, and this is proved to be impossible by drawing any radius cutting both loops.

2. Since the circles cannot intersect, one must be *entirely* within the other.

3. Therefore the outer circle must be greater than the inner, and the radius of the outer greater than that of the inner.

4. Now, if *F* be the centre of the greater and *G* of the inner circle, and if *FG* produced beyond *G* does *not* pass through *A*, the given point of contact, then there are three possible hypotheses.

(*a*) *A* may lie on *GF* produced beyond *F*.

(*b*) A may lie outside the line FG altogether, in which case FG produced beyond G must, in consequence of result 2 above, either

(i) meet the circles in a point common to both, or

(ii) meet the circles in two points, of which that which is on the inner circle is nearer to G than the other is.

(*a*) is then proved to be impossible by means of the fact that the radius of the inner circle is less than the radius of the outer.

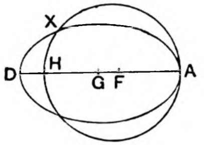
(*b*) (ii) is Euclid's case; and his proof holds equally of (*b*) (i), the hypothesis, namely, that D and H in the figure coincide.

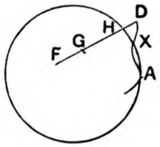
Thus all alternative hypotheses are successively shown to be impossible, and the proposition is completely established.

I think, however, that this procedure may be somewhat shortened in the following manner.

In order to make Euclid's proof absolutely conclusive we have only (1) to take care to produce FG beyond G, the centre of the "inner" circle, and then (2) to prove that the point in which FG so produced meets the "inner" circle is *not further* from G than is the point in which it meets the other circle. Euclid's proof is equally valid whether the first point is nearer to G than the second or the first point and the second coincide.

If FG produced beyond G does not pass through A, there are two conceivable hypotheses: (*a*) A may lie on GF produced beyond F, or (*b*) A may be outside FG produced either way. In either case, if FG produced meets the "inner" circle in D and the other in H, and if GD is greater than GH, then the "inner" circle must cut the "outer" circle at some point between A and D, say X.





But if two circles have a common point X lying on one side of the line of centres, they must have another corresponding point on the other side of the line of centres. This is clear from III. 7, 8; for the point is determined by drawing from F and G, on the opposite side to that where X is, straight lines FY, GY making with FD angles equal to the angles DFX, DGX respectively.

Hence the two circles will have at least three points common: which is impossible.

Therefore GD cannot be greater than GH; accordingly GD must be either equal to, or less than, GH, and Euclid's proof is valid.

The particular hypothesis in which FG is supposed to be in the same straight line with A but G is on the side of F away from A is easily disposed of, and would in any case have been left to the reader by Euclid.

For GD is either equal to or less than GH.

Therefore GD is less than FH, and therefore less than FA.

But GD is equal to GA, and therefore greater than FA: which is impossible.

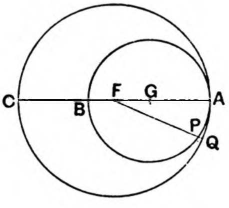
Subject to the same preliminary investigation as that required by Euclid's proof, the proposition can also be proved directly from III. 7.

For, by III. 7, GH is the shortest straight line that can be drawn from G to the circle with centre F;

therefore GH is less than GA,

and therefore less than GD: which is absurd.

This proposition is the crucial one as regards circles which touch internally; and, when it is once established, the relative position of the circles can be completely elucidated by means of it and the propositions which have preceded it. Thus, in the annexed figure, if F be the centre of the outer circle and G the centre of the inner, and if any radius FQ of the outer circle meet the two circles in Q, P respectively, it follows, from III. 7, III. 8, or the corresponding theorem with reference to a point *on* the circumference, that FA is the maximum straight line from F to the circumference of the inner circle, FP is less than FA, and FP diminishes in length as FQ moves round from FA until FP reaches its minimum length FB. Hence the circles do not meet at any other point than A, and the distance PQ cut off between them on any radius FQ of the outer circle becomes greater and greater as FQ moves round from FA to FC and is a maximum when FQ coincides with FC, after which it diminishes again on the other side of FC.



The same consideration gives the partial converse of III. 11 which forms the 6th lemma of Pappus to the first book of the *Tactiones* of Apollonius (Pappus, VII. p. 826). This is to the effect that, *if* AB, AC *are in one straight line, and on one side of* A, *the circles described on* AB, AC *as diameters touch* (*internally at the point* A). Pappus concludes this from the fact that the circles have a common tangent at A; but the truth of it is clear from the fact that FP diminishes as FQ moves away from FA on either side; whence the circles meet at A but do not cut one another.

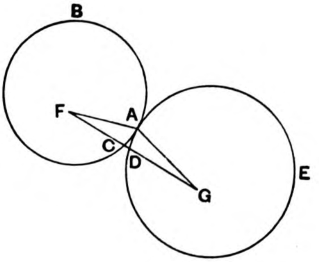
Pappus' 5th lemma (VII. p. 824) is another partial converse, namely that, *given two circles touching internally at* A, *and a line* ABC *drawn from* A *cutting both, then, if the centre of the outer circle lies on* ABC, *so does the centre of the inner.* Pappus himself proves this, by means of the common tangent to the circles at A, in two ways. (1) The tangent is at right angles to AC and therefore to AB: therefore the centre of the inner circle lies on AB. (2) By III. 32, the angles in the alternate segments of both circles are right angles, so that ABC is a diameter of both.

[PROPOSITION 12.

If two circles touch one another externally, the straight line joining their centres will pass through the point of contact.

For let the two circles ABC, ADE touch one another externally at the point A, and let the centre F of ABC, and the centre G of ADE, be taken;

I say that the straight line joined from *F* to *G* will pass through the point of contact at *A*.



For suppose it does not, but, if possible, let it pass as *FCDG*, and let *AF*, *AG* be joined.

Then, since the point *F* is the centre of the circle *ABC*,

FA is equal to *FC*.

Again, since the point *G* is the centre of the circle *ADE*,

GA is equal to *GD*.

But *FA* was also proved equal to *FC*;

therefore *FA*, *AG* are equal to *FC*, *GD*,

so that the whole *FG* is greater than *FA*, *AG*;

but it is also less [I. 20]: which is impossible.

Therefore the straight line joined from *F* to *G* will not fail to pass through the point of contact at *A*;

therefore it will pass through it.

Therefore etc. Q. E. D.]

23. **will not fail to pass.** The Greek has the double negative, οὐκ ἄρα ἡ...εὐθεῖα... οὐκ ἐλεύσεται, literally "the straight line...will not *not*-pass...."

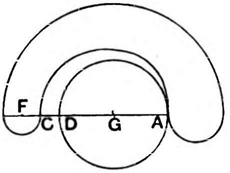
Heron says on III. 11: "Euclid in proposition 11 has supposed the two circles to touch internally, made his proposition deal with this case and proved what was sought in it. *But I will show how it is to be proved if the contact is external.*" He then gives substantially the proof and figure of III. 12. It seems clear that neither Heron nor an-Nairīzī had III. 12 in this place.

Campanus and the Arabic edition of Naṣīraddīn aṭ-Ṭūsī have nothing more of III. 12 than the following addition to III. 11. "In the case of external contact the two lines *ae* and *eb* will be greater than *ab*, whence *ad* and *cb* will be greater than the whole *ab*, which is false." (The points *a*, *b*, *c*, *d*, *e* correspond respectively to *G*, *F*, *C*, *D*, *A* in the above figure.) It is most probable that Theon or some other editor added Heron's proof in his edition and made Prop. 12 out of it (an-Nairīzī, ed. Curtze, pp. 121—2). An-Nairīzī and Campanus, conformably with what has been said, number Prop. 13 of Heiberg's text Prop. 12, and so on through the Book.

What was said in the note on the last proposition applies, *mutatis mutandis*, to this. Camerer proceeds in the same manner as before; and we may use the same alternative argument in this case also.

Euclid's proof is valid provided only that, if *FG*, joining the assumed centres, meets the circle with centre *F* in *C* and the other circle in *D*, *C* is not within the circle *ADE* and *D* is not within the circle *ABC*. (The proof is equally valid whether *C*, *D* coincide or the successive points are, as drawn in the figure, in the order *F*, *C*, *D*, *G*.) Now, if *C* is within the circle *ADE* and D within the circle ABC, the circles must have cut between A and C and between A and D. Hence, as before, they must also have another corresponding point common on the other side of CD. That is, the circles must have *three* common points: which is impossible.

Hence Euclid's proof is valid if F, A, G form a triangle, and the only hypothesis which has still to be disproved is the hypothesis which he would in any case have left to the reader, namely that A does not lie on FG but on *FG produced* in either direction. In this case, as before, either C, D must coincide or C is nearer F than D is. Then the radius FC must be equal to FA: which is impossible, since FC cannot be greater than FD, and must therefore be *less* than FA.



Given the same preliminaries, III. 12 can be proved by means of III. 8.

Again, when the proposition III. 12 is once proved, III. 8 helps us to prove at once that the circles lie entirely outside each other and have no other common point than the point of contact.

Among Pappus' lemmas to Apollonius' *Tactiones* are the two partial converses of this proposition corresponding to those given in the last note. Lemma 4 (VII. p. 824) is to the effect that, *if* AB, AC *be in one straight line*, B *and* C *being on opposite sides of* A, *the circles drawn on* AB, AC *as diameters touch externally at* A. Lemma 3 (VII. p. 822) states that, *if two circles touch externally at* A *and* BAC *is drawn through* A *cutting both circles and containing the centre of one*, BAC *will also contain the centre of the other.* The proofs, as before, use the common tangent at A.

Mr H. M. Taylor gets over the difficulties involved by III. 11, 12 in a manner which is most ingenious but not Euclidean. He first proves that, *if two circles meet at a point not in the same straight line with their centres, the circles intersect at that point*; this is very easily established by means of III. 7, 8 and the third similar theorem. Then he gives as a corollary the statement that, *if two circles touch, the point of contact is in the same straight line with their centres.* It is not explained how this is inferred from the substantive proposition; it seems, however, to be a *logical* inference simply. By the proposition, every A (circles meeting at a point not in the same straight line with the centre) is B (circles which intersect); therefore every not-B is not-A, i.e. circles which do not intersect do not meet at a point not in the same straight line with the centres. Now non-intersecting circles may either meet (i.e. touch) or not meet. In the former case they must meet *on* the line of centres: for, if they met at a point not in that line, they would intersect. But such a purely *logical* inference is foreign to Euclid's manner. As De Morgan says, "Euclid may have been ignorant of the identity of 'Every X is Y' and 'Every not-Y is not-X,' for anything that appears in his writings; he makes the one follow from the other by a new proof each time" (quoted in Keynes' *Formal Logic*, p. 81).

There is no difficulty in proving, by means of I. 20, Mr Taylor's next proposition that, *if two circles meet at a point which lies in the same straight line as their centres and is between the centres, the circles touch at that point, and each circle lies without the other.* But the similar proof, by means of I. 20, of the corresponding theorem for internal contact seems to be open to the same objection as Euclid's proof of III. 11 in that it assumes without proof that the circle which has its centre nearest to the point of meeting is the "inner" circle. Lastly, in order to prove that, *if two circles have a point of contact, they* *do not meet at any other point*, Mr Taylor uses the questionable corollary. Therefore in any case his alternative procedure does not seem preferable to Euclid's.

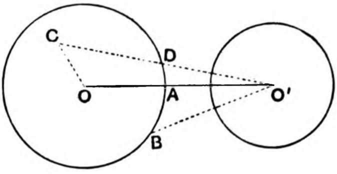
The alternative to Eucl. III. 11—13 which finds most favour in modern continental text-books (e.g. Legendre, Baltzer, Henrici and Treutlein, Veronese, Ingrami, Enriques and Amaldi) connects the number, position and nature of the coincidences between points on two circles with the relation in which the distance between their centres stands to the length of their radii. Enriques and Amaldi, whose treatment of the different cases is typical, give the following propositions (Veronese gives them in the converse form).

1. *If the distance between the centres of two circles is greater than the sum of the radii, the two circles have no point common and are external to one another.*

Let O, O' be the centres of the circles (which we will call "the circles O, O'"), r, r' their radii respectively.

Since then $OO' > r + r'$, *a fortiori* $OO' > r$, and O' is therefore exterior to the circle O.

Next, the circumference of the circle O intersects OO' in a point A, and since $OO' > r + r'$, $AO' > r'$, and A is external to the circle O'.



But $O'A$ is less than any straight line, as $O'B$, drawn to the circumference of the circle O [III. 8]; hence all points, as B, on the circumference of the circle O are external to the circle O'.

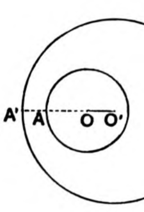
Lastly, if C be any point internal to the circle O, the sum of OC, $O'C$ is greater than $O'O$, and *a fortiori* greater than $r + r'$.

But OC is less than r: therefore $O'C$ is greater than r', or C is external to O.

Similarly we prove that any point on or within the circumference of the circle O' is external to the circle O.

2. *If the distance between the centres of two unequal circles is less than the difference of the radii, the two circumferences have no common point and the lesser circle is entirely within the greater.*

Let O, O' be the centres of the two circles, r, r' their radii respectively $(r < r')$.



Since $OO' < r' - r$, *a fortiori* $OO' < r'$, so that O is internal to the circle O'.

If A, A' be the points in which the straight line $O'O$ intersects respectively the circumferences of the circles O, O',

$$O'O \text{ is less than } O'A' - OA,$$

so that $O'O + OA$, or $O'A$, is less than $O'A'$, and therefore A is internal to the circle O'.

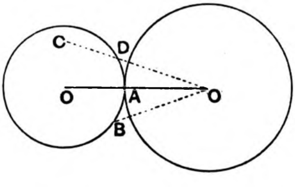
But, of all the straight lines from O' to the circumference of the circle O, $O'A$ passing through the centre O is the greatest [III. 7]; whence all the points of the circumference of O are internal to the circle O'.

A similar argument to the preceding will show that all points within the circle O are internal to the circle O'.

3. *If the distance between the centres of two circles is equal to the sum of the radii, the two circumferences have one point common and one only, and that point is on the line of centres. Each circle is external to the other.*

Let O, O' be the centres, r, r' the radii of the circles, so that OO' is equal to $r + r'$.

Thus OO' is greater than r, so that O' is external to the circle O, and the circumference of the circle O cuts OO' in a point A.



And, since OO' is equal to $r + r'$, and OA to r, it follows that $O'A$ is equal to r', so that A belongs also to the circumference of the circle O'.

The proof that all other points on, and all points within, the circumference of the circle O are external to the circle O' follows the similar proof of prop. 1 above. And similarly all points (except A) on, and all points within, the circumference of the circle O' are external to the circle O.

The two circles, having one common point only, *touch* at that point, which lies, as shown, on the line of centres. And, since the circles are external to one another, they touch *externally*.

4. *If the distance between the centres of two unequal circles is equal to the difference between the radii, the two circumferences have one point and one only in common, and that point lies on the line of centres. The lesser circle is within the other.*

The proof is that of prop. 2 above, *mutatis mutandis*.

The circles here touch *internally* at the point on the line of centres.

5. *If the distance between the centres of two circles is less than the sum, and greater than the difference, of the radii, the two circumferences have two common points symmetrically situated with respect to the line of centres but not lying on that line.*

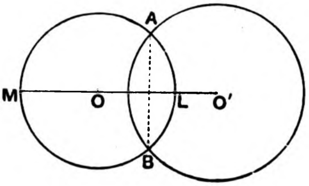
Let O, O' be the centres of the two circles, r, r' their radii, r' being the greater, so that

$$r' - r < OO' < r + r'.$$

It follows that in any case $OO + r > r'$, so that, if OM be taken on $O'O$ produced equal to r (so that M is on the circumference of the circle O), M is external to the circle O'.

We have to use the same Postulate as in Eucl. I. 1 that

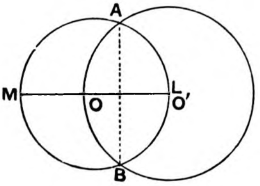
An arc of a circle which has one extremity within and the other without a given circle has one point common with the latter and only one; from which it follows, if we consider two such arcs making a complete circumference, that, *if a circumference of a circle passes through one point internal to, and one point external to a given circle, it cuts the latter circle in two points.*



We have then to prove that the circle O, besides having one point M of its circumference external to the circle O', has one other point of its circumference (L) internal to the latter circle.

Three cases have to be distinguished according as OO' is greater than, equal to, or less than, the radius r of the lesser circle.

(1) $OO' > r$. (See the preceding figure.)



Measure OL along OO' equal to r, so that L lies on the circumference of the circle O.

Then, since $OO' < r + r'$, $O'L$ will be less than r', so that L is within the circle O'.

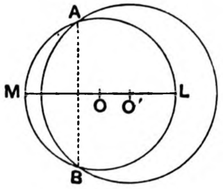
(2) $OO' = r$.

In this case the circumference of the circle O passes through O', or L coincides with O'.

(3) $OO' < r$.

If we measure OL along OO' equal to r, the point L will lie on the circumference of the circle O.

Then $O'L = r - OO'$,



so that $O'L < r$, and *a fortiori* $O'L < r'$, so that L lies within the circle O'.

Thus, in all three cases, since the circumference of O passes through one point (M) external to, and one point (L) internal to, the circle O', the two circumferences intersect in two points A, B [Post.]

And A, B cannot lie on the line of centres OO', since this straight line intersects the circle O in L, M only, and of these points one is inside, the other outside, the circle O'.

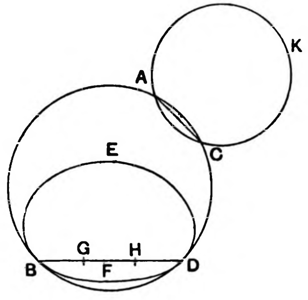
Since AB is a common chord of both circles, the straight line bisecting it at right angles passes through both centres, i.e. is identical with OO'.

And again by means of III. 7, 8 we prove that all points except A, B on the arc ALB lie within the circle O', and all points except A, B on the arc AMB outside that circle; and so on.

PROPOSITION 13.

A circle does not touch a circle at more points than one, whether it touch it internally or externally.

For, if possible, let the circle $ABDC$ touch the circle $EBFD$, first internally, at more points than one, namely D, B.



Let the centre G of the circle $ABDC$, and the centre H of $EBFD$, be taken.

Therefore the straight line joined from G to H will fall on B, D. [III. 11]

Let it so fall, as $BGHD$.

Then, since the point G is the centre of the circle $ABCD$,

BG is equal to GD;

therefore *BG* is greater than *HD*;

therefore *BH* is much greater than *HD*.

Again, since the point *H* is the centre of the circle *EBFD*,

BH is equal to *HD*;

but it was also proved much greater than it: which is impossible.

Therefore a circle does not touch a circle internally at more points than one.

I say further that neither does it so touch it externally.

For, if possible, let the circle *ACK* touch the circle *ABDC* at more points than one, namely *A*, *C*,

and let *AC* be joined.

Then, since on the circumference of each of the circles *ABDC*, *ACK* two points *A*, *C* have been taken at random, the straight line joining the points will fall within each circle; [III. 2]

but it fell within the circle *ABCD* and outside *ACK* [III. Def. 3]: which is absurd.

Therefore a circle does not touch a circle externally at more points than one.

And it was proved that neither does it so touch it internally.

Therefore etc. Q. E. D.

3, 7, 14, 27, 30, 33. **ABDC.** Euclid writes *ABCD* (here and in the next proposition), notwithstanding the order in which the points are placed in the figure.

25, 37. **does it so touch it.** It is necessary to supply these words which the Greek (ὅτι οὐδὲ ἐκτὸς and ὅτι οὐδὲ ἐντός) leaves to be understood.

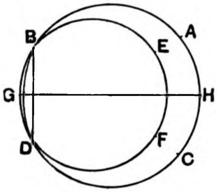
The difficulties which have been felt in regard to the proofs of this proposition need not trouble us now, because they have already been disposed of in the discussion of the more crucial propositions III. 11, 12.

Euclid's proof of the first part of the proposition differs from Simson's; and we will deal with Euclid's first. On this Camerer remarks that it is assumed that the supposed second point of contact lies on the line of centres *produced beyond the centre of the "outer" circle*, whereas all that is proved in III. 11 is that the line of centres *produced beyond the centre of the "inner" circle* passes through a point of contact. But, by the same argument as that given on III. 11, we show that the circles cannot have a point of contact, or even any common point, outside the line of centres, because, if there were such a point, there would be a corresponding common point on the other side of the line, and the circles would have *three* common points. Hence the only hypothesis left is that the second point of contact may be *on* the line of centres but in the direction of the centre of the "*outer*" circle; and Euclid's proof disposes of this hypothesis.

Heron (in an-Nairīzī, ed. Curtze, pp. 122—4), curiously enough, does not question Euclid's assumption that the line of centres passes through both points of contact (if double contact is possible); but he devotes some space to proving that the centre of the "outer" circle must lie within the "inner" circle, a fact which he represents Euclid as asserting ("sicut dixit Euclides"), though there is no such assertion in our text. The proof of the fact is of course easy. If the line of centres passes through *both* points of contact, and the centre of the "outer" circle lies either on or outside the "inner" circle, the line of centres must cut the "inner" circle in *three* points in all: which is impossible, as Heron shows by the lemma, which he places here (and proves by I. 16), that *a straight line cannot cut the circumference of a circle in more points than two.*

Simson's proof is as follows (there is no real need for giving two figures as he does).

"If it be possible, let the circle EBF touch the circle ABC in more points than one, and first on the inside, in the points B, D; join BD, and draw GH bisecting BD at right angles.



Therefore, because the points B, D are in the circumference of each of the circles, the straight line BD falls within each of them: And their centres are in the straight line GH which bisects BD at right angles:

Therefore GH passes through the point of contact [III. 11]; but it does not pass through it, because the points B, D are without the straight line GH: which is absurd.

Therefore one circle cannot touch another on the inside in more points than one."

On this Camerer remarks that, unless III. 11 be more completely elucidated than it is by Euclid's demonstration, which Simson has, it is not sufficiently clear that, besides the point of contact in which GH meets the circles, they cannot have another point of contact either (1) on GH or (2) outside it. Here again the latter supposition (2) is rendered impossible because in that case there would be a third common point on the opposite side of GH; and the former supposition (1) is that which Euclid's proof destroys.

Simson retains Euclid's proof of the second part of the proposition, though his own proof of the first part would apply to the second part also if a reference to III. 12 were substituted for the reference to III. 11. Euclid might also have proved the second part by the same method as that which he employs for the first part.

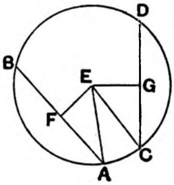
PROPOSITION 14.

In a circle equal straight lines are equally distant from the centre, and those which are equally distant from the centre are equal to one another.

Let $ABDC$ be a circle, and let AB, CD be equal straight lines in it;

I say that AB, CD are equally distant from the centre.

For let the centre of the circle $ABDC$ be taken [III. 1], and let it be *E*; from *E* let *EF*, *EG* be drawn perpendicular to *AB*, *CD*, and let *AE*, *EC* be joined.



Then, since a straight line *EF* through the centre cuts a straight line *AB* not through the centre at right angles, it also bisects it. [III. 3]

Therefore *AF* is equal to *FB*;

therefore *AB* is double of *AF*.

For the same reason

CD is also double of *CG*;

and *AB* is equal to *CD*;

therefore *AF* is also equal to *CG*.

And, since *AE* is equal to *EC*,

the square on *AE* is also equal to the square on *EC*.

But the squares on *AF*, *EF* are equal to the square on *AE*, for the angle at *F* is right;

and the squares on *EG*, *GC* are equal to the square on *EC*, for the angle at *G* is right; [I. 47]

therefore the squares on *AF*, *FE* are equal to the squares on *CG*, *GE*,

of which the square on *AF* is equal to the square on *CG*, for *AF* is equal to *CG*;

therefore the square on *FE* which remains is equal to the square on *EG*,

therefore *EF* is equal to *EG*.

But in a circle straight lines are said to be equally distant from the centre when the perpendiculars drawn to them from the centre are equal; [III. Def. 4]

therefore *AB*, *CD* are equally distant from the centre.

Next, let the straight lines *AB*, *CD* be equally distant from the centre; that is, let *EF* be equal to *EG*.

I say that *AB* is also equal to *CD*.

For, with the same construction, we can prove, similarly, that *AB* is double of *AF*, and *CD* of *CG*.

And, since *AE* is equal to *CE*,

the square on *AE* is equal to the square on *CE*.

But the squares on *EF*, *FA* are equal to the square on *AE*, and the squares on *EG*, *GC* equal to the square on *CE*. [I. 47]

Therefore the squares on EF, FA are equal to the squares on EG, GC,

of which the square on EF is equal to the square on EG, for EF is equal to EG;

therefore the square on AF which remains is equal to the square on CG;

therefore AF is equal to CG.

And AB is double of AF, and CD double of CG;

therefore AB is equal to CD.

Therefore etc.

Q. E. D.

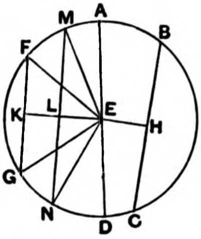
Heron (an-Nairīzī, pp. 125—7) has an elaborate addition to this proposition in which he proves, first by *reductio ad absurdum*, and then directly, that the centre of the circle falls between the two chords.

PROPOSITION 15.

Of straight lines in a circle the diameter is greatest, and of the rest the nearer to the centre is always greater than the more remote.

Let $ABCD$ be a circle, let AD be its diameter and E the centre; and let BC be nearer to the diameter AD, and FG more remote;

I say that AD is greatest and BC greater than FG.



For from the centre E let EH, EK be drawn perpendicular to BC, FG.

Then, since BC is nearer to the centre and FG more remote, EK is greater than EH. [III. Def. 5]

Let EL be made equal to EH, through L let LM be drawn at right angles to EK and carried through to N, and let ME, EN, FE, EG be joined.

Then, since EH is equal to EL,

BC is also equal to MN. [III. 14]

Again, since AE is equal to EM, and ED to EN,

AD is equal to ME, EN.

But ME, EN are greater than MN, [I. 20]
and MN is equal to BC;

therefore AD is greater than BC.

And, since the two sides ME, EN are equal to the two sides FE, EG,
and the angle MEN greater than the angle FEG,
therefore the base MN is greater than the base FG. [I. 24]

But MN was proved equal to BC.

Therefore the diameter AD is greatest and BC greater than FG.

Therefore etc. Q. E. D.

1. **Of straight lines.** The Greek leaves these words to be understood.
5. **Nearer to the diameter AD.** As BC, FG are not in general parallel to AD, Euclid should have said "nearer to the centre."

It will be observed that Euclid's proof differs from that given in our text-books (which is Simson's) in that Euclid introduces another line MN, which is drawn so as to be equal to BC but at right angles to EK and therefore parallel to FG. Simson dispenses with MN and bases his proof on a similar proof by Theodosius (*Sphaerica* I. 6). He proves that the sum of the squares on EH, HB is equal to the sum of the squares on EK, KF; whence he infers that, since the square on EH is less than the square on EK, the square on BH is greater than the square on FK. It may be that Euclid would have regarded this as too complicated an inference to make without explanation or without an increase in the number of his axioms. But, on the other hand, Euclid himself assumes that the angle subtended at the centre by MN is greater than the angle subtended by FG, or, in other words, that M, N both fall outside the triangle FEG. This is a similar assumption to that made in III. 7, 8, as already noticed; and its truth is obvious because EM, EN, being radii of the circle, are greater than the distances from E to the points in which MN cuts EF, EG, and therefore the latter points are nearer than M, N are to L, the foot of the perpendicular from E to MN.

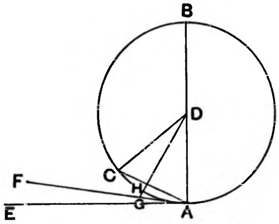
Simson adds the converse of the proposition, proving it in the same way as he proves the proposition itself.

PROPOSITION 16.

The straight line drawn at right angles to the diameter of a circle from its extremity will fall outside the circle, and into the space between the straight line and the circumference another straight line cannot be interposed; further the angle of the semicircle is greater, and the remaining angle less, than any acute rectilineal angle.

Let ABC be a circle about D as centre and AB as diameter;

I say that the straight line drawn from A at right angles to AB from its extremity will fall outside the circle.



For suppose it does not, but, if possible, let it fall within as CA, and let DC be joined.

Since DA is equal to DC,

the angle DAC is also equal to the angle ACD. [I. 5]

But the angle DAC is right;

therefore the angle ACD is also right:

thus, in the triangle ACD, the two angles DAC, ACD are equal to two right angles: which is impossible. [I. 17]

Therefore the straight line drawn from the point A at right angles to BA will not fall within the circle.

Similarly we can prove that neither will it fall on the circumference;

therefore it will fall outside.

Let it fall as AE;

I say next that into the space between the straight line AE and the circumference CHA another straight line cannot be interposed.

For, if possible, let another straight line be so interposed, as FA, and let DG be drawn from the point D perpendicular to FA.

Then, since the angle AGD is right,

and the angle DAG is less than a right angle,

AD is greater than DG. [I. 19]

But DA is equal to DH;

therefore DH is greater than DG, the less than the greater: which is impossible.

Therefore another straight line cannot be interposed into the space between the straight line and the circumference.

I say further that the angle of the semicircle contained by the straight line BA and the circumference CHA is greater than any acute rectilineal angle,

and the remaining angle contained by the circumference CHA and the straight line AE is less than any acute rectilineal angle.

For, if there is any rectilineal angle greater than the angle contained by the straight line BA and the circumference

CHA, and any rectilineal angle less than the angle contained by the circumference *CHA* and the straight line *AE*, then into the space between the circumference and the straight line *AE* a straight line will be interposed such as will make an angle contained by straight lines which is greater than the angle contained by the straight line *BA* and the circumference *CHA*, and another angle contained by straight lines which is less than the angle contained by the circumference *CHA* and the straight line *AE*.

But such a straight line cannot be interposed;

therefore there will not be any acute angle contained by straight lines which is greater than the angle contained by the straight line *BA* and the circumference *CHA*, nor yet any acute angle contained by straight lines which is less than the angle contained by the circumference *CHA* and the straight line *AE*.—

PORISM. From this it is manifest that the straight line drawn at right angles to the diameter of a circle from its extremity touches the circle.

Q. E. D.

4. **cannot be interposed**, literally "will not fall in between" (οὐ παρεμπεσεῖται).

This proposition is historically interesting because of the controversies to which the last part of it gave rise from the 13th to the 17th centuries. History was here repeating itself, for it is certain that, in ancient Greece, both before and after Euclid's time, there had been a great deal of the same sort of contention about the nature of the "angle of a semicircle" and the "remaining angle" between the circumference of the semicircle and the tangent at its extremity. As we have seen (note on I. Def. 8), the latter angle had a recognised name, κερατοειδὴς γωνία, *horn-like* or *cornicular* angle; though this term does not appear in Euclid, it is often used by Proclus, evidently as a term well understood. While it is from Proclus that we get the best idea of the ancient controversies on this subject, we may, I think, infer their prevalence in Euclid's time from this solitary appearance of the two "angles" in the *Elements*. Along with the definition of the angle *of* a segment, it seems to show that, although these angles are only mentioned to be dropped again immediately, and are of no use in elementary geometry, or even at all, Euclid thought that an allusion to them would be expected of him; it is as if he merely meant to guard himself against appearing to ignore a subject which the geometers of his time regarded with interest. If this conjecture is right, the mention of these angles would correspond to the insertion of definitions of which he makes no use, e.g. those of a rhombus and a rhomboid.

Proclus has no hesitation in speaking of the "angle of a semicircle" and the "horn-like angle" as true *angles*. Thus he says that "angles are contained by a straight line and a circumference in two ways; for they are either contained by a straight line and a convex circumference, like that of the semicircle, or by a straight line and a concave circumference, like the κερατοειδής" (p. 127, 11—14). "There are *mixed* lines, as spirals, and angles, as the angle of a semicircle and the κερατοειδής" (p. 104, 16—18). The difficulty which the ancients felt arose from the very fact which Euclid embodies in this proposition. Since an angle can be divided by a line, it would seem to be a magnitude; "but if it is a magnitude, and all homogeneous magnitudes which are finite have a ratio to one another, then all homogeneous angles, or rather all those on surfaces, will have a ratio to one another, so that the *cornicular* will also have a ratio to the rectilineal. But things which have a ratio to one another can, if multiplied, exceed one another. Therefore the *cornicular* angle will also sometime exceed the rectilineal; which is impossible, for it is proved that the former is less than any rectilineal angle" (Proclus, p. 121, 24—122, 6). The nature of contact between straight lines and circles was also involved in the question, and that this was the subject of controversy before Euclid's time is clear from the title of a work attributed to Democritus (fl. 420—400 B.C.) περὶ διαφορῆς γνώμονος ἢ περὶ ψαύσιος κύκλου καὶ σφαίρης, *On a difference in a gnomon or on contact of a circle and a sphere.* There is, however, another reading of the first words of this title as given by Diogenes Laertius (IX. 47), namely περὶ διαφορῆς γνώμης. *On a difference of opinion*, etc. May it not be that neither reading is correct, but that the words should be περὶ διαφορῆς γωνίης ἢ περὶ ψαύσιος κύκλου καὶ σφαίρης, *On a difference in an angle or on contact with a circle and a sphere*? There would, of course, hardly be any "angle" in connexion with the sphere; but I do not think that this constitutes any difficulty, because the sphere might easily be tacked on as a kindred subject to the circle. A curiously similar collocation of words appears in a passage of Proclus, though this may be an accident. He says (p. 50, 4) πῶς δὲ γωνιῶν διαφορὰς λέγομεν καὶ αὐξήσεις αὐτῶν ... and then, in the next line but one, πῶς δὲ τὰς ἁφὰς τῶν κύκλων ἢ τῶν εὐθειῶν, "In what sense do we speak of *differences of angles and of increases of them* ... and in what sense of the *contacts* (or meetings) *of circles* or of straight lines?" I cannot help thinking that this subject of *cornicular* angles would have had a fascination for Democritus as being akin to the question of infinitesimals, and very much of the same character as the other question which Plutarch (*On Common Notions*, XXXIX. 3) says that he raised, namely that of the relation between the base of a cone and a section of it by a plane parallel to the base and apparently, to judge by the context, infinitely near to it: "if a cone were cut by a plane parallel to its base, what must we think of the surfaces of the sections, that they are equal or unequal? For, if they are unequal, they will make the cone irregular, as having many indentations like steps, and unevennesses; but, if they are equal, the sections will be equal, and the cone will appear to have the property of the cylinder, as being made up of equal and not unequal circles, which is the height of absurdity."

The contributions by Democritus to such investigations are further attested by a passage in the *Method* of Archimedes discovered by Heiberg in 1906 (*Archimedes*, ed. Heiberg, Vol. II. 1913, p. 430; T. L. Heath, *The* Method *of Archimedes*, 1912, p. 13), which says that, though Eudoxus was the first to discover the scientific proof of the propositions (attributed to him) that the cone and the pyramid are one-third of the cylinder and prism respectively which have the same base and equal height, they were first *stated*, without proof, by Democritus.

A full history of the later controversies about the cornicular "angle" cannot be given here; more on the subject will be found in Camerer's Euclid (Excursus IV. on III. 16) or in Cantor's *Geschichte der Mathematik*.

Vol. II. (see *Contingenzwinkel* in the index). But the following short note about the attitude of certain well-known mathematicians to the question will perhaps not be out of place. Johannes Campanus, who edited Euclid in the 13th century, inferred from III. 16 that there was a flaw in the principle that *the transition from the less to the greater, or vice versâ, takes place through all intermediate quantities and therefore through the equal.* If a diameter of a circle, he says, be moved about its extremity until it takes the position of the tangent to that circle, then, as long as it cuts the circle, it makes an acute angle *less* than the "angle of a semicircle"; but the moment it ceases to cut, it makes a right angle *greater* than the same "angle of a semicircle." The rectilineal angle is never, during the transition, *equal* to the "angle of a semicircle." There is therefore an apparent inconsistency with x. 1, and Campanus could only observe (as he does on that proposition), in explanation of the paradox, that "these are not angles in the same sense (univoce), for the curved and the straight are not things of the same kind without qualification (simpliciter)." The argument assumes, of course, that the right angle *is* greater than the "angle of a semicircle."

Very similar is the statement of the paradox by Cardano (1501—1576), who observed that *a quantity may continually increase without limit, and another diminish without limit; and yet the first, however increased, may be less than the second, however diminished.* The first quantity is of course the *angle of contact*, as he calls it, which may be "increased" indefinitely by drawing smaller and smaller circles touching the same straight line at the same point, but will always be less than any acute rectilineal angle however small.

We next come to the French geometer, Peletier (Peletarius), who edited the *Elements* in 1557, and whose views on this subject seem to mark a great advance. Peletier's opinions and arguments are most easily accessible in the account of them given by Clavius (Christoph Klau [?], 1537—1612) in the 1607 edition of his Euclid. The violence of the controversy between the two will be understood from the fact that the arguments and counter-arguments (which sometimes run into other matters than the particular question at issue) cover, in that book, 26 pages of small print. Peletier held that the "angle of contact" was not an angle at all, that the "contact of two circles," i.e. the "angle" between the circumferences of two circles touching one another internally or externally, is not a *quantity*, and that the "contact of a straight line with a circle" is not a *quantity* either; that angles contained by a diameter and a circumference whether inside or outside the circle are *right angles* and equal to rectilineal right angles, and that angles contained by a diameter and the circumference in *all* circles are *equal.* The proof which Peletier gave of the latter proposition in a letter to Cardano is sufficiently ingenious. If a greater and a less semicircle be placed with their diameters terminating at a common point and lying in a straight line, then (1) the *angle of* the larger obviously cannot be *less* than the *angle of* the smaller. Neither (2) can the former be *greater* than the latter; for, if it were, we could obtain another *angle of* a semicircle greater still by drawing a still larger semicircle, and so on, until we should ultimately have an *angle of* a semicircle greater than a right angle: which is impossible. Hence the *angles of* semicircles must all be *equal*, and the differences between them *nothing.* Having satisfied himself that all *angles of contact* are *not*-angles, *not*-quantities, and therefore *nothings*, Peletier holds the difficulty about x. 1 to be at an end. He adds the interesting remark that the essence of an angle is in *cutting*, not contact, and that a tangent is not *inclined* to the circle at the point of contact but is, as it were, *immersed* in it at that point, just as much as if the circle did not diverge from it on either side.

The reply of Clavius need not detain us. He argues, evidently appealing to the eye, that the angle of contact can be *divided* by the arc of a circle greater than the given one, that the angles of two semicircles of different sizes cannot be equal, since they do not coincide if they are applied to one another, that there is nothing to prevent *angles of contact* from being *quantities*, it being only necessary, in view of x. 1, to admit that they are not of the same kind as rectilineal angles; lastly that, if the angle of contact had been a *nothing*, Euclid would not have given himself so much trouble to prove that it is less than any acute angle. (The word is *desudasset*, which is certainly an exaggeration as applied to what is little more than an *obiter dictum* in iii. 16.)

Vieta (1540—1603) ranged himself on the side of Peletier, maintaining that the *angle of contact* is no angle; only he uses a new method of proof. The circle, he says, may be regarded as a plane figure with an infinite number of sides and angles; but *a straight line touching a straight line, however short it may be, will coincide with that straight line and will not make an angle.* Never before, says Cantor (ii$_1$, p. 540), had it been so plainly declared what exactly was to be understood by *contact*.

Galileo Galilei (1564—1642) seems to have held the same view as Vieta and to have supported it by a very similar argument derived from the comparison of the circle and an inscribed polygon with an infinite number of sides.

The last writer on the question who must be mentioned is John Wallis (1616—1703). He published in 1656 a paper entitled *De angulo contactus et semicirculi tractatus* in which he also maintained that the so-called angle was not a true angle, and was not a *quantity*. Vincent Leotaud (1595—1672) took up the cudgels for Clavius in his *Cyclomathia* which appeared in 1663. This brought a reply from Wallis in a letter to Leotaud dated 17 February, 1667, but not apparently published till it appeared in *A defense of the treatise of the angle of contact* which, with a separate title-page, and date 1684, was included in the English edition of his Algebra dated 1685. The essence of Wallis' position may be put as follows. According to Euclid's definition, a plane angle is an *inclination* of two lines; therefore two lines forming an angle must *incline* to one another, and, if two lines meet without being *inclined* to one another at the point of meeting (which is the case when a circumference is touched by a straight line), the lines do not form an *angle*. The "angle of contact" is therefore no angle, because *at the point of contact* the straight line is not inclined to the circle but lies on it ἀκλινῶς, or is coincident with it. Again, as a point is not a line but a *beginning* of a line, and a line is not a surface but a *beginning* of a surface, so an angle is not the distance between two lines, but their initial tendency towards separation: *Angulus* (*seu gradus divaricationis*) *Distantia non est sed Inceptivus distantiae.* How far lines, which at their point of meeting do not form an angle, separate from one another as they pass on depends on the *degree of curvature* (gradus curvitatis), and it is the latter which has to be compared in the case of two lines so meeting. The arc of a smaller circle is more curved as having as much curvature in a lesser length, and is therefore curved in a greater degree. Thus what Clavius called *angulus contactus* becomes with Wallis *gradus curvitatis*, the use of which expression shows that curvature and curvature can be compared according to one and the same standard. A straight line has the least possible curvature; but of the "angle" made by it with a curve which it touches we cannot say that it is greater or less than the "angle" which a second curve touching the same straight line at the same point makes with the first curve; for in both cases there is no true angle at all (cf. Cantor iii$_1$, p. 24).

The words usually given as a part of the corollary "and that a straight line touches a circle at one point only, since in fact the straight line meeting it in two points was proved to fall within it" are omitted by Heiberg as being an undoubted addition of Theon's. It was Simson who added the further remark that "it is evident that there can be but one straight line which touches the circle at the same point."

PROPOSITION 17.

From a given point to draw a straight line touching a given circle.

Let *A* be the given point, and *BCD* the given circle; thus it is required to draw from the point *A* a straight line touching the circle *BCD*.

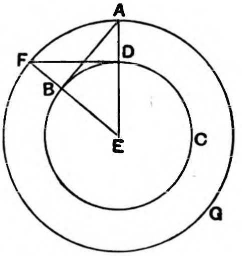
For let the centre *E* of the circle be taken; [III. 1]

let *AE* be joined, and with centre *E* and distance *EA* let the circle *AFG* be described;

from *D* let *DF* be drawn at right angles to *EA*,

and let *EF*, *AB* be joined;

I say that *AB* has been drawn from the point *A* touching the circle *BCD*.



For, since *E* is the centre of the circles *BCD*, *AFG*,

EA is equal to *EF*, and *ED* to *EB*;

therefore the two sides *AE*, *EB* are equal to the two sides *FE*, *ED*:

and they contain a common angle, the angle at *E*;

therefore the base *DF* is equal to the base *AB*,

and the triangle *DEF* is equal to the triangle *BEA*,

and the remaining angles to the remaining angles; [I. 4]

therefore the angle *EDF* is equal to the angle *EBA*.

But the angle *EDF* is right;

therefore the angle *EBA* is also right.

Now *EB* is a radius;

and the straight line drawn at right angles to the diameter of a circle, from its extremity, touches the circle; [III. 16, Por.]

therefore *AB* touches the circle *BCD*.

Therefore from the given point *A* the straight line *AB* has been drawn touching the circle *BCD*.

The construction shows, of course, that two straight lines can be drawn from a given external point to touch a given circle; and it is equally obvious that these two straight lines are equal in length and equally inclined to the straight line joining the external point to the centre of the given circle. These facts are given by Heron (an-Nairīzī, p. 130).

It is true that Euclid leaves out the case where the given point lies *on* the circumference of the circle, doubtless because the construction is so directly indicated by III. 16, Por. as to be scarcely worth a separate statement.

An easier solution is of course possible as soon as we know (III. 31) that the angle in a semicircle is a right angle; for we have only to describe a circle on AE as diameter, and this circle cuts the given circle in the two points of contact.

PROPOSITION 18.

If a straight line touch a circle, and a straight line be joined from the centre to the point of contact, the straight line so joined will be perpendicular to the tangent.

For let a straight line DE touch the circle ABC at the point C, let the centre F of the circle ABC be taken, and let FC be joined from F to C;

I say that FC is perpendicular to DE.

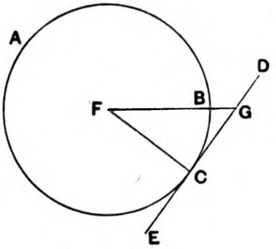
For, if not, let FG be drawn from F perpendicular to DE.

Then, since the angle FGC is right,

the angle FCG is acute; [I. 17]

and the greater angle is subtended by the greater side; [I. 19]

therefore FC is greater than FG.



But FC is equal to FB;

therefore FB is also greater than FG,

the less than the greater: which is impossible.

Therefore FG is not perpendicular to DE.

Similarly we can prove that neither is any other straight line except FC;

therefore FC is perpendicular to DE.

Therefore etc.

Q. E. D.

3. **the tangent,** ἡ ἐφαπτομένη.

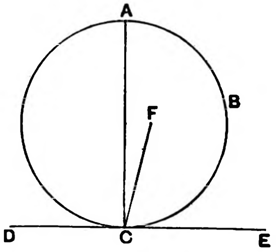
Just as III. 3 contains two *partial* converses of the Porism to III. 1, so the present proposition and the next give two partial converses of the corollary to III. 16. We may show their relation thus: suppose three things, (1) a tangent at a point of a circle, (2) a straight line drawn from the centre to the point of contact, (3) right angles made at the point of contact [with (1) or (2) as the case may be]. Then the corollary to III. 16 asserts that (2) and (3) together give (1), III. 18 that (1) and (2) give (3), and III. 19 that (1) and (3) give (2), i.e. that the straight line drawn from the point of contact at right angles to the tangent passes through the centre.

PROPOSITION 19.

If a straight line touch a circle, and from the point of contact a straight line be drawn at right angles to the tangent, the centre of the circle will be on the straight line so drawn.

For let a straight line *DE* touch the circle *ABC* at the point *C*, and from *C* let *CA* be drawn at right angles to *DE*;

I say that the centre of the circle is on *AC*.



For suppose it is not, but, if possible, let *F* be the centre, and let *CF* be joined.

Since a straight line *DE* touches the circle *ABC*,

and *FC* has been joined from the centre to the point of contact,

FC is perpendicular to *DE*; [III. 18]

therefore the angle *FCE* is right.

But the angle *ACE* is also right;

therefore the angle *FCE* is equal to the angle *ACE*,

the less to the greater: which is impossible.

Therefore *F* is not the centre of the circle *ABC*.

Similarly we can prove that neither is any other point except a point on *AC*.

Therefore etc.

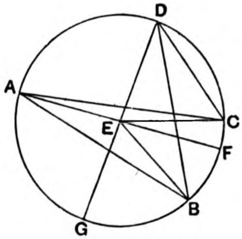
Q. E. D.

We may also regard III. 19 as a partial converse of III. 18. Thus suppose (1) a straight line through the centre, (2) a straight line through the point of contact, and suppose (3) to mean perpendicular to the tangent; then III. 18 asserts that (1) and (2) combined produce (3), and III. 19 that (2) and (3) produce (1); while again we may enunciate a second partial converse of III. 18, corresponding to the statement that (1) and (3) produce (2), to the effect that a straight line drawn through the centre perpendicular to the tangent passes through the point of contact.

We may add at this point, or even after the Porism to III. 16, the theorem that *two circles which touch one another internally or externally have a common tangent at their point of contact.* For the line joining their centres, produced if necessary, passes through their point of contact, and a straight line drawn through that point at right angles to the line of centres is a tangent to both circles.

PROPOSITION 20.

In a circle the angle at the centre is double of the angle at the circumference, when the angles have the same circumference as base.



Let *ABC* be a circle, let the angle *BEC* be an angle at its centre, and the angle *BAC* an angle at the circumference, and let them have the same circumference *BC* as base;

I say that the angle *BEC* is double of the angle *BAC*.

For let *AE* be joined and drawn through to *F*.

Then, since *EA* is equal to *EB*,

the angle *EAB* is also equal to the angle *EBA*; [I. 5]

therefore the angles *EAB*, *EBA* are double of the angle *EAB*.

But the angle *BEF* is equal to the angles *EAB*, *EBA*; [I. 32]

therefore the angle *BEF* is also double of the angle *EAB*.

For the same reason

the angle *FEC* is also double of the angle *EAC*.

Therefore the whole angle *BEC* is double of the whole angle *BAC*.

Again let another straight line be inflected, and let there be another angle *BDC*; let *DE* be joined and produced to *G*.

Similarly then we can prove that the angle *GEC* is double of the angle *EDC*,

of which the angle *GEB* is double of the angle *EDB*;

therefore the angle *BEC* which remains is double of the angle *BDC*.

Therefore etc. Q. E. D.

25. **let another straight line be inflected,** κεκλάσθω δὴ πάλιν (without εὐθεῖα). The verb κλάω (to *break off*) was the regular technical term for drawing from a point a (broken) straight line which first meets another straight line or curve and is then *bent back* from it to another point, or (in other words) for drawing straight lines from two points meeting at a point on a curve or another straight line. κεκλάσθαι is one of the geometrical terms the definition of which must according to Aristotle be assumed (*Anal. Post.* I. 10, 76 b 9).

The early editors, Tartaglia, Commandinus, Peletarius, Clavius and others, gave the extension of this proposition to the case where the segment is less than a semicircle, and where accordingly the "angle" corresponding to Euclid's "angle at the centre" is greater than two right angles. The convenience of the extension is obvious, and the proof of it is the same as the first part of Euclid's proof. By means of the extension III. 21 is demonstrated without making two cases; III. 22 will follow immediately from the fact that the sum of the "angles at the centre" for two segments making up a whole circle is equal to four right angles; also III. 31 follows immediately from the extended proposition.

But all the editors referred to were forestalled in this matter by Heron, as we now learn from the commentary of an-Nairīzī (ed. Curtze, p. 131 sqq.). Heron gives the extension of Euclid's proposition which, he says, it had been left for him to make, but which is necessary in order that the caviller may not be able to say that the next proposition (about the equality of the angles in any segment) is not established generally, i.e. in the case of a segment less than a semicircle as well as in the case of a segment greater than a semicircle, inasmuch as III. 20, as given by Euclid, only enables us to prove it in the latter case. Heron's enunciation is important as showing how he describes what we should now call an "angle" greater than two right angles. (The language of Gherard's translation is, in other respects, a little obscure; but the meaning is made clear by what follows.)

"The angle," Heron says, "which is at the centre of any circle is double of the angle which is at the circumference of it *when one arc is the base of both angles*; and *the remaining angles which are at the centre, and fill up the four right angles*, are double of the angle at the circumference of the arc which is subtended by the [original] angle which is at the centre."

Thus the "angle greater than two right angles" is for Heron the sum of certain "angles" in the Euclidean sense of angles less than two right angles. The particular method of splitting up which Heron adopts will be seen from his proof, which is in substance as follows.

Let *CDB* be an *angle* at the centre, *CAB* that at the circumference.

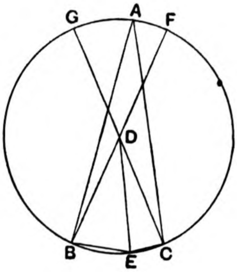
Produce *BD*, *CD* to *F*, *G*;

take any point *E* on *BC*, and join *BE*, *EC*, *ED*.

Then any angle in the segment *BAC* is half of the angle *BDC*; and *the sum of the angles* BDG, GDF, FDC *is double of any angle in the segment* BEC.

Proof. Since CD is equal to ED,

the angles DCE, DEC are equal.



Therefore the exterior angle GDE is equal to twice the angle DEC.
Similarly the exterior angle FDE is equal to twice the angle DEB.
By addition, the angles GDE, FDE are double of the angle BEC.

But

the angle BDC is equal to the angle FDG,

therefore *the sum of the angles* BDG, GDF, FDC *is double of the angle* BEC.

And Euclid has proved the first part of the proposition, namely that the angle BDC is double of the angle BAC.

Now, says Heron, BAC is *any* angle in the segment BAC, and therefore *any* angle in the segment BAC is half of the angle BDC.

Therefore all the angles in the segment BAC are equal.

Again, BEC is *any* angle in the segment BEC and is equal to *half the sum of the angles* BDG, GDF, FDC.

Therefore all the angles in the segment BEC are equal.

Hence III. 21 is proved *generally*.

Lastly, says Heron,

since *the sum of the angles* BDG, GDF, FDC is double of the angle BEC,

and the angle BDC is double of the angle BAC,

therefore, by addition, the *sum of four right angles* is double of the sum of the angles BAC, BEC.

Hence the angles BAC, BEC are together equal to two right angles, and III. 22 is proved.

The above notes of Heron show conclusively, if proof were wanted, that Euclid had no idea of III. 20 applying *in terms* (either as a matter of enunciation or proof) to the case where the angle at the circumference, or the angle in the segment, is *obtuse*. He would not have recognised the "angle" greater than two right angles or the so-called "straight angle" as being an *angle* at all. This is indeed clear from his definition of an angle as the *inclination* κ.τ.ἑ., and from the language used by other later Greek mathematicians where there would be an opportunity for introducing the extension. Thus Proclus' notion of a "four-sided triangle" (cf. the note above on the definition of a triangle) shows that he did not count a re-entrant angle as an angle, and Zenodorus' application to the same figure of the word "hollow-angled" shows that in that case it was the exterior angle only which he would have called an angle. Further it would have been inconvenient to have introduced at the beginning of the *Elements* an "angle" equal to or greater than two right angles, because other definitions, e.g. that of a *right angle*, would have needed a qualification. If an "angle" might be equal to two right angles, one straight line in a straight line with another would have satisfied Euclid's definition of a right angle. This is noticed by Dodgson (p. 160), but it is practically brought out by Proclus on I. 13. "For he did not merely say that 'any straight line standing on a straight line either makes two right angles or angles equal to two right angles' but '*if it make angles.*'

If it stand on the straight line at its extremity and make one angle, is it possible for this to be equal to two right angles? It is of course impossible; *for every rectilineal angle is less than two right angles*, as every solid angle is less than four right angles (p. 292, 13—20)." [It is true that it has been generally held that the meaning of "angle" is tacitly extended in VI. 33, but there is no real ground for this view. See the note on the proposition.]

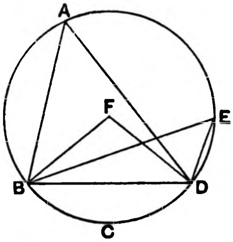
It will be observed that, following his usual habit, Euclid omits the demonstration of the case which some editors, e.g. Clavius, have thought it necessary to give separately, the case namely where one of the lines forming the angle in the segment passes through the centre. Euclid's proof gives so obviously the means of proving this that it is properly left out.

Todhunter observes, what Clavius had also remarked, that there are two assumptions in the proof of III. 20, namely that, if A is double of B and C double of D, then the sum, or difference, of A and C is equal to double the sum, or difference, of B and D respectively, the assumptions being particular cases of V. 1 and V. 5. But of course it is easy to satisfy ourselves of the correctness of the assumption without any recourse to Book V.

PROPOSITION 21.

In a circle the angles in the same segment are equal to one another.

Let $ABCD$ be a circle, and let the angles BAD, BED be angles in the same segment $BAED$;



I say that the angles BAD, BED are equal to one another.

For let the centre of the circle $ABCD$ be taken, and let it be F; let BF, FD be joined.

Now, since the angle BFD is at the centre,

and the angle BAD at the circumference,

and they have the same circumference BCD as base,

therefore the angle BFD is double of the angle BAD. [III. 20]

For the same reason

the angle BFD is also double of the angle BED;

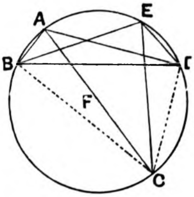
therefore the angle BAD is equal to the angle BED.

Therefore etc.

Q. E. D.

Under the restriction that the "angle at the centre" used in III. 20 must be less than two right angles, Euclid's proof of this proposition only applies to the case of a segment greater than a semicircle, and the case of a segment equal to or less than a semicircle has to be considered separately. The simplest proof, of many, seems to be that of Simson.

"But, if the segment $BAED$ be not greater than a semicircle, let BAD, BED be angles in it: these also are equal to one another.



Draw AF to the centre, and produce it to C, and join CE.

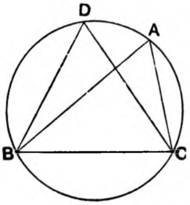
Therefore the segment $BADC$ is greater than a semicircle, and the angles in it BAC, BEC are equal, by the first case.

For the same reason, because $CBED$ is greater than a semicircle,

the angles CAD, CED are equal.

Therefore the whole angle BAD is equal to the whole angle BED."

We can prove, by means of *reductio ad absurdum*, the important converse of this proposition, namely that, *if there be any two triangles on the same base and on the same side of it, and with equal vertical angles, the circle passing through the extremities of the base and the vertex of one triangle will pass through the vertex of the other triangle also.* That a circle can be thus described about a triangle is clear from Euclid's construction in III. 9, which shows how to draw a circle passing through any three points, though it is in IV. 5 only that we have the problem stated. Now, suppose a circle BAC drawn through the angular points of a triangle BAC, and let BDC be another triangle with the same base BC and on the same side of it, and having its vertical angle D equal to the angle A. Then shall the circle pass through D.



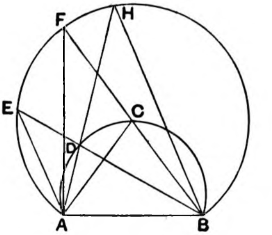
For, if it does not, it must pass through some point E on BD or on BD produced. If then EC be joined, the angle BEC is equal to the angle BAC, by III. 21, and therefore equal to the angle BDC. Therefore an exterior angle of a triangle is equal to the interior and opposite angle: which is impossible, by I. 16.

Therefore D lies on the circle BAC.

Similarly for any other triangle on the base BC and with vertical angle equal to A. Thus, *if any number of triangles be constructed on the same base and on the same side of it, with equal vertical angles, the vertices will all lie on the circumference of a segment of a circle.*

A useful theorem derivable from III. 21 is given by Serenus (*De sectione coni*, Props. 52, 53).

If ADB be any segment of a circle, and C be such a point on the circumference that AC is equal to CB, and if there be described with C as centre and radius CA or CB the circle AHB, then, ADB being any other angle in the segment ACB, and BD being produced to meet the outer segment in E, the sum of AD, DB is equal to BE.



If BC be produced to meet the outer segment in F, and FA be joined,

CA, CB, CF are by hypothesis equal.

Therefore the angle FAC is equal to the angle AFC.

Also, by III. 21, the angles ACB, ADB are equal;

therefore their supplements, the angles *ACF*, *ADE*, are equal.

Further, by III. 21, the angles *AEB*, *AFB* are equal.

Hence in the triangles *ACF*, *ADE* two angles are respectively equal;

therefore the third angles *EAD*, *FAC* are equal.

But the angle *FAC* is equal to the angle *AFC*, and therefore equal to the angle *AED*.

Therefore the angles *AED*, *EAD* are equal, or the triangle *DEA* is isosceles,

and *AD* is equal to *DE*.

Adding *BD* to both, we see that

BE is equal to the sum of *AD* and *DB*.

Now, *BF* being a diameter of the circle of which the outer segment is a part,

BF is greater than *BE*;

therefore *AC*, *CB* are together greater than *AD*, *DB*.

And, generally, *of all triangles on the same base and on the same side of it which have equal vertical angles, the isosceles triangle is that which has the greatest perimeter, and of the others that has the lesser perimeter which is further from being isosceles.*

The theorem of Serenus gives us the means of solving the following problem given in Todhunter's Euclid, p. 324.

To find a point in the circumference of a given segment of a circle such that the straight lines which join the point to the extremities of the straight line on which the segment stands may be together equal to a given straight line (the length of which is of course subject to limits).

Let *ACB* in the above figure be the given segment. Find, by bisecting *AB* at right angles, a point *C* on it such that *AC* is equal to *CB*.

Then with centre *C* and radius *CA* or *CB* describe the segment of a circle *AHB* on the same side of *AB*.

Lastly, with *A* or *B* as centre and radius equal to the given straight line describe a circle. This circle will, if the given straight line be greater than *AB* and less than twice *AC*, meet the outer segment in two points, and if we join those points to the centre of the circle last drawn (whether *A* or *B*), the joining straight lines will cut the inner segment in points satisfying the given condition. If the given straight line be *equal* to twice *AC*, *C* is of course the required point. If the given straight line be greater than twice *AC*, there is no possible solution.

PROPOSITION 22.

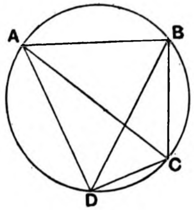
The opposite angles of quadrilaterals in circles are equal to two right angles.

Let *ABCD* be a circle, and let *ABCD* be a quadrilateral in it;

I say that the opposite angles are equal to two right angles.

Let *AC*, *BD* be joined.

Then, since in any triangle the three angles are equal to two right angles, [I. 32] the three angles *CAB*, *ABC*, *BCA* of the triangle *ABC* are equal to two right angles.



But the angle *CAB* is equal to the angle *BDC*, for they are in the same segment *BADC*; [III. 21]

and the angle *ACB* is equal to the angle *ADB*, for they are in the same segment *ADCB*;

therefore the whole angle *ADC* is equal to the angles *BAC*, *ACB*.

Let the angle *ABC* be added to each;

therefore the angles *ABC*, *BAC*, *ACB* are equal to the angles *ABC*, *ADC*.

But the angles *ABC*, *BAC*, *ACB* are equal to two right angles;

therefore the angles *ABC*, *ADC* are also equal to two right angles.

Similarly we can prove that the angles *BAD*, *DCB* are also equal to two right angles.

Therefore etc.

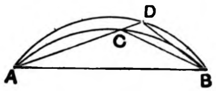
Q. E. D.

As Todhunter remarks, the converse of this proposition is true and very important: *if two opposite angles of a quadrilateral be together equal to two right angles, a circle may be circumscribed about the quadrilateral.* We can, by the method of III. 9, or by IV. 5, circumscribe a circle about the triangle *ABC*; and we can then prove, by *reductio ad absurdum*, that the circle passes through the fourth angular point *D*.

PROPOSITION 23.

On the same straight line there cannot be constructed two similar and unequal segments of circles on the same side.

For, if possibie, on the same straight line *AB* let two similar and unequal segments of circles *ACB*, *ADB* be constructed on the same side;



let *ACD* be drawn through, and let *CB*, *DB* be joined.

Then, since the segment *ACB* is similar to the segment *ADB*, and similar segments of circles are those which admit equal angles, [III. Def. 11]

the angle *ACB* is equal to the angle *ADB*, the exterior to the interior: which is impossible. [I. 16]

Therefore etc.

Q. E. D.

1. **cannot be constructed**, *οὐ συσταθήσεται*, the same phrase as in I. 7.

Clavius and the other early editors point out that, while the words "on the same side" in the enunciation are necessary for Euclid's proof, it is equally true that neither can there be two similar and unequal segments on *opposite* sides of the same straight line; this is at once made clear by causing one of the segments to revolve round the base till it is on the same side with the other.

Simson observes with reason that, while Euclid in the following proposition, III. 24, thinks it necessary to dispose of the hypothesis that, if two similar segments on equal bases are applied to one another with the bases coincident, the segments cannot cut in any other point than the extremities of the base (since otherwise two circles would cut one another in more points than two), this remark is an equally necessary preliminary to III. 23, in order that we may be justified in drawing the segments as being one inside the other. Simson accordingly begins his proof of III. 23 thus:

"Then, because the circle *ACB* cuts the circle *ADB* in the two points *A*, *B*, they cannot cut one another in any other point:

One of the segments must therefore fall within the other.

Let *ACB* fall within *ADB* and draw the straight line *ACD*, etc."

Simson has also substituted "not coinciding with one another" for "unequal" in Euclid's enunciation.

Then in III. 24 Simson leaves out the words referring to the hypothesis that the segment *AEB* when applied to the other *CFD* may be "otherwise placed as *CGD*"; in fact, after stating that *AB* must coincide with *CD*, he merely adds words quoting the result of III. 23: "Therefore, the straight line *AB* coinciding with *CD*, the segment *AEB* must coincide with the segment *CFD*, and is therefore equal to it."

PROPOSITION 24.

Similar segments of circles on equal straight lines are equal to one another.

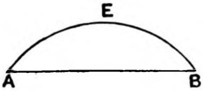
For let *AEB*, *CFD* be similar segments of circles on equal straight lines *AB*, *CD*;

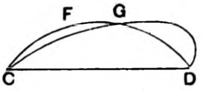
I say that the segment *AEB* is equal to the segment *CFD*.

For, if the segment *AEB* be applied to *CFD*, and if the point *A* be placed on *C* and the straight line *AB* on *CD*, the point *B* will also coincide with the point *D*, because *AB* is equal to *CD*;

and, *AB* coinciding with *CD*,

the segment *AEB* will also coincide with *CFD*.





For, if the straight line *AB* coincide with *CD* but the segment *AEB* do not coincide with *CFD*,

it will either fall within it, or outside it;

or it will fall awry, as *CGD*, and a circle cuts a circle at more points than two: which is impossible. [III. 10]

Therefore, if the straight line *AB* be applied to *CD*, the segment *AEB* will not fail to coincide with *CFD* also;

therefore it will coincide with it and will be equal to it.

Therefore etc.

Q. E. D.

15. **fall awry, παραλλάξει,** the same word as used in the like case in I. 8. The word implies that the applied figure will partly fall short of, and partly overlap, the figure to which it is applied.

Compare the note on the last proposition. I have put a semicolon instead of the comma which the Greek text has after "outside it," in order the better to indicate that the inference "and a circle cuts a circle in more points than two" only refers to the third hypothesis that the applied segment is "otherwise placed (παραλλάξει) as *CGD*." The first two hypotheses are disposed of by a *tacit* reference to the preceding proposition III. 23.

PROPOSITION 25.

Given a segment of a circle, to describe the complete circle of which it is a segment.

Let *ABC* be the given segment of a circle;

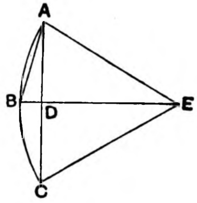
thus it is required to describe the complete circle belonging to the segment *ABC*, that is, of which it is a segment.

For let *AC* be bisected at *D*, let *DB* be drawn from the point *D* at right angles to *AC*, and let *AB* be joined;

the angle *ABD* is then greater than, equal to, or less than the angle *BAD*.

First let it be greater;

and on the straight line *BA*, and at the point *A* on it, let the angle *BAE* be constructed equal to the angle *ABD*; let *DB* be drawn through to *E*, and let *EC* be joined.



Then, since the angle *ABE* is equal to the angle *BAE*,

the straight line *EB* is also equal to *EA*. [I. 6]

And, since *AD* is equal to *DC*, and *DE* is common,

the two sides *AD*, *DE* are equal to the two sides *CD*, *DE* respectively;

and the angle *ADE* is equal to the angle *CDE*, for each is right;

therefore the base *AE* is equal to the base *CE*.

But *AE* was proved equal to *BE*;

therefore *BE* is also equal to *CE*;

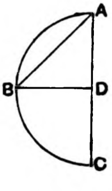
therefore the three straight lines *AE*, *EB*, *EC* are equal to one another.

Therefore the circle drawn with centre *E* and distance one of the straight lines *AE*, *EB*, *EC* will also pass through the remaining points and will have been completed. [III. 9]

Therefore, given a segment of a circle, the complete circle has been described.

And it is manifest that the segment *ABC* is less than a semicircle, because the centre *E* happens to be outside it.

Similarly, even if the angle *ABD* be equal to the angle *BAD*,



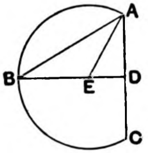
AD being equal to each of the two *BD*, *DC*,

the three straight lines *DA*, *DB*, *DC* will be equal to one another,

D will be the centre of the completed circle,

and *ABC* will clearly be a semicircle.

But, if the angle ABD be less than the angle BAD, and if we construct, on the straight line BA and at the point A on it, an angle equal to the angle ABD, the centre will fall on DB within the segment ABC, and the segment ABC will clearly be greater than a semicircle.



Therefore, given a segment of a circle, the complete circle has been described.

Q. E. F.

1. **to describe the complete circle**, *προσαναγράψαι τὸν κύκλον*, literally "to describe the circle *on to it*.'

It will be remembered that Simson takes first the case in which the angles ABD, BAD are equal to one another, and then takes the other two cases together, telling us to "produce BD, if necessary." This is a little shorter than Euclid's procedure, though Euclid does not repeat the proof of the first case in giving the third, but only refers to it as equally applicable.

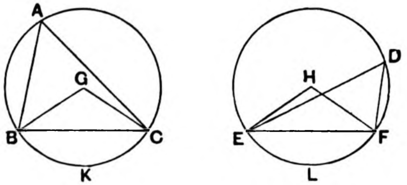
Campanus, Peletarius and others give the solution of this problem in which we take two chords not parallel and bisect each at right angles by straight lines, which must meet in the centre, since each contains the centre and they only intersect in one point. Clavius, Billingsley, Barrow and others give the rather simpler solution in which the two chords have one extremity common (cf. Euclid's proofs of III. 9, 10). This method De Morgan favours, and (as noted on III. 1 above) would make III. 1, this proposition, and IV. 5 all *corollaries* of the theorem that "the line which bisects a chord perpendicularly must contain the centre." Mr H. M. Taylor practically adopts this order and method, though he finds the centre of a circle by means of any two non-parallel chords; but he finds *the centre of the circle of which a given arc is a part* (his proposition corresponding to III. 25) by bisecting at right angles first the base and then the chord joining one extremity of the base to the point in which the line bisecting the base at right angles meets the circumference of the segment. Under De Morgan's alternative the relation between Euclid III. 1 and the Porism to it would be reversed, and Euclid's notion of a Porism or *corollary* would have to be considerably extended.

If the problem is solved after the manner of IV. 5, it is still desirable to state, as Euclid does, after proving AE, EB, EC to be all equal, that "the circle drawn with centre E and distance one of the straight lines AE, EB, EC *will also pass through the remaining points of the segment*" [III. 9], in order to show that part of the circle described actually coincides with the given segment. This is not so clear if the centre is determined as the intersection of the straight lines bisecting at right angles chords which join pairs of *four* different points.

PROPOSITION 26.

In equal circles equal angles stand on equal circumferences, whether they stand at the centres or at the circumferences.

Let *ABC*, *DEF* be equal circles, and in them let there be equal angles, namely at the centres the angles *BGC*, *EHF*, and at the circumferences the angles *BAC*, *EDF*;
I say that the circumference *BKC* is equal to the circumference *ELF*.



For let *BC*, *EF* be joined.

Now, since the circles *ABC*, *DEF* are equal,
the radii are equal.

Thus the two straight lines *BG*, *GC* are equal to the two straight lines *EH*, *HF*;
and the angle at *G* is equal to the angle at *H*;
therefore the base *BC* is equal to the base *EF*. [I. 4]

And, since the angle at *A* is equal to the angle at *D*,
the segment *BAC* is similar to the segment *EDF*; [III. Def. 11]
and they are upon equal straight lines.

But similar segments of circles on equal straight lines are equal to one another; [III. 24]
therefore the segment *BAC* is equal to *EDF*.

But the whole circle *ABC* is also equal to the whole circle *DEF*;
therefore the circumference *BKC* which remains is equal to the circumference *ELF*.

Therefore etc. Q. E. D.

As in III. 21, if Euclid's proof is to cover all cases, it requires us to take cognisance of "angles at the centre" which are equal to or greater than two right angles. Otherwise we must deal separately with the cases where the angle at the circumference is equal to or greater than a right angle. The case of an *obtuse* angle at the circumference can of course be reduced by means of III. 22 to the case of an acute angle at the circumference; and, in case the angle at the circumference is right, it is readily proved, by drawing the radii to the vertex of the angle and to the other extremities of the lines containing it, that the latter two radii are in a straight line, whence they make equal bases in the two circles as in Euclid's proof.

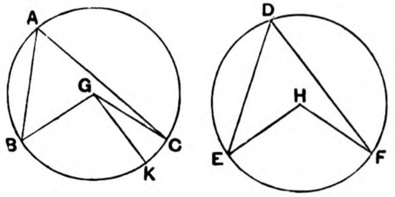
Lardner has another way of dealing with the right angle or obtuse angle at the circumference. In either case, he says, "bisect them, and the halves of them are equal, and it can be proved, as above, that the arcs upon which these halves stand are equal, whence it follows that the arcs on which the given angles stand are equal."

PROPOSITION 27.

In equal circles angles standing on equal circumferences are equal to one another, whether they stand at the centres or at the circumferences.

For in equal circles *ABC*, *DEF*, on equal circumferences *BC*, *EF*, let the angles *BGC*, *EHF* stand at the centres *G*, *H*, and the angles *BAC*, *EDF* at the circumferences;

I say that the angle *BGC* is equal to the angle *EHF*, and the angle *BAC* is equal to the angle *EDF*.



For, if the angle *BGC* is unequal to the angle *EHF*, one of them is greater.

Let the angle *BGC* be greater: and on the straight line *BG*, and at the point *G* on it, let the angle *BGK* be constructed equal to the angle *EHF*. [I. 23]

Now equal angles stand on equal circumferences, when they are at the centres; [III. 26]

therefore the circumference *BK* is equal to the circumference *EF*.

But *EF* is equal to *BC*;

therefore *BK* is also equal to *BC*, the less to the greater: which is impossible.

Therefore the angle *BGC* is not unequal to the angle *EHF*;

therefore it is equal to it.

And the angle at A is half of the angle BGC,

and the angle at D half of the angle EHF; [III. 20]

therefore the angle at A is also equal to the angle at D.

Therefore etc.

Q. E. D.

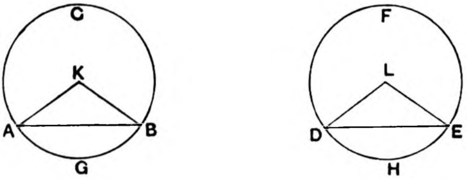
This proposition is the converse of the preceding one, and the remarks about the method of treating the different cases apply here also.

PROPOSITION 28.

In equal circles equal straight lines cut off equal circumferences, the greater equal to the greater and the less to the less.

Let ABC, DEF be equal circles, and in the circles let AB, DE be equal straight lines cutting off ACB, DFE as greater circumferences and AGB, DHE as lesser;

I say that the greater circumference ACB is equal to the greater circumference DFE, and the less circumference AGB to DHE.



For let the centres K, L of the circles be taken, and let AK, KB, DL, LE be joined.

Now, since the circles are equal,

the radii are also equal;

therefore the two sides AK, KB are equal to the two sides DL, LE;

and the base AB is equal to the base DE;

therefore the angle AKB is equal to the angle DLE. [I. 8]

But equal angles stand on equal circumferences, when they are at the centres; [III. 26]

therefore the circumference AGB is equal to DHE.

And the whole circle *ABC* is also equal to the whole circle *DEF*;

therefore the circumference *ACB* which remains is also equal to the circumference *DFE* which remains.

Therefore etc.

Q. E. D.

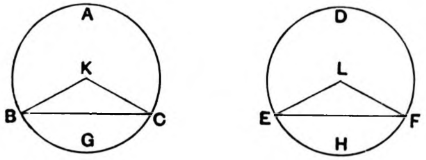
Euclid's proof does not in terms cover the particular case in which the chord in one circle passes through its centre; but indeed this was scarcely worth giving, as the proof can easily be supplied. Since the chord in one circle passes through its centre, the chord in the second circle must also be a diameter of that circle, for equal circles are those which have equal diameters, and all other chords in any circle are less than its diameter [III. 15]; hence the segments cut off in each circle are semicircles, and these must be equal because the circles are equal.

PROPOSITION 29.

In equal circles equal circumferences are subtended by equal straight lines.

Let *ABC*, *DEF* be equal circles, and in them let equal circumferences *BGC*, *EHF* be cut off; and let the straight lines *BC*, *EF* be joined;

I say that *BC* is equal to *EF*.



For let the centres of the circles be taken, and let them be *K*, *L*; let *BK*, *KC*, *EL*, *LF* be joined.

Now, since the circumference *BGC* is equal to the circumference *EHF*,

the angle *BKC* is also equal to the angle *ELF*. [III. 27]

And, since the circles *ABC*, *DEF* are equal,

the radii are also equal;

therefore the two sides *BK*, *KC* are equal to the two sides *EL*, *LF*; and they contain equal angles;

therefore the base *BC* is equal to the base *EF*. [I. 4]

Therefore etc.

The particular case of this converse of III. 28 in which the given arcs are arcs of semicircles is even easier than the corresponding case of III. 28 itself.

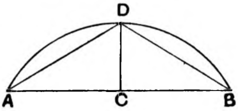
The propositions III. 26—29 are of course equally true if the *same* circle is taken instead of *two equal* circles.

PROPOSITION 30.

To bisect a given circumference.

Let ADB be the given circumference;

thus it is required to bisect the circumference ADB.



Let AB be joined and bisected at C; from the point C let CD be drawn at right angles to the straight line AB, and let AD, DB be joined.

Then, since AC is equal to CB, and CD is common,

the two sides AC, CD are equal to the two sides BC, CD; and the angle ACD is equal to the angle BCD, for each is right;

therefore the base AD is equal to the base DB. [I. 4]

But equal straight lines cut off equal circumferences, the greater equal to the greater, and the less to the less; [III. 28]

and each of the circumferences AD, DB is less than a semicircle;

therefore the circumference AD is equal to the circumference DB.

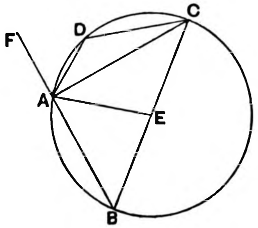
Therefore the given circumference has been bisected at the point D.

Q. E. F.

PROPOSITION 31.

In a circle the angle in the semicircle is right, that in a greater segment less than a right angle, and that in a less segment greater than a right angle; and further the angle of the greater segment is greater than a right angle, and the angle of the less segment less than a right angle.

Let *ABCD* be a circle, let *BC* be its diameter, and *E* its centre, and let *BA*, *AC*, *AD*, *DC* be joined;



I say that the angle *BAC* in the semicircle *BAC* is right,

the angle *ABC* in the segment *ABC* greater than the semicircle is less than a right angle,

and the angle *ADC* in the segment *ADC* less than the semicircle is greater than a right angle.

Let *AE* be joined, and let *BA* be carried through to *F*.

Then, since *BE* is equal to *EA*,

the angle *ABE* is also equal to the angle *BAE*. [I. 5]

Again, since *CE* is equal to *EA*,

the angle *ACE* is also equal to the angle *CAE*. [I. 5]

Therefore the whole angle *BAC* is equal to the two angles *ABC*, *ACB*.

But the angle *FAC* exterior to the triangle *ABC* is also equal to the two angles *ABC*, *ACB*; [I. 32]

therefore the angle *BAC* is also equal to the angle *FAC*;

therefore each is right; [I. Def. 10]

therefore the angle *BAC* in the semicircle *BAC* is right.

Next, since in the triangle *ABC* the two angles *ABC*, *BAC* are less than two right angles, [I. 17]

and the angle *BAC* is a right angle,

the angle *ABC* is less than a right angle;

and it is the angle in the segment *ABC* greater than the semicircle.

Next, since *ABCD* is a quadrilateral in a circle,

and the opposite angles of quadrilaterals in circles are equal to two right angles, [III. 22]

while the angle *ABC* is less than a right angle,

therefore the angle *ADC* which remains is greater than a right angle;

and it is the angle in the segment *ADC* less than the semicircle.

I say further that the angle of the greater segment, namely that contained by the circumference *ABC* and the straight line *AC*, is greater than a right angle;

and the angle of the less segment, namely that contained by the circumference *ADC* and the straight line *AC*, is less than a right angle.

This is at once manifest.

For, since the angle contained by the straight lines *BA*, *AC* is right,

the angle contained by the circumference *ABC* and the straight line *AC* is greater than a right angle.

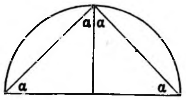
Again, since the angle contained by the straight lines *AC*, *AF* is right,

the angle contained by the straight line *CA* and the circumference *ADC* is less than a right angle.

Therefore etc. Q. E. D.

As already stated, this proposition is immediately deducible from III. 20 if that theorem is extended so as to include the case where the segment is equal to or less than a semicircle, and where consequently the "angle at the centre" is equal to two right angles or greater than two right angles respectively.

There are indications in Aristotle that the proof of the first part of the theorem in use before Euclid's time proceeded on different lines. Two passages of Aristotle refer to the proposition that the angle in a semicircle is a right angle. The first passage is *Anal. Post.* II. 11, 94 a 28: "Why is the angle in a semicircle a right angle? Or what makes it a right angle? (τίνος ὄντος ὀρθή;) Suppose *A* to be a right angle, *B* half of two right angles, *C* the angle in a semicircle. Then *B* is the cause of *A*, the right angle, being an attribute of *C*, the angle in the semicircle. For *B* is equal to *A*, and *C* to *B*; for *C* is half of two right angles. Therefore it is in virtue of *B* being half of two right angles that *A* is an attribute of *C*; and the latter means the fact that the angle in a semicircle is right." Now this passage by itself would be consistent with a proof like Euclid's or the alternative interpolated proof next to be mentioned. But the second passage throws a different light on the subject. This is *Metaph.* 1051 a 26: "Why is the angle in a semicircle a right angle invariably (καθόλου)? Because, *if there be three straight lines, two forming the base, and the third set up at right angles at its middle point*, the fact is obvious by simple inspection to any one who knows the property referred to" (ἐκεῖνο is the property that the angles of a triangle are together equal to two right angles, mentioned two lines before). That is to say, the angle *at the middle point* of the circumference of the semicircle was taken and proved, by means of the two isosceles right-angled triangles, to be the sum of two angles each equal to one-fourth of the sum of the angles of the large triangle in the figure, or of two right angles; and the proof must have been completed by means of the theorem of III. 21 (that angles



in the same segment are equal), which Euclid's more general proof does not need.

In the Greek texts before that of August there is an alternative proof that the angle *BAC* (in a semicircle) is right. August and Heiberg relegate it to an Appendix.

"Since the angle *AEC* is double of the angle *BAE* (for it is equal to the two interior and opposite angles), while the angle *AEB* is also double of the angle *EAC*,

the angles *AEB*, *AEC* are double of the angle *BAC*.

But the angles *AEB*, *AEC* are equal to two right angles;

therefore the angle *BAC* is right."

Lardner gives a slightly different proof of the second part of the theorem.

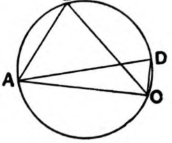
If *ABC* be a segment greater than a semicircle, draw the diameter *AD*, and join *CD*, *CA*.

Then, in the triangle *ACD*, the angle *ACD* is right (being the angle in a semicircle);

therefore the angle *ADC* is acute.

But the angle *ADC* is equal to the angle *ABC* in the same segment;

therefore the angle *ABC* is acute.



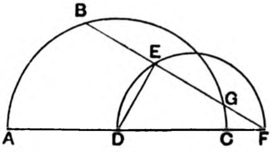
Euclid's references in this proposition to the angle *of* a segment greater or less than a semicircle respectively seem, like the part of III. 16 relating to the angle *of* a semicircle, to be a survival of ancient controversies and not to be put in deliberately as being an essential part of elementary geometry. Cf. the notes on III. Def. 7 and III. 16.

The corollary ordinarily attached to this proposition is omitted by Heiberg as an interpolation of date later than Theon. It is to this effect: "From this it is manifest that, if one angle of a triangle be equal to the other two, the first angle is right because the exterior angle to it is also equal to the same angles, and if the adjacent angles be equal, they are right." No doubt the corollary is rightly suspected, because there is no necessity for it here, and the words ὅπερ ἔδει δεῖξαι come before it, not after it, as is usual with Euclid. But, on the other hand, as the fact stated does appear in the proof of III. 31, the Porism would be a Porism after the usual type, and I do not quite follow Heiberg's argument that, "if Euclid had wished to add it, he ought to have placed it after I. 32."

It has already been mentioned above (p. 44) that this proposition supplies us with an alternative construction for the problem in III. 17 of drawing the two tangents to a circle from an external point.

Two theorems of some historical interest which follow directly from III. 31 may be mentioned.

The first is a lemma of Pappus on "the 24th problem" of the second Book of Apollonius' lost treatise on νεύσεις (Pappus VII. p. 812) and is to this effect. If a circle, as *DEF*, pass through *D*, the centre of a circle *ABC*, and if through *F*, the other point in which the line of centres meets the circle *DEF*, any straight line be drawn (and produced if necessary) meeting the circle *DEF* in *E* and the circle *ABC* in *B*, *G*,

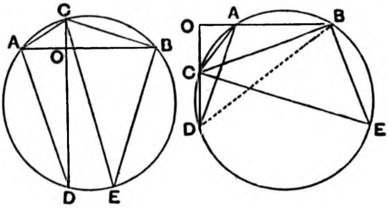


then E is the middle point of BG. For, if DE be joined, the angle DEF (in a semicircle) is a right angle [III. 31]; and DE, being at right angles to the chord BG of the circle ABC, also bisects it [III. 3].

The second is a proposition in the *Liber Assumptorum*, attributed (no doubt erroneously as regards much of it) to Archimedes, which has reached us through the Arabic (Archimedes, ed. Heiberg, II. pp. 520—521).

If two chords AB, CD *in a circle intersect at right angles in a point* O, *then the sum of the squares on* AO, BO, CO, DO *is equal to the square on the diameter.*

For draw the diameter CE, and join AC, CB, AD, BE.



Then the angle CAO is equal to the angle CEB. (This follows, in the first figure, from III. 21 and, in the second, from I. 13 and III. 22.) Also the angle COA, being right, is equal to the angle CBE which, being the angle in a semicircle, is also right [III. 31].

Therefore the triangles AOC, EBC have two angles equal respectively; whence the third angles ACO, ECB are equal. (In the second figure the angle ACO is, by I. 13 and III. 22, equal to the angle ABD, and therefore the angles ABD, ECB are equal.)

Therefore, in both figures, the arcs AD, BE, and consequently the chords AD, BE subtended by them, are equal. [III. 26, 29]

Now the squares on AO, DO are equal to the square on AD [I. 47], that is, to the square on BE.

And the squares on CO, BO are equal to the square on BC.

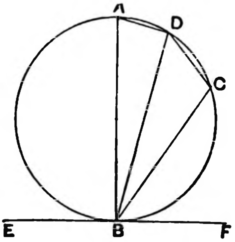
Therefore, by addition, the squares on AO, BO, CO, DO are equal to the squares on EB, BC, i.e. to the square on CE. [I. 47]

PROPOSITION 32.

If a straight line touch a circle, and from the point of contact there be drawn across, in the circle, a straight line cutting the circle, the angles which it makes with the tangent will be equal to the angles in the alternate segments of the circle.

For let a straight line EF touch the circle $ABCD$ at the point B, and from the point B let there be drawn across, in the circle $ABCD$, a straight line BD cutting it;

I say that the angles which BD makes with the tangent EF will be equal to the angles in the alternate segments of the circle, that is, that the angle *FBD* is equal to the angle constructed in the segment *BAD*, and the angle *EBD* is equal to the angle constructed in the segment *DCB*.



For let *BA* be drawn from *B* at right angles to *EF*,

let a point *C* be taken at random on the circumference *BD*,

and let *AD*, *DC*, *CB* be joined.

Then, since a straight line *EF* touches the circle *ABCD* at *B*,

and *BA* has been drawn from the point of contact at right angles to the tangent,

the centre of the circle *ABCD* is on *BA*. [III. 19]

Therefore *BA* is a diameter of the circle *ABCD*;

therefore the angle *ADB*, being an angle in a semicircle, is right. [III. 31]

Therefore the remaining angles *BAD*, *ABD* are equal to one right angle. [I. 32]

But the angle *ABF* is also right;

therefore the angle *ABF* is equal to the angles *BAD*, *ABD*.

Let the angle *ABD* be subtracted from each;

therefore the angle *DBF* which remains is equal to the angle *BAD* in the alternate segment of the circle.

Next, since *ABCD* is a quadrilateral in a circle,

its opposite angles are equal to two right angles. [III. 22]

But the angles *DBF*, *DBE* are also equal to two right angles;

therefore the angles *DBF*, *DBE* are equal to the angles *BAD*, *BCD*,

of which the angle *BAD* was proved equal to the angle *DBF*;

therefore the angle *DBE* which remains is equal to the angle *DCB* in the alternate segment *DCB* of the circle.

Therefore etc. Q. E. D.

The converse of this theorem is true, namely that, *If a straight line drawn through one extremity of a chord of a circle make with that chord angles equal respectively to the angles in the alternate segments of the circle, the straight line so drawn touches the circle.*

This can, as Camerer and Todhunter remark, be proved indirectly; or we may prove it, with Clavius, directly. Let *BD* be the given chord, and let *EF* be drawn through *B* so that it makes with *BD* angles equal to the angles in the alternate segments of the circle respectively.

Let *BA* be the diameter through *B*, and let *C* be any point on the circumference of the segment *DCB* which does not contain *A*. Join *AD*, *DC*, *CB*.

Then, since, by hypothesis, the angle *FBD* is equal to the angle *BAD*, let the angle *ABD* be added to both;

therefore the angle *ABF* is equal to the angles *ABD*, *BAD*.

But the angle *BDA*, being the angle in a semicircle, is a right angle;

therefore the remaining angles *ABD*, *BAD* in the triangle *ABD* are equal to a right angle.

Therefore the angle *ABF* is right;

hence, since *BA* is the diameter through *B*,

EF touches the circle at *B*. [III. 16, Por.]

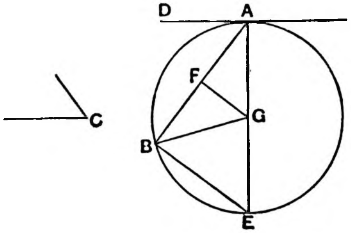
Pappus assumes in one place (IV. p. 196) the consequence of this proposition that, *If two circles touch, any straight line drawn through the point of contact and terminated by both circles cuts off segments in each which are respectively similar.* Pappus also shows how to prove this (VII. p. 826) by drawing the common tangent at the point of contact and using this proposition, III. 32.

PROPOSITION 33.

On a given straight line to describe a segment of a circle admitting an angle equal to a given rectilineal angle.

Let *AB* be the given straight line, and the angle at *C* the given rectilineal angle;

thus it is required to describe on the given straight line *AB* a segment of a circle admitting an angle equal to the angle at *C*.



The angle at *C* is then acute, or right, or obtuse.

First let it be acute, and, as in the first figure, on the straight line *AB*, and at the point *A*, let the angle *BAD* be constructed equal to the angle at *C*;

therefore the angle *BAD* is also acute.

Let *AE* be drawn at right angles to *DA*, let *AB* be bisected at *F*, let *FG* be drawn from the point *F* at right angles to *AB*, and let *GB* be joined.

Then, since *AF* is equal to *FB*,
and *FG* is common,
the two sides *AF*, *FG* are equal to the two sides *BF*, *FG*;
and the angle *AFG* is equal to the angle *BFG*;
therefore the base *AG* is equal to the base *BG*. [I. 4]

Therefore the circle described with centre *G* and distance *GA* will pass through *B* also.

Let it be drawn, and let it be *ABE*;
let *EB* be joined.

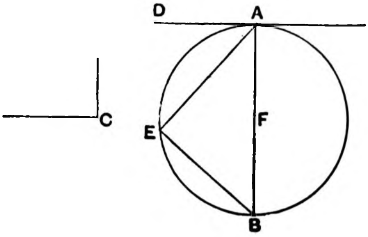
Now, since *AD* is drawn from *A*, the extremity of the diameter *AE*, at right angles to *AE*,
therefore *AD* touches the circle *ABE*. [III. 16, Por.]

Since then a straight line *AD* touches the circle *ABE*, and from the point of contact at *A* a straight line *AB* is drawn across in the circle *ABE*,
the angle *DAB* is equal to the angle *AEB* in the alternate segment of the circle. [III. 32]

But the angle *DAB* is equal to the angle at *C*;
therefore the angle at *C* is also equal to the angle *AEB*.

Therefore on the given straight line *AB* the segment *AEB* of a circle has been described admitting the angle *AEB* equal to the given angle, the angle at *C*.

Next let the angle at *C* be right;



and let it be again required to describe on *AB* a segment of a circle admitting an angle equal to the right angle at *C*.

Let the angle *BAD* be constructed equal to the right angle at *C*, as is the case in the second figure;

let *AB* be bisected at *F*, and with centre *F* and distance either *FA* or *FB* let the circle *AEB* be described.

Therefore the straight line *AD* touches the circle *ABE*, because the angle at *A* is right. [III. 16, Por.]

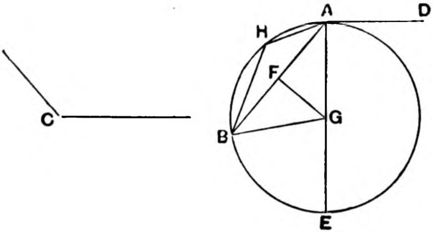
And the angle *BAD* is equal to the angle in the segment *AEB*, for the latter too is itself a right angle, being an angle in a semicircle. [III. 31]

But the angle *BAD* is also equal to the angle at *C*.

Therefore the angle *AEB* is also equal to the angle at *C*.

Therefore again the segment *AEB* of a circle has been described on *AB* admitting an angle equal to the angle at *C*.

Next, let the angle at *C* be obtuse;



and on the straight line *AB*, and at the point *A*, let the angle *BAD* be constructed equal to it, as is the case in the third figure;

let *AE* be drawn at right angles to *AD*, let *AB* be again bisected at *F*, let *FG* be drawn at right angles to *AB*, and let *GB* be joined.

Then, since *AF* is again equal to *FB*,

and *FG* is common,

the two sides *AF*, *FG* are equal to the two sides *BF*, *FG*;

and the angle *AFG* is equal to the angle *BFG*;

therefore the base *AG* is equal to the base *BG*. [I. 4]

Therefore the circle described with centre *G* and distance *GA* will pass through *B* also; let it so pass, as *AEB*.

Now, since *AD* is drawn at right angles to the diameter *AE* from its extremity,

AD touches the circle *AEB*. [III. 16, Por.]

And *AB* has been drawn across from the point of contact at *A*;

therefore the angle *BAD* is equal to the angle constructed in the alternate segment *AHB* of the circle. [III. 32]

But the angle *BAD* is equal to the angle at *C*.

Therefore the angle in the segment *AHB* is also equal to the angle at *C*.

Therefore on the given straight line *AB* the segment *AHB* of a circle has been described admitting an angle equal to the angle at *C*.

Q. E. F.

Simson remarks truly that the first and third cases, those namely in which the given angle is acute and obtuse respectively, have exactly the same construction and demonstration, so that there is no advantage in repeating them. Accordingly he deals with the cases as one, merely drawing two different figures. It is also true, as Simson says, that the demonstration of the second case in which the given angle is a right angle "is done in a roundabout way," whereas, as Clavius showed, the problem can be more easily solved by merely bisecting *AB* and describing a semicircle on it. A glance at Euclid's figure and proof will however show a more curious fact, namely that he does not, in the proof of the second case, use the *angle in the alternate segment*, as he does in the other two cases. He might have done so after proving that *AD* touches the circle; this would only have required his point *E* to be placed on the side of *AB* opposite to *D*. Instead of this, he uses III. 31, and proves that the angle *AEB* is equal to the angle *C*, because the former is an *angle in a semicircle*, and is therefore a right angle as *C* is.

The difference of procedure is no doubt owing to the fact that he has not, in III. 32, distinguished the case in which the cutting and touching straight lines are at right angles, i.e. in which the two alternate segments are semicircles. To prove this case would also have required III. 31, so that nothing would have been gained by stating it separately in III. 32 and then quoting the result as part of III. 32, instead of referring directly to III. 31.

It is assumed in Euclid's proof of the first and third cases that *AE* and *FG* will meet; but of course there is no difficulty in satisfying ourselves of this.

PROPOSITION 34.

From a given circle to cut off a segment admitting an angle equal to a given rectilineal angle.

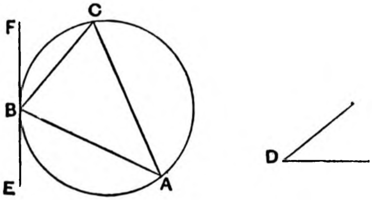
Let *ABC* be the given circle, and the angle at *D* the given rectilineal angle;

thus it is required to cut off from the circle *ABC* a segment admitting an angle equal to the given rectilineal angle, the angle at *D*.

Let *EF* be drawn touching *ABC* at the point *B*, and on the straight line *FB*, and at the point *B* on it, let the angle *FBC* be constructed equal to the angle at *D*. [I. 23]

Then, since a straight line *EF* touches the circle *ABC*, and *BC* has been drawn across from the point of contact at *B*,

the angle *FBC* is equal to the angle constructed in the alternate segment *BAC*. [III. 32]



But the angle *FBC* is equal to the angle at *D*;

therefore the angle in the segment *BAC* is equal to the angle at *D*.

Therefore from the given circle *ABC* the segment *BAC* has been cut off admitting an angle equal to the given rectilineal angle, the angle at *D*.

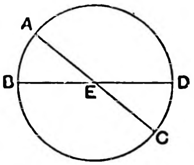
Q. E. F.

An alternative construction here would be to make an "angle at the centre" (in the extended sense, if necessary) double of the given angle; and, if the given angle is right, it is only necessary to draw a diameter of the circle.

PROPOSITION 35.

If in a circle two straight lines cut one another, the rectangle contained by the segments of the one is equal to the rectangle contained by the segments of the other.

For in the circle *ABCD* let the two straight lines *AC*, *BD* cut one another at the point *E*;

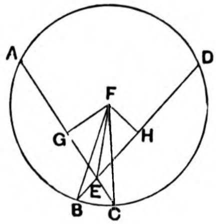


I say that the rectangle contained by *AE*, *EC* is equal to the rectangle contained by *DE*, *EB*.

If now *AC*, *BD* are through the centre, so that *E* is the centre of the circle *ABCD*, it is manifest that, *AE*, *EC*, *DE*, *EB* being equal,

the rectangle contained by *AE*, *EC* is also equal to the rectangle contained by *DE*, *EB*.

Next let *AC*, *DB* not be through the centre; let the centre of *ABCD* be taken, and let it be *F*;



from *F* let *FG*, *FH* be drawn perpendicular to the straight lines *AC*, *DB*, and let *FB*, *FC*, *FE* be joined.

Then, since a straight line *GF* through the centre cuts a straight line *AC* not through the centre at right angles,

it also bisects it; [III. 3]

therefore *AG* is equal to *GC*.

Since, then, the straight line *AC* has been cut into equal parts at *G* and into unequal parts at *E*, the rectangle contained by *AE*, *EC* together with the square on *EG* is equal to the square on *GC*; [II. 5]

Let the square on *GF* be added;

therefore the rectangle *AE*, *EC* together with the squares on *GE*, *GF* is equal to the squares on *CG*, *GF*.

But the square on *FE* is equal to the squares on *EG*, *GF*, and the square on *FC* is equal to the squares on *CG*, *GF*; [I. 47]

therefore the rectangle *AE*, *EC* together with the square on *FE* is equal to the square on *FC*.

And *FC* is equal to *FB*;

therefore the rectangle *AE*, *EC* together with the square on *EF* is equal to the square on *FB*.

For the same reason, also,

the rectangle *DE*, *EB* together with the square on *FE* is equal to the square on *FB*.

But the rectangle *AE*, *EC* together with the square on *FE* was also proved equal to the square on *FB*;

therefore the rectangle *AE*, *EC* together with the square on *FE* is equal to the rectangle *DE*, *EB* together with the square on *FE*.

Let the square on *FE* be subtracted from each;

therefore the rectangle contained by *AE*, *EC* which remains is equal to the rectangle contained by *DE*, *EB*.

Therefore etc.

In addition to the two cases in Euclid's text, Simson (following Campanus) gives two intermediate cases, namely (1) that in which one chord passes through the centre and bisects the other which does not pass through the centre at right angles, and (2) that in which one passes through the centre and cuts the other which does not pass through the centre but not at right angles. Simson then reduces Euclid's second case, the most general one, to the second of the two intermediate cases by drawing the diameter through E. His note is as follows: "As the 25th and 33rd propositions are divided into more cases, so this 35th is divided into fewer cases than are necessary. Nor can it be supposed that Euclid omitted them because they are easy; as he has given the case which by far is the easiest of them all, viz. that in which both the straight lines pass through the centre: And in the following proposition he separately demonstrates the case in which the straight line passes through the centre, and that in which it does not pass through the centre: So that it seems Theon, or some other, has thought them too long to insert: But cases that require different demonstrations should not be left out in the Elements, as was before taken notice of: These cases are in the translation from the Arabic and are now put into the text." Notwithstanding the ingenuity of the argument based on the separate mention by Euclid of the simplest case of all, I think the conclusion that Euclid himself gave *four* cases is unsafe; in fact, in giving the simplest and most difficult cases only, he seems to be following quite consistently his habit of avoiding *too great* multiplicity of cases, while not ignoring their existence.

The deduction from the next proposition (III. 36) which Simson, following Clavius and others, gives as a corollary to it, namely that, *If from any point without a circle there be drawn two straight lines cutting it, the rectangles contained by the whole lines and the parts of them without the circle are equal to one another*, can of course be combined with III. 35 in one enunciation.

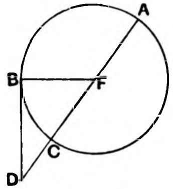
As remarked by Todhunter, a large portion of the proofs of III. 35, 36 amounts to proving the proposition, *If any point be taken on the base, or the base produced, of an isosceles triangle, the rectangle contained by the segments of the base* (*i.e. the respective distances of the ends of the base from the point*) *is equal to the difference between the square on the straight line joining the point to the vertex and the square on one of the equal sides of the triangle.* This is of course an immediate consequence of I. 47 combined with II. 5 or II. 6.

The converse of III. 35 and Simson's corollary to III. 36 may be stated thus. *If two straight lines* AB, CD, *produced if necessary, intersect at* O, *and if the rectangle* AO, OB *be equal to the rectangle* CO, OD, *the circumference of a circle will pass through the four points* A, B, C, D. The proof is indirect. We describe a circle through three of the points, as A, B, C (by the method used in Euclid's proofs of III. 9, 10), and then we prove, by the aid of III. 35 and the corollary to III. 36, that the circle cannot but pass through D also.

PROPOSITION 36.

If a point be taken outside a circle and from it there fall on the circle two straight lines, and if one of them cut the circle and the other touch it, the rectangle contained by the whole of the straight line which cuts the circle and the straight *line intercepted on it outside between the point and the convex circumference will be equal to the square on the tangent.*

For let a point D be taken outside the circle ABC, and from D let the two straight lines DCA, DB fall on the circle ABC; let DCA cut the circle ABC and let BD touch it;



I say that the rectangle contained by AD, DC is equal to the square on DB.

Then DCA is either through the centre or not through the centre.

First let it be through the centre, and let F be the centre of the circle ABC; let FB be joined;

therefore the angle FBD is right. [III. 18]

And, since AC has been bisected at F, and CD is added to it,

the rectangle AD, DC together with the square on FC is equal to the square on FD. [II. 6]

But FC is equal to FB;

therefore the rectangle AD, DC together with the square on FB is equal to the square on FD.

And the squares on FB, BD are equal to the square on FD; [I. 47]

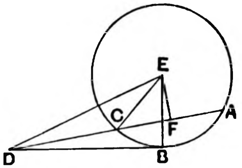
therefore the rectangle AD, DC together with the square on FB is equal to the squares on FB, BD.

Let the square on FB be subtracted from each;

therefore the rectangle AD, DC which remains is equal to the square on the tangent DB.

Again, let DCA not be through the centre of the circle ABC;

let the centre E be taken, and from E let EF be drawn perpendicular to AC; let EB, EC, ED be joined.



Then the angle EBD is right. [III. 18]

And, since a straight line EF through the centre cuts a straight line AC not through the centre at right angles, it also bisects it; [III. 3]

therefore AF is equal to FC.

Now, since the straight line *AC* has been bisected at the point *F*, and *CD* is added to it,
the rectangle contained by *AD*, *DC* together with the square on *FC* is equal to the square on *FD*. [II. 6]

Let the square on *FE* be added to each;
therefore the rectangle *AD*, *DC* together with the squares on *CF*, *FE* is equal to the squares on *FD*, *FE*.

But the square on *EC* is equal to the squares on *CF*, *FE*, for the angle *EFC* is right; [I. 47]
and the square on *ED* is equal to the squares on *DF*, *FE*;
therefore the rectangle *AD*, *DC* together with the square on *EC* is equal to the square on *ED*.

And *EC* is equal to *EB*;
therefore the rectangle *AD*, *DC* together with the square on *EB* is equal to the square on *ED*.

But the squares on *EB*, *BD* are equal to the square on *ED*, for the angle *EBD* is right; [I. 47]
therefore the rectangle *AD*, *DC* together with the square on *EB* is equal to the squares on *EB*, *BD*.

Let the square on *EB* be subtracted from each;
therefore the rectangle *AD*, *DC* which remains is equal to the square on *DB*.

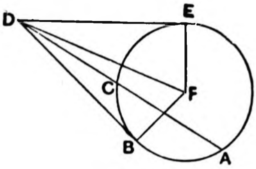
Therefore etc. Q. E. D.

Cf. note on the preceding proposition. Observe that, whereas it would be natural with us to prove first that, if *A* is an external point, and two straight lines *AEB*, *AFC* cut the circle in *E*, *B* and *F*, *C* respectively, the rectangle *BA*, *AE* is equal to the rectangle *CA*, *AF*, and thence that, the tangent from *A* being *a straight line like* AEB *in its limiting position when* E *and* B *coincide*, either rectangle is equal to the square on the tangent (cf. Mr H. M. Taylor, p. 253), Euclid and the Greek geometers generally did not allow themselves to infer the truth of a proposition in a *limiting case* directly from the general case including it, but preferred a separate proof of the limiting case (cf. *Apollonius of Perga*, p. 40, 139—140). This accounts for the form of III. 36.

PROPOSITION 37.

If a point be taken outside a circle and from the point there fall on the circle two straight lines, if one of them cut the circle, and the other fall on it, and if further the rectangle contained by the whole of the straight line which cuts *the circle and the straight line intercepted on it outside between the point and the convex circumference be equal to the square on the straight line which falls on the circle, the straight line which falls on it will touch the circle.*

For let a point *D* be taken outside the circle *ABC*; from *D* let the two straight lines *DCA*, *DB* fall on the circle *ACB*; let *DCA* cut the circle and *DB* fall on it; and let the rectangle *AD*, *DC* be equal to the square on *DB*.



I say that *DB* touches the circle *ABC*.

For let *DE* be drawn touching *ABC*; let the centre of the circle *ABC* be taken, and let it be *F*; let *FE*, *FB*, *FD* be joined.

Thus the angle *FED* is right. [III. 18]

Now, since *DE* touches the circle *ABC*, and *DCA* cuts it, the rectangle *AD*, *DC* is equal to the square on *DE*. [III. 36]

But the rectangle *AD*, *DC* was also equal to the square on *DB*;

therefore the square on *DE* is equal to the square on *DB*;

therefore *DE* is equal to *DB*.

And *FE* is equal to *FB*;

therefore the two sides *DE*, *EF* are equal to the two sides *DB*, *BF*;

and *FD* is the common base of the triangles;

therefore the angle *DEF* is equal to the angle *DBF*. [I. 8]

But the angle *DEF* is right;

therefore the angle *DBF* is also right.

And *FB* produced is a diameter;

and the straight line drawn at right angles to the diameter of a circle, from its extremity, touches the circle; [III. 16, Por.]

therefore *DB* touches the circle.

Similarly this can be proved to be the case even if the centre be on *AC*.

Therefore etc. Q. E. D.

De Morgan observes that there is here the same defect as in I. 48, i.e. an apparent avoidance of indirect demonstration by drawing the tangent *DE* on the opposite side of *DF* from *DB*. The case is similar to the ***apparently*** direct proof which Campanus gave. He drew the straight line from *D* passing through the centre, and then (without drawing a second tangent) proved by the aid of II. 6 that the square on *DF* is equal to the sum of the squares on *DB*, *BF*; whence (by I. 48) the angle *DBF* is a right angle. But this proof uses I. 48, the very proposition to which De Morgan's original remark relates.

The undisguised indirect proof is easy. If *DB* does not touch the circle, it must cut it if produced, and it follows that the square on *DB* must be equal to the rectangle contained by *DB* and a longer line: which is absurd.

BOOK IV.

DEFINITIONS.

1. A rectilineal figure is said to be **inscribed in a rectilineal figure** when the respective angles of the inscribed figure lie on the respective sides of that in which it is inscribed.

2. Similarly a figure is said to be **circumscribed about a figure** when the respective sides of the circumscribed figure pass through the respective angles of that about which it is circumscribed.

3. A rectilineal figure is said to be **inscribed in a circle** when each angle of the inscribed figure lies on the circumference of the circle.

4. A rectilineal figure is said to be **circumscribed about a circle**, when each side of the circumscribed figure touches the circumference of the circle.

5. Similarly a circle is said to be **inscribed in a figure** when the circumference of the circle touches each side of the figure in which it is inscribed.

6. A circle is said to be **circumscribed about a figure** when the circumference of the circle passes through each angle of the figure about which it is circumscribed.

7. A straight line is said to be **fitted into a circle** when its extremities are on the circumference of the circle.

DEFINITIONS 1—7.

I append, as usual, the Greek text of the definitions.

1. *Σχῆμα εὐθύγραμμον εἰς σχῆμα εὐθύγραμμον ἐγγράφεσθαι λέγεται, ὅταν ἑκάστη τῶν τοῦ ἐγγραφομένου σχήματος γωνιῶν ἑκάστης πλευρᾶς τοῦ, εἰς ὃ ἐγγράφεται, ἅπτηται.*

2. Σχῆμα δὲ ὁμοίως περὶ σχῆμα περιγράφεσθαι λέγεται, ὅταν ἑκάστη πλευρὰ τοῦ περιγραφομένου ἑκάστης γωνίας τοῦ, περὶ ὃ περιγράφεται, ἅπτηται.

3. Σχῆμα εὐθύγραμμον εἰς κύκλον ἐγγράφεσθαι λέγεται, ὅταν ἑκάστη γωνία τοῦ ἐγγραφομένου ἅπτηται τῆς τοῦ κύκλου περιφερείας.

4. Σχῆμα δὲ εὐθύγραμμον περὶ κύκλον περιγράφεσθαι λέγεται, ὅταν ἑκάστη πλευρὰ τοῦ περιγραφομένου ἐφάπτηται τῆς τοῦ κύκλου περιφερείας.

5. Κύκλος δὲ εἰς σχῆμα ὁμοίως ἐγγράφεσθαι λέγεται, ὅταν ἡ τοῦ κύκλου περιφέρεια ἑκάστης πλευρᾶς τοῦ, εἰς ὃ ἐγγράφεται, ἅπτηται.

6. Κύκλος δὲ περὶ σχῆμα περιγράφεσθαι λέγεται, ὅταν ἡ τοῦ κύκλου περιφέρεια ἑκάστης γωνίας τοῦ, περὶ ὃ περιγράφεται, ἅπτηται.

7. Εὐθεῖα εἰς κύκλον ἐναρμόζεσθαι λέγεται, ὅταν τὰ πέρατα αὐτῆς ἐπὶ τῆς περιφερείας ᾖ τοῦ κύκλου.

In the first two definitions an English translation, if it is to be clear, must depart slightly from the exact words used in the Greek, where "each side" of one figure is said to pass through "each angle" of another, or "each angle" (i.e. angular point) of one lies on "each side" of another (ἑκάστη πλευρά, ἑκάστη γωνία).

It is also necessary, in the five definitions 1, 2, 3, 5 and 6, to translate the same Greek word ἅπτηται in three different ways. It was observed on III. Def. 2 that the usual meaning of ἅπτεσθαι in Euclid is to *meet*, in contradistinction to ἐφάπτεσθαι, which means to *touch*. Exceptionally, as in Def. 5, ἅπτεσθαι has the meaning of *touch*. But two new meanings of the word appear, the first being to *lie on*, as in Deff. 1 and 3, the second to *pass through*, as in Deff. 2 and 6; "each angle" lies on (ἅπτεται) a side or on a circle, and "each side," or a circle, passes through (ἅπτεται) an angle or "each angle." The first meaning of *lying on* is exemplified in the phrase of Pappus ἅψεται τὸ σημεῖον θέσει δεδομένης εὐθείας, "will lie on a straight line given in position"; the meaning of *passing through* seems to be much rarer (I have not seen it in Archimedes or Pappus), but, as pointed out on III. Def. 2, Aristotle uses the compound ἐφάπτεσθαι in this sense.

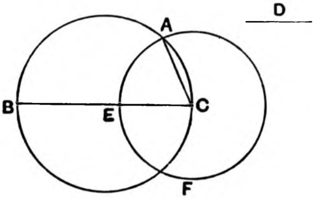
Simson proposed to read ἐφάπτηται in the case (Def. 5) where ἅπτηται means *touches*. He made the like suggestion as regards the Greek text of III. 11, 12, 13, 18, 19; in the first four of these cases there seems to be MS. authority for the compound verb, and in the fifth Heiberg adopts Simson's correction.

BOOK IV. PROPOSITIONS

PROPOSITION 1.

Into a given circle to fit a straight line equal to a given straight line which is not greater than the diameter of the circle.

Let ABC be the given circle, and D the given straight line not greater than the diameter of the circle;



thus it is required to fit into the circle ABC a straight line equal to the straight line D.

Let a diameter BC of the circle ABC be drawn.

Then, if BC is equal to D, that which was enjoined will have been done; for BC has been fitted into the circle ABC equal to the straight line D.

But, if BC is greater than D,

let CE be made equal to D, and with centre C and distance CE let the circle EAF be described;

let CA be joined.

Then, since the point C is the centre of the circle EAF,

CA is equal to CE.

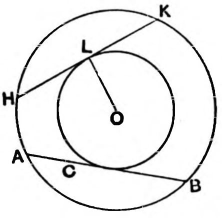
But CE is equal to D;

therefore D is also equal to CA.

Therefore into the given circle ABC there has been fitted CA equal to the given straight line D.

Of this problem as it stands there are of course an infinite number of solutions; and, if a particular point be chosen as one extremity of the chord to be "fitted in," there are two solutions. More difficult cases of "fitting into" a circle a chord of given length are arrived at by adding some further condition, e.g. (1) that the chord is to be parallel to a given straight line, or (2) that the chord, produced if necessary, shall pass through a given point. The former problem is solved by Pappus (III. p. 132); instead of drawing the chord as a tangent to a circle concentric with the given circle and having as radius a straight line the square on which is equal to the difference between the squares on the radius of the given circle and on half the given length, he merely draws the diameter of the circle which is parallel to the given direction, measures from the centre along it in each direction a length equal to half the given length, and then draws, on one side of the diameter, perpendiculars to it through the two points so determined.

The second problem of drawing a chord of given length, being less than the diameter of the circle, and passing through a given point, is more important as having been one of the problems discussed by Apollonius in his work entitled νεύσεις, now lost. Pappus states the problem thus (VII. p. 670): "A circle being given in position, to fit into it a straight line given in magnitude and verging (νεύουσαν) towards a given (point)." To do this we have only to place any chord *HK* in the given circle (with centre *O*) equal to the given length, take *L* the middle point of it, with *O* as centre and *OL* as radius describe a circle, and lastly through the given point *C* draw a tangent to this circle meeting the given circle in *A*, *B*. *AB* is then one of *two* chords which can be drawn satisfying the given conditions, if *C* is outside the inner circle; if *C* is *on* the inner circle, there is one solution only; and, if *C* is within the inner circle, there is no solution. Thus, if *C* is within the outer (given) circle, besides the condition that the given length must not be greater than the diameter of the circle, there is another necessary condition of the possibility of a solution, viz. that the given length must not be *less* than double of the straight line the square on which is equal to the difference between the squares (1) on the radius of the given circle and (2) on the distance between its centre and the given point.



PROPOSITION 2.

In a given circle to inscribe a triangle equiangular with a given triangle.

Let *ABC* be the given circle, and *DEF* the given triangle;

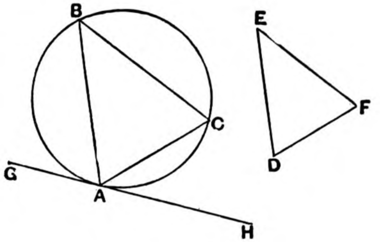
thus it is required to inscribe in the circle *ABC* a triangle equiangular with the triangle *DEF*.

Let *GH* be drawn touching the circle *ABC* at *A* [III. 16, Por.];

on the straight line *AH*, and at the point *A* on it, let the angle *HAC* be constructed equal to the angle *DEF*,

and on the straight line *AG*, and at the point *A* on it, let the angle *GAB* be constructed equal to the angle *DFE*; [I. 23]

let *BC* be joined.



Then, since a straight line *AH* touches the circle *ABC*, and from the point of contact at *A* the straight line *AC* is drawn across in the circle,

therefore the angle *HAC* is equal to the angle *ABC* in the alternate segment of the circle. [III. 32]

But the angle *HAC* is equal to the angle *DEF*;

therefore the angle *ABC* is also equal to the angle *DEF*.

For the same reason

the angle *ACB* is also equal to the angle *DFE*;

therefore the remaining angle *BAC* is also equal to the remaining angle *EDF*. [I. 32]

Therefore in the given circle there has been inscribed a triangle equiangular with the given triangle. Q. E. F.

Here again, since any point on the circle may be taken as an angular point of the triangle, there are an infinite number of solutions. Even when a particular point has been chosen to form one angular point, the required triangle may be constructed in six ways. For any one of the three angles may be placed at the point; and, whichever is placed there, the positions of the two others relatively to it may be interchanged. The sides of the triangle will, in all the different solutions, be of the same length respectively; only their relative positions will be different.

This problem can of course be reduced (as it was by Borelli) to III. 34, namely the problem of cutting off from a given circle a segment containing an angle equal to a given angle. It can also be solved by the alternative method applicable to III. 34 of drawing "angles at the centre" equal to double the angles of the given triangle respectively; and by this method we can easily solve this problem, or III. 34, with the further condition that one side of the required triangle, or the base of the required segment, respectively, shall be parallel to a given straight line.

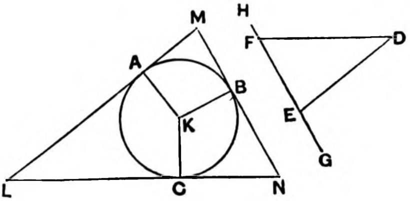
As a particular case, we can, by the method of this proposition, describe an *equilateral* triangle in any circle after we have first constructed any equilateral triangle by the aid of I. 1. The possibility of this is assumed in IV. 16. It is of course equivalent to dividing the circumference of a circle into *three equal parts*. As De Morgan says, the idea of dividing a revolution into equal parts should be kept prominent in considering Book IV.; this aspect of the construction of regular polygons is obvious enough, and the reason why the division of the circle into *three* equal parts is not given by Euclid is that it happens to be as easy to divide the circle into three parts which are in the ratio of the angles of any triangle as to divide it into three equal parts.

PROPOSITION 3.

About a given circle to circumscribe a triangle equiangular with a given triangle.

Let *ABC* be the given circle, and *DEF* the given triangle;

thus it is required to circumscribe about the circle *ABC* a triangle equiangular with the triangle *DEF*.



Let *EF* be produced in both directions to the points *G*, *H*,

let the centre *K* of the circle *ABC* be taken [III. 1], and let the straight line *KB* be drawn across at random;

on the straight line *KB*, and at the point *K* on it, let the angle *BKA* be constructed equal to the angle *DEG*,

and the angle *BKC* equal to the angle *DFH*; [I. 23]

and through the points *A*, *B*, *C* let *LAM*, *MBN*, *NCL* be drawn touching the circle *ABC*. [III. 16, Por.]

Now, since *LM*, *MN*, *NL* touch the circle *ABC* at the points *A*, *B*, *C*,

and *KA*, *KB*, *KC* have been joined from the centre *K* to the points *A*, *B*, *C*, therefore the angles at the points *A*, *B*, *C* are right. [III. 18]

And, since the four angles of the quadrilateral *AMBK* are equal to four right angles, inasmuch as *AMBK* is in fact divisible into two triangles,

and the angles *KAM*, *KBM* are right,

therefore the remaining angles *AKB*, *AMB* are equal to two right angles.

But the angles *DEG*, *DEF* are also equal to two right angles; [I. 13]

therefore the angles *AKB*, *AMB* are equal to the angles *DEG*, *DEF*,

of which the angle *AKB* is equal to the angle *DEG*;

therefore the angle *AMB* which remains is equal to the angle *DEF* which remains.

Similarly it can be proved that the angle *LNB* is also equal to the angle *DFE*;

therefore the remaining angle *MLN* is equal to the angle *EDF*. [I. 32]

Therefore the triangle *LMN* is equiangular with the triangle *DEF*; and it has been circumscribed about the circle *ABC*.

Therefore about a given circle there has been circumscribed a triangle equiangular with the given triangle.

Q. E. F.

10. **at random,** literally "as it may chance," ὡς ἔτυχεν. The same expression is used in III. 1 and commonly.

22. **is in fact divisible,** καὶ διαιρεῖται, literally "is actually divided."

The remarks as to the number of ways in which Prop. 2 can be solved apply here also.

Euclid leaves us to satisfy ourselves that the three tangents *will* meet and form a triangle. This follows easily from the fact that each of the angles *AKB*, *BKC*, *CKA* is less than two right angles. The first two are so by construction, being the supplements of two angles of the given triangle respectively, and, since all three angles round *K* are together equal to four right angles, it follows that the third, the angle *AKC*, is equal to the sum of the two angles *E*, *F* of the triangle, i.e. to the supplement of the angle *D*, and is therefore less than two right angles.

Peletarius and Borelli gave an alternative solution, first inscribing a triangle equiangular to the given triangle, by IV. 2, and then drawing tangents to the circle parallel to the sides of the inscribed triangle respectively. This method will of course give two solutions, since two tangents can be drawn parallel to each of the sides of the inscribed triangle.

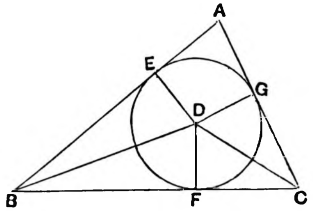
If the three pairs of parallel tangents be drawn and produced far enough, they will form *eight* triangles, two of which are the triangles *circumscribed* to the circle in the manner required in the proposition. The other six triangles are so related to the circle that the circle touches two of the sides in each *produced*, i.e. the circle is an *escribed* circle to each of the six triangles.

PROPOSITION 4.

In a given triangle to inscribe a circle.

Let *ABC* be the given triangle;

thus it is required to inscribe a circle in the triangle *ABC*.



Let the angles *ABC*, *ACB* be bisected by the straight lines *BD*, *CD* [I. 9], and let these meet one another at the point *D*;

from *D* let *DE*, *DF*, *DG* be drawn perpendicular to the straight lines *AB*, *BC*, *CA*.

Now, since the angle *ABD* is equal to the angle *CBD*,

and the right angle *BED* is also equal to the right angle *BFD*,

EBD, *FBD* are two triangles having two angles equal to two angles and one side equal to one side, namely that subtending one of the equal angles, which is *BD* common to the triangles;

therefore they will also have the remaining sides equal to the remaining sides; [I. 26]

therefore *DE* is equal to *DF*.

For the same reason

DG is also equal to *DF*.

Therefore the three straight lines *DE*, *DF*, *DG* are equal to one another;

therefore the circle described with centre *D* and distance one of the straight lines *DE*, *DF*, *DG* will pass also through the remaining points, and will touch the straight lines *AB*, *BC*, *CA*, because the angles at the points *E*, *F*, *G* are right.

For, if it cuts them, the straight line drawn at right angles to the diameter of the circle from its extremity will be found to fall within the circle: which was proved absurd; [III. 16] therefore the circle described with centre *D* and distance one of the straight lines *DE*, *DF*, *DG* will not cut the straight lines *AB*, *BC*, *CA*;

therefore it will touch them, and will be the circle inscribed in the triangle *ABC*. [IV. Def. 5]

Let it be inscribed, as *FGE*.

Therefore in the given triangle *ABC* the circle *EFG* has been inscribed.

Q. E. F.

26, 34. **and distance one of the (straight lines D)E, (D)F, (D)G.** The words and letters here shown in brackets are put in to fill out the rather careless language of the Greek. Here and in several other places in Book IV. Euclid says literally "and with distance one of the (points) *E*, *F*, *G*" (καὶ διαστήματι ἑνὶ τῶν E, Z, H) and the like. In one case (IV. 13) he actually has "with distance one of the *points G*, *H*, *K*, *L*, *M*" (διαστήματι ἑνὶ τῶν H, Θ, K, Λ, M σημείων). Heiberg notes "Graecam locutionem satis miram et negligentem," but, in view of its frequent occurrence in good MSS., does not venture to correct it.

Euclid does not think it necessary to prove that *BD*, *CD will* meet; this is indeed obvious, for the angles *DBC*, *DCB* are together half of the angles *ABC*, *ACB*, which themselves are together less than two right angles, and therefore the two bisectors of the angles *B*, *C* must meet, by Post. 5.

It follows from the proof of this proposition that, if the bisectors of two angles *B*, *C* of a triangle meet in *D*, the line joining *D* to *A* also bisects the third angle *A*, or the bisectors of the three angles of a triangle meet in a point.

It will be observed that Euclid uses the *indirect* form of proof when showing that the circle touches the three sides of the triangle. Simson proves it directly, and points out that Euclid does the same in III. 17, 33 and 37, whereas in IV. 8 and 13 as well as here he uses the *indirect* form. The difference is unimportant, being one of form and not of substance; the indirect proof refers back to III. 16, whereas the direct refers back to the Porism to that proposition.

We may state this problem in the more general form: *To describe a circle touching three given straight lines which do not all meet in one point, and of which not more than two are parallel.*

In the case (1) where two of the straight lines are parallel and the third cuts them, two pairs of interior angles are formed, one on each side of the third straight line. If we bisect each of the interior angles on one side, the bisectors will meet in a point, and this point will be the centre of a circle which can be drawn touching each of the three straight lines, its radius being the perpendicular from the point on any one of the three. Since the *alternate* angles are equal, two equal circles can be drawn in this manner satisfying the given condition.

In the case (2) where the three straight lines form a triangle, suppose each straight line produced indefinitely. Then each straight line will make two pairs of interior angles with the other two, one pair forming two angles of the triangle, and the other pair being their supplements. By bisecting each angle of either pair we obtain, in the manner of the proposition, two circles satisfying the conditions, one of them being the inscribed circle of the triangle and the other being a circle *escribed* to it, i.e. touching one side and the other two sides *produced.* Next, taking the pairs of interior angles formed by a second side with the other two produced indefinitely, we get two circles satisfying the conditions, one of which is the same inscribed circle that we had before, while the other is a second escribed circle. Similarly with the third side. Hence we have the inscribed circle, and three escribed circles (one opposite each angle of the triangle), i.e. four circles in all, satisfying the conditions of the problem.

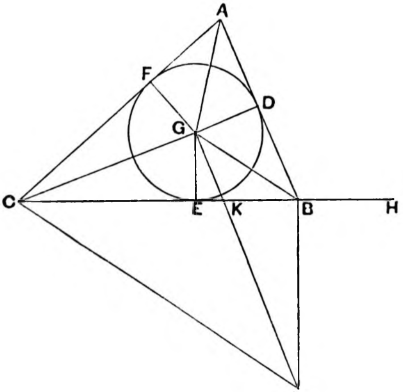
It may perhaps not be inappropriate to give at this point Heron's elegant proof of the formula for the area of a triangle in terms of the sides, which we usually write thus:

$$\triangle = \sqrt{s(s-a)(s-b)(s-c)},$$

although it requires the theory of proportions and uses some ungeometrical expressions, e.g. the product of two areas and the "side" of such a product, where of course the areas are so many square units of length. The proof is given in the *Metrica*, I. 8, and in the *Dioptra*, 30 (Heron, Vol. III., Teubner, 1903, pp. 20—24 and pp. 280—4, or Heron, ed. Hultsch, pp. 235—7).

Suppose the sides of the triangle ABC to be given in length.

Inscribe the circle DEF, and let G be its centre.



Join AG, BG, CG, DG, EG, FG.

Then $$BC \,.\, EG = 2 \,.\, \triangle BGC,$$
$$CA \,.\, FG = 2 \,.\, \triangle ACG,$$
$$AB \,.\, DG = 2 \,.\, \triangle ABG.$$

Therefore, by addition,
$$p \,.\, EG = 2 \,.\, \triangle ABC,$$
where p is the perimeter.

Produce CB to H, so that $BH = AD$.

Then, since $AD = AF$, $DB = BE$, $FC = CE$,
$$CH = \tfrac{1}{2}p.$$

Hence $$CH \,.\, EG = \triangle ABC.$$

But $CH.EG$ is the "side" of the product $CH^2.EG^2$, that is $\sqrt{CH^2.EG^2}$;

therefore $$(\triangle ABC)^2 = CH^2.EG^2.$$

Draw GL at right angles to CG, and BL at right angles to CB, meeting at L. Join CL.

Then, since each of the angles CGL, CBL is right, $CGBL$ is a quadrilateral in a circle.

Therefore the angles CGB, CLB are equal to two right angles.

Now the angles CGB, AGD are equal to two right angles, since AG, BG, CG bisect the angles at G, and the angles CGB, AGD are equal to the angles AGC, DGB, while the sum of all four is equal to four right angles.

Therefore the angles AGD, CLB are equal.

So are the right angles ADG, CBL.

Therefore the triangles AGD, CLB are similar.

Hence $$\begin{aligned} BC:BL &= AD:DG \\ &= BH:EG, \end{aligned}$$

and, alternately, $$\begin{aligned} CB:BH &= BL:EG \\ &= BK:KE, \end{aligned}$$

whence, *componendo*, $$CH:HB = BE:EK.$$

It follows that $$\begin{aligned} CH^2:CH.HB &= BE.EC:CE.EK \\ &= BE.EC:EG^2 \end{aligned}$$

Therefore

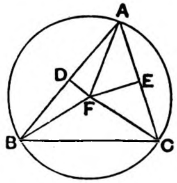
$$\begin{aligned} (\triangle ABC)^2 = CH^2.EG^2 &= CH.HB.CE.EB \\ &= \tfrac{1}{2}p\,(\tfrac{1}{2}p - BC)\,(\tfrac{1}{2}p - AB)\,(\tfrac{1}{2}p - AC). \end{aligned}$$

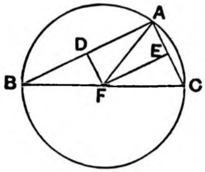
PROPOSITION 5.

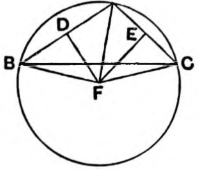
About a given triangle to circumscribe a circle.

Let ABC be the given triangle;

thus it is required to circumscribe a circle about the given triangle ABC.







Let the straight lines AB, AC be bisected at the points D, E [I. 10], and from the points D, E let DF, EF be drawn at right angles to AB, AC;

they will then meet within the triangle ABC, or on the straight line BC, or outside BC.

First let them meet within at F, and let FB, FC, FA be joined.

Then, since AD is equal to DB,
and DF is common and at right angles,
therefore the base AF is equal to the base FB. [I. 4]

Similarly we can prove that

CF is also equal to AF;

so that FB is also equal to FC;

therefore the three straight lines FA, FB, FC are equal to one another.

Therefore the circle described with centre F and distance one of the straight lines FA, FB, FC will pass also through the remaining points, and the circle will have been circumscribed about the triangle ABC.

Let it be circumscribed, as ABC.

Next, let DF, EF meet on the straight line BC at F, as is the case in the second figure; and let AF be joined.

Then, similarly, we shall prove that the point F is the centre of the circle circumscribed about the triangle ABC.

Again, let DF, EF meet outside the triangle ABC at F, as is the case in the third figure, and let AF, BF, CF be joined.

Then again, since AD is equal to DB,
and DF is common and at right angles,
therefore the base AF is equal to the base BF. [I. 4]

Similarly we can prove that

CF is also equal to AF;

so that BF is also equal to FC;

therefore the circle described with centre F and distance one of the straight lines FA, FB, FC will pass also through the remaining points, and will have been circumscribed about the triangle ABC.

Therefore about the given triangle a circle has been circumscribed.

Q. E. F.

And it is manifest that, when the centre of the circle falls within the triangle, the angle BAC, being in a segment greater than the semicircle, is less than a right angle;

when the centre falls on the straight line *BC*, the angle *BAC*, being in a semicircle, is right;

and when the centre of the circle falls outside the triangle, the angle *BAC*, being in a segment less than the semicircle, is greater than a right angle. [III. 31]

Simson points out that Euclid does not prove that *DF*, *EF* will meet, and he inserts in the text the following argument to supply the omission.

"*DF*, *EF* produced meet one another. For, if they do not meet, they are parallel, wherefore *AB*, *AC*, which are at right angles to them, are parallel [or, he should have added, in a straight line]: which is absurd."

This assumes, of course, that straight lines which are at right angles to two parallels are themselves parallel; but this is an obvious deduction from I. 28.

On the assumption that *DF*, *EF* will meet Todhunter has this note: "It has been proposed to show this in the following way: join *DE*; then the angles *EDF* and *DEF* are together less than the angles *ADF* and *AEF*, that is, they are together less than two right angles; and therefore *DF* and *EF* will meet, by Axiom 12 [Post. 5]. This assumes that *ADE* and *AED* are *acute* angles; it may, however, be easily shown that *DE* is parallel to *BC*, so that the triangle *ADE* is equiangular to the triangle *ABC*; and we must therefore select the two sides *AB* and *AC* such that *ABC* and *ACB* may be acute angles."

This is, however, unsatisfactory. Euclid makes no such selection in III. 9 and III. 10, where the same assumption is tacitly made; and it is unnecessary, because it is easy to prove that the straight lines *DF*, *EF* meet in *all* cases, by considering the different possibilities separately and drawing a separate figure for each case.

Simson thinks that Euclid's demonstration had been spoiled by some unskilful hand both because of the omission to prove that the perpendicular bisectors meet, and because "without any reason he divides the proposition into three cases, whereas one and the same construction and demonstration serves for them all, as Campanus has observed." However, up to the usual words ὅπερ ἔδει ποιῆσαι there seems to be no doubt about the text. Heiberg suggests that Euclid gave separately the case where *F* falls on *BC* because, in that case, only *AF* needs to be drawn and not *BF*, *CF* as well.

The addition, though given in Simson and the text-books as a "corollary," has no heading πόρισμα in the best MSS.; it is an explanation like that which is contained in the penultimate paragraph of III. 25.

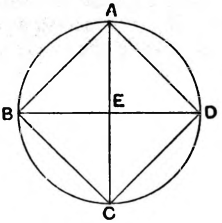
The Greek text has a further addition, which is rejected by Heiberg as not genuine, "So that, further, when the given angle happens to be less than a right angle, *DF*, *EF* will fall within the triangle, when it is right, on *BC*, and, when it is greater than a right angle, outside *BC*. (being) what it was required to do." Simson had already observed that the text here is vitiated "where mention is made of a given angle, though there neither is, nor can be, anything in the proposition relating to a given angle."

PROPOSITION 6.

In a given circle to inscribe a square.

Let *ABCD* be the given circle;
thus it is required to inscribe a square in the circle *ABCD*.

Let two diameters *AC*, *BD* of the circle *ABCD* be drawn at right angles to one another, and let *AB*, *BC*, *CD*, *DA* be joined.



Then, since *BE* is equal to *ED*, for *E* is the centre,
and *EA* is common and at right angles,
therefore the base *AB* is equal to the base *AD*. [I. 4]

For the same reason
each of the straight lines *BC*, *CD* is also equal to each of the straight lines *AB*, *AD*;

therefore the quadrilateral *ABCD* is equilateral.

I say next that it is also right-angled.

For, since the straight line *BD* is a diameter of the circle *ABCD*,
therefore *BAD* is a semicircle;

therefore the angle *BAD* is right. [III. 31]

For the same reason
each of the angles *ABC*, *BCD*, *CDA* is also right;

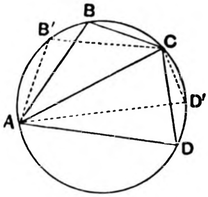
therefore the quadrilateral *ABCD* is right-angled.

But it was also proved equilateral;
therefore it is a square; [I. Def. 22]
and it has been inscribed in the circle *ABCD*.

Therefore in the given circle the square *ABCD* has been inscribed.

Q. E. F.

Euclid here proceeds to consider problems corresponding to those in Props. 2—5 with reference to figures of four or more sides, but with the difference that, whereas he dealt with triangles of any form, he confines himself henceforth to regular figures. It happened to be as easy to divide a circle into *three* parts which are in the ratio of the angles, or of the supplements of the angles, of a triangle as into three *equal* parts. But, when it is required to inscribe in a circle a figure equiangular to a given *quadrilateral*, this can only be done provided that the quadrilateral has either pair of opposite angles equal to two right angles. Moreover, in this case, the problem *may* be solved in the same way as that of IV. 2, i.e. by simply inscribing a triangle equiangular to one of the triangles into which the quadrilateral is divided by either diagonal, and then drawing on the side corresponding to the diagonal as base another triangle equiangular to the other triangle contained in the quadrilateral. But this is not the *only* solution; there are an infinite number of other solutions in which the inscribed quadrilateral will, unlike that found by this particular method, not be of the same *form* as the given quadrilateral. For suppose *ABCD* to be the quadrilateral inscribed in the circle by the method of IV. 2. Take *any* point *B′* on *AB*, join *AB′*, and then make the angle *DAD′* (measured towards *AC*) equal to the angle *BAB′*. Join *B′C*, *CD′*. Then *AB′CD′* is also *equiangular* to the given quadrilateral, but not of the same *form*. Hence the problem is indeterminate in the case of the general quadrilateral. It is equally so if the given quadrilateral is a rectangle; and it is determinate only when the given quadrilateral is a *square*.



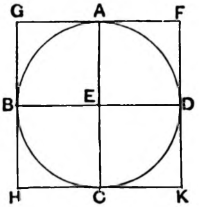
PROPOSITION 7.

About a given circle to circumscribe a square.

Let *ABCD* be the given circle;

thus it is required to circumscribe a square about the circle *ABCD*.

Let two diameters *AC*, *BD* of the circle *ABCD* be drawn at right angles to one another, and through the points *A*, *B*, *C*, *D* let *FG*, *GH*, *HK*, *KF* be drawn touching the circle *ABCD*. [III. 16, Por.]



Then, since *FG* touches the circle *ABCD*,

and *EA* has been joined from the centre *E* to the point of contact at *A*,

therefore the angles at *A* are right. [III. 18]

For the same reason

the angles at the points *B*, *C*, *D* are also right.

Now, since the angle *AEB* is right,

and the angle *EBG* is also right,

therefore *GH* is parallel to *AC*. [I. 28]

For the same reason

AC is also parallel to *FK*,

so that *GH* is also parallel to *FK*. [I. 30]

Similarly we can prove that

each of the straight lines *GF*, *HK* is parallel to *BED*.

Therefore *GK*, *GC*, *AK*, *FB*, *BK* are parallelograms;

therefore *GF* is equal to *HK*, and *GH* to *FK*. [I. 34]

And, since *AC* is equal to *BD*,

and *AC* is also equal to each of the straight lines *GH*, *FK*,

while *BD* is equal to each of the straight lines *GF*, *HK*, [I. 34]

therefore the quadrilateral *FGHK* is equilateral.

I say next that it is also right-angled.

For, since *GBEA* is a parallelogram,

and the angle *AEB* is right,

therefore the angle *AGB* is also right. [I. 34]

Similarly we can prove that

the angles at *H*, *K*, *F* are also right.

Therefore *FGHK* is right-angled.

But it was also proved equilateral;

therefore it is a square;

and it has been circumscribed about the circle *ABCD*.

Therefore about the given circle a square has been circumscribed.

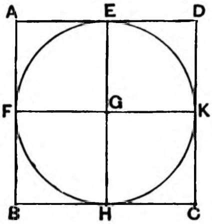
Q. E. F.

It is just as easy to describe about a given circle a polygon equiangular to any given polygon as it is to describe a square about a given circle. We have only to use the method of IV. 3, i.e. to take any radius of the circle, to measure round the centre successive angles in one and the same direction equal to the supplements of the successive angles of the given polygon and, lastly, to draw tangents to the circle at the extremities of the several radii so determined; but again the polygon would in general not be of the same *form* as the given one; it would only be so if the given polygon happened to be such that a circle could be inscribed in it. To take the case of a *quadrilateral* only: it is easy to prove that, if a quadrilateral be described about a circle, the sum of one pair of opposite sides must be equal to the sum of the other pair. It may be proved, *conversely*, that, if a quadrilateral has the sums of the pairs of opposite sides equal, a circle can be inscribed in it. If then a given quadrilateral has the sums of the pairs of opposite sides equal, a quadrilateral can be described about any given circle not only equiangular with it but having the same *form* or, in the words of Book VI., *similar* to it.

PROPOSITION 8.

In a given square to inscribe a circle.

Let $ABCD$ be the given square;
thus it is required to inscribe a circle in the given square $ABCD$.



Let the straight lines AD, AB be bisected at the points E, F respectively [I. 10],
through E let EH be drawn parallel to either AB or CD, and through F let FK be drawn parallel to either AD or BC; [I. 31]
therefore each of the figures AK, KB, AH, HD, AG, GC, BG, GD is a parallelogram, and their opposite sides are evidently equal. [I. 34]

Now, since AD is equal to AB,
and AE is half of AD, and AF half of AB,
therefore AE is equal to AF,
so that the opposite sides are also equal;
therefore FG is equal to GE.

Similarly we can prove that each of the straight lines GH, GK is equal to each of the straight lines FG, GE;
therefore the four straight lines GE, GF, GH, GK are equal to one another.

Therefore the circle described with centre G and distance one of the straight lines GE, GF, GH, GK will pass also through the remaining points.

And it will touch the straight lines AB, BC, CD, DA, because the angles at E, F, H, K are right.

For, if the circle cuts AB, BC, CD, DA, the straight line drawn at right angles to the diameter of the circle from its extremity will fall within the circle: which was proved absurd; [III. 16]
therefore the circle described with centre G and distance one of the straight lines GE, GF, GH, GK will not cut the straight lines AB, BC, CD, DA.

Therefore it will touch them, and will have been inscribed in the square $ABCD$.

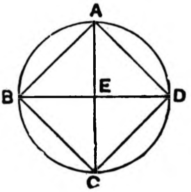
Therefore in the given square a circle has been inscribed.

As was remarked in the last note, a circle can be inscribed in any *quadrilateral* which has the sum of one pair of opposite sides equal to the sum of the other pair. In particular, it follows that a circle can be inscribed in a *square* or a *rhombus*, but not in a rectangle or a rhomboid.

PROPOSITION 9.

About a given square to circumscribe a circle.

Let *ABCD* be the given square;
thus it is required to circumscribe a circle about the square *ABCD*.



For let *AC*, *BD* be joined, and let them cut one another at *E*.

Then, since *DA* is equal to *AB*,
and *AC* is common,
therefore the two sides *DA*, *AC* are equal to the two sides *BA*, *AC*;
and the base *DC* is equal to the base *BC*;
therefore the angle *DAC* is equal to the angle *BAC*. [I. 8]

Therefore the angle *DAB* is bisected by *AC*.

Similarly we can prove that each of the angles *ABC*, *BCD*, *CDA* is bisected by the straight lines *AC*, *DB*.

Now, since the angle *DAB* is equal to the angle *ABC*,
and the angle *EAB* is half the angle *DAB*,
and the angle *EBA* half the angle *ABC*,
therefore the angle *EAB* is also equal to the angle *EBA*;
so that the side *EA* is also equal to *EB*. [I. 6]

Similarly we can prove that each of the straight lines *EA*, *EB* is equal to each of the straight lines *EC*, *ED*.

Therefore the four straight lines *EA*, *EB*, *EC*, *ED* are equal to one another.

Therefore the circle described with centre *E* and distance one of the straight lines *EA*, *EB*, *EC*, *ED* will pass also through the remaining points;
and it will have been circumscribed about the square *ABCD*.

Let it be circumscribed, as *ABCD*.

Therefore about the given square a circle has been circumscribed.

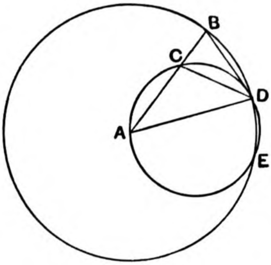
PROPOSITION 10.

To construct an isosceles triangle having each of the angles at the base double of the remaining one.

Let any straight line *AB* be set out, and let it be cut at the point *C* so that the rectangle contained by *AB*, *BC* is equal to the square on *CA*; [II. 11]

with centre *A* and distance *AB* let the circle *BDE* be described,

and let there be fitted in the circle *BDE* the straight line *BD* equal to the straight line *AC* which is not greater than the diameter of the circle *BDE*. [IV. 1]



Let *AD*, *DC* be joined, and let the circle *ACD* be circumscribed about the triangle *ACD*. [IV. 5]

Then, since the rectangle *AB*, *BC* is equal to the square on *AC*,

and *AC* is equal to *BD*,

therefore the rectangle *AB*, *BC* is equal to the square on *BD*.

And, since a point *B* has been taken outside the circle *ACD*,

and from *B* the two straight lines *BA*, *BD* have fallen on the circle *ACD*, and one of them cuts it, while the other falls on it,

and the rectangle *AB*, *BC* is equal to the square on *BD*,

therefore *BD* touches the circle *ACD*. [III. 37]

Since, then, *BD* touches it, and *DC* is drawn across from the point of contact at *D*,

therefore the angle *BDC* is equal to the angle *DAC* in the alternate segment of the circle. [III. 32]

Since, then, the angle *BDC* is equal to the angle *DAC*, let the angle *CDA* be added to each;

therefore the whole angle *BDA* is equal to the two angles *CDA*, *DAC*.

But the exterior angle *BCD* is equal to the angles *CDA*, *DAC*; [I. 32]

therefore the angle *BDA* is also equal to the angle *BCD*.

But the angle *BDA* is equal to the angle *CBD*, since the side *AD* is also equal to *AB*; [I. 5]

so that the angle *DBA* is also equal to the angle *BCD*.

Therefore the three angles *BDA*, *DBA*, *BCD* are equal to one another.

And, since the angle *DBC* is equal to the angle *BCD*,

the side *BD* is also equal to the side *DC*. [I. 6]

But *BD* is by hypothesis equal to *CA*;

therefore *CA* is also equal to *CD*,

so that the angle *CDA* is also equal to the angle *DAC*; [I. 5]

therefore the angles *CDA*, *DAC* are double of the angle *DAC*.

But the angle *BCD* is equal to the angles *CDA*, *DAC*;

therefore the angle *BCD* is also double of the angle *CAD*.

But the angle *BCD* is equal to each of the angles *BDA*, *DBA*;

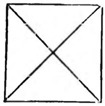
therefore each of the angles *BDA*, *DBA* is also double of the angle *DAB*.

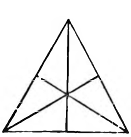
Therefore the isosceles triangle *ABD* has been constructed having each of the angles at the base *DB* double of the remaining one.

Q. E. F.

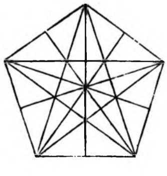
There is every reason to conclude that the connexion of the triangle constructed in this proposition with the regular pentagon, and the construction of the triangle itself, were the discovery of the Pythagoreans. In the first place the Scholium IV. No. 2 (Heiberg, Vol. V. p. 273) says "this Book is the discovery of the Pythagoreans." Secondly, the summary in Proclus (p. 65, 20) says that Pythagoras discovered "the construction of the cosmic figures," by which must be understood the five regular solids. Thirdly, Iamblichus (*Vit. Pyth.* c. 18, s. 88) quotes a story of Hippasus, "that he was one of the Pythagoreans but, owing to his being the first to publish and write down (the construction of) the sphere arising from the twelve pentagons (τὴν ἐκ τῶν δώδεκα πενταγώνων), perished by shipwreck for his impiety, having got credit for the discovery all the same, whereas everything belonged to HIM (ἐκείνου τοῦ ἀνδρός), for it is thus that they refer to Pythagoras, and they do not call him by his name." Cantor has (I_3, pp. 176 sqq.) collected notices which help us to form an idea how the discovery of the Euclidean construction for a regular pentagon may have been arrived at by the Pythagoreans.

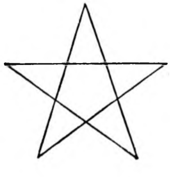
Plato puts into the mouth of Timaeus a description of the formation from right-angled triangles of the figures which are the faces of the first four regular solids. The face of the cube is the square which is formed from isosceles right-angled triangles by placing four of these triangles contiguously so that the four right angles are in contact at the centre. The equilateral triangle, however, which is the form of the faces of the tetrahedron, the octahedron and the icosahedron, cannot be constructed from isosceles right-angled triangles, but is constructed from a particular scalene right-angled triangle which Timaeus (54 A, B) regards as the most beautiful of all scalene right-angled triangles, namely that in which the square on one of the sides about the right angle is three times the square on the other. This is, of course, the triangle forming half of an equilateral triangle bisected by the perpendicular from one angular point on the opposite side. The Platonic Timaeus does not construct his equilateral triangle from two such triangles but from six, by placing the latter contiguously round a point so that the hypotenuses and the smaller of the sides about the right angles respectively adjoin, and all of them meet at the common centre, as shown in the figure (*Timaeus*, 54 D, E.). The probability that this exposition was Pythagorean is confirmed by the independent testimony of Proclus (pp. 304—5), who attributes to the Pythagoreans the theorem that six equilateral triangles, or three hexagons, or four squares, placed contiguously with one angular point of each at a common point, will just fill up the four right angles round that point, and that no other regular polygons in any numbers have this property.





How then would it be proposed to split up into triangles, or to make up out of triangles, the face of the remaining solid, the dodecahedron? It would easily be seen that the pentagon could not be constructed by means of the two right-angled triangles which were used for constructing the square and the equilateral triangle respectively. But attempts would naturally be made to split up the pentagon into elementary triangles, and traces of such attempts are actually forthcoming. Plutarch has in two passages spoken of the division of the faces of the dodecahedron into triangles, remarking in one place (*Quaest. Platon.* V. 1) that each of the twelve faces is made up of 30 elementary scalene triangles, so that, taken together, they give 360 such triangles, and in another (*De defectu oraculorum*, c. 33) that the elementary triangle of the dodecahedron must be different from that of the tetrahedron, octahedron and icosahedron. Another writer of the 2nd cent., Alcinous, has, in his introduction to the study of Plato (*De doctrina Platonis*, c. 11), spoken similarly of the 360 elements which are produced when every one of the pentagons is divided into 5 isosceles triangles, and each of the latter into 6 scalene triangles. Now, if we proceed to draw lines in a pentagon separating it into this number of small triangles as shown in the above figure, the figure





which stands out most prominently in the mass of lines is the "star-pentagon," as drawn separately, which then (if the consecutive corners be joined) suggests the drawing, as part of a pentagon, of a triangle of a definite character. Now we are expressly told by Lucian and the scholiast to the *Clouds* of Aristophanes (see Bretschneider, pp. 85—86) that the triple interwoven triangle, the pentagram (*τὸ τριπλοῦν τρίγωνον, τὸ δι' ἀλλήλων, τὸ πεντάγραμμον*), was used by the Pythagoreans as a symbol of recognition between the members of the same school (*συμβόλῳ πρὸς τοὺς ὁμοδόξους ἐχρῶντο*), and was called by them Health. There seems to be therefore no room for doubt that the construction of a pentagon by means of an isosceles triangle having each of its base angles double of the vertical angle was due to the Pythagoreans.

The construction of this triangle depends upon II. 11, or the problem of dividing a straight line so that the rectangle contained by the whole and one of the parts is equal to the square on the other part. This problem of course appears again in Eucl. VI. 30 as the problem of cutting a given straight line *in extreme and mean ratio*, i.e. the problem of the *golden section*, which is no doubt "the section" referred to in the passage of the summary given by Proclus (p. 67, 6) which says that Eudoxus "greatly added to the number of the theorems which Plato originated regarding the section." This idea that Plato began the study of the "golden section" as a subject in itself is not in the least inconsistent with the supposition that the problem of Eucl. II. 11 was solved by the Pythagoreans. The very fact that Euclid places it among other propositions which are clearly Pythagorean in origin is significant, as is also the fact that its solution is effected by "applying to a straight line a rectangle equal to a given square and exceeding by a square," while Proclus says plainly (p. 419, 15) that, according to Eudemus, "the application of areas, their *exceeding* and their falling short, are ancient and discoveries of the Muse of the Pythagoreans."

We may suppose the construction of IV. 10 to have been arrived at by analysis somewhat as follows (Todhunter's Euclid, p. 325).

Suppose the problem solved, i.e. let *ABD* be an isosceles triangle having each of its base angles double of the vertical angle.

Bisect the angle *ADB* by the straight line *DC* meeting *AB* in *C*. [I. 9]

Therefore the angle *BDC* is equal to the angle *BAD*; and the angle *CDA* is also equal to the angle *BAD*,

so that *DC* is equal to *CA*.

Again, since, in the triangles *BCD*, *BDA*,

the angle *BDC* is equal to the angle *BAD*,

and the angle *B* is common,

the third angle *BCD* is equal to the third angle *BDA*, and therefore to the angle *DBC*.

Therefore *DC* is equal to *DB*.

Now, if a circle be described about the triangle *ACD* [IV. 5], since the angle *BDC* is equal to the angle in the segment *CAD*,

BD must touch the circle [by the converse of III. 32 easily proved from it by *reductio ad absurdum*].

Hence [III. 36] the square on *BD* and therefore the square on *CD*, or *AC*, is equal to the rectangle *AB*, *BC*.

Thus the problem is reduced to that of cutting *AB* at *C* so that the rectangle *AB*, *BC* is equal to the square on *AC*. [II. 11]

When this is done, we have only to draw a circle with centre *A* and radius *AB* and place in it a chord *BD* equal in length to *AC*. [IV. 1]

Since each of the angles *ABD*, *ADB* is double of the angle *BAD*, the latter is equal to one-fifth of the sum of all three, i.e. is one-fifth of two right angles, or two-fifths of a right angle, and each of the base angles is four-fifths of a right angle.

If we bisect the angle *BAD*, we obtain an angle equal to one-fifth of a right angle, so that the proposition enables us *to divide a right angle into five equal parts*.

It will be observed that *BD* is the side of a regular *decagon* inscribed in the larger circle.

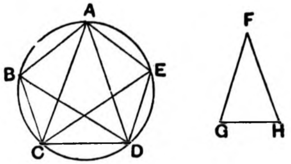
Proclus, as remarked above (Vol. I. p. 130), gives IV. 10 as an instance in which two of the six formal divisions of a proposition, the *setting-out* and the "*definition*," are left out, and explains that they are unnecessary because there is no *datum* in the enunciation. This is however no more than formally true, because Euclid does begin his proposition by *setting out* "any straight line *AB*," and he constructs an isosceles triangle having *AB* for one of its equal sides, i.e. he does practically imply a datum in the enunciation, and a corresponding *setting-out* and "*definition*" in the proposition itself.

PROPOSITION 11.

In a given circle to inscribe an equilateral and equiangular pentagon.

Let *ABCDE* be the given circle;

thus it is required to inscribe in the circle *ABCDE* an equilateral and equiangular pentagon.



Let the isosceles triangle *FGH* be set out having each of the angles at *G*, *H* double of the angle at *F*; [IV. 10]

let there be inscribed in the circle *ABCDE* the triangle *ACD* equiangular with the triangle *FGH*, so that the angle *CAD* is equal to the angle at *F* and the angles at *G*, *H* respectively equal to the angles *ACD*, *CDA*; [IV. 2]

therefore each of the angles *ACD*, *CDA* is also double of the angle *CAD*.

Now let the angles *ACD*, *CDA* be bisected respectively by the straight lines *CE*, *DB* [I. 9], and let *AB*, *BC*, *DE*, *EA* be joined.

Then, since each of the angles *ACD*, *CDA* is double of the angle *CAD*,

and they have been bisected by the straight lines *CE*, *DB*, therefore the five angles *DAC*, *ACE*, *ECD*, *CDB*, *BDA* are equal to one another.

But equal angles stand on equal circumferences; [III. 26]

therefore the five circumferences *AB*, *BC*, *CD*, *DE*, *EA* are equal to one another.

But equal circumferences are subtended by equal straight lines; [III. 29]

therefore the five straight lines *AB*, *BC*, *CD*, *DE*, *EA* are equal to one another;

therefore the pentagon *ABCDE* is equilateral.

I say next that it is also equiangular.

For, since the circumference *AB* is equal to the circumference *DE*, let *BCD* be added to each;

therefore the whole circumference *ABCD* is equal to the whole circumference *EDCB*.

And the angle *AED* stands on the circumference *ABCD*, and the angle *BAE* on the circumference *EDCB*;

therefore the angle *BAE* is also equal to the angle *AED*. [III. 27]

For the same reason

each of the angles *ABC*, *BCD*, *CDE* is also equal to each of the angles *BAE*, *AED*;

therefore the pentagon *ABCDE* is equiangular.

But it was also proved equilateral;

therefore in the given circle an equilateral and equiangular pentagon has been inscribed.

Q. E. F.

De Morgan remarks that "the method of IV. 11 is not so natural as making a direct use of the angle obtained in the last." On the other hand, if we look at the figure and notice that it shows the whole of the *pentagram-star* except one line (that connecting *B* and *E*), I think we shall conclude that the method is nearer to that used by the Pythagoreans, and therefore of much more historical interest.

Another method would of course be to use IV. 10 to describe a *decagon* in the circle, and then to join any vertex to the next alternate one, the latter to the next alternate one, and so on.

Mr H. M. Taylor gives "a complete geometrical construction for inscribing a regular decagon or pentagon in a given circle," as follows.

"Find *O* the centre.

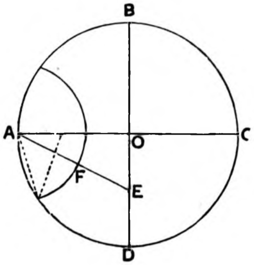
Draw two diameters *AOC*, *BOD* at right angles to one another.

Bisect *OD* in *E*.

Draw *AE* and cut off *EF* equal to *OE*.

Place round the circle ten chords equal to *AF*.

These chords will be the sides of a regular decagon. Draw the chords joining five alternate vertices of the decagon; they will be the sides of a regular pentagon."



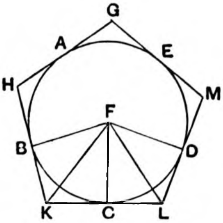
The construction is of course only a combination of those in II. 11 and IV. 1; and the proof would have to follow that in IV. 10.

PROPOSITION 12.

About a given circle to circumscribe an equilateral and equiangular pentagon.

Let *ABCDE* be the given circle;

thus it is required to circumscribe an equilateral and equiangular pentagon about the circle *ABCDE*.



Let *A*, *B*, *C*, *D*, *E* be conceived to be the angular points of the inscribed pentagon, so that the circumferences *AB*, *BC*, *CD*, *DE*, *EA* are equal; [IV. 11]

through *A*, *B*, *C*, *D*, *E* let *GH*, *HK*, *KL*, *LM*, *MG* be drawn touching the circle; [III. 16, Por.]

let the centre *F* of the circle *ABCDE* be taken [III. 1], and let *FB*, *FK*, *FC*, *FL*, *FD* be joined.

Then, since the straight line *KL* touches the circle *ABCDE* at *C*,

and *FC* has been joined from the centre *F* to the point of contact at *C*,

therefore *FC* is perpendicular to *KL*; [III. 18]

therefore each of the angles at *C* is right.

For the same reason

the angles at the points *B*, *D* are also right.

And, since the angle *FCK* is right,
therefore the square on *FK* is equal to the squares on *FC*, *CK*.

For the same reason [I. 47]
the square on *FK* is also equal to the squares on *FB*, *BK*;

so that the squares on *FC*, *CK* are equal to the squares on *FB*, *BK*,

of which the square on *FC* is equal to the square on *FB*;
therefore the square on *CK* which remains is equal to the square on *BK*.

Therefore *BK* is equal to *CK*.

And, since *FB* is equal to *FC*,
and *FK* common,

the two sides *BF*, *FK* are equal to the two sides *CF*, *FK*;
and the base *BK* equal to the base *CK*;

therefore the angle *BFK* is equal to the angle *KFC*, [I. 8]

and the angle *BKF* to the angle *FKC*.

Therefore the angle *BFC* is double of the angle *KFC*,

and the angle *BKC* of the angle *FKC*.

For the same reason

the angle *CFD* is also double of the angle *CFL*,

and the angle *DLC* of the angle *FLC*.

Now, since the circumference *BC* is equal to *CD*,
the angle *BFC* is also equal to the angle *CFD*. [III. 27]

And the angle *BFC* is double of the angle *KFC*, and the angle *DFC* of the angle *LFC*;

therefore the angle *KFC* is also equal to the angle *LFC*.

But the angle *FCK* is also equal to the angle *FCL*;
therefore *FKC*, *FLC* are two triangles having two angles equal to two angles and one side equal to one side, namely *FC* which is common to them;
therefore they will also have the remaining sides equal to the remaining sides, and the remaining angle to the remaining angle; [I. 26]

therefore the straight line *KC* is equal to *CL*,
and the angle *FKC* to the angle *FLC*.

And, since *KC* is equal to *CL*,
therefore *KL* is double of *KC*.

For the same reason it can be proved that

HK is also double of BK.

And BK is equal to KC;

therefore HK is also equal to KL.

Similarly each of the straight lines HG, GM, ML can also be proved equal to each of the straight lines HK, KL;

therefore the pentagon $GHKLM$ is equilateral.

I say next that it is also equiangular.

For, since the angle FKC is equal to the angle FLC, and the angle HKL was proved double of the angle FKC,

and the angle KLM double of the angle FLC,

therefore the angle HKL is also equal to the angle KLM.

Similarly each of the angles KHG, HGM, GML can also be proved equal to each of the angles HKL, KLM;

therefore the five angles GHK, HKL, KLM, LMG, MGH are equal to one another.

Therefore the pentagon $GHKLM$ is equiangular.

And it was also proved equilateral; and it has been circumscribed about the circle $ABCDE$.

Q. E. F.

De Morgan remarks that IV. 12, 13, 14 supply the place of the following: *Having given a regular polygon of any number of sides inscribed in a circle, to describe the same about the circle; and, having given the polygon, to inscribe and circumscribe a circle.* For the method can be applied generally, as indeed Euclid practically says in the Porism to IV. 15 about the regular hexagon and in the remark appended to IV. 16 about the regular fifteen-angled figure.

The *conclusion* of this proposition, "therefore about the given circle an equilateral and equiangular pentagon has been circumscribed," is omitted in the MSS.

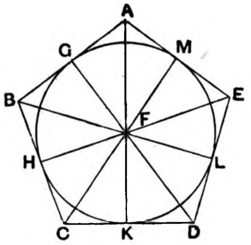
PROPOSITION 13.

In a given pentagon, which is equilateral and equiangular, to inscribe a circle.

Let $ABCDE$ be the given equilateral and equiangular pentagon;

thus it is required to inscribe a circle in the pentagon $ABCDE$.

For let the angles BCD, CDE be bisected by the straight lines CF, DF respectively; and from the point F, at which the straight lines *CF*, *DF* meet one another, let the straight lines *FB*, *FA*, *FE* be joined.



Then, since *BC* is equal to *CD*,
and *CF* common,
the two sides *BC*, *CF* are equal to the two sides *DC*, *CF*;
and the angle *BCF* is equal to the angle *DCF*;
therefore the base *BF* is equal to the base *DF*,
and the triangle *BCF* is equal to the triangle *DCF*,
and the remaining angles will be equal to the remaining angles, namely those which the equal sides subtend. [I. 4]

Therefore the angle *CBF* is equal to the angle *CDF*.

And, since the angle *CDE* is double of the angle *CDF*,
and the angle *CDE* is equal to the angle *ABC*,
while the angle *CDF* is equal to the angle *CBF*;
therefore the angle *CBA* is also double of the angle *CBF*;
therefore the angle *ABF* is equal to the angle *FBC*;
therefore the angle *ABC* has been bisected by the straight line *BF*.

Similarly it can be proved that
the angles *BAE*, *AED* have also been bisected by the straight lines *FA*, *FE* respectively.

Now let *FG*, *FH*, *FK*, *FL*, *FM* be drawn from the point *F* perpendicular to the straight lines *AB*, *BC*, *CD*, *DE*, *EA*.

Then, since the angle *HCF* is equal to the angle *KCF*,
and the right angle *FHC* is also equal to the angle *FKC*,
FHC, *FKC* are two triangles having two angles equal to two angles and one side equal to one side, namely *FC* which is common to them and subtends one of the equal angles;
therefore they will also have the remaining sides equal to the remaining sides; [I. 26]
therefore the perpendicular *FH* is equal to the perpendicular *FK*.

Similarly it can be proved that
each of the straight lines *FL*, *FM*, *FG* is also equal to each of the straight lines *FH*, *FK*;

therefore the five straight lines *FG*, *FH*, *FK*, *FL*, *FM* are equal to one another.

Therefore the circle described with centre *F* and distance one of the straight lines *FG*, *FH*, *FK*, *FL*, *FM* will pass also through the remaining points;

and it will touch the straight lines *AB*, *BC*, *CD*, *DE*, *EA*, because the angles at the points *G*, *H*, *K*, *L*, *M* are right.

For, if it does not touch them, but cuts them,

it will result that the straight line drawn at right angles to the diameter of the circle from its extremity falls within the circle: which was proved absurd. [III. 16]

Therefore the circle described with centre *F* and distance one of the straight lines *FG*, *FH*, *FK*, *FL*, *FM* will not cut the straight lines *AB*, *BC*, *CD*, *DE*, *EA*;

therefore it will touch them.

Let it be described, as *GHKLM*.

Therefore in the given pentagon, which is equilateral and equiangular, a circle has been inscribed.

Q. E. F.

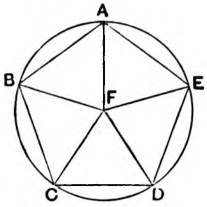
PROPOSITION 14.

About a given pentagon, which is equilateral and equiangular, to circumscribe a circle.

Let *ABCDE* be the given pentagon, which is equilateral and equiangular;

thus it is required to circumscribe a circle about the pentagon *ABCDE*.

Let the angles *BCD*, *CDE* be bisected by the straight lines *CF*, *DF* respectively, and from the point *F*, at which the straight lines meet, let the straight lines *FB*, *FA*, *FE* be joined to the points *B*, *A*, *E*.



Then in manner similar to the preceding it can be proved that the angles *CBA*, *BAE*, *AED* have also been bisected by the straight lines *FB*, *FA*, *FE* respectively.

Now, since the angle *BCD* is equal to the angle *CDE*,
and the angle *FCD* is half of the angle *BCD*,
and the angle *CDF* half of the angle *CDE*,
therefore the angle *FCD* is also equal to the angle *CDF*,
so that the side *FC* is also equal to the side *FD*. [I. 6]

Similarly it can be proved that
each of the straight lines *FB*, *FA*, *FE* is also equal to each of the straight lines *FC*, *FD*;
therefore the five straight lines *FA*, *FB*, *FC*, *FD*, *FE* are equal to one another.

Therefore the circle described with centre *F* and distance one of the straight lines *FA*, *FB*, *FC*, *FD*, *FE* will pass also through the remaining points, and will have been circumscribed.

Let it be circumscribed, and let it be *ABCDE*.

Therefore about the given pentagon, which is equilateral and equiangular, a circle has been circumscribed.

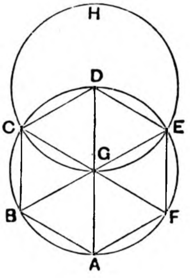
Q. E. F.

PROPOSITION 15.

In a given circle to inscribe an equilateral and equiangular hexagon.

Let *ABCDEF* be the given circle;
thus it is required to inscribe an equilateral and equiangular hexagon in the circle *ABCDEF*.

Let the diameter *AD* of the circle *ABCDEF* be drawn;
let the centre *G* of the circle be taken, and with centre *D* and distance *DG* let the circle *EGCH* be described;
let *EG*, *CG* be joined and carried through to the points *B*, *F*,
and let *AB*, *BC*, *CD*, *DE*, *EF*, *FA* be joined.



I say that the hexagon *ABCDEF* is equilateral and equiangular.

For, since the point *G* is the centre of the circle *ABCDEF*, *GE* is equal to *GD*.

Again, since the point D is the centre of the circle GCH,

DE is equal to DG.

But GE was proved equal to GD;

therefore GE is also equal to ED;

therefore the triangle EGD is equilateral;

and therefore its three angles EGD, GDE, DEG are equal to one another, inasmuch as, in isosceles triangles, the angles at the base are equal to one another. [I. 5]

And the three angles of the triangle are equal to two right angles; [I. 32]

therefore the angle EGD is one-third of two right angles.

Similarly, the angle DGC can also be proved to be one-third of two right angles.

And, since the straight line CG standing on EB makes the adjacent angles EGC, CGB equal to two right angles,

therefore the remaining angle CGB is also one-third of two right angles.

Therefore the angles EGD, DGC, CGB are equal to one another;

so that the angles vertical to them, the angles BGA, AGF, FGE are equal. [I. 15]

Therefore the six angles EGD, DGC, CGB, BGA, AGF, FGE are equal to one another.

But equal angles stand on equal circumferences; [III. 26]
therefore the six circumferences AB, BC, CD, DE, EF, FA are equal to one another.

And equal circumferences are subtended by equal straight lines; [III. 29]

therefore the six straight lines are equal to one another;

therefore the hexagon $ABCDEF$ is equilateral.

I say next that it is also equiangular.

For, since the circumference FA is equal to the circumference ED,

let the circumference $ABCD$ be added to each;

therefore the whole $FABCD$ is equal to the whole $EDCBA$;

and the angle *FED* stands on the circumference *FABCD*, and the angle *AFE* on the circumference *EDCBA*;

therefore the angle *AFE* is equal to the angle *DEF*. [III. 27]

Similarly it can be proved that the remaining angles of the hexagon *ABCDEF* are also severally equal to each of the angles *AFE*, *FED*;

therefore the hexagon *ABCDEF* is equiangular.

But it was also proved equilateral;

and it has been inscribed in the circle *ABCDEF*.

Therefore in the given circle an equilateral and equiangular hexagon has been inscribed.

Q. E. F.

PORISM. From this it is manifest that the side of the hexagon is equal to the radius of the circle.

And, in like manner as in the case of the pentagon, if through the points of division on the circle we draw tangents to the circle, there will be circumscribed about the circle an equilateral and equiangular hexagon in conformity with what was explained in the case of the pentagon.

And further by means similar to those explained in the case of the pentagon we can both inscribe a circle in a given hexagon and circumscribe one about it.

Q. E. F.

Heiberg, I think with good reason, considers the Porism to this proposition to be referred to in the instance which Proclus (p. 304, 2) gives of a porism following a *problem*. As the text of Proclus stands, "the (porism) found in the second Book (τὸ δὲ ἐν τῷ δευτέρῳ βιβλίῳ κείμενον) is a porism to a problem"; but this is not true of the only porism that we find in the second Book, namely the porism to II. 4. Hence Heiberg thinks that for τῷ δευτέρῳ βιβλίῳ should be read τῷ δ′ βιβλίῳ, i.e. the fourth Book. Moreover Proclus speaks of *the* porism in the particular Book, from which we gather that there was only *one* porism in Book IV. as he knew it, and therefore that he did not regard as a *porism* the addition to IV. 5. Cf. note on that proposition.

It appears that Theon substituted for the first words of the Porism to IV. 15 "And in like manner as in the case of the pentagon" (ὁμοίως δὲ τοῖς ἐπὶ τοῦ πενταγώνου) the simple word "and" or "also" (καί), apparently thinking that the words had the same meaning as the similar words lower down. This is however not the case, the meaning being that "if, as in the case of the pentagon, we draw tangents, we can prove, also as was done in the case of the pentagon, that the figure so formed is a circumscribed regular hexagon."

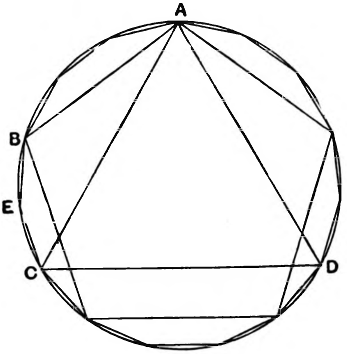
PROPOSITION 16.

In a given circle to inscribe a fifteen-angled figure which shall be both equilateral and equiangular.

Let *ABCD* be the given circle; thus it is required to inscribe in the circle *ABCD* a fifteen-angled figure which shall be both equilateral and equiangular.

In the circle *ABCD* let there be inscribed a side *AC* of the equilateral triangle inscribed in it, and a side *AB* of an equilateral pentagon;

therefore, of the equal segments of which there are fifteen in the circle *ABCD*, there will be five in the circumference *ABC* which is one-third of the circle, and there will be three in the circumference *AB* which is one-fifth of the circle;



therefore in the remainder *BC* there will be two of the equal segments.

Let *BC* be bisected at *E*; [III. 30]

therefore each of the circumferences *BE*, *EC* is a fifteenth of the circle *ABCD*.

If therefore we join *BE*, *EC* and fit into the circle *ABCD* straight lines equal to them and in contiguity, a fifteen-angled figure which is both equilateral and equiangular will have been inscribed in it.

Q. E. F.

And, in like manner as in the case of the pentagon, if through the points of division on the circle we draw tangents to the circle, there will be circumscribed about the circle a fifteen-angled figure which is equilateral and equiangular.

And further, by proofs similar to those in the case of the pentagon, we can both inscribe a circle in the given fifteen-angled figure and circumscribe one about it.

Here, as in III. 10, we have the term "circle" used by Euclid in its exceptional sense of the *circumference* of a circle, instead of the "*plane figure* contained by one line" of I. Def. 15. Cf. the note on that definition (Vol. I. pp. 184—5).

Proclus (p. 269) refers to this proposition in illustration of his statement that Euclid gave proofs of a number of propositions with an eye to their use in astronomy. "With regard to the last proposition in the fourth Book in which he inscribes the side of the fifteen-angled figure in a circle, for what object does anyone assert that he propounds it except for the reference of this problem to astronomy? For, when we have inscribed the fifteen-angled figure in the circle through the poles, we have the distance from the poles both of the equator and the zodiac, since they are distant from one another by the side of the fifteen-angled figure." This agrees with what we know from other sources, namely that up to the time of Eratosthenes (*circa* 284—204 B.C.) 24 was generally accepted as the correct measurement of the obliquity of the ecliptic. This measurement, and the construction of the fifteen-angled figure, were probably due to the Pythagoreans, though it would appear that the former was not known to Oenopides of Chios (fl. *circa* 460 B.C.), as we learn from Theon of Smyrna (pp. 198—9, ed. Hiller), who gives Dercyllides as his authority, that Eudemus (fl. *circa* 320 B.C.) stated in his *ἀστρολογίαι* that, while Oenopides discovered certain things, and Thales, Anaximander and Anaximenes others, it was the rest (*οἱ λοιποί*) who added other discoveries to these and, among them, that "the axes of the fixed stars and of the planets respectively are distant from one another by the side of a fifteen-angled figure." Eratosthenes evaluated the angle to $\frac{11}{83}$rds of 180°, i.e. about 23° 51′ 20″, which measurement was apparently not improved upon in antiquity (cf. Ptolemy, *Syntaxis*, ed. Heiberg, p. 68).

Euclid has now shown how to describe regular polygons with 3, 4, 5, 6 and 15 sides. Now, when any regular polygon is given, we can construct a regular polygon with twice the number of sides by first describing a circle about the given polygon and then bisecting all the smaller arcs subtended by the sides. Applying this process any number of times, we see that we can by Euclid's methods construct regular polygons with $3 . 2^n$, $4 . 2^n$, $5 . 2^n$, $15 . 2^n$ sides, where n is zero or any positive integer.

BOOK V.

INTRODUCTORY NOTE.

The anonymous author of a scholium to Book v. (Euclid, ed. Heiberg, Vol. v. p. 280), who is perhaps Proclus, tells us that "some say" this Book, containing the general theory of proportion which is equally applicable to geometry, arithmetic, music, and all mathematical science, "is the discovery of Eudoxus, the teacher of Plato." Not that there had been no theory of proportion developed before his time; on the contrary, it is certain that the Pythagoreans had worked out such a theory with regard to *numbers*, by which must be understood commensurable and even whole numbers (a number being a "multitude made up of units," as defined in Eucl. VII). Thus we are told that the Pythagoreans distinguished three sorts of *means*, the arithmetic, the geometric and the harmonic mean, the geometric mean being called proportion (ἀναλογία) *par excellence*; and further Iamblichus speaks of the "most perfect proportion consisting of four terms and specially called *harmonic*," in other words, the proportion

$$a : \frac{a+b}{2} = \frac{2ab}{a+b} : b,$$

which was said to be a discovery of the Babylonians and to have been first introduced into Greece by Pythagoras (Iamblichus, *Comm. on Nicomachus*, p. 118). Now the principle of similitude is one which is presupposed by all the arts of design from their very beginnings; it was certainly known to the Egyptians, and it must certainly have been thoroughly familiar to Pythagoras and his school. This consideration, together with the evidence of the employment by him of the *geometric proportion*, makes it indubitable that the Pythagoreans used the theory of proportion, in the form in which it was known to them, i.e. as applicable to commensurables only, in their geometry. But the discovery, also by the Pythagoreans, of the incommensurable would of course be seen to render the proofs which depended on the theory of proportion as then understood inconclusive; as Tannery observes (*La Géométrie grecque*, p. 98), "the discovery of incommensurability must have caused a veritable logical scandal in geometry and, in order to avoid it, they were obliged to restrict as far as possible the use of the principle of similitude, pending the discovery of a means of establishing it on the basis of a theory of proportion independent of commensurability." The glory of the latter discovery belongs then most probably to Eudoxus. Certain it is that the complete theory was already familiar to Aristotle, as we shall see later.

It seems probable, as indicated by Tannery (*loc. cit.*), that the theory of proportions and the principle of similitude took, in the earliest Greek geometry, an earlier place than they do in Euclid, but that, in consequence of the discovery of the incommensurable, the treatment of the subject was fundamentally remodelled in the period between Pythagoras and Eudoxus. An indication of this is afforded by the clever device used in Euclid I. 44 for applying to a given straight line a parallelogram equal to a given triangle; the equality of the "complements" in a parallelogram is there used for doing what is practically finding a fourth proportional to three given straight lines. Thus Euclid was no doubt following for the subject-matter of Books I.—IV. what had become the traditional method, and this is probably one of the reasons why proportions and similitude are postponed till as late as Books V., VI.

It is a remarkable fact that the theory of proportions is twice treated in Euclid, in Book V. with reference to magnitudes in general, and in Book VII. with reference to the particular case of numbers. The latter exposition referring only to commensurables may be taken to represent fairly the theory of proportions at the stage which it had reached before the great extension of it made by Eudoxus. The differences between the definitions etc. in Books V. and VII. will appear as we go on; but the question naturally arises, why did Euclid not save himself so much repetition and treat numbers merely as a particular case of magnitude, referring back to the corresponding more general propositions of Book V. instead of proving the same propositions over again for numbers? It could not have escaped him that numbers fall under the conception of magnitude. Aristotle had plainly indicated that magnitudes may be numbers when he observed (*Anal. post.* I. 7, 75 b 4) that you cannot adapt the arithmetical method of proof to the properties of magnitudes if the magnitudes are not numbers. Further Aristotle had remarked (*Anal. post.* I. 5, 74 a 17) that the proposition that the terms of a proportion can be taken alternately was at one time proved separately for numbers, lines, solids and times, though it was possible to prove it for all by one demonstration; but, because there was no common name comprehending them all, namely numbers, lengths, times and solids, and their character was different, they were taken separately. Now however, he adds, the proposition is proved generally. Yet Euclid says nothing to connect the two theories of proportion even when he comes in X. 5 to a proportion two terms of which are magnitudes and two are numbers ("Commensurable magnitudes have to one another the ratio which a number has to a number"). The probable explanation of the phenomenon is that Euclid simply followed tradition and gave the two theories as he found them. This would square with the remark in Pappus (VII. p. 678) as to Euclid's fairness to others and his readiness to give them credit for their work.

DEFINITIONS.

1. A magnitude is a **part** of a magnitude, the less of the greater, when it measures the greater.

2. The greater is a **multiple** of the less when it is measured by the less.

3. A **ratio** is a sort of relation in respect of size between two magnitudes of the same kind.

4. Magnitudes are said to **have a ratio** to one another which are capable, when multiplied, of exceeding one another.

5. Magnitudes are said to **be in the same ratio**, the first to the second and the third to the fourth, when, if any equimultiples whatever be taken of the first and third, and any equimultiples whatever of the second and fourth, the former equimultiples alike exceed, are alike equal to, or alike fall short of, the latter equimultiples respectively taken in corresponding order.

6. Let magnitudes which have the same ratio be called **proportional**.

7. When, of the equimultiples, the multiple of the first magnitude exceeds the multiple of the second, but the multiple of the third does not exceed the multiple of the fourth, then the first is said to **have a greater ratio** to the second than the third has to the fourth.

8. A proportion in three terms is the least possible.

9. When three magnitudes are proportional, the first is said to have to the third the **duplicate ratio** of that which it has to the second.

10. When four magnitudes are < continuously > proportional, the first is said to have to the fourth the **triplicate ratio** of that which it has to the second, and so on continually, whatever be the proportion.

11. The term **corresponding magnitudes** is used of antecedents in relation to antecedents, and of consequents in relation to consequents.

12. **Alternate ratio** means taking the antecedent in relation to the antecedent and the consequent in relation to the consequent.

13. **Inverse ratio** means taking the consequent as antecedent in relation to the antecedent as consequent.

14. **Composition of a ratio** means taking the antecedent together with the consequent as one in relation to the consequent by itself.

15. **Separation of a ratio** means taking the excess by which the antecedent exceeds the consequent in relation to the consequent by itself.

16. **Conversion of a ratio** means taking the antecedent in relation to the excess by which the antecedent exceeds the consequent.

17. A ratio **ex aequali** arises when, there being several magnitudes and another set equal to them in multitude which taken two and two are in the same proportion, as the first is to the last among the first magnitudes, so is the first to the last among the second magnitudes;

Or, in other words, it means taking the extreme terms by virtue of the removal of the intermediate terms.

18. A **perturbed proportion** arises when, there being three magnitudes and another set equal to them in multitude, as antecedent is to consequent among the first magnitudes, so is antecedent to consequent among the second magnitudes, while, as the consequent is to a third among the first magnitudes, so is a third to the antecedent among the second magnitudes.

DEFINITION 1.

Μέρος ἐστὶ μέγεθος μεγέθους τὸ ἔλασσον τοῦ μείζονος, ὅταν καταμετρῇ τὸ μεῖζον.

The word *part* (μέρος) is here used in the restricted sense of a *submultiple* or an *aliquot part* as distinct from the more general sense in which it is used in the Common Notion (5) which says that "the whole is greater than the part." It is used in the same restricted sense in VII. Def. 3, which is the same definition as this with "number" (ἀριθμός) substituted for "magnitude." VII. Def. 4, keeping up the restriction, says that, when a number does not measure another number, it is *parts* (in the plural), not *a part* of it. Thus, 1, 2, or 3, is *a part* of 6, but 4 is not *a part* of 6 but *parts*. The same distinction between the restricted and the more general sense of the word *part* appears in Aristotle, *Metaph.* 1023 b 12: "In one sense a part is that into which quantity (τὸ ποσόν) can anyhow be divided; for that which is taken away from quantity, *quâ* quantity, is always called a 'part' of it, as e.g. two is said to be in a sense a part of three. But in another sense a 'part' is only *what measures* (τὰ καταμετροῦντα) such quantities. Thus two is in one sense said to be a part of three, in the other not."

DEFINITION 2.

Πολλαπλάσιον δὲ τὸ μεῖζον τοῦ ἐλάττονος, ὅταν καταμετρῆται ὑπὸ τοῦ ἐλάττονος.

DEFINITION 3.

Λόγος ἐστὶ δύο μεγεθῶν ὁμογενῶν ἡ κατὰ πηλικότητα ποιὰ σχέσις.

The best explanation of the definitions of *ratio* and *proportion* that I have seen is that of De Morgan, which will be found in the articles under those titles in the *Penny Cyclopaedia*, Vol. XIX. (1841); and in the following notes I shall draw largely from these articles. Very valuable also are the notes on the definitions of Book V. given by Hankel (fragment on Euclid published as an appendix to his work *Zur Geschichte der Mathematik in Alterthum und Mittelalter*, 1874).

There has been controversy as to what is the proper translation of the word πηλικότης in the definition. σχέσις κατὰ πηλικότητα has generally been translated "relation in respect of *quantity*." Upon this De Morgan remarks that it makes nonsense of the definition; "for magnitude has hardly a different meaning from quantity, and a relation of magnitudes with respect to quantity may give a clear idea to those who want a word to convey a notion of architecture with respect to building or of battles with respect to fighting, and to no others." The true interpretation De Morgan, following Wallis and Gregory, takes to be *quantuplicity*, referring to the number of times one magnitude is contained in the other. For, he says, we cannot describe magnitude in language without quantuplicitative reference to other magnitude; hence he supposes that the definition simply conveys the fact that the mode of expressing quantity in terms of quantity is entirely based upon the notion of quantuplicity or that relation of which we take cognizance when we find how many times one is contained in the other. While all the rest of De Morgan's observations on the definition are admirable, it seems to me that on this question of the proper translation of πηλικότης he is in error. He supports his view mainly by reference (1) to the definition of a compounded ratio usually given as the 5th definition of Book VI., which speaks of the πηλικότητες of two ratios being multiplied together, and (2) to the comments of Eutocius and a scholiast on this definition. Eutocius says namely (Archimedes, ed. Heiberg, III. p. 120) that "the term πηλικότης is evidently used of the number from which the given ratio is called, as (among others) Nicomachus says in his first book on music and Heron in his commentary on the Introduction to Arithmetic." But it now appears certain that this definition is an interpolation; it is never used, it is not found in Campanus, and Peyrard's MS. only has it in the margin. At the same time it is clear that, if the definition is admitted at all, any commentator would be obliged to explain it in the way that Eutocius does, whether the explanation was consistent with the proper meaning of πηλικότης or not. Hence we must look elsewhere for the meaning of πηλίκος and πηλικότης. If we do this, I think we shall find no case in which the words have the sense attributed to them by De Morgan. The real meaning of πηλίκος is *how great*. It is so used by Aristotle, e.g. in *Eth. Nic.* V. 10, 1134 b 11, where he speaks of a man's child being as it were a part of him so long as he is of a certain age (ἕως ἂν ᾖ πηλίκον). Again Nicomachus, to whom Eutocius appeals, himself (I. 2, 5, p. 5, ed. Hoche) distinguishes πηλίκος as referring to *magnitude*, while ποσός refers to *multitude*. So does Iamblichus in his commentary on Nicomachus (p. 8, 3—5); besides which Iamblichus distinguishes πηλίκον as the subject of geometry, being *con-*

tinuous, and ποσόν as the subject of arithmetic, being *discrete*, and speaks of a point being the origin of πηλίκον as a unit is of ποσόν, and so on. Similarly, Ptolemy (*Syntaxis*, ed. Heiberg, p. 31) speaks of the *size* (πηλικότης) of the chords in a circle (περὶ τῆς πηλικότητος τῶν ἐν τῷ κύκλῳ εὐθειῶν). Consequently I think we can only translate πηλικότης in the definition as *size*. This corresponds to Hankel's translation of it as "Grösse," though he uses this same word for a concrete "magnitude" as well; *size* seems to me to give the proper distinction between πηλικότης and μέγεθος, as *size* is the attribute, and a *magnitude* (in its ordinary mathematical sense) is the thing which possesses the attribute of size.

The view that "relation in respect of *size*" is meant by the words in the text is also confirmed, I think, by a later remark of De Morgan himself, namely that a synonym for the word *ratio* may be found in the more intelligible term *relative magnitude*. In fact σχέσις in the definition corresponds to *relative* and πηλικότης to *magnitude*. (By magnitude De Morgan here means the attribute and not the thing possessing it.)

Of the definition as a whole Simson and Hankel express the opinion that it is an interpolation. Hankel points to the fact that it is unnecessary and moreover so vague as to be of no practical use, while the very use of the expression κατὰ πηλικότητα seems to him suspicious, since the only other place in which the word πηλικότης occurs in Euclid is the 5th definition of Book VI., which is admittedly not genuine. Yet the definition of ratio appears in all the MSS., the only variation being that some add the words πρὸς ἄλληλα, "to one another," which are rejected by Heiberg as an interpolation of Theon; and on the whole there seems to be no sufficient ground for regarding it as other than genuine. The true explanation of its presence would appear to be substantially that given by Barrow (*Lectiones Cantabrig.*, London, 1684, Lect. III. of 1666), namely that Euclid inserted it for completeness' sake, more for ornament than for use, intending to give the learner a general notion of ratio by means of a metaphysical, rather than a mathematical definition; "for metaphysical it is and not, properly speaking, mathematical, since nothing depends on it or is deduced from it by mathematicians, nor, as I think, can anything be deduced." This is confirmed by the fact that there is no definition of λόγος in Book VII., and it could equally have been dispensed with here. Similarly De Morgan observes that Euclid never attempts this vague sort of definition except when, dealing with a well-known term of common life, he wishes to bring it into geometry with something like an expressed meaning which may aid the conception of the thing, though it does not furnish a perfect criterion. Thus we may compare the definition with that of a straight line, where Euclid merely calls the reader's attention to the well-known term εὐθεῖα γραμμή, tries how far he can present the conception which accompanies it in other words, and trusts for the correct use of the term to the axioms (or postulates) which the universal conception of a straight line makes self-evident.

We have now to trace as clearly as possible the development of the conception of λόγος, *ratio*, or relative magnitude. In its primitive sense λόγος was only used of a ratio between commensurables, i.e. a ratio which could be *expressed*, and the manner of expressing it is indicated in the proposition, Eucl. X. 5, which proves that *commensurable magnitudes have to one another the ratio which a number has to a number*. That this was the primitive meaning of λόγος is proved by the use of the term ἄλογος for the incommensurable, which means *irrational* in the sense of *not having a ratio* to something taken as rational (ῥητός).

Euclid himself shows us how we are to set about finding the ratio, or relative magnitude, of two commensurable magnitudes. He gives, in X. 3, practically our ordinary method of finding the greatest common measure. If A, B be two magnitudes of which B is the less, we cut off from A a part equal to B, from the remainder a part equal to B, and so on, until we leave a remainder less than B, say R_1. We measure off R_1 from B in the same way until a remainder R_2 is left which is less than R_1. We repeat the process with R_1, R_2, and so on, until we find a remainder which is contained in the preceding remainder a certain number of times exactly. If account is taken of the number of times each magnitude is contained (with something over, except at the last) in that upon which it is measured, we can calculate how many times the last remainder is contained in A and how many times the last remainder is contained in B; and we can thus express the ratio of A to B as the ratio of one number to another.

But it may happen that the two magnitudes have no common measure, i.e. are incommensurable, in which case the process described would never come to an end and the means of expression would fail; the magnitudes would then *have no ratio* in the primitive sense. But the word λόγος, ratio, acquires in Euclid, Book V., a wider sense covering the *relative magnitude* of incommensurables as well as commensurables; as stated in Euclid's 4th definition, "magnitudes are said to have a *ratio* to one another which can, when multiplied, exceed one another," and finite incommensurables have this property as much as commensurables. De Morgan explains the manner of transition from the narrower to the wider signification of *ratio* as follows. "Since the relative magnitude of two quantities is always shown by the quantuplicitative mode of expression, when that is possible, and since proportional quantities (pairs which have the same relative magnitude) are pairs which have the same mode (if possible) of expression by means of each other; in all such cases sameness of relative magnitude leads to sameness of mode of expression; or proportion is sameness of ratios (in the primitive sense). But sameness of relative magnitude may exist where quantuplicitative expression is impossible; thus the diagonal of a larger square is the same compared with its side as the diagonal of a smaller square compared with *its* side. It is an easy transition to speak of sameness of ratio even in this case; that is, to use the term ratio in the sense of relative magnitude, that word having originally only a reference to the mode of expressing relative magnitude, in cases which allow of a particular mode of expression. The word *irrational* (ἄλογος) does not make any corresponding change but continues to have its primitive meaning, namely, incapable of quantuplicitative expression."

It remains to consider how we are to describe the *relative magnitude* of two incommensurables of the same kind. That they have a definite relation is certain. Suppose, for precision, that S is the side of a square, D its diagonal; then, if S is given, any alteration in D or any error in D would make the figure cease to be a square. At the same time, a person altogether ignorant of the relative magnitude of D and S might say that drawing two straight lines of length S so as to form a right angle and joining the ends by a straight line, the length of which would accordingly be D, does not help him to realise the relative magnitude, but that he would like to know how many diagonals make an exact number of sides. We should have to reply that no number of diagonals whatever makes an exact number of sides; but that he may mention any fraction of the side, a hundredth, a thousandth or a millionth, and that we will then express the diagonal with an error not so great as that fraction. We then tell him that 1,000,000 diagonals exceed

1,414,213 sides but fall short of 1,414,214 sides; consequently the diagonal lies between 1·414213 and 1·414214 times the side, and these differ only by one-millionth of the side, so that the error in the diagonal is less still. To enable him to continue the process further, we show him how to perform the arithmetical operation of approximating to the value of $\sqrt{2}$. This gives the means of carrying the approximation to any degree of accuracy that may be desired. In the power, then, of carrying approximations of this kind as far as we please lies that of expressing the ratio, so far as expression is possible, and of comparing the ratio with others as accurately as if expression had been possible.

Euclid was of course aware of this, as were probably others before him; though the actual approximations to the values of ratios of incommensurables of which we find record in the works of the great Greek geometers are very few. The history of such approximations up to Archimedes is, so far as material was available, sketched in *The Works of Archimedes* (pp. lxxvii and following); and it is sufficient here to note the facts (1) that Plato, and even the Pythagoreans, were familiar with $\frac{7}{5}$ as an approximation to $\sqrt{2}$, (2) that the method of finding any number of successive approximations by the system of *side-* and *diagonal-*numbers described by Theon of Smyrna was also Pythagorean (cf. the note above on Euclid, II. 9, 10), (3) that Archimedes, without a word of preliminary explanation, gives out that

$$\tfrac{1351}{780} > \sqrt{3} > \tfrac{265}{153},$$

gives approximate values for the square roots of several large numbers, and proves that the ratio of the circumference of a circle to its diameter is less than $3\frac{1}{7}$ but greater than $3\frac{10}{71}$, (4) that the first approach to the rapidity with which the decimal system enables us to approximate to the value of surds was furnished by the method of sexagesimal fractions, which was almost as convenient to work with as the method of decimals, and which appears fully developed in Ptolemy's σύνταξις. A number consisting of a whole number and any fraction was under this system represented as so many units, so many of the fractions which we should denote by $\frac{1}{60}$, so many of those which we should write $(\frac{1}{60})^2$, $(\frac{1}{60})^3$, and so on. Theon of Alexandria shows us how to extract the square root of 4500 in this sexagesimal system, and, to show how effective it was, it is only necessary to mention that Ptolemy gives $\dfrac{103}{60} + \dfrac{55}{60^2} + \dfrac{23}{60^3}$ as an approximation to $\sqrt{3}$, which approximation is equivalent to 1·7320509 in the ordinary decimal notation and is therefore correct to 6 places.

Between Def. 3 and Def. 4 two manuscripts and Campanus insert "Proportion is the sameness of ratios" (ἀναλογία δὲ ἡ τῶν λόγων ταὐτότης), and even the best MS. has it in the margin. It would be altogether out of place, since it is not till Def. 5 that it is explained what sameness of ratios is. The words are an interpolation later than Theon (Heiberg, Vol. v. pp. xxxv, lxxxix), and are no doubt taken from arithmetical works (cf. Nicomachus and Theon of Smyrna). It is true that Aristotle says similarly, "Proportion is equality of ratios" (*Eth. Nic.* v. 6, 1131 a 31), and he appears to be quoting from the Pythagoreans; but the reference is to *numbers*.

Similarly two MSS. (inferior) insert after Def. 7 "Proportion is the similarity (ὁμοιότης) of ratios." Here too we have a mere interpolation.

DEFINITION 4.

Λόγον ἔχειν πρὸς ἄλληλα μεγέθη λέγεται, ἃ δύναται πολλαπλασιαζόμενα ἀλλήλων ὑπερέχειν.

This definition supplements the last one. De Morgan says that it amounts to saying that the magnitudes are of the same species. But this can hardly be all; the definition seems rather to be meant, on the one hand, to exclude the relation of a finite magnitude to a magnitude of the same kind which is either infinitely great or infinitely small, and, even more, to emphasise the fact that the term *ratio*, as defined in the preceding definition, and about to be used throughout the book, includes the relation between any two *incommensurable* as well as between any two commensurable finite magnitudes of the same kind. Hence, while De Morgan seems to regard the extension of the meaning of *ratio* to include the relative magnitude of incommensurables as, so to speak, taking place between Def. 3 and Def. 5, the 4th definition appears to show that it is ratio in its extended sense that is being defined in Def. 3.

DEFINITION 5.

Ἐν τῷ αὐτῷ λόγῳ μεγέθη λέγεται εἶναι πρῶτον πρὸς δεύτερον καὶ τρίτον πρὸς τέταρτον, ὅταν τὰ τοῦ πρώτου καὶ τρίτου ἰσάκις πολλαπλάσια τῶν τοῦ δευτέρου καὶ τετάρτου ἰσάκις πολλαπλασίων καθ᾽ ὁποιονοῦν πολλαπλασιασμὸν ἑκάτερον ἑκατέρου ἢ ἅμα ὑπερέχῃ ἢ ἅμα ἴσα ἢ ἅμα ἐλλείπῃ ληφθέντα κατάλληλα.

In my translation of this definition I have compromised between an attempted literal translation and the more expanded version of Simson. The difficulty in the way of an exactly literal translation is due to the fact that the words (καθ᾽ ὁποιονοῦν πολλαπλασιασμὸν) signifying that the equimultiples *in each case* are any equimultiples *whatever* occur only once in the Greek, though they apply *both* to τὰ...ἰσάκις πολλαπλάσια in the nominative and τῶν...ἰσάκις πολλαπλασίων in the genitive. I have preferred "alike" to "simultaneously" as a translation of ἅμα because "simultaneously" might suggest that time was of the essence of the matter, whereas what is meant is that any particular comparison made between the equimultiples must be made between *the same* equimultiples of the two pairs respectively, not that they need to be compared *at the same time.*

Aristotle has an allusion to a definition of "the same ratio" in *Topics* VIII. 3, 158 b 29: "In mathematics too some things appear to be not easy to prove (γράφεσθαι) for want of a definition, e.g. that the parallel to the side which cuts a plane [a parallelogram] divides the straight line [the other side] and the area similarly. But, when the definition is expressed, the said property is immediately manifest; for the areas and the straight lines *have the same* ἀνταναίρεσις, *and this is the definition of 'the same ratio.'*" Upon this passage Alexander says similarly, "This is the definition of proportionals which the ancients used: *magnitudes are proportional to one another which have* (or show) *the same* ἀνθυφαίρεσις, and Aristotle has called the latter ἀνταναίρεσις." Heiberg (*Mathematisches zu Aristoteles*, p. 22) thinks that Aristotle is alluding to the fact that the proposition referred to could not be rigorously proved so long as the Pythagorean definition applicable to commensurable magnitudes only was adhered to, and is quoting the definition belonging to the complete theory of Eudoxus; whence, in view of the positive statement of Aristotle that the definition quoted *is* the definition of "the same ratio," it would appear that the Euclidean definition (which Heiberg describes as a careful and exact paraphrase of ἀνταναίρεσις) is Euclid's own. I do not feel able to subscribe to this view, which seems to me to involve very grave difficulties. The Euclidean definition is regularly appealed to in Book V. as the criterion of magnitudes being in proportion, and the use of it would appear to constitute the whole essence of the new general theory of proportion; if then this theory is due to Eudoxus, it seems impossible to believe that the definition was not also due to him. Certainly the definition given by Aristotle would be no substitute for it; ἀνθυφαίρεσις and ἀνταναίρεσις are words almost as vague and "metaphysical" (as Barrow would say) as the words used to define *ratio*, and it is difficult to see how any mathematical facts could be deduced from such a definition. Consider for a moment the etymology of the words. ὑφαίρεσις or ἀναίρεσις means "removal," "taking away" or "destruction" of a thing; and the prefix ἀντὶ indicates that the "taking away" from one magnitude *answers to*, corresponds with, alternates with, the "taking away" from the other. So far therefore as the etymology goes, the word seems rather to suggest the "taking away" of corresponding *fractions*, and therefore to suit the old imperfect theory of proportion rather than the new one. Thus Waitz (*ad loc.*) paraphrases the definition as meaning that "as many parts as are taken from one magnitude, so many are at the same time taken from the other as well." A possible explanation would seem to be that, though Eudoxus had formulated the new definition, the old one was still current in the text-books of Aristotle's time, and was taken by him as being a good enough illustration of what he wished to bring out in the passage of the *Topics* referred to.

From the revival of learning in Europe onwards the Euclidean definition of proportion was the subject of much criticism. Campanus had failed to understand it, had in fact misinterpreted it altogether, and he may have misled others such as Ramus (1515—72), always a violently hostile critic of Euclid. Among the objectors to it was no less a person than Galileo. For particulars of the controversies on the subject down to Thomas Simpson (*Elem. of Geometry*, Lond. 1800) the reader is referred to the Excursus at the end of the second volume of Camerer's Euclid (1825). For us it is interesting to note that the unsoundness of the usual criticisms of the definition was never better exposed than by Barrow. Some of the objections, he pointed out (*Lect. Cantabr.* VII. of 1666), are due to misconception on the part of their authors as to the nature of a definition. Thus Euclid is required by these objectors (e.g. Tacquet) to do the impossible and to show that what is predicated in the definition is true of the thing defined, as if any one should be required to show that the name "circle" was applicable to those figures alone which have their radii all equal! As we are entitled to assign to such figures and such figures only the name of "circle," so Euclid is entitled ("quamvis non temere nec imprudenter at certis de causis iustis illis et idoneis") to describe a certain property which four magnitudes may have, and to call magnitudes possessing that property magnitudes "in the same ratio." Others had argued from the occurrence of the other definition of proportion in VII. Def. 20 that Euclid was dissatisfied with the present one; Barrow pointed out that, on the contrary, it was the fact that VII. Def. 20 was not adequate to cover the case of incommensurables which made Euclid adopt the present definition here. Lastly, he maintains, against those who descant on the "obscurity" of V. Def. 5, that the supposed obscurity is due, partly no doubt to the inherent difficulty of the subject of incommensurables, but also to faulty translators, and most of all to lack of effort in the learner to grasp thoroughly the meaning of words which, in themselves, are as clearly expressed as they could be.

To come now to the merits of the case, the best defence and explanation of the definition that I have seen is that given by De Morgan. He first translates it, observes that it applies equally to commensurable or incommensurable quantities because no attempt is made to measure one by an aliquot part of another, and then proceeds thus.

"The two questions which must be asked, and satisfactorily answered, previously to its [the definition's] reception, are as follows:

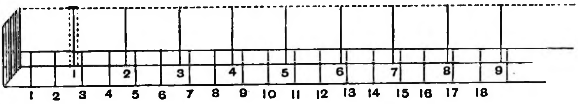
1. What right had Euclid, or any one else, to expect that the preceding most prolix and unwieldy statement should be received by the beginner as the definition of a relation the perception of which is one of the most common acts of his mind, since it is performed on every occasion where similarity or dissimilarity of figure is looked for or presents itself?

2. If the preceding question should be clearly answered, how can the definition of proportion ever be used; or how is it possible to compare every one of the infinite number of multiples of A with every one of the multiples of B?

To the first question we reply that not only is the test proposed by Euclid tolerably simple, when more closely examined, but that it is, or might be made to appear, an easy and natural consequence of those fundamental perceptions with which it may at first seem difficult to compare it."

To elucidate this De Morgan gives the following illustration.

Suppose there is a straight colonnade composed of equidistant columns (which we will understand to mean the vertical lines forming the axes of the columns respectively), the first of which is at a distance from a bounding wall equal to the distance between consecutive columns. In front of the colonnade let there be a straight row of equidistant railings (regarded as meaning their axes), the first being at a distance from the bounding wall equal to the distance between consecutive railings. Let the columns be numbered from the wall, and also the railings. We suppose of course that the column distance (say, C) and the railing distance (say, R) are different and that they may bear to each other any ratio, commensurable or incommensurable; i.e. that there need not go any exact number of railings to any exact number of columns.



If the construction be supposed carried on to any extent, a spectator can, by mere inspection, and without measurement, compare C with R to any degree of accuracy. For example, since the 10th railing falls between the 4th and 5th columns, $10R$ is greater than $4C$ and less than $5C$, and therefore R lies between $\frac{4}{10}$ths of C and $\frac{5}{10}$ths of C. To get a more accurate notion, the ten-thousandth railing may be taken; suppose it falls between the 4674th and 4675th columns. Therefore $10,000R$ lies between $4674C$ and $4675C$, or R lies between $\frac{4674}{10000}$ and $\frac{4675}{10000}$ of C. There is no limit to the degree of accuracy thus obtainable; and the ratio of R to C is determined when the order of distribution of the railings among the columns is assigned *ad infinitum*; or, in other words, when the position of *any* giver railing can be found, as to the numbers of the columns between which it lies. Any alteration, however small, in the place of the first railing must at last affect the order of distribution. Suppose e.g. that the first railing is moved from the wall by one part in a thousand of the distance between the columns; then the second railing is pushed forward by $\frac{2}{1000}C$, the third by $\frac{3}{1000}C$, and so on, so that the railings after the thousandth are pushed forward by more than C; i.e. the order with respect to the columns is disarranged.

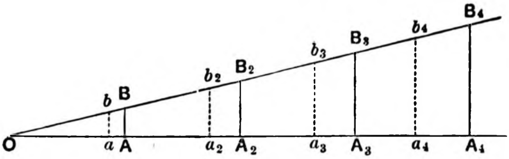
Now let it be proposed to make a model of the preceding construction in which c shall be the column distance and r the railing distance. It needs no definition of proportion, nor anything more than the conception which we have of that term prior to definition (and with which we must show the agreement of any definition that we may adopt), to assure us that C must be to R in the same proportion as c to r if the model be truly formed. Nor is it drawing too largely on that conception of proportion to assert that the distribution of the railings among the columns in the model must be everywhere the same as in the original; for example, that the model would be *out of proportion* if its 37th railing fell between the 18th and 19th columns, while the 37th railing of the original fell between the 17th and 18th columns. Thus the dependence of Euclid's definition upon common notions is settled; for the obvious relation between the construction and its model which has just been described contains the collection of conditions, the fulfilment of which, according to Euclid, constitutes proportion. According to Euclid, whenever mC exceeds, equals, or falls short of nR, then mc must exceed, equal, or fall short of nr; and, by the most obvious property of the constructions, according as the mth column comes after, opposite to, or before the nth railing in the original, the mth column must come after, opposite to, or before the nth railing in the correct model.

Thus the test proposed by Euclid is necessary. It is also sufficient. For admitting that, to a given original with a given column-distance in the model, there is one correct model railing distance (which must therefore be that which distributes the railings among the columns as in the original), we have seen that any other railing distance, however slightly different, would at last give a different distribution; that is, the correct distance, and the correct distance only, satisfies all the conditions required by Euclid's definition.

The use of the word *distribution* having been well learnt, says De Morgan, the following way of stating the definition will be found easier than that of Euclid. "Four magnitudes, A and B of one kind, and C and D of the same or another kind, are proportional when all the multiples of A can be distributed among the multiples of B in the same intervals as the corresponding multiples of C among those of D." Or, whatever numbers m, n may be, if mA lies between nB and $(n+1)B$, mC lies between nD and $(n+1)D$.

It is important to note that, if the test be always satisfied from and after any given multiples of A and C, it must be satisfied before those multiples. For instance, let the test be always satisfied from and after $100A$ and $100C$; and let $5A$ and $5C$ be instances for examination. Take any multiple of 5 which will exceed 100, say 50 times five; and let it be found on examination that $250A$ lies between $678B$ and $679B$; then $250C$ lies between $678D$ and $679D$. Divide by 50, and it follows that $5A$ lies between $13\frac{28}{50}B$ and $13\frac{29}{50}B$, and *a fortiori* between $13B$ and $14B$. Similarly, $5C$ lies between $13\frac{28}{50}D$ and $13\frac{29}{50}D$, and therefore between $13D$ and $14D$. Or $5A$ lies in the same interval among the multiples of B in which $5C$ lies among the multiples of D. And so for any multiple of A, C less than $100A$, $100C$.

There remains the second question relating to the infinite character of the definition; four magnitudes A, B, C, D are not to be called proportional until it is shown that *every* multiple of A falls in the same intervals among the multiples of B in which the same multiple of C is found among the multiples of D. Suppose that the distribution of the railings among the columns should be found to agree in the model and the original as far as the millionth railing. This proves only that the railing distance of the model does not err by the millionth part of the corresponding column distance. We can thus fix limits to the disproportion, if any, and we may make those limits as small as we please, by carrying on the method of *observation*; but we cannot *observe* an infinite number of cases and *so* enable ourselves to affirm proportion absolutely. Mathematical methods however enable us to avoid the difficulty. We can take *any multiples whatever* and work with them as if they were particular multiples. De Morgan gives, as an instance to show that the definition of proportion can in practice be used, notwithstanding its infinite character, the following proof of a proposition to the same effect as Eucl. VI. 2.



"Let OAB be a triangle to one side AB of which ab is drawn parallel, and on OA produced set off AA_2, A_2A_3 etc. equal to OA, and aa_2, a_2a_3 etc. equal to Oa.

Through every one of the points so obtained draw parallels to AB, meeting OB produced in b_2, B_2 etc.

Then it is easily proved that bb_2, b_2b_3 etc. are severally equal to Ob, and BB_2, B_2B_3 etc. to OB.

Consequently a distribution of the multiples of OA among the multiples of Oa is made on one line, and of OB among those of Ob on the other.

The examination of this distribution in all its extent (which is impossible, and hence the apparent difficulty of using the definition) is rendered unnecessary by the known property of parallel lines. For, since A_3 lies between a_3 and a_4, B_3 must lie between b_3 and b_4; for, if not, the line A_3B_3 would cut either a_3b_3 or a_4b_4.

Hence, without inquiring where A_m *does* fall, we know that, if it fall between a_n and a_{n+1}, B_m must fall between b_n and b_{n+1}; or, if $m \,.\, OA$ fall in magnitude between $n \,.\, Oa$ and $(n+1)Oa$, then $m \,.\, OB$ must fall between $n \,.\, Ob$ and $(n+1)Ob$."

Max Simon remarks (*Euclid und die sechs planimetrischen Bücher*, p. 110), after Zeuthen, that Euclid's definition of equal ratios is word for word the same as Weierstrass' definition of equal numbers. So far from agreeing in the usual view that the Greeks saw in the irrational no *number*, Simon thinks it is clear from Eucl. V. that they possessed a notion of number in all its generality as clearly defined as, nay almost identical with, Weierstrass' conception of it.

Certain it is that there is an exact correspondence, almost coincidence, between Euclid's definition of equal ratios and the modern theory of irrationals due to Dedekind. Premising the ordinal arrangement of natural numbers in ascending order, then enlarging the sphere of numbers by including (1) negative numbers as well as positive, (2) fractions, as a/b, where a, b may be any natural numbers, provided that b is not zero, and arranging the fractions ordinally among the other numbers according to the definition:

$$\text{let } \frac{a}{b} \text{ be } <=> \frac{c}{d} \text{ according as } ad \text{ is } <=> bc,$$

Dedekind arrives at the following definition of an irrational number.

An irrational number α is defined whenever a law is stated which will assign every given rational number to one and only one of two classes A and B such that (1) every number in A precedes every number in B, and (2) there is no last number in A and no first number in B; the definition of α being that it is the one number which lies between all numbers in A and all numbers in B.

Now let x/y and x'/y' be equal ratios in Euclid's sense.

Then $\frac{x}{y}$ will divide all rational numbers into two groups A and B;

$\frac{x'}{y'}$ " " " A' and B'.

Let $\frac{a}{b}$ be any rational number in A, so that

$$\frac{a}{b} < \frac{x}{y}.$$

This means that $ay < bx$.

But Euclid's definition asserts that in that case $ay' < bx'$ also.

Hence also $$\frac{a}{b} < \frac{x'}{y'};$$

therefore every member of group A is also a member of group A'.

Similarly every member of group B is a member of group B'.

For, if $\frac{a}{b}$ belong to B,

$$\frac{a}{b} > \frac{x}{y},$$

which means that $ay > bx$.

But in that case, by Euclid's definition, $ay' > bx'$;

therefore also $$\frac{a}{b} > \frac{x'}{y'}.$$

Thus, in other words, A and B are coextensive with A' and B' respectively;

therefore $\frac{x}{y} = \frac{x'}{y'}$, according to Dedekind, as well as according to Euclid.

If x/y, x'/y' happen to be *rational*,

then one of the groups, say A, includes x/y,

and one of the groups, say A', includes x'/y'.

In this case $\frac{a}{b}$ might *coincide* with $\frac{x}{y}$;

that is $$\frac{a}{b} = \frac{x}{y},$$

which means that $$ay = bx.$$

Therefore, by Euclid's definition, $ay' = bx'$;

so that $$\frac{a}{b} = \frac{x'}{y'}.$$

Thus the groups are again coextensive.

In a word, Euclid's definition divides all rational numbers into two *coextensive* classes, and therefore defines equal ratios in a manner exactly corresponding to Dedekind's theory.

Alternatives for Eucl. V. Def. 5.

Saccheri records in his *Euclides ob omni naevo vindicatus* that a distinguished geometer of his acquaintance proposed to substitute for Euclid's the following definition:

"A first magnitude has to a second the same ratio that a third has to a fourth when the first contains the aliquot parts of the second, *according to any number* [i.e. with any denominator] *whatever*, the same number of times as the number of times the third contains the same aliquot parts of the fourth"; on which Saccheri remarks that he sees no advantage in this definition, which presupposes the notion of *division*, over that of Euclid which uses *multiplication* and the notions of *greater*, *equal*, and *less*.

This definition was, however, practically adopted by Faifofer (*Elementi di geometria*, 3 ed., 1882) in the following form:

"Four magnitudes taken in a certain order form a proportion when, by measuring the first and the third respectively by any equi-submultiples whatever of the second and of the fourth, equal quotients are obtained."

Ingrami (*Elementi di geometria*, 1904) takes multiples of the first and third instead of submultiples of the second and fourth:

"Given four magnitudes in predetermined order, the first two homogeneous with one another, and likewise also the last two, the magnitudes are said to form a proportion (or to be in proportion) when *any multiple* of the first contains the second the same number of times that the equimultiple of the third contains the fourth."

Veronese's definition (*Elementi di geometria*, Pt. II., 1905) is like that of Faifofer; Enriques and Amaldi (*Elementi di geometria*, 1905) adhere to Euclid's.

Proportionals of VII. Def. 20 a particular case.

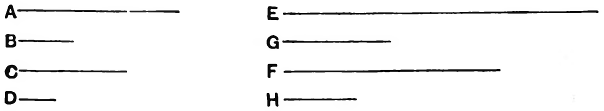
It has already been observed that Euclid has nowhere proved (though the fact cannot have escaped him) that the proportion of numbers is included in the proportion of magnitudes as a special case. This is proved by Simson as being necessary to the 5th and 6th propositions of Book X. Simson's proof is contained in his propositions C and D inserted in the text of Book V. and in the notes thereon. Proposition C and the note on it prove that, *if four magnitudes are proportionals according to* VII. *Def.* 20, *they are also proportionals according to* V. *Def.* 5. Prop. D and the note prove the partial converse, namely that, if four magnitudes are proportionals according to the 5th definition of Book V., *and if the first be any multiple, or any part, or parts, of the second*, the third is the same multiple, part, or parts, of the fourth. The proofs use certain results obtained in Book V.

Prop. C is as follows:

If the first be the same multiple of the second, or the same part of it, that the third is of the fourth, the first is to the second as the third to the fourth.

Let the first A be the same *multiple* of B the second that C the third is of the fourth D;

A is to B as C is to D.



Take of A, C any equimultiples whatever E, F; and of B, D any equimultiples whatever G, H.

Then, because A is the same multiple of B that C is of D,

and E is the same multiple of A that F is of C,

E is the same multiple of B that F is of D. [v. 3]

Therefore E, F are the same multiples of B, D.

But G, H are equimultiples of B, D;

therefore, if E be a greater multiple of B than G is, F is a greater multiple of D than H is of D;

that is, if E be greater than G, F is greater than H.

In like manner,

if E be equal to G, or less, F is equal to H, or less than it.

But E, F are equimultiples, any whatever, of A, C;

and G, H any equimultiples whatever of B, D.

Therefore A is to B as C is to D. [v. Def. 5]

Next, let the first A be the same *part* of the second B that the third C is of the fourth D:

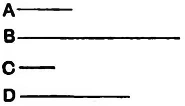
A is to B as C is to D.

For B is the same multiple of A that D is of C;

wherefore, by the preceding case,

B is to A as D is to C;

and, *inversely*, A is to B as C is to D.



[For this last inference Simson refers to his Proposition B. That proposition is very simply proved by taking any equimultiples E, F of B, D and any equimultiples G, H of A, C and then arguing as follows:

Since A is to B as C is to D,

G, H are *simultaneously* greater than, equal to, or less than E, F respectively; so that

E, F are *simultaneously* less than, equal to, or greater than G, H respectively,

and therefore [Def. 5] B is to A as D is to C.]

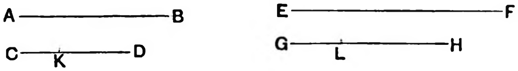
We have now only to add to Prop. C the case where AB contains the *same parts* of CD that EF does of GH:

in this case likewise AB is to CD as EF to GH.

Let CK be a part of CD, and GL the same part of GH; let AB be the same multiple of CK that EF is of GL.

Therefore, by Prop. C,

AB is to CK as EF to GL.



And CD, GH are equimultiples of CK, GL, the second and fourth.

Therefore AB is to CD as EF to GH [Simson's Cor. to v. 4, which however is the particular case of v. 4 in which the "equimultiples" of one pair are the pair itself, i.e. the pair multiplied by unity].

To prove the partial converse we begin with Prop. D.

If the first be to the second as the third to the fourth, and if the first be a multiple or part of the second, the third is the same multiple or the same part of the fourth.

Let A be to B as C is to D;

and, first, let A be a multiple of B;

C is the same multiple of D.

Take E equal to A, and whatever multiple A or E is of B, make F the same multiple of D.

Then, because A is to B as C is to D,

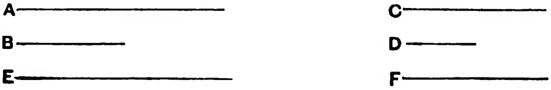
and of B the second and D the fourth equimultiples have been taken E and F,

A is to E as C is to F. [v. 4, Cor.]

But A is equal to E;

therefore C is equal to F.

[In support of this inference Simson cites his Prop. A, which however we can directly deduce from v. Def. 5 by taking any, but *the same*, equimultiples of all four magnitudes.]



Now F is the same multiple of D that A is of B;

therefore C is the same multiple of D that A is of B.

Next, let the first A be a part of the second B;

C the third is the same part of the fourth D.

Because A is to B as C is to D,

inversely, B is to A as D is to C. [Prop. B]

But A is a part of B; therefore B is a multiple of A;

and, by the preceding case, D is the same multiple of C,

that is, C is the same part of D that A is of B.

We have, again, only to add to Prop D the case where AB contains any parts of CD, and AB is to CD as EF to GH;

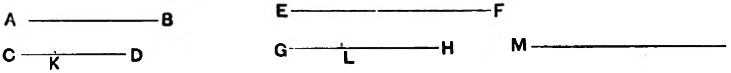
then shall EF contain the same parts of GH that AB does of CD.

For let CK be a part of CD, and GL the same part of GH; and let AB be a multiple of CK.

EF shall be the same multiple of GL.

Take M the same multiple of GL that AB is of CK;

therefore AB is to CK as M is to GL. [Prop. C]



And CD, GH are equimultiples of CK, GL;

therefore AB is to CD as M is to GH.

But, by hypothesis, AB is to CD as EF is to GH;

therefore M is equal to EF, [v. 9]

and consequently EF is the same multiple of GL that AB is of CK.

DEFINITION 6.

Τὰ δὲ τὸν αὐτὸν ἔχοντα λόγον μεγέθη ἀνάλογον καλείσθω.

Ἀνάλογον, though usually written in one word, is equivalent to ἀνὰ λόγον, *in proportion*. It comes however in Greek mathematics to be used practically as an indeclinable adjective, as here; cf. αἱ τέσσαρες εὐθεῖαι ἀνάλογον ἔσονται, "the four straight lines will be proportional," τρίγωνα τὰς πλευρὰς ἀνάλογον ἔχοντα, "triangles having their sides proportional." Sometimes it is used adverbially: ἀνάλογον ἄρα ἐστὶν ὡς ἡ ΒΑ πρὸς τὴν ΑΓ, οὕτως ἡ ΗΔ πρὸς τὴν ΔΖ, "proportionally therefore, as BA is to AC, so is GD to DF"; so too, apparently, in the expression ἡ μέση ἀνάλογον (εὐθεῖα), "the mean proportional." I do not follow the objection of Max Simon (Euclid, p. 110) to "proportional" as a translation of ἀνάλογον. "We ask," he says, "in vain, what is proportional to what? We say e.g. that weight is proportional to price because double, treble etc. weight corresponds to double, treble etc. price. But here the meaning must be 'standing in a relation of proportion.'" Yet he admits that the Latin word *proportionalis* is an adequate expression. He translates by "in proportion" in the text of this definition. But I do not see that "in proportion" is better than "proportional." The fact is that both expressions are elliptical when used of four magnitudes "in proportion"; but there is surely no harm in using either when the meaning is so well understood.

The use of the word καλείσθω, "*let* magnitudes having the same ratio *be called* proportional," seems to indicate that this definition is Euclid's own.

DEFINITION 7.

Ὅταν δὲ τῶν ἰσάκις πολλαπλασίων τὸ μὲν τοῦ πρώτου πολλαπλάσιον ὑπερέχῃ τοῦ τοῦ δευτέρου πολλαπλασίου, τὸ δὲ τοῦ τρίτου πολλαπλάσιον μὴ ὑπερέχῃ τοῦ τοῦ τετάρτου πολλαπλασίου, τότε τὸ πρῶτον πρὸς τὸ δεύτερον μείζονα λόγον ἔχειν λέγεται, ἤπερ τὸ τρίτον πρὸς τὸ τέταρτον.

As De Morgan observes, the practical test of *disproportion* is simpler than that of proportion. For, whereas no examination of individual cases, however extensive, will enable an observer of the construction and its model (the illustration by means of columns and railings described above) to affirm proportion or deny disproportion, and all it enables us to do is to fix limits (as small as we please) to the disproportion (if any), a single instance may enable us to deny proportion or affirm disproportion, and also to state which way the disproportion lies. Let the 19th railing in the original fall beyond the 11th column, while the 19th railing of the (so-called) model does not come up to the 11th column. It follows from this one instance that the railing distance of the model is too small relatively to the column distance, or that the column distance is too great relatively to the railing distance. That is, the ratio of r to c is less than that of R to C, or the ratio of c to r is greater than that of C to R.

Saccheri (*op. cit.*) remarks (as Commandinus had done) that the ratio of the first magnitude to the second will also be greater than that of the third to the fourth if, while the multiple of the first is *equal* to the multiple of the second, the multiple of the third is *less* than that of the fourth: a case not mentioned in Euclid's definition. Saccheri speaks of this case being included in Clavius' interpretation of the definition. I have, however, failed to find a reference to the case in Clavius, though he adds, as a sort of corollary, in his note on the definition, that if, on the other hand, the multiple of the first is *less* than the multiple of the second, while the multiple of the third is *not less* than that of the fourth, the ratio of the first to the second is *less* than that of the third to the fourth.

Euclid presumably left out the second possible criterion for a greater ratio, and the definition of a less ratio, because he was anxious to reduce the definitions to the minimum necessary for his purpose, and to leave the rest to be inferred as soon as the development of the propositions of Book v. enabled this to be done without difficulty.

Saccheri tried to reduce the second possible criterion for a greater ratio to that given by Euclid in his definition without recourse to anything coming later in the Book, but, in order to do this, he has to use "multiples" produced by multipliers which are not integral numbers, but integral numbers *plus* proper fractions, so that Euclid's Def. 7 becomes inapplicable.

De Morgan notes that "proof should be given that the same pair of magnitudes can never offer both tests [i.e. the test in the definition for a greater ratio and the corresponding test for a less ratio, with "less" substituted for "greater" in the definition] to another pair; that is, the test of greater ratio from one set of multiples, and that of less ratio from another." In other words, if m, n, p, q are integers and A, B, C, D four magnitudes, none of the pairs of equations

$$(1)\quad mA > nB,\quad mC = \text{or} < nD,$$
$$(2)\quad mA = nB,\quad mC < nD$$

can be satisfied simultaneously with any one of the pairs of equations

$$(3)\quad pA = qB,\quad pC > qD,$$
$$(4)\quad pA < qB,\quad pC > \text{or} = qD.$$

There is no difficulty in proving this with the help of two simple assumptions which are indeed obvious.

We need only take in illustration one of the numerous cases. Suppose, if possible, that the following pairs of equations are simultaneously true:

$$(1)\quad mA > nB,\quad mC < nD$$
and
$$(2)\quad pA < qB,\quad pC > qD.$$

Multiply (1) by q and (2) by n.

(We need here to assume that, where rX, rY are any equimultiples of any magnitudes X, Y,

$$\text{according as } X >=< Y, \quad rX >=< rY.$$

This is contained in Simson's Axioms 1 and 3.)

We have then the pairs of equations

$$mqA > nqB, \quad mqC < nqD,$$
$$npA < nqB, \quad npC > nqD.$$

From the second equations in each pair it follows that

$$mqC < npC.$$

(We now need to assume that, if rX, sX are any multiples of X, and rY, sY the same multiples of Y, then,

$$\text{according as } rX >=< sX, \quad rY >=< sY.$$

Simson uses this same assumption in his proof of v. 18.)

Therefore $$mqA < npA.$$

But it follows from the first equations in each pair that

$$mqA > npA:$$

which is impossible.

Nor can Euclid's criterion for a greater ratio coexist with that for equal ratios.

DEFINITION 8.

Ἀναλογία δὲ ἐν τρισὶν ὅροις ἐλαχίστη ἐστίν.

This is the reading of Heiberg and Camerer (who follow Peyrard's MS.) and is that translated above. The other reading has ἐλαχίστοις, which can only be translated "consists in three terms *at least*." Hankel regards the definition as a later interpolation, because it is superfluous, and because the word ὅρος for a *term* in a proportion is nowhere else used by Euclid, though it is common in later writers such as Nicomachus and Theon of Smyrna. The genuineness of the definition is however supported by the fact that Aristotle not only uses ὅρος in this sense (*Eth. Nic.* v. 6, 7, 1131 b 5, 9), but has a similar remark (*ibid.* 1131 a 31) that a "proportion is in *four* terms at least." The difference from Euclid is only formal; for Aristotle proceeds: "The *discrete* (διῃρημένη) (proportion) is clearly in four (terms), but so also is the *continuous* (συνεχής). For it uses one as two and mentions it twice, e.g. (in stating) that, as α is to β, so also is β to γ; thus β is mentioned twice, so that, if β be twice put down, the proportionals are four." The distinction between *discrete* and *continuous* seems to have been Pythagorean (cf. Nicomachus, II. 21, 5; 23, 2, 3; where however συνημμένη is used instead of συνεχής); Euclid does not use the words διῃρημένη and συνεχής in this connexion.

So far as they go, the first words of the next definition (9), "When three magnitudes are proportionals," which seemingly refer to Def. 8, also support the view that the latter is, at least in substance, genuine.

DEFINITIONS 9, 10.

9. Ὅταν δὲ τρία μεγέθη ἀνάλογον ᾖ, τὸ πρῶτον πρὸς τὸ τρίτον διπλασίονα λόγον ἔχειν λέγεται ἤπερ πρὸς τὸ δεύτερον.

10. Ὅταν δὲ τέσσαρα μεγέθη ἀνάλογον ᾖ, τὸ πρῶτον πρὸς τὸ τέταρτον τριπλασίονα λόγον ἔχειν λέγεται ἤπερ πρὸς τὸ δεύτερον, καὶ ἀεὶ ἑξῆς ὁμοίως, ὡς ἂν ἡ ἀναλογία ὑπάρχῃ.

Here, and in connexion with the definitions of duplicate, triplicate, etc. ratios, would be the place to expect a definition of "*compound* ratio." None such is however forthcoming, and the only "definition" of it that we find is that forming VI. Def. 5, which is an interpolation made, perhaps, even before Theon's time. According to the interpolated definition, "A ratio is said to be compounded of ratios when the sizes (πηλικότητες) of the ratios multiplied together make some (? ratio)." But the multiplication of the *sizes* (or magnitudes) of two ratios of incommensurable, and even of commensurable, magnitudes is an operation unknown to the classical Greek geometers. Eutocius (Archimedes, ed. Heiberg, III. p. 120) is driven to explain the definition by making πηλικότης mean the *number* from which the given ratio is called, or, in other words, the number which multiplied into the consequent of the ratio gives the antecedent. But he is only able to work out his idea with reference to ratios between numbers, or between *commensurable* magnitudes; and indeed the definition is quite out of place in Euclid's theory of proportion.

There is then only one statement in Euclid's text as we have it indicating what is meant by compound ratio; this is in VI. 23, where he says abruptly "But the ratio of *K* to *M* is compounded of the ratio of *K* to *L* and that of *L* to *M*." Simson accordingly gives a definition (A of Book V.) of compound ratio directly suggested by the statement in VI. 23 just quoted.

"When there are any number of magnitudes of the same kind, the first is said to have to the last of them the ratio compounded of the ratio which the first has to the second, and of the ratio which the second has to the third, and of the ratio which the third has to the fourth, and so on unto the last magnitude.

For example, if *A*, *B*, *C*, *D* be four magnitudes of the same kind, the first *A* is said to have to the last *D* the ratio compounded of the ratio of *A* to *B*, and of the ratio of *B* to *C*, and of the ratio of *C* to *D*; or the ratio of *A* to *D* is said to be compounded of the ratios of *A* to *B*, *B* to *C*, and *C* to *D*.

And if *A* has to *B* the same ratio which *E* has to *F*; and *B* to *C* the same ratio that *G* has to *H*; and *C* to *D* the same that *K* has to *L*; then, by this definition, *A* is said to have to *D* the ratio compounded of ratios which are the same with the ratios of *E* to *F*, *G* to *H*, and *K* to *L*: and the same thing is to be understood when it is more briefly expressed, by saying, *A* has to *D* the ratio compounded of the ratios of *E* to *F*, *G* to *H*, and *K* to *L*.

In like manner, the same things being supposed, if *M* has to *N* the same ratio which *A* has to *D*; then, for shortness' sake, *M* is said to have to *N* the ratio compounded of the ratios of *E* to *F*, *G* to *H*, and *K* to *L*."

De Morgan has some admirable remarks on compound ratio, which not only give a very clear view of what is meant by it but at the same time supply a plausible explanation of the *origin* of the term. "Treat ratio," says De Morgan, "as an engine of operation. Let that of *A* to *B* suggest the power of altering any magnitude in that ratio." (It is true that it is not yet proved that, *B* being any magnitude, and *P* and *Q* two magnitudes of the same kind, there does exist a magnitude *A* which is to *B* in the same ratio as *P* to *Q*. It is not till VI. 12 that this is proved, by construction, in the particular case where the three magnitudes are straight lines. The proof in the Greek text of V. 18 which assumes the truth of the more general proposition is, by reason of that assumption, open to objection; see the note on that proposition.) Now "every alteration of a magnitude is alteration in some ratio, two or more successive alterations are jointly equivalent to but one, and the ratio of the initial magnitude to the terminal one is as properly said to be the *compound* ratio of alteration as 13 to be the compound addend in lieu of 8 and 5, or 28 the compound multiple for 7 and 4. Composition is used here, as elsewhere, for the process of detecting one single alteration which produces the joint effect of two or more. The composition of the ratios of *P* to *R*, *R* to *S*, *T* to *U*, is performed by assuming *A*, altering it in the first ratio into *B*, altering *B* in the second ratio into *C*, and *C* in the third ratio into *D*. The joint effect turns *A* into *D*, and the ratio of *A* to *D* is the compounded ratio."

Another word for *compounded* ratio is συνημμένος (συνάπτω) which is common in Archimedes and later writers.

It is clear that duplicate ratio, triplicate ratio etc. defined in V. Deff. 9 and 10 are merely particular cases of compound ratio, being in fact the ratios compounded of two, three etc. *equal* ratios. The use which the Greek geometers made of compounded, duplicate, triplicate ratios etc. is well illustrated by the discovery of Hippocrates that the problem of the duplication of the cube (or, more generally, the construction of a cube which shall be to a given cube in any given ratio) reduces to that of finding "two mean proportionals in continued proportion." This amounted to seeing that, if x, y are two mean proportionals in continued proportion between any two lines a, b, in other words, if a is to x as x to y, and x is to y as y to b, then a cube with side a is to a cube with side x as a is to b; and this is equivalent to saying that a has to b the triplicate ratio of a to x.

Euclid is careful to use the forms διπλασίων, τριπλασίων, etc. to express what we translate as *duplicate*, *triplicate* etc. ratios; the Greek mathematicians, however, commonly used διπλάσιος λόγος, "double ratio," τριπλάσιος λόγος, "triple ratio" etc. in the sense of the ratios of 2 to 1, 3 to 1 etc. The effort, if such it was, to keep the one form for the one signification and the other for the other was only partially successful, as there are several instances of the contrary use, e.g. in Archimedes, Nicomachus and Pappus.

The expression for having the ratio which is "duplicate (triplicate) of that which it has to the second" is curious—διπλασίονα (τριπλασίονα) λόγον ἔχειν ἤπερ πρὸς τὸ δεύτερον—ἤπερ being used as if διπλασίονα or τριπλασίονα were a sort of comparative, in the same way as it is used after μείζονα or ἐλάσσονα. Another way of expressing the same thing is to say λόγος διπλασίων (τριπλασίων) τοῦ, ὃν ἔχει... the ratio "duplicate of that (ratio) which..." The explanation of both constructions would seem to be that διπλάσιος or διπλασίων is, as Hultsch translates it in his edition of Pappus (cf. p. 59, 17), *duplo maior*, where the ablative *duplo* implies not a difference but a proportion.

The four magnitudes in Def. 10 must of course be in *continued* proportion (κατὰ τὸ συνεχές). The Greek text as it stands does not state this.

DEFINITION 11.

Ὁμόλογα μεγέθη λέγεται τὰ μὲν ἡγούμενα τοῖς ἡγουμένοις τὰ δὲ ἑπόμενα τοῖς ἑπομένοις.

It is difficult to express the meaning of the Greek in as few words. A translation more literal, but conveying less, would be, "Antecedents are called *corresponding magnitudes* to antecedents, and consequents to consequents."

I have preferred to translate ὁμόλογος by "corresponding" rather than by "homologous." I do not agree with Max Simon when he says (Euclid, p. 111) that the technical term "homologous" is not the adjective ὁμόλογος, and does not mean "corresponding," "agreeing," but "like in respect of the proportion" ("ähnlich in Bezug auf das Verhältniss"). The definition seems to me to be for the purpose of appropriating to a technical use precisely the ordinary adjective ὁμόλογος, "agreeing" or "corresponding."

Antecedents, ἡγούμενα, are literally "*leading* (terms)," and *consequents*, ἑπόμενα, "*following* (terms)."

DEFINITION 12.

Ἐναλλὰξ λόγος ἐστὶ λῆψις τοῦ ἡγουμένου πρὸς τὸ ἡγούμενον καὶ τοῦ ἑπομένου πρὸς τὸ ἑπόμενον.

We now come to a number of expressions for the transformation of ratios or proportions. The first is ἐναλλάξ, *alternately*, which would be better described with reference to a proportion of four terms than with reference to a ratio. But probably Euclid defined all the terms in Deff. 12—16 with reference to *ratios* because to define them with reference to proportions would look like assuming what ought to be proved, namely the legitimacy of the various transformations of proportions (cf. v. 16, 7 Por., 18, 17, 19 Por.). The word ἐναλλάξ is of course a common term which has no exclusive reference to mathematics. But this same use of it with reference to proportions already occurs in Aristotle: *Anal. post.* I. 5, 74 a 18, καὶ τὸ ἀνάλογον ὅτι ἐναλλάξ, "and that a proportion (is true) alternately, or *alternando*." Used with λόγος, as here, the adverb ἐναλλάξ has the sense of an adjective, "alternate"; we have already had it similarly used of "alternate angles" (αἱ ἐναλλὰξ γωνίαι) in the theory of parallels.

DEFINITION 13.

Ἀνάπαλιν λόγος ἐστὶ λῆψις τοῦ ἑπομένου ὡς ἡγουμένου πρὸς τὸ ἡγούμενον ὡς ἑπόμενον.

Ἀνάπαλιν, "inversely," "the other way about," is also a general term with no exclusive reference to mathematics. For this use of it with reference to proportion cf. Aristotle, *De Caelo* I. 6, 273 b 32 τὴν ἀναλογίαν ἣν τὰ βάρη ἔχει, οἱ χρόνοι ἀνάπαλιν ἕξουσιν, "the proportion which the weights have, the times will have *inversely*." As here used with λόγος, ἀνάπαλιν is, exceptionally, adjectival.

DEFINITION 14.

Σύνθεσις λόγου ἐστὶ λῆψις τοῦ ἡγουμένου μετὰ τοῦ ἑπομένου ὡς ἑνὸς πρὸς αὐτὸ τὸ ἑπόμενον.

The *composition* of a ratio is to be distinguished from the *compounding* of ratios and *compounded ratio* (συγκείμενος λόγος) as explained above in the note on Deff. 9, 10. The fact is that συντίθημι and what serves for the passive of it (σύγκειμαι) are used for *adding* as well as *compounding* in the sense of compounding ratios. In order to distinguish the two senses, I have always used the word *componendo* where the sense is that of this definition, though this requires a slight departure from the literal rendering of some passages. Thus the enunciation of V. 17 says, literally, "if magnitudes compounded be in proportion they will also be in proportion separated" (ἐὰν συγκείμενα μεγέθη ἀνάλογον ᾖ, καὶ διαιρεθέντα ἀνάλογον ἔσται). This practically means that, if $A + B$ is to B as $C + D$ is to D, then A is to B as C is to D. I have accordingly translated as follows: "if magnitudes be proportional *componendo*, they will also be proportional *separando*." (It will be observed that *separando*, a term explained in the next note, is here used, not relatively to the proportion A is to B as C is to D, but relatively to the proportion *componendo*, viz. $A + B$ is to B as $C + D$ is to D.) The corresponding term for *componendo* in the Greek mathematicians is συνθέντι, literally "to one who has compounded," i.e. "if we compound." (For this absolute use of the dative of the participle cf. Nicomachus I. 8, 9 ἀπὸ μονάδος...κατὰ τὸν διπλάσιον λόγον προχωροῦντι μέχρις ἀπείρου, ὅσοι καὶ ἂν γένωνται, οὗτοι πάντες ἀρτιάκις ἄρτιοί εἰσιν. A very good instance from Aristotle is *Eth. Nic.* I. 5, 1097 b 12 ἐπεκτείνοντι γὰρ ἐπὶ τοὺς γονεῖς καὶ τοὺς ἀπογόνους καὶ τῶν φίλων τοὺς φίλους εἰς ἄπειρον πρόεισιν.) A variation for συνθέντι found in Archimedes is κατὰ σύνθεσιν. Perhaps the more exclusive use of the form συνθέντι by geometers later than Euclid to denote the *composition of a ratio*, as compared with Euclid's more general use of σύνθεσις and other parts of the verb συντίθημι or σύγκειμαι, may point to a desire to get rid of ambiguity of terms and to make the terminology of geometry more exact.

DEFINITION 15.

Διαίρεσις λόγου ἐστὶ λῆψις τῆς ὑπεροχῆς, ᾗ ὑπερέχει τὸ ἡγούμενον τοῦ ἑπομένου, πρὸς αὐτὸ τὸ ἑπόμενον.

As *composition of a ratio* means the transformation, e.g., of the ratio of A to B into the ratio of $A+B$ to B, so the *separation of a ratio* indicates the transformation of it into the ratio of $A-B$ to B. Thus, as the new antecedent is in one case got by *adding* the original antecedent to the original consequent, so the antecedent in the other case is obtained by *subtracting* the original consequent from the original antecedent (it being assumed that the latter is greater than the former). Hence the literal translations of διαίρεσις λόγου, "division of a ratio," and of διελόντι (the corresponding term to συνθέντι) as *dividendo*, scarcely give a sufficiently obvious explanation of the meaning. Heiberg accordingly translates by "subtractio rationis," which again may be thought to depart too far from the Greek. Perhaps "separation" and *separando* may serve as a compromise.

DEFINITION 16.

Ἀναστροφὴ λόγου ἐστὶ λῆψις τοῦ ἡγουμένου πρὸς τὴν ὑπεροχήν, ᾗ ὑπερέχει τὸ ἡγούμενον τοῦ ἑπομένου.

Conversion of a ratio means taking, e.g., instead of the ratio of A to B, the ratio of A to $A-B$ (A being again supposed greater than B). As ἀναστροφή is used for *conversion*, so ἀναστρέψαντι is used for *convertendo* (corresponding to the terms συνθέντι and διελόντι).

DEFINITION 17.

Δι' ἴσου λόγος ἐστὶ πλειόνων ὄντων μεγεθῶν καὶ ἄλλων αὐτοῖς ἴσων τὸ πλῆθος σύνδυο λαμβανομένων καὶ ἐν τῷ αὐτῷ λόγῳ, ὅταν ᾖ ὡς ἐν τοῖς πρώτοις μεγέθεσι τὸ πρῶτον πρὸς τὸ ἔσχατον, οὕτως ἐν τοῖς δευτέροις μεγέθεσι τὸ πρῶτον πρὸς τὸ ἔσχατον· ἢ ἄλλως· Λῆψις τῶν ἄκρων καθ' ὑπεξαίρεσιν τῶν μέσων.

δι' ἴσου, *ex aequali*, must apparently mean *ex aequali distantia*, at an equal distance or interval, i.e. after an equal number of intervening terms. The wording of the definition suggests that it is rather a *proportion ex aequali* than a *ratio ex aequali* which is being defined (cf. Def. 12). The meaning is clear enough. If a, b, c, d... be one set of magnitudes, and A, B, C, D... another set of magnitudes, such that

a is to b as A is to B,
b is to c as B is to C,

and so on, the last proportion being, e.g.,

k is to l, as K is to L,

then the inference *ex aequali* is that

a is to l as A is to L.

The *fact* that this is so, or the *truth* of the inference from the hypothesis, is not proved until V. 22. The definition is therefore merely verbal; it gives a convenient *name* to a certain inference which is of constant application in mathematics. But *ex aequali* could not be intelligibly defined except with reference to two sets of ratios respectively equal.

DEFINITION 18.

Τεταραγμένη δὲ ἀναλογία ἐστίν, ὅταν τριῶν ὄντων μεγεθῶν καὶ ἄλλων αὐτοῖς ἴσων τὸ πλῆθος γίνηται ὡς μὲν ἐν τοῖς πρώτοις μεγέθεσιν ἡγούμενον πρὸς ἑπόμενον, οὕτως ἐν τοῖς δευτέροις μεγέθεσιν ἡγούμενον πρὸς ἑπόμενον, ὡς δὲ ἐν τοῖς πρώτοις μεγέθεσιν ἑπόμενον πρὸς ἄλλο τι, οὕτως ἐν τοῖς δευτέροις ἄλλο τι πρὸς ἡγούμενον.

Though the words δι' ἴσου, *ex aequali*, are not in this definition, it gives a description of a case in which the inference *ex aequali* is still true, as will be hereafter proved in V. 23. A *perturbed proportion* is an expression for the case when, there being three magnitudes a, b, c and three others A, B, C,

a is to b as B is to C,
and b is to c as A is to B.

Another description of this case is found in Archimedes, "*the ratios being dissimilarly ordered*" (ἀνομοίως τεταγμένων τῶν λόγων). The full description of the *inference* in this case (as proved in V. 23), namely that

a is to c as A is to C,

is *ex aequali in perturbed proportion* (δι' ἴσου ἐν τεταραγμένῃ ἀναλογίᾳ). Archimedes sometimes omits the δι' ἴσου, first giving the two proportions and proceeding thus: "therefore, the proportions being dissimilarly ordered, a has to c the same ratio as A has to C."

The fact that Def. 18 describes a particular case in which the inference δι' ἴσου will be proved true seems to have suggested to some one after Theon's time the interpolation of another definition between 17 and 18 to describe the *ordinary* case where the argument *ex aequali* holds good. The interpolated definition runs thus: "an *ordered proportion* (τεταγμένη ἀναλογία) arises when, as antecedent is to consequent, so is antecedent to consequent, and, as consequent is to something else, so is consequent to something else." This case needed no description after Def. 17 itself; and the supposed definition is never used.

After the definitions of Book v. Simson supplies the following *axioms*.

1. Equimultiples of the same or of equal magnitudes are equal to one another.

2. Those magnitudes of which the same or equal magnitudes are equimultiples are equal to one another.

3. A multiple of a greater magnitude is greater than the same multiple of a less.

4. That magnitude of which a multiple is greater than the same multiple of another is greater than that other magnitude.

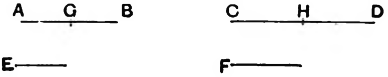
BOOK V. PROPOSITIONS.

PROPOSITION 1.

If there be any number of magnitudes whatever which are, respectively, equimultiples of any magnitudes equal in multitude, then, whatever multiple one of the magnitudes is of one, that multiple also will all be of all.

Let any number of magnitudes whatever *AB*, *CD* be respectively equimultiples of any magnitudes *E*, *F* equal in multitude;

I say that, whatever multiple *AB* is of *E*, that multiple will *AB*, *CD* also be of *E*, *F*.



For, since *AB* is the same multiple of *E* that *CD* is of *F*, as many magnitudes as there are in *AB* equal to *E*, so many also are there in *CD* equal to *F*.

Let *AB* be divided into the magnitudes *AG*, *GB* equal to *E*,

and *CD* into *CH*, *HD* equal to *F*;

then the multitude of the magnitudes *AG*, *GB* will be equal to the multitude of the magnitudes *CH*, *HD*.

Now, since *AG* is equal to *E*, and *CH* to *F*,

therefore *AG* is equal to *E*, and *AG*, *CH* to *E*, *F*.

For the same reason

GB is equal to *E*, and *GB*, *HD* to *E*, *F*;

therefore, as many magnitudes as there are in *AB* equal to *E*, so many also are there in *AB*, *CD* equal to *E*, *F*;

therefore, whatever multiple *AB* is of *E*, that multiple will *AB*, *CD* also be of *E*, *F*.

Therefore etc.

Q. E. D.

De Morgan remarks of V. 1—6 that they are "simple propositions of concrete arithmetic, covered in language which makes them unintelligible to modern ears. The first, for instance, states no more than that *ten* acres and *ten* roods make *ten* times as much as one acre and one rood." One aim therefore of notes on these as well as the other propositions of Book V. should be to make their purport clearer to the learner by setting them side by side with the same truths expressed in the much shorter and more familiar modern (algebraical) notation. In doing so, we shall express magnitudes by the first letters of the alphabet, a, b, c etc., adopting small instead of capital letters so as to avoid confusion with Euclid's lettering; and we shall use the small letters m, n, p etc. to represent integral numbers. Thus ma will always mean m times a or the m^{th} multiple of a (counting $1\,.\,a$ as the first, $2\,.\,a$ as the second multiple, and so on).

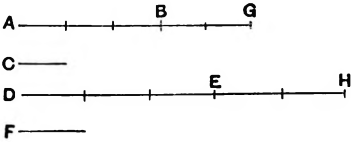
Prop. 1 then asserts that, if ma, mb, mc etc. be any equimultiples of a, b, c etc., then

$$ma + mb + mc + \ldots = m\,(a + b + c + \ldots).$$

PROPOSITION 2.

If a first magnitude be the same multiple of a second that a third is of a fourth, and a fifth also be the same multiple of the second that a sixth is of the fourth, the sum of the first and fifth will also be the same multiple of the second that the sum of the third and sixth is of the fourth.

Let a first magnitude, *AB*, be the same multiple of a second, *C*, that a third, *DE*, is of a fourth, *F*, and let a fifth, *BG*, also be the same multiple of the second, *C*, that a sixth, *EH*, is of the fourth *F*;



I say that the sum of the first and fifth, *AG*, will be the same multiple of the second, *C*, that the sum of the third and sixth, *DH*, is of the fourth, *F*.

For, since *AB* is the same multiple of *C* that *DE* is of *F*, therefore, as many magnitudes as there are in *AB* equal to *C*, so many also are there in *DE* equal to *F*.

For the same reason also,

as many as there are in *BG* equal to *C*, so many are there also in *EH* equal to *F*;

therefore, as many as there are in the whole *AG* equal to *C*, so many also are there in the whole *DH* equal to *F*.

Therefore, whatever multiple *AG* is of *C*, that multiple also is *DH* of *F*.

Therefore the sum of the first and fifth, *AG*, is the same multiple of the second, *C*, that the sum of the third and sixth, *DH*, is of the fourth, *F*.

Therefore etc.

Q. E. D.

To find the corresponding formula for the result of this proposition, we may suppose a to be the "second" magnitude and b the "fourth." If now the "first" magnitude is ma, the "third" is, by hypothesis, mb; and, if the "fifth" magnitude is na, the "sixth" is nb. The proposition then asserts that $ma + na$ is the same multiple of a that $mb + nb$ is of b.

More generally, if pa, qa... and pb, qb... be any further equimultiples of a, b respectively, $ma + na + pa + qa + \ldots$ is the same multiple of a that $mb + nb + pb + qb + \ldots$ is of b. This extension is stated in Simson's corollary to v. 2 thus:

"From this it is plain that, if any number of magnitudes *AB*, *BG*, *GH* be multiples of another *C*; and as many *DE*, *EK*, *KL* be the same multiples of *F*, each of each; the whole of the first, viz. *AH*, is the same multiple of *C* that the whole of the last, viz. *DL*, is of *F*."

The course of the proof, which separates m into its units and also n into its units, practically tells us that the multiple of a arrived at by adding the *two* multiples is the $(m + n)$th multiple; or practically we are shown that

$$ma + na = (m + n)a,$$

or, more generally, that

$$ma + na + pa + \ldots = (m + n + p + \ldots)a.$$

PROPOSITION 3.

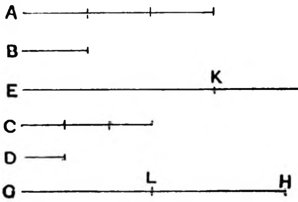
If a first magnitude be the same multiple of a second that a third is of a fourth, and if equimultiples be taken of the first and third, then also ex aequali *the magnitudes taken will be equimultiples respectively, the one of the second and the other of the fourth.*

Let a first magnitude *A* be the same multiple of a second *B* that a third *C* is of a fourth *D*, and let equimultiples *EF*, *GH* be taken of *A*, *C*;

I say that *EF* is the same multiple of *B* that *GH* is of *D*.

For, since *EF* is the same multiple of *A* that *GH* is of *C*, therefore, as many magnitudes as there are in *EF* equal to *A*, so many also are there in *GH* equal to *C*.

Let EF be divided into the magnitudes EK, KF equal to A, and GH into the magnitudes GL, LH equal to C; then the multitude of the magnitudes EK, KF will be equal to the multitude of the magnitudes GL, LH.



And, since A is the same multiple of B that C is of D, while EK is equal to A, and GL to C, therefore EK is the same multiple of B that GL is of D.

For the same reason

KF is the same multiple of B that LH is of D.

Since, then, a first magnitude EK is the same multiple of a second B that a third GL is of a fourth D, and a fifth KF is also the same multiple of the second B that a sixth LH is of the fourth D, therefore the sum of the first and fifth, EF, is also the same multiple of the second B that the sum of the third and sixth, GH, is of the fourth D. [V. 2]

Therefore etc.

Q. E. D.

Heiberg remarks of the use of *ex aequali* in the enunciation of this proposition that, strictly speaking, it has no reference to the definition (17) of a *ratio ex aequali*. But the uses of the expression here and in the definition are, I think, sufficiently parallel, as may be seen thus. The proposition asserts that, if

na, nb are equimultiples of a, b,

and if $m \cdot na$, $m \cdot nb$ are equimultiples of na, nb,

then $m \cdot na$ is the same multiple of a that $m \cdot nb$ is of b. Clearly the proposition can be extended by taking further equimultiples of the last equimultiples and so on; and we can prove that

$p \cdot q \ldots m \cdot na$ is the same multiple of a that $p \cdot q \ldots m \cdot nb$ is of b,

where the series of numbers $p \cdot q \ldots m \cdot n$ is exactly the same in both expressions;

and *ex aequali* (δι' ἴσου) expresses the fact that the equimultiples are *at the same distance* from a, b in the series na, $m \cdot na \ldots$ and nb, $m \cdot nb \ldots$ respectively.

Here again the proof breaks m into its units, and then breaks n into its units; and we are practically shown that the multiple of a arrived at, viz. $m.na$, is the multiple denoted by the product of the numbers m, n, i.e. the (mn)th multiple, or in other words that

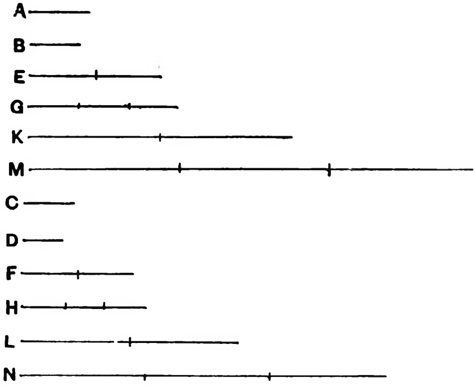
$$m.na = mn.a.$$

PROPOSITION 4.

If a first magnitude have to a second the same ratio as a third to a fourth, any equimultiples whatever of the first and third will also have the same ratio to any equimultiples whatever of the second and fourth respectively, taken in corresponding order.

For let a first magnitude A have to a second B the same ratio as a third C to a fourth D; and let equimultiples E, F be taken of A, C, and G, H other, chance, equimultiples of B, D;

I say that, as E is to G, so is F to H.



For let equimultiples K, L be taken of E, F, and other, chance, equimultiples M, N of G, H.

Since E is the same multiple of A that F is of C, and equimultiples K, L of E, F have been taken, therefore K is the same multiple of A that L is of C. [v. 3]

For the same reason

M is the same multiple of B that N is of D.

And, since, as *A* is to *B*, so is *C* to *D*, and of *A*, *C* equimultiples *K*, *L* have been taken, and of *B*, *D* other, chance, equimultiples *M*, *N*, therefore, if *K* is in excess of *M*, *L* also is in excess of *N*, if it is equal, equal, and if less, less. [v. Def. 5]

And *K*, *L* are equimultiples of *E*, *F*, and *M*, *N* other, chance, equimultiples of *G*, *H*; therefore, as *E* is to *G*, so is *F* to *H*. [v. Def. 5]

Therefore etc.

Q. E. D.

This proposition shows that, if a, b, c, d are proportionals, then

$$ma \text{ is to } nb \text{ as } mc \text{ is to } nd;$$

and the proof is as follows:

Take pma, pmc any equimultiples of ma, mc, and qnb, qnd any equimultiples of nb, nd.

Since $a : b = c : d$, it follows [v. Def. 5] that,

$$\text{according as } pma > = < qnb,\ pmc > = < qnd.$$

But the p- and q-equimultiples are *any* equimultiples;

therefore [v. Def. 5]

$$ma : nb = mc : nd.$$

It will be observed that Euclid's phrase for taking *any* equimultiples of *A*, *C* and *any* other equimultiples of *B*, *D* is "let there be taken equimultiples *E*, *F* of *A*, *C*, and *G*, *H* other, *chance*, equimultiples of *B*, *D*," *E*, *F* being called ἰσάκις πολλαπλάσια simply, and *G*, *H* ἄλλα, ἃ ἔτυχεν, ἰσάκις πολλαπλάσια. And similarly, when *any* equimultiples (*K*, *L*) of *E*, *F* come to be taken, and *any* other equimultiples (*M*, *N*) of *G*, *H*. But later on Euclid uses the same phrases about the *new* equimultiples with reference to the original magnitudes, reciting that "there have been taken, of *A*, *C*, equimultiples *K*, *L* and of *B*, *D*, other, *chance*, equimultiples *M*, *N*"; whereas *M*, *N* are not *any equimultiples whatever* of *B*, *D*, but are *any* equimultiples of the *particular* multiples (*G*, *H*) which have been taken of *B*, *D* respectively, though *these latter* have been taken at random. Simson would, in the first place, add ἃ ἔτυχεν in the passages where *any* equimultiples *E*, *F* are taken of *A*, *C* and *any* equimultiples *K*, *L* are taken of *E*, *F*, because the words are "wholly necessary" and, in the second place, would leave them out where *M*, *N* are called ἄλλα, ἃ ἔτυχεν, ἰσάκις πολλαπλάσια of *B*, *D*, because it is not true that of *B*, *D* have been taken "any equimultiples whatever (ἃ ἔτυχε), *M*, *N*." Simson adds: "And it is strange that neither Mr Briggs, who did right to leave out these words in one place of Prop. 13 of this book, nor Dr Gregory, who changed them into the word 'some' in three places, and left them out in a fourth of that same Prop. 13, did not also leave them out in this place of Prop. 4 and in the second of the two places where they occur in Prop. 17 of this book, in neither of which they can stand consistent with truth: And in none of all these places, even in those which they corrected in their Latin translation, have they cancelled the words ἃ ἔτυχε in the Greek text, as they ought to have done. The same words ἃ ἔτυχε are found in four places of Prop. 11 of this book, in the first and last of which they are necessary, but in the second and third, though they are true, they are quite superfluous; as they likewise are in the second of the two places in which they are found in the 12th prop. and in the like places of Prop. 22, 23 of this book; but are wanting in the last place of Prop. 23, as also in Prop. 25, Book XI."

As will be seen, Simson's emendations amount to alterations of the text so considerable as to suggest doubt whether we should be justified in making them in the absence of MS. authority. The phrase "equimultiples of *A*, *C* and other, chance, equimultiples of *B*, *D*" recurs so constantly as to suggest that it was for Euclid a quasi-stereotyped phrase, and that it is equally genuine wherever it occurs. Is it then absolutely necessary to insert ἃ ἔτυχε in places where it does not occur, and to leave it out in the places where Simson holds it to be wrong? I think the text can be defended as it stands. In the first place to say "take equimultiples of *A*, *C*" is a fair enough way of saying take *any* equimultiples *whatever* of *A*, *C*. The other difficulty is greater, but may, I think, be only due to the adoption of *any whatever* as the translation of ἃ ἔτυχε. As a matter of fact, the words only mean *chance* equimultiples, equimultiples which are the result of random selection. Is it not justifiable to describe the product of two chance numbers, numbers selected at random, as being a "*chance* number," since it is the result of two random selections? I think so, and I have translated ἃ ἔτυχε accordingly as implying, in the case in question, "other equimultiples whatever they may happen to be."

To this proposition Theon added the following:

"Since then it was proved that, if *K* is in excess of *M*, *L* is also in excess of *N*, if it is equal, (the other is) equal, and if less, less,

it is clear also that,

if *M* is in excess of *K*, *N* is also in excess of *L*, if it is equal, (the other is) equal, and if less, less;

and for this reason,

as *G* is to *E*, so also is *H* to *F*.

PORISM. From this it is manifest that, if four magnitudes be proportional, they will also be proportional inversely."

Simson rightly pointed out that the demonstration of what Theon intended to prove, viz. that, if *E*, *G*, *F*, *H* be proportionals, they are proportional inversely, i.e. *G* is to *E* as *H* is to *F*, does not in the least depend upon this 4th proposition or the proof of it; for, when it is said that, "if *K* exceeds *M*, *L* also exceeds *N* etc.," this is not proved from the fact that *E*, *G*, *F*, *H* are proportionals (which is the conclusion of Prop. 4), but from the fact that *A*, *B*, *C*, *D* are proportionals.

The proposition that, if *A*, *B*, *C*, *D* are proportionals, they are also proportionals inversely is not given by Euclid, but Simson supplies the proof in his Prop. B. The fact is really obvious at once from the 5th definition of Book V. (cf. p. 127 above), and Euclid probably omitted the proposition as unnecessary.

Simson added, in place of Theon's corollary, the following:

"Likewise, if the first has the same ratio to the second which the third has to the fourth, then also any equimultiples whatever of the first and third have the same ratio to the second and fourth: And, in like manner, the first and the third have the same ratio to any equimultiples whatever of the second and fourth."

The proof, of course, follows exactly the method of Euclid's proposition itself, with the only difference that, instead of one of the two pairs of equimultiples, the magnitudes themselves are taken. In other words, the conclusion that

$$ma \text{ is to } nb \text{ as } mc \text{ is to } nd$$

is equally true when either m or n is equal to unity.

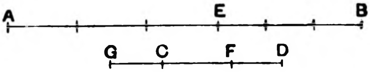
As De Morgan says, Simson's corollary is only necessary to those who will not admit M into the list M, $2M$, $3M$ etc.; the exclusion is grammatical and nothing else. The same may be said of Simson's Prop. A to the effect that, "If the first of four magnitudes has to the second the same ratio which the third has to the fourth: then, if the first be greater than the second, the third is also greater than the fourth; and if equal, equal; if less, less." This is needless to those who believe *once* A to be a proper component of the list of multiples, in spite of *multus* signifying many.

PROPOSITION 5.

If a magnitude be the same multiple of a magnitude that a part subtracted is of a part subtracted, the remainder will also be the same multiple of the remainder that the whole is of the whole.

For let the magnitude AB be the same multiple of the magnitude CD that the part AE subtracted is of the part CF subtracted;

I say that the remainder EB is also the same multiple of the remainder FD that the whole AB is of the whole CD.



For, whatever multiple AE is of CF, let EB be made that multiple of CG.

Then, since AE is the same multiple of CF that EB is of GC,

therefore AE is the same multiple of CF that AB is of GF. [V. 1]

But, by the assumption, AE is the same multiple of CF that AB is of CD.

Therefore AB is the same multiple of each of the magnitudes GF, CD;

therefore GF is equal to CD.

Let CF be subtracted from each;

therefore the remainder GC is equal to the remainder FD.

And, since *AE* is the same multiple of *CF* that *EB* is of *GC*,
and *GC* is equal to *DF*,
therefore *AE* is the same multiple of *CF* that *EB* is of *FD*.

But, by hypothesis,

AE is the same multiple of *CF* that *AB* is of *CD*;
therefore *EB* is the same multiple of *FD* that *AB* is of *CD*.

That is, the remainder *EB* will be the same multiple of the remainder *FD* that the whole *AB* is of the whole *CD*.

Therefore etc.

Q. E. D.

10. **let EB be made that multiple of CG,** *τοσαυταπλάσιον γεγονέτω καὶ τὸ* ΕΒ *τοῦ* ΓΗ. From this way of stating the construction one might suppose that *CG* was given and *EB* had to be found equal to a certain multiple of it. But in fact *EB* is what is given and *CG* has to be found, i.e. *CG* has to be constructed as a certain *sub*multiple of *EB*.

This proposition corresponds to v. 1, with subtraction taking the place of addition. It proves the formula

$$ma - mb = m(a - b).$$

Euclid's construction assumes that, if *AE* is any multiple of *CF*, and *EB* is any other magnitude, a fourth straight line can be found such that *EB* is the same multiple of it that *AE* is of *CF*, or in other words that, given any magnitude, we can divide it into any number of equal parts. This is however not proved, even of straight lines, much less other magnitudes, until vi. 9. Peletarius had already seen this objection to the construction. The difficulty is not got over by regarding it merely as a *hypothetical* construction; for hypothetical constructions are not in Euclid's manner. The remedy is to substitute the alternative construction given by Simson, after Peletarius and Campanus' translation from the Arabic, which only requires us to add a magnitude to itself a certain number of times. The demonstration follows Euclid's line exactly.

"Take *AG* the same multiple of *FD* that *AE* is of *CF*;
therefore *AE* is the same multiple of *CF* that *EG* is of *CD*. [v. 1]

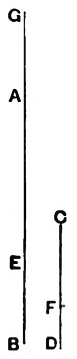
But *AE*, by hypothesis, is the same multiple of *CF* that *AB* is of *CD*; therefore *EG* is the same multiple of *CD* that *AB* is of *CD*;

wherefore *EG* is equal to *AB*.

Take from them the common magnitude *AE*; the remainder *AG* is equal to the remainder *EB*.

Wherefore, since *AE* is the same multiple of *CF* that *AG* is of *FD*, and since *AG* is equal to *EB*,
therefore *AE* is the same multiple of *CF* that *EB* is of *FD*.

But *AE* is the same multiple of *CF* that *AB* is of *CD*;
therefore *EB* is the same multiple of *FD* that *AB* is of *CD*."



Euclid's proof amounts to this.

Suppose a magnitude x taken such that

$$ma - mb = mx, \text{ say.}$$

Add mb to each side, whence (by v. 1)

$$ma = m(x + b).$$

Therefore $$a = x + b, \text{ or } x = a - b,$$

so that $$ma - mb = m(a - b).$$

Simson's proof, on the other hand, argues thus.

Take $x = m(a - b)$, the same multiple of $(a - b)$ that mb is of b.

Then, by addition of mb to both sides, we have [v. 1]

$$x + mb = ma,$$

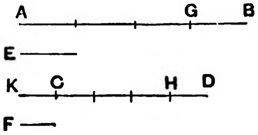
or $$x = ma - mb.$$

That is, $$ma - mb = m(a - b).$$

PROPOSITION 6.

If two magnitudes be equimultiples of two magnitudes, and any magnitudes subtracted from them be equimultiples of the same, the remainders also are either equal to the same or equimultiples of them.

For let two magnitudes *AB*, *CD* be equimultiples of two magnitudes *E*, *F*, and let *AG*, *CH* subtracted from them be equimultiples of the same two *E*, *F*; I say that the remainders also, *GB*, *HD*, are either equal to *E*, *F* or equimultiples of them.



For, first, let *GB* be equal to *E*; I say that *HD* is also equal to *F*.

For let *CK* be made equal to *F*.

Since *AG* is the same multiple of *E* that *CH* is of *F*, while *GB* is equal to *E* and *KC* to *F*, therefore *AB* is the same multiple of *E* that *KH* is of *F*. [v. 2]

But, by hypothesis, *AB* is the same multiple of *E* that *CD* is of *F*; therefore *KH* is the same multiple of *F* that *CD* is of *F*.

Since then each of the magnitudes *KH*, *CD* is the same multiple of *F*,

therefore *KH* is equal to *CD*.

Let CH be subtracted from each;
therefore the remainder KC is equal to the remainder HD.

But F is equal to KC;
therefore HD is also equal to F.

Hence, if GB is equal to E, HD is also equal to F.

Similarly we can prove that, even if GB be a multiple of E, HD is also the same multiple of F.

Therefore etc.

Q. E. D.

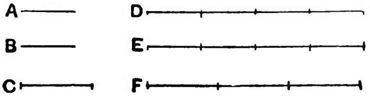
This proposition corresponds to v. 2, with subtraction taking the place of addition. It asserts namely that, if n is less than m, $ma - na$ is the same multiple of a that $mb - nb$ is of b. The enunciation distinguishes the cases in which $m - n$ is equal to 1 and greater than 1 respectively.

Simson observes that, while only the first case (the simpler one) is proved in the Greek, both are given in the Latin translation from the Arabic; and he supplies accordingly the proof of the second case, which Euclid leaves to the reader. The fact is that it is exactly the same as the other except that, in the construction, CK is made the same multiple of F that GB is of E, and at the end, when it has been proved that KC is equal to HD, instead of concluding that HD is equal to F, we have to say "Because GB is the same multiple of E that KC is of F, and KC is equal to HD, therefore HD is the same multiple of F that GB is of E."

Proposition 7.

Equal magnitudes have to the same the same ratio, as also has the same to equal magnitudes.

Let A, B be equal magnitudes and C any other, chance, magnitude;
I say that each of the magnitudes A, B has the same ratio to C, and C has the same ratio to each of the magnitudes A, B.



For let equimultiples D, E of A, B be taken, and of C another, chance, multiple F.

Then, since D is the same multiple of A that E is of B, while A is equal to B,

therefore D is equal to E.

But F is another, chance, magnitude.

If therefore D is in excess of F, E is also in excess of F, if equal to it, equal; and, if less, less.

And D, E are equimultiples of A, B,
while F is another, chance, multiple of C;

therefore, as A is to C, so is B to C. [V. Def. 5]

I say next that C also has the same ratio to each of the magnitudes A, B.

For, with the same construction, we can prove similarly that D is equal to E;
and F is some other magnitude.

If therefore F is in excess of D, it is also in excess of E, if equal, equal; and, if less, less.

And F is a multiple of C, while D, E are other, chance, equimultiples of A, B;

therefore, as C is to A, so is C to B. [V. Def. 5]

Therefore etc.

PORISM. From this it is manifest that, if any magnitudes are proportional, they will also be proportional inversely.

Q. E. D.

In this proposition there is a similar use of ὃ ἔτυχεν to that which has been discussed under Prop. 4. *Any* multiple F of C is taken and then, four lines lower down, we are told that "F is another, chance, magnitude." It is of course not *any magnitude whatever*, and Simson leaves out the sentence, but this time without calling attention to it.

Of the Porism to this proposition Heiberg says that it is properly put here in the best MS.; for, as August had already observed, if it was in its right place where Theon put it (at the end of V. 4), the second part of the proof of this proposition would be unnecessary. But the truth is that the Porism is no more in place here. The most that the proposition proves is that, if A, B are equal, and C any other magnitude, then two conclusions are simultaneously established, (1) that A is to C as B is to C and (2) that C is to A as C is to B. The second conclusion is not established from the first conclusion (as it ought to be in order to justify the inference in the Porism), but from a hypothesis on which the first conclusion itself depends; and moreover it is not a proportion in its general form, i.e. between *four* magnitudes, that is in question, but only the particular case in which the consequents are equal.

Aristotle tacitly assumes *inversion* (combined with the solution of the problem of Eucl. VI. 11) in *Meteorologica* III. 5, 376 a 14—16.

PROPOSITION 8.

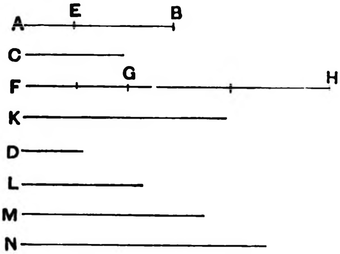
Of unequal magnitudes, the greater has to the same a greater ratio than the less has; and the same has to the less a greater ratio than it has to the greater.

Let AB, C be unequal magnitudes, and let AB be greater; let D be another, chance, magnitude;

I say that AB has to D a greater ratio than C has to D, and D has to C a greater ratio than it has to AB.

For, since AB is greater than C, let BE be made equal to C;

then the less of the magnitudes AE, EB, if multiplied, will sometime be greater than D. [v. Def. 4]



[*Case* 1.]

First, let AE be less than EB;

let AE be multiplied, and let FG be a multiple of it which is greater than D;

then, whatever multiple FG is of AE, let GH be made the same multiple of EB and K of C;

and let L be taken double of D, M triple of it, and successive multiples increasing by one, until what is taken is a multiple of D and the first that is greater than K. Let it be taken, and let it be N which is quadruple of D and the first multiple of it that is greater than K.

Then, since K is less than N first,

therefore K is not less than M.

And, since FG is the same multiple of AE that GH is of EB,

therefore FG is the same multiple of AE that FH is of AB. [v. 1]

But FG is the same multiple of AE that K is of C;

therefore FH is the same multiple of AB that K is of C;

therefore FH, K are equimultiples of AB, C.

Again, since GH is the same multiple of EB that K is of C,

and EB is equal to C,

therefore GH is equal to K.

But K is not less than M;

therefore neither is GH less than M.

And FG is greater than D;

therefore the whole FH is greater than D, M together.

But D, M together are equal to N, inasmuch as M is triple of D, and M, D together are quadruple of D, while N is also quadruple of D; whence M, D together are equal to N.

But FH is greater than M, D;

therefore FH is in excess of N,

while K is not in excess of N.

And FH, K are equimultiples of AB, C, while N is another, chance, multiple of D;

therefore AB has to D a greater ratio than C has to D. [v. Def. 7]

I say next, that D also has to C a greater ratio than D has to AB.

For, with the same construction, we can prove similarly that N is in excess of K, while N is not in excess of FH.

And N is a multiple of D,

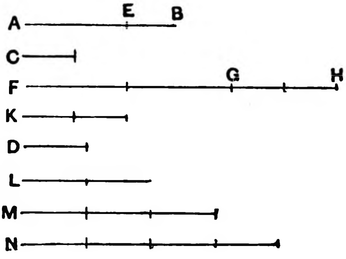
while FH, K are other, chance, equimultiples of AB, C;

therefore D has to C a greater ratio than D has to AB. [v. Def. 7]

[*Case* 2.]

Again, let AE be greater than EB.

Then the less, EB, if multiplied, will sometime be greater than D. [v. Def. 4]



Let it be multiplied, and let GH be a multiple of EB and greater than D;

and, whatever multiple GH is of EB, let FG be made the same multiple of AE, and K of C.

Then we can prove similarly that FH, K are equimultiples of AB, C;

and, similarly, let N be taken a multiple of D but the first that is greater than FG,

so that FG is again not less than M.

But GH is greater than D;
therefore the whole FH is in excess of D, M, that is, of N.

Now K is not in excess of N, inasmuch as FG also, which is greater than GH, that is, than K, is not in excess of N.

And in the same manner, by following the above argument, we complete the demonstration.

Therefore etc.

Q. E. D.

The two separate cases found in the Greek text of the demonstration can practically be compressed into one. Also the expositor of the two cases makes them differ more than they need. It is necessary in each case to select the *smaller* of the two segments AE, EB of AB with a view to taking a multiple of it which is greater than D; in the first case therefore AE is taken, in the second EB. But, while in the first case successive multiples of D are taken in order to find the first multiple that is greater than GH (or K), in the second case the multiple is taken which is the first that is greater than FG. This difference is not necessary; the first multiple of D that is greater than GH would equally serve in the second case. Lastly, the use of the magnitude K might have been dispensed with in both cases; it is of no practical use and only lengthens the proofs. For these reasons Simson considers that Theon, or some other unskilful editor, has vitiated the proposition. This however seems an unsafe assumption; for, while it was not the habit of the great Greek geometers to discuss separately a number of different cases (e.g. in I. 7 and I. 35 Euclid proves one case and leaves the others to the reader), there are many exceptions to prove the rule, e.g. Eucl. III. 25 and 33; and we know that many fundamental propositions, afterwards proved generally, were first discovered in relation to particular cases and then generalised, so that Book V., presenting a comparatively new theory, might fairly be expected to exhibit more instances than the earlier books do of unnecessary subdivision. The use of the K is no more conclusive against the genuineness of the proofs.

Nevertheless Simson's version of the proof is certainly shorter, and moreover it takes account of the case in which AE is *equal* to EB, and of the case in which AE, EB are both greater than D (though these cases are scarcely worth separate mention).

"If the magnitude which is not the greater of the two AE, EB be (1) not less than D, take FG, GH the doubles of AE, EB.

But if that which is not the greater of the two AE, EB be (2) less than D, this magnitude can be multiplied so as to become greater than D whether it be AE or EB.

Let it be multiplied until it becomes greater than D, and let the other be multiplied as often; let FG be the multiple thus taken of AE and GH the same multiple of EB,

therefore FG and GH are each of them greater than D.

And, in every one of the cases, take L the double of D, M its triple and so on, till the multiple of D be that which first becomes greater than GH.

Let N be that multiple of D which is first greater than GH, and M the multiple of D which is next less than N.

Then, because N is the multiple of D which is the first that becomes greater than GH,

the next preceding multiple is not greater than GH;

that is, GH is not less than M.

And, since FG is the same multiple of AE that GH is of EB,

GH is the same multiple of EB that FH is of AB; [v. 1]

wherefore FH, GH are equimultiples of AB, EB.

And it was shown that GH was not less than M;

and, by the construction, FG is greater than D;

therefore the whole FH is greater than M, D together.

But M, D together are equal to N;

therefore FH is greater than N.

But GH is not greater than N;

and FH, GH are equimultiples of AB, BE,

and N is a multiple of D;

therefore AB has to D a greater ratio than BE (or C) has to D. [v. Def. 7]

Also D has to BE a greater ratio than it has to AB.

For, having made the same construction, it may be shown, in like manner, that N is greater than GH but that it is not greater than FH;

and N is a multiple of D,

and GH, FH are equimultiples of EB, AB;

Therefore D has to EB a greater ratio than it has to AB." [v. Def. 7]

The proof may perhaps be more readily grasped in the more symbolical form thus.

Take the mth equimultiples of C, and of the excess of AB over C (that is, of AE), such that each is greater than D;

and, of the multiples of D, let pD be the first that is greater than mC, and nD the next less multiple of D.

Then, since mC is not less than nD,

and, by the construction, $m(AE)$ is greater than D,

the sum of mC and $m(AE)$ is greater than the sum of nD and D.

That is, $m(AB)$ is greater than pD.

And, by the construction, mC is less than pD.

Therefore [v. Def. 7] AB has to D a greater ratio than C has to D.

Again, since pD is less than $m(AB)$,

and pD is greater than mC,

D has to C a greater ratio than D has to AB.

PROPOSITION 9.

Magnitudes which have the same ratio to the same are equal to one another; and magnitudes to which the same has the same ratio are equal.

For let each of the magnitudes *A*, *B* have the same ratio to *C*;

I say that *A* is equal to *B*.

For, otherwise, each of the magnitudes *A*, *B* would not have had the same ratio to *C*; [v. 8]

but it has;

therefore *A* is equal to *B*.

Again, let *C* have the same ratio to each of the magnitudes *A*, *B*;

I say that *A* is equal to *B*.

For, otherwise, *C* would not have had the same ratio to each of the magnitudes *A*, *B*; [v. 8]

but it has;

therefore *A* is equal to *B*.

Therefore etc.

Q. E. D.

If *A* is to *C* as *B* is to *C*,

or if *C* is to *A* as *C* is to *B*, then *A* is equal to *B*.

Simson gives a more explicit proof of this proposition which has the advantage of referring back to the fundamental 5th and 7th definitions, instead of quoting the results of previous propositions, which, as will be seen from the next note, may be, in the circumstances, unsafe.

"Let *A*, *B* have each of them the same ratio to *C*;

A is equal to *B*.

For, if they are not equal, one of them is greater than the other;

let *A* be the greater.

Then, by what was shown in the preceding proposition, there are some equimultiples of *A* and *B*, and some multiple of *C*, such that the multiple of *A* is greater than the multiple of *C*, but the multiple of *B* is not greater than that of *C*.

Let such multiples be taken, and let *D*, *E* be the equimultiples of *A*, *B*, and *F* the multiple of *C*, so that *D* may be greater than *F*, and *E* not greater than *F*.

But, because *A* is to *C* as *B* is to *C*,

and of *A*, *B* are taken equimultiples *D*, *E*, and of *C* is taken a multiple *F*, and *D* is greater than *F*,

E must also be greater than *F*. [v. Def. 5]

But *E* is not greater than *F*: which is impossible.

Next, let *C* have the same ratio to each of the magnitudes *A* and *B*;

A is equal to *B*.

For, if not, one of them is greater than the other;

let *A* be the greater.

Therefore, as was shown in Prop. 8, there is some multiple F of C, and some equimultiples E and D of B and A, such that F is greater than E and not greater than D.

But, because C is to B as C is to A,
and F the multiple of the first is greater than E the multiple of the second,
F the multiple of the third is greater than D the multiple of the fourth. [v. Def. 5]

But F is not greater than D: which is impossible.

Therefore A is equal to B."

PROPOSITION 10.

Of magnitudes which have a ratio to the same, that which has a greater ratio is greater; and that to which the same has a greater ratio is less.

For let A have to C a greater ratio than B has to C;
I say that A is greater than B.

For, if not, A is either equal to B or less.

Now A is not equal to B;
for in that case each of the magnitudes A, B would have had the same ratio to C; [v. 7]
but they have not;
therefore A is not equal to B.

Nor again is A less than B;
for in that case A would have had to C a less ratio than B has to C; [v. 8]
but it has not;
therefore A is not less than B.

But it was proved not to be equal either;
therefore A is greater than B.

Again, let C have to B a greater ratio than C has to A;
I say that B is less than A.

For, if not, it is either equal or greater.

Now B is not equal to A;
for in that case C would have had the same ratio to each of the magnitudes A, B; [v. 7]
but it has not;
therefore A is not equal to B.

Nor again is *B* greater than *A*;
for in that case *C* would have had to *B* a less ratio than it has to *A*; [v. 8]
but it has not;
therefore *B* is not greater than *A*.
But it was proved that it is not equal either;
therefore *B* is less than *A*.
Therefore etc. Q. E. D.

No better example can, I think, be found of the acuteness which Simson brought to bear in his critical examination of the *Elements*, and of his great services to the study of Euclid, than is furnished by the admirable note on this proposition where he points out a serious flaw in the proof as given in the text.

For the first time Euclid is arguing about *greater* and *less* ratios, and it will be found by an examination of the steps of the proof that he assumes more with regard to the meaning of the terms than he is entitled to assume, having regard to the fact that the definition of greater ratio (Def. 7) is all that, as yet, he has to go upon. That we cannot argue, at present, about *greater* and *less* as applied to *ratios* in the same way as about the same terms in relation to *magnitudes* is indeed sufficiently indicated by the fact that Euclid does not assume for ratios what is in Book I. an axiom, viz. that things which are equal to the same thing are equal to one another; on the contrary, he *proves*, in Prop. 11, that ratios which are the same with the same ratio are the same with one another.

Let us now examine the steps of the proof in the text. First we are told that

"*A* is greater than *B*.
For, if not, it is either equal to *B* or less than it.
Now *A* is not equal to *B*;
for in that case each of the two magnitudes *A*, *B* would have had the same ratio to *C*: [v. 7]
but they have not:
therefore *A* is not equal to *B*."

As Simson remarks, the force of this reasoning is as follows.

If *A* has to *C* the same ratio as *B* has to *C*,

then—supposing any equimultiples of *A*, *B* to be taken and any multiple of *C*—

by Def. 5, if the multiple of *A* be greater than the multiple of *C*, the multiple of *B* is also greater than that of *C*.

But it follows from the hypothesis (that *A* has a greater ratio to *C* than *B* has to *C*) that,

by Def. 7, there must be *some* equimultiples of *A*, *B* and *some* multiple of *C* such that the multiple of *A* is greater than the multiple of *C*, but the multiple of *B* is *not* greater than the same multiple of *C*.

And this directly contradicts the preceding deduction from the supposition that *A* has to *C* the same ratio as *B* has to *C*;

therefore that supposition is impossible.

The proof now goes on thus:

"Nor again is A less than B;

for, in that case, A would have had to C a less ratio than B has to C; [v. 8]

but it has not;

therefore A is not less than B."

It is here that the difficulty arises. As before, we must use Def. 7. "A would have had to C a less ratio than B has to C," or the equivalent statement that B would have had to C a greater ratio than A has to C, means that there would have been *some* equimultiples of B, A and *some* multiple of C such that

(1) the multiple of B is greater than the multiple of C, but

(2) the multiple of A is *not* greater than the multiple of C,

and it ought to have been proved that this can never happen if the hypothesis of the proposition is true, viz. that A has to C a greater ratio than B has to C: that is, it should have been proved that, in the latter case, the multiple of A is *always* greater than the multiple of C whenever the multiple of B is greater than the multiple of C (for, when this is demonstrated, it will be evident that B cannot have a greater ratio to C than A has to C). But this is not proved (cf. the remark of De Morgan quoted in the note on v. Def. 7, p. 130), and hence it is not proved that the above inference from the supposition that A is less than B is inconsistent with the hypothesis in the enunciation. The proof therefore fails.

Simson suggests that the proof is not Euclid's, but the work of some one who apparently "has been deceived in applying what is manifest, when understood of magnitudes, unto ratios, viz. that a magnitude cannot be both greater and less than another."

The proof substituted by Simson is satisfactory and simple.

"Let A have to C a greater ratio than B has to C;

A is greater than B.

For, because A has a greater ratio to C than B has to C, there are some equimultiples of A, B and some multiple of C such that

the multiple of A is greater than the multiple of C, but the multiple of B is *not* greater than it. [v. Def. 7]

Let them be taken, and let D, E be equimultiples of A, B, and F a multiple of C, such that

D is greater than F,

but E is not greater than F.

Therefore D is greater than E.

And, because D and E are equimultiples of A and B, and D is greater than E,

therefore A is greater than B. [Simson's 4th Ax.]

Next, let C have a greater ratio to B than it has to A;

B is less than A.

For there is some multiple F of C and some equimultiples E and D of B and A such that

F is greater than E but not greater than D. [v. Def. 7]

Therefore E is less than D;

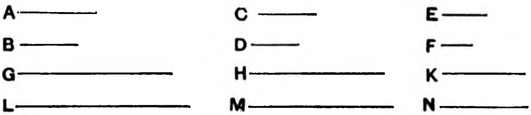
and, because E and D are equimultiples of B and A,

therefore B is less than A."

PROPOSITION 11.

Ratios which are the same with the same ratio are also the same with one another.

For, as *A* is to *B*, so let *C* be to *D*,
and, as *C* is to *D*, so let *E* be to *F*;
I say that, as *A* is to *B*, so is *E* to *F*.



For of *A*, *C*, *E* let equimultiples *G*, *H*, *K* be taken, and of *B*, *D*, *F* other, chance, equimultiples *L*, *M*, *N*.

Then since, as *A* is to *B*, so is *C* to *D*,
and of *A*, *C* equimultiples *G*, *H* have been taken,
and of *B*, *D* other, chance, equimultiples *L*, *M*,
therefore, if *G* is in excess of *L*, *H* is also in excess of *M*,
if equal, equal,
and if less, less.

Again, since, as *C* is to *D*, so is *E* to *F*,
and of *C*, *E* equimultiples *H*, *K* have been taken,
and of *D*, *F* other, chance, equimultiples *M*, *N*,
therefore, if *H* is in excess of *M*, *K* is also in excess of *N*,
if equal, equal,
and if less, less.

But we saw that, if *H* was in excess of *M*, *G* was also in excess of *L*; if equal, equal; and if less, less;
so that, in addition, if *G* is in excess of *L*, *K* is also in excess of *N*,
if equal, equal,
and if less, less.

And *G*, *K* are equimultiples of *A*, *E*,
while *L*, *N* are other, chance, equimultiples of *B*, *F*;
therefore, as *A* is to *B*, so is *E* to *F*.

Therefore etc.

Algebraically, if $a : b = c : d$,
and $c : d = e : f$,
then $a : b = e : f$.

The idiomatic use of the imperfect in quoting a result previously obtained is noteworthy. Instead of saying "But *it was proved that*, if *H is* in excess of *M*, *G is* also in excess of *L*," the Greek text has "But if *H was* in excess of *M*, *G* was also in excess of *L*," ἀλλὰ εἰ ὑπερεῖχε τὸ Θ τοῦ Μ, ὑπερεῖχε καὶ τὸ Η τοῦ Λ.

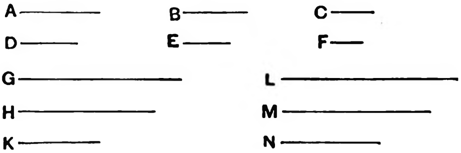
This proposition is tacitly used in combination with v. 16 and v. 24 in the geometrical passage in Aristotle, *Meteorologica* III. 5, 376 a 22—26.

PROPOSITION 12.

If any number of magnitudes be proportional, as one of the antecedents is to one of the consequents, so will all the antecedents be to all the consequents.

Let any number of magnitudes *A*, *B*, *C*, *D*, *E*, *F* be proportional, so that, as *A* is to *B*, so is *C* to *D* and *E* to *F*;

I say that, as *A* is to *B*, so are *A*, *C*, *E* to *B*, *D*, *F*.



For of *A*, *C*, *E* let equimultiples *G*, *H*, *K* be taken,
and of *B*, *D*, *F* other, chance, equimultiples *L*, *M*, *N*.

Then since, as *A* is to *B*, so is *C* to *D*, and *E* to *F*,
and of *A*, *C*, *E* equimultiples *G*, *H*, *K* have been taken,
and of *B*, *D*, *F* other, chance, equimultiples *L*, *M*, *N*,
therefore, if *G* is in excess of *L*, *H* is also in excess of *M*, and *K* of *N*,
if equal, equal,
and if less, less;
so that, in addition,
if *G* is in excess of *L*, then *G*, *H*, *K* are in excess of *L*, *M*, *N*,
if equal, equal,
and if less, less.

Now G and G, H, K are equimultiples of A and A, C, E, since, if any number of magnitudes whatever are respectively equimultiples of any magnitudes equal in multitude, whatever multiple one of the magnitudes is of one, that multiple also will all be of all. [V. 1]

For the same reason

L and L, M, N are also equimultiples of B and B, D, F;

therefore, as A is to B, so are A, C, E to B, D, F. [V. Def. 5]

Therefore etc.

Q. E. D.

Algebraically, if $a : a' = b : b' = c : c'$ etc., each ratio is equal to the ratio $(a + b + c + \ldots) : (a' + b' + c' + \ldots)$.

This theorem is quoted by Aristotle, *Eth. Nic.* V. 7, 1131 b 14, in the shortened form "the whole is to the whole what each part is to each part (respectively)."

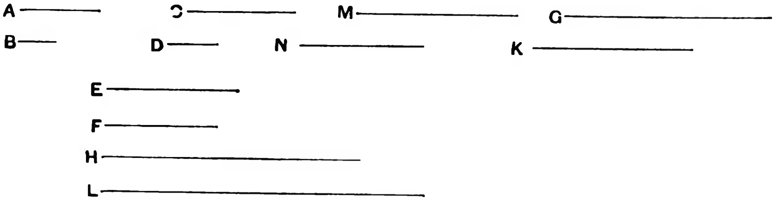
PROPOSITION 13.

If a first magnitude have to a second the same ratio as a third to a fourth, and the third have to the fourth a greater ratio than a fifth has to a sixth, the first will also have to the second a greater ratio than the fifth to the sixth.

For let a first magnitude A have to a second B the same ratio as a third C has to a fourth D,

and let the third C have to the fourth D a greater ratio than a fifth E has to a sixth F;

I say that the first A will also have to the second B a greater ratio than the fifth E to the sixth F.



For, since there are some equimultiples of C, E, and of D, F other, chance, equimultiples, such that the multiple of C is in excess of the multiple of D,

while the multiple of E is not in excess of the multiple of F, [v. Def. 7]
let them be taken,
and let G, H be equimultiples of C, E,
and K, L other, chance, equimultiples of D, F,
so that G is in excess of K, but H is not in excess of L;
and, whatever multiple G is of C, let M be also that multiple of A,
and, whatever multiple K is of D, let N be also that multiple of B.

Now, since, as A is to B, so is C to D,
and of A, C equimultiples M, G have been taken,
and of B, D other, chance, equimultiples N, K,
therefore, if M is in excess of N, G is also in excess of K,
if equal, equal,
and if less, less. [v. Def. 5]

But G is in excess of K;
therefore M is also in excess of N.

But H is not in excess of L;
and M, H are equimultiples of A, E,
and N, L other, chance, equimultiples of B, F;

therefore A has to B a greater ratio than E has to F. [v. Def. 7]

Therefore etc.

Q. E. D.

Algebraically, if $a : b = c : d$,
and $c : d > e : f$,
then $a : b > e : f$.

After the words "for, since" in the first line of the proof, Theon added "C has to D a greater ratio than E has to F," so that "there are some equimultiples" began, with him, the principal sentence.

The Greek text has, after "of D, F other, chance, equimultiples," "*and* the multiple of C is in excess of the multiple of D...." The meaning being "such that," I have substituted this for "and," after Simson.

The following will show the method of Euclid's proof.

Since $c : d > e : f$,

there will be some equimultiples mc, me of c, e, and some equimultiples nd, nf of d, f, such that

$$mc > nd, \text{ while } me \not> nf.$$

But, since $a : b = c : d$,

therefore, according as $ma >=< nb$, $mc >=< nd$.

And $mc > nd$;

therefore $ma > nb$, while (from above) $me \not> nf$.

Therefore $a : b > e : f$.

Simson adds as a corollary the following:

"If the first have a greater ratio to the second than the third has to the fourth, but the third the same ratio to the fourth which the fifth has to the sixth, it may be demonstrated in like manner that the first has a greater ratio to the second than the fifth has to the sixth."

This however scarcely seems to be worth separate statement, since it only amounts to changing the order of the two parts of the hypothesis.

PROPOSITION 14.

If a first magnitude have to a second the same ratio as a third has to a fourth, and the first be greater than the third, the second will also be greater than the fourth; if equal, equal; and if less, less.

For let a first magnitude A have the same ratio to a second B as a third C has to a fourth D; and let A be greater than C;

I say that B is also greater than D.

For, since A is greater than C,

and B is another, chance, magnitude,

therefore A has to B a greater ratio than C has to B. [v. 8]

But, as A is to B, so is C to D;

therefore C has also to D a greater ratio than C has to B. [v. 13]

But that to which the same has a greater ratio is less; [v. 10]

therefore D is less than B;

so that B is greater than D.

Similarly we can prove that, if A be equal to C, B will also be equal to D;

and, if A be less than C, B will also be less than D.

Therefore etc.

Algebraically, if $a : b = c : d$,
then, according as $a >=< c$, $b >=< d$.

Simson adds the specific proof of the second and third parts of this proposition, which Euclid dismisses with "Similarly we can prove...."

"Secondly, if A be equal to C, B is equal to D; for A is to B as C, that is A, is to D;

therefore B is equal to D. [v. 9]

Thirdly, if A be less than C, B shall be less than D.

For C is greater than A;

and, because C is to D as A is to B,

D is greater than B, by the first case.

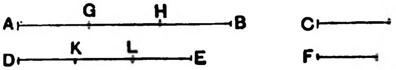
Wherefore B is less than D."

Aristotle, *Meteorol.* III. 5, 376 a 11—14, quotes the equivalent proposition that, if $a > b$, $c > d$.

PROPOSITION 15.

Parts have the same ratio as the same multiples of them taken in corresponding order.

For let AB be the same multiple of C that DE is of F; I say that, as C is to F, so is AB to DE.



For, since AB is the same multiple of C that DE is of F, as many magnitudes as there are in AB equal to C, so many are there also in DE equal to F.

Let AB be divided into the magnitudes AG, GH, HB equal to C,
and DE into the magnitudes DK, KL, LE equal to F;
then the multitude of the magnitudes AG, GH, HB will be equal to the multitude of the magnitudes DK, KL, LE.

And, since AG. GH, HB are equal to one another,
and DK, KL, LE are also equal to one another,
therefore, as AG is to DK, so is GH to KL, and HB to LE. [v. 7]

Therefore, as one of the antecedents is to one of the consequents, so will all the antecedents be to all the consequents; [v. 12]

therefore, as AG is to DK, so is AB to DE.

But AG is equal to C and DK to F;

therefore, as C is to F, so is AB to DE.

Therefore etc. Q. E. D.

Algebraically, $a : b = ma : mb$.

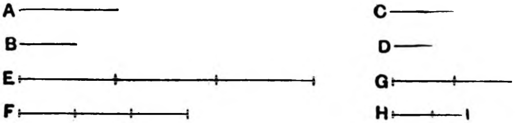
PROPOSITION 16.

If four magnitudes be proportional, they will also be proportional alternately.

Let A, B, C, D be four proportional magnitudes,

so that, as A is to B, so is C to D;

I say that they will also be so alternately, that is, as A is to C, so is B to D.



For of A, B let equimultiples E, F be taken,

and of C, D other, chance, equimultiples G, H.

Then, since E is the same multiple of A that F is of B,

and parts have the same ratio as the same multiples of them, [v. 15]

therefore, as A is to B, so is E to F.

But as A is to B, so is C to D;

therefore also, as C is to D, so is E to F. [v. 11]

Again, since G, H are equimultiples of C, D,

therefore, as C is to D, so is G to H. [v. 15]

But, as C is to D, so is E to F;

therefore also, as E is to F, so is G to H. [v. 11]

But, if four magnitudes be proportional, and the first be greater than the third,

the second will also be greater than the fourth;

if equal, equal;

and if less, less. [v. 14]

Therefore, if E is in excess of G, F is also in excess of H,

if equal, equal,

and if less, less.

Now E, F are equimultiples of A, B,

and G, H other, chance, equimultiples of C, D;

therefore, as A is to C, so is B to D. [v. Def. 5]

Therefore etc.

Q. E. D.

3. **"Let A, B, C, D be four proportional magnitudes, so that, as A is to B, so is C to D."** In a number of expressions like this it is absolutely necessary, when translating into English, to interpolate words which are not in the Greek. Thus the Greek here is: Ἔστω τέσσαρα μεγέθη ἀνάλογον τὰ Α, Β, Γ, Δ, ὡς τὸ Α πρὸς τὸ Β, οὕτως τὸ Γ πρὸς τὸ Δ, literally "Let A, B, C, D be four proportional magnitudes, as A to B, so C to D." The same remark applies to the corresponding expressions in the next propositions, V. 17, 18, and to other forms of expression in V. 20—23 and later propositions: e.g. in V. 20 we have a phrase meaning literally "Let there be magnitudes...which taken two and two are in the same ratio, as A to B, so D to E," etc.: in V. 21 "(magnitudes)...which taken two and two are in the same ratio, and let the proportion of them be perturbed, as A to B, so E to F," etc. In all such cases (where the Greek is so terse as to be almost ungrammatical) I shall insert the words necessary in English, without further remark.

Algebraically, if $a : b = c : d$,

then $a : c = b : d$.

Taking equimultiples ma, mb of a, b, and equimultiples nc, nd of c, d, we have, by V. 15,

$$a : b = ma : mb,$$

$$c : d = nc : nd.$$

And, since $a : b = c : d$,

we have [V. 11] $ma : mb = nc : nd$.

Therefore [V. 14], according as $ma >=< nc$, $mb >=< nd$,

so that $a : c = b : d$.

Aristotle tacitly uses the theorem in *Meteorologica* III. 5, 376 a 22—24.

The four magnitudes in this proposition must all be *of the same kind*, and Simson inserts "of the same kind" in the enunciation.

This is the first of the propositions of Eucl. V. which Smith and Bryant (*Euclid's Elements of Geometry*, 1901, pp. 298 sqq.) prove by means of VI. 1 so far as the only geometrical magnitudes in question are *straight lines* or *rectilineal areas*; and certainly the proofs are more easy to follow than Euclid's. The proof of this proposition is as follows.

To prove that, *If four magnitudes of the same kind* [straight lines or rectilineal areas] *be proportionals, they will be proportionals when taken alternately.*

Let P, Q, R, S be the four magnitudes of the same kind such that

$$P : Q = R : S;$$

then it is required to prove that

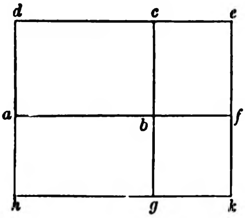
$$P : R = Q : S.$$

First, let all the magnitudes be areas.

Construct a rectangle $abcd$ equal to the area P, and to bc apply the rectangle $bcef$ equal to Q.

Also to ab, bf apply rectangles ag, bk equal to R, S respectively.

Then, since the rectangles ac, be have the same height, they are to one another as their bases. [vi. 1]



Hence $P : Q = ab : bf$.

But $P : Q = R : S$.

Therefore $R : S = ab : bf$, [v. 11]

i.e. rect. ag : rect. $bk = ab : bf$.

Hence (by the converse of vi. 1) the rectangles ag, bk have the same height, so that k is on the line hg.

Hence the rectangles ac, ag have the same height, namely ab; also be, bk have the same height, namely bf.

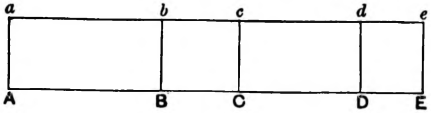
Therefore rect. ac : rect. $ag = bc : bg$,

and rect. be : rect. $bk = bc : bg$. [vi. 1]

Therefore rect. ac : rect. ag = rect. be : rect. bk. [v. 11]

That is, $P : R = Q : S$.

Secondly, let the magnitudes be straight lines AB, BC, CD, DE.
Construct the rectangles Ab, Bc, Cd, De with the same height.



Then $Ab : Bc = AB : BC$,

and $Cd : De = CD : DE$. [vi. 1]

But $AB : BC = CD : DE$.

Therefore $Ab : Bc = Cd : De$. [v. 11]

Hence, by the first case,

$Ab : Cd = Bc : De$,

and, since these rectangles have the same height,

$AB : CD = BC : DE$.

PROPOSITION 17.

If magnitudes be proportional componendo, *they will also be proportional* separando.

Let AB, BE, CD, DF be magnitudes proportional *componendo*, so that, as AB is to BE, so is CD to DF;

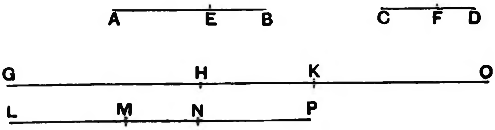
I say that they will also be proportional *separando*, that is, as AE is to EB, so is CF to DF.

For of AE, EB, CF, FD let equimultiples GH, HK, LM, MN be taken,

and of EB, FD other, chance, equimultiples, KO, NP,

Then, since *GH* is the same multiple of *AE* that *HK* is of *EB*,
therefore *GH* is the same multiple of *AE* that *GK* is of *AB*. [v. 1]

But *GH* is the same multiple of *AE* that *LM* is of *CF*;
therefore *GK* is the same multiple of *AB* that *LM* is of *CF*.



Again, since *LM* is the same multiple of *CF* that *MN* is of *FD*,
therefore *LM* is the same multiple of *CF* that *LN* is of *CD*. [v. 1]

But *LM* was the same multiple of *CF* that *GK* is of *AB*;
therefore *GK* is the same multiple of *AB* that *LN* is of *CD*.

Therefore *GK*, *LN* are equimultiples of *AB*, *CD*.

Again, since *HK* is the same multiple of *EB* that *MN* is of *FD*,

and *KO* is also the same multiple of *EB* that *NP* is of *FD*,
therefore the sum *HO* is also the same multiple of *EB* that *MP* is of *FD*. [v. 2]

And, since, as *AB* is to *BE*, so is *CD* to *DF*,
and of *AB*, *CD* equimultiples *GK*, *LN* have been taken,
and of *EB*, *FD* equimultiples *HO*, *MP*,
therefore, if *GK* is in excess of *HO*, *LN* is also in excess of *MP*,
if equal, equal,
and if less, less.

Let *GK* be in excess of *HO*;
then, if *HK* be subtracted from each,

GH is also in excess of *KO*.

But we saw that, if *GK* was in excess of *HO*, *LN* was also in excess of *MP*;

therefore *LN* is also in excess of *MP*, and, if MN be subtracted from each,

LM is also in excess of NP;

so that, if GH is in excess of KO, LM is also in excess of NP.

Similarly we can prove that,

if GH be equal to KO, LM will also be equal to NP,
and if less, less.

And GH, LM are equimultiples of AE, CF,
while KO, NP are other, chance, equimultiples of EB, FD;

therefore, as AE is to EB, so is CF to FD.

Therefore etc.

Q. E. D.

Algebraically, if $$a : b = c : d,$$
then $$(a-b) : b = (c-d) : d.$$

I have already noted the somewhat strange use of the participles of συγκεῖσθαι and διαιρεῖσθαι to convey the sense of the technical σύνθεσις and διαίρεσις λόγου, or what we denote by *componendo* and *separando*. ἐὰν συγκείμενα μεγέθη ἀνάλογον ᾖ, καὶ διαιρεθέντα ἀνάλογον ἔσται is, literally, "if magnitudes compounded be proportional, they will also be proportional separated," by which is meant "if one magnitude made up of two parts is to one of its parts as another magnitude made up of two parts is to one of its parts, the remainder of the first whole is to the part of it first taken as the remainder of the second whole is to the part of it first taken." In the algebraical formula above a, c are the wholes and b, $a-b$ and d, $c-d$ are the parts and remainders respectively. The formula might also be stated thus:

If $$a+b : b = c+d : d,$$
then $$a : b = c : d,$$
in which case $a+b$, $c+d$ are the wholes and b, a and d, c the parts and remainders respectively. Looking at the last formula, we observe that "separated," διαιρεθέντα, is used with reference not to the magnitudes a, b, c, d but to the *compounded* magnitudes $a+b$, b, $c+d$, d.

As the proof is somewhat long, it will be useful to give a conspectus of it in the more symbolical form. To avoid minuses, we will take for the hypothesis the form

$a+b$ is to b as $c+d$ is to d.

Take any equimultiples of the four magnitudes a, b, c, d, viz.

$$ma, \quad mb, \quad mc, \quad md,$$

and any other equimultiples of the consequents, viz.

nb and nd.

Then, by V. 1, $m(a+b)$, $m(c+d)$ are equimultiples of $a+b$, $c+d$,

and, by V. 2, $(m+n)b$, $(m+n)d$ are equimultiples of b, d.

Therefore, by Def. 5, since $a+b$ is to b as $c+d$ is to d,

according as $m(a+b) > = < (m+n)b$, $m(c+d) > = < (m+n)d$.

Subtract from $m(a+b)$, $(m+n)b$ the common part mb, and from $m(c+d)$, $(m+n)d$ the common part md; and we have,

according as $ma >=< nb$, $mc >=< nd$.

But ma, mc are any equimultiples of a, c, and nb, nd any equimultiples of b, d,

therefore, by v. Def. 5,

a is to b as c is to d.

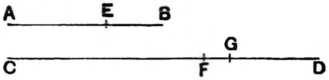
Smith and Bryant's proof follows, *mutatis mutandis*, their alternative proof of the next proposition (see pp. 173—4 below).

PROPOSITION 18.

If magnitudes be proportional separando, *they will also be proportional* componendo.

Let *AE*, *EB*, *CF*, *FD* be magnitudes proportional *separando*, so that, as *AE* is to *EB*, so is *CF* to *FD*;

I say that they will also be proportional *componendo*, that is, as *AB* is to *BE*, so is *CD* to *FD*.



For, if *CD* be not to *DF* as *AB* to *BE*,

then, as *AB* is to *BE*, so will *CD* be either to some magnitude less than *DF* or to a greater.

First, let it be in that ratio to a less magnitude *DG*.

Then, since, as *AB* is to *BE*, so is *CD* to *DG*,

they are magnitudes proportional *componendo*;

so that they will also be proportional *separando*. [v. 17]

Therefore, as *AE* is to *EB*, so is *CG* to *GD*.

But also, by hypothesis,

as *AE* is to *EB*, so is *CF* to *FD*.

Therefore also, as *CG* is to *GD*, so is *CF* to *FD*. [v. 11]

But the first *CG* is greater than the third *CF*;

therefore the second *GD* is also greater than the fourth *FD*. [v. 14]

But it is also less: which is impossible.

Therefore, as *AB* is to *BE*, so is not *CD* to a less magnitude than *FD*.

Similarly we can prove that neither is it in that ratio to a greater;

it is therefore in that ratio to *FD* itself.

Therefore etc.

Q. E. D.

Algebraically, if $a : b = c : d$,

then $(a + b) : b = (c + d) : d$.

In the enunciation of this proposition there is the same special use of διῃρημένα and συντεθέντα as there was of συγκείμενα and διαιρεθέντα in the last enunciation. Practically, as the algebraical form shows, διῃρημένα might have been left out.

The following is the method of proof employed by Euclid.

Given that $a : b = c : d$,

suppose, if possible, that

$$(a + b) : b = (c + d) : (d \pm x).$$

Therefore, *separando* [v. 17],

$$a : b = (c \mp x) : (d \pm x),$$

whence, by v. 11, $(c \mp x) : (d \pm x) = c : d$.

But $(c - x) < c$, while $(d + x) > d$,

and $(c + x) > c$, while $(d - x) < d$,

which relations respectively contradict v. 14.

Simson pointed out (as Saccheri before him saw) that Euclid's demonstration is not legitimate, because it assumes without proof that *to any three magnitudes, two of which, at least, are of the same kind, there exists a fourth proportional.* Clavius and, according to him, other editors made this an axiom. But it is far from axiomatic; it is not till vi. 12 that Euclid shows, by construction, that it is true even in the particular case where the three given magnitudes are all straight lines.

In order to remove the defect it is necessary either (1) to prove beforehand the proposition thus assumed by Euclid or (2) to prove v. 18 independently of it.

Saccheri ingeniously proposed that the assumed proposition should be proved, for areas and straight lines, by means of Euclid vi. 1, 2 and 12. As he says, there was nothing to prevent Euclid from interposing these propositions immediately after v. 17 and then proving v. 18 by means of them. vi. 12 enables us to *construct* the fourth proportional when the three given magnitudes are straight lines; and vi. 12 depends only on vi. 1 and 2. "Now," says Saccheri, "when we have once found the means of constructing a straight line which is a fourth proportional to three given straight lines, we obviously have the solution of the general problem 'To construct a straight line which shall have to a given straight line the same ratio which two *polygons* have (to one another).'" For it is sufficient to transform the polygons into two triangles of equal height and then to construct a straight line which shall be a fourth proportional to the bases of the triangles and the given straight line.

The method of Saccheri is, as will be seen, similar to that adopted by

Smith and Bryant (*loc. cit.*) in proving the theorems of Euclid v. 16, 17, 18, 22, so far as straight lines and rectilineal areas are concerned, by means of vi. 1.

De Morgan gives a sketch of a general proof of the assumed proposition that, B being any magnitude, and P and Q two magnitudes of the same kind, there does exist a magnitude A which is to B in the same ratio as P to Q.

"The right to reason upon any aliquot part of any magnitude is assumed; though, in truth, aliquot parts obtained by continual bisection would suffice: and it is taken as previously proved that the tests of greater and of less ratio are never both presented in any one scale of relation as compared with another" (see note on v. Def. 7 *ad fin.*).

"(1) If M be to B in a greater ratio than P to Q, so is every magnitude greater than M, and so are *some less magnitudes*; and if M be to B in a less ratio than P to Q, so is every magnitude less than M, and so are *some greater magnitudes*. Part of this is in every system: the rest is proved thus. If M be to B in a greater ratio than P to Q, say, for instance, we find that $15M$ lies between $22B$ and $23B$, while $15P$ lies before $22Q$. Let $15M$ exceed $22B$ by Z; then, if N be less than M by anything less than the 15th part of Z, $15N$ is between $22B$ and $23B$: or N, less than M, is in a greater ratio to B than P to Q. And similarly for the other case.

(2) M can certainly be taken so small as to be in a less ratio to B than P to Q, and so large as to be in a greater; and since we can never pass from the greater ratio back again to the smaller by increasing M, it follows that, while we pass from the first designated value to the second, we come upon an intermediate magnitude A such that every smaller is in a less ratio to B than P to Q, and every greater in a greater ratio. Now A cannot be in a less ratio to B than P to Q, for then some greater magnitudes would also be in a less ratio; nor in a greater ratio, for then some less magnitudes would be in a greater ratio; therefore A is in the same ratio to B as P to Q. The previously proved proposition above mentioned shows the three alternatives to be the only ones."

Alternative proofs of V. 18.

Simson bases his alternative on v. 5, 6. As the 18th proposition is the converse of the 17th, and the latter is proved by means of v. 1 and 2, of which v. 5 and 6 are converses, the proof of v. 18 by v. 5 and 6 would be natural; and Simson holds that Euclid must have proved v. 18 in this way because "the 5th and 6th do not enter into the demonstration of any proposition in this book as we have it, nor can they be of any use in any proposition of the Elements," and "the 5th and 6th have undoubtedly been put into the 5th book for the sake of some propositions in it, as all the other propositions about equimultiples have been."

Simson's proof is however, as it seems to me, intolerably long and difficult to follow unless it be put in the symbolical form as follows.

Suppose that a is to b as c is to d;

it is required to prove that $a+b$ is to b as $c+d$ is to d.

Take any equimultiples of the last four magnitudes, say

$$m(a+b), \quad mb, \quad m(c+d), \quad md,$$

and any equimultiples of b, d, as

$$nb, \quad nd.$$

Clearly, if nb is greater than mb,
nd is greater than md;
if equal, equal; and if less, less.

I Suppose nb not greater than mb, so that nd is also not greater than md.

Now $m(a+b)$ is greater than mb:
therefore $m(a+b)$ is greater than nb.
Similarly $m(c+d)$ is greater than nd.

II. Suppose nb greater than mb.

Since $m(a+b)$, mb, $m(c+d)$, md are equimultiples of $(a+b)$, b, $(c+d)$, d,
ma is the same multiple of a that $m(a+b)$ is of $(a+b)$,
and mc is the same multiple of c that $m(c+d)$ is of $(c+d)$,
so that ma, mc are equimultiples of a, c. [V. 5]

Again nb, nd are equimultiples of b, d,
and so are mb, md;
therefore $(n-m)b$, $(n-m)d$ are equimultiples of b, d and, whether $n-m$ is equal to unity or to any other integer [V. 6], it follows, by Def. 5, that, since a, b, c, d are proportionals,
if ma is greater than $(n-m)b$,
then mc is greater than $(n-m)d$;
if equal, equal; and if less, less.

(1) If now $m(a+b)$ is greater than nb, subtracting mb from each, we have
ma is greater than $(n-m)b$;
therefore mc is greater than $(n-m)d$,
and, if we add md to each,
$m(c+d)$ is greater than nd.

(2) Similarly it may be proved that,
if $m(a+b)$ is equal to nb,
then $m(c+d)$ is equal to nd,
and (3) that, if $m(a+b)$ is less than nb,
then $m(c+d)$ is less than nd.

But (under I. above) it was proved that, in the case where nb is not greater than mb,
$m(a+b)$ is always greater than nb,
and $m(c+d)$ is always greater than nd.

Hence, whatever be the values of m and n, $m(c+d)$ is always greater than, equal to, or less than nd according as $m(a+b)$ is greater than, equal to, or less than nb.

Therefore, by Def. 5,
$a+b$ is to b as $c+d$ is to d.

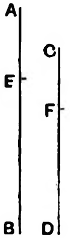
Todhunter gives the following short demonstration from Austin (*Examination of the first six books of Euclid's Elements*).

"Let AE be to EB as CF is to FD:
AB shall be to BE as CD is to DF.

For, because AE is to EB as CF is to FD,
therefore, alternately,

AE is to CF as EB is to FD. [v. 16]

And, as one of the antecedents is to its consequent, so is the sum of the antecedents to the sum of the consequents: [v. 12]



therefore, as EB is to FD, so are AE, EB together to CF, FD together;

that is, AB is to CD as EB is to FD.

Therefore, alternately,

AB is to BE as CD is to FD."

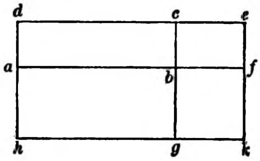
The objection to this proof is that it is only valid in the case where the proposition v. 16 used in it is valid, i.e. where all four magnitudes are of the same kind.

Smith and Bryant's proof avails where all four magnitudes are straight lines, where all four magnitudes are rectilineal areas, or where one antecedent and its consequent are straight lines and the other antecedent and its consequent rectilineal areas.

Suppose that $A : B = C : D$.

First, let all the magnitudes be areas.

Construct a rectangle *abcd* equal to A, and to *bc* apply the rectangle *bcef* equal to B.



Also to *ab*, *bf* apply the rectangles *ag*, *bk* equal to C, D respectively.

Then, since the rectangles *ac*, *be* have equal heights *bc*, they are to one another as their bases. [vi. 1]

$$\begin{aligned} \text{Hence}\quad ab : bf &= \text{rect. } ac : \text{rect. } be \\ &= A : B \\ &= C : D \\ &= \text{rect. } ag : \text{rect. } bk. \end{aligned}$$

Therefore [vi. 1, converse] the rectangles *ag*, *bk* have the same height, so that *k* is on the straight line *hg*.

$$\begin{aligned} \text{Hence}\quad A + B : B &= \text{rect. } ae : \text{rect. } be \\ &= af : bf \\ &= \text{rect. } ak : \text{rect. } bk \\ &= C + D : D. \end{aligned}$$

Secondly, let the magnitudes A, B be straight lines and the magnitudes C, D areas.

Let *ab*, *bf* be equal to the straight lines A, B, and to *ab*, *bf* apply the rectangles *ag*, *bk* equal to C, D respectively.

Then, as before, the rectangles *ag*, *bk* have the same height.

$$\begin{aligned} \text{Now}\quad A + B : B &= af : bf \\ &= \text{rect. } ak : \text{rect. } bk \\ &= C + D : D. \end{aligned}$$

Thirdly, let all the magnitudes be straight lines.

Apply to the straight lines C, D rectangles P, Q having the same height.

Then $P : Q = C : D$. [VI. 1]

Hence, by the second case,

$$A + B : B = P + Q : Q.$$

Also $P + Q : Q = C + D : D$.

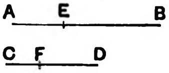
Therefore $A + B : B = C + D : D$.

PROPOSITION 19

If, as a whole is to a whole, so is a part subtracted to a part subtracted, the remainder will also be to the remainder as whole to whole.

For, as the whole *AB* is to the whole *CD*, so let the part *AE* subtracted be to the part *CF* subtracted;

I say that the remainder *EB* will also be to the remainder *FD* as the whole *AB* to the whole *CD*.



For since, as *AB* is to *CD*, so is *AE* to *CF*,

alternately also, as *BA* is to *AE*, so is *DC* to *CF*. [V. 16]

And, since the magnitudes are proportional *componendo*, they will also be proportional *separando*, [V. 17]

that is, as *BE* is to *EA*, so is *DF* to *CF*,

and, alternately,

as *BE* is to *DF*, so is *EA* to *FC*. [V. 16]

But, as *AE* is to *CF*, so by hypothesis is the whole *AB* to the whole *CD*.

Therefore also the remainder *EB* will be to the remainder *FD* as the whole *AB* is to the whole *CD*. [V. 11]

Therefore etc.

[PORISM. From this it is manifest that, if magnitudes be proportional *componendo*, they will also be proportional *convertendo*.]

Q. E. D.

Algebraically, if $a : b = c : d$ (where $c < a$ and $d < b$), then

$$(a - c) : (b - d) = a : b.$$

The "Porism" at the end of this proposition is led up to by a few lines which Heiberg brackets because it is not Euclid's habit to explain a Porism, and indeed a Porism, from its very nature, should not need any explanation, being a sort of by-product appearing without effort or trouble, ἀπραγματεύτως (Proclus, p. 303, 6). But Heiberg thinks that Simson does wrong in finding fault with the argument leading to the "Porism," and that it does contain the true demonstration of *conversion* of a ratio. In this it appears to me that Heiberg is clearly mistaken, the supposed proof on the basis of Prop. 19 being no more correct than the similar attempt to prove the *inversion* of a ratio from Prop. 4. The words are: "And since it was proved that, as AB is to CD, so is EB to FD,

alternately also, as AB is to BE, so is CD to FD:

therefore magnitudes when compounded are proportional.

But it was proved that, as BA is to AE, so is DC to CF, and this is *convertendo*."

It will be seen that this amounts to proving *from the hypothesis* $a : b = c : d$ that the following transformations are simultaneously true, viz.:

$$a : a - c = b : b - d,$$

and

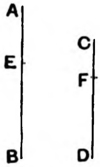
$$a : c = b : d.$$

The former is not proved from the latter as it ought to be if it were intended to prove *conversion*.

The inevitable conclusion is that both the "Porism" and the argument leading up to it are interpolations, though no doubt made, as Heiberg says, before Theon's time.

The *conversion* of ratios does not depend upon v. 19 at all but, as Simson shows in his Proposition E (containing a proof already given by Clavius), on Props. 17 and 18. Prop. E is as follows.

If four magnitudes be proportionals, they are also proportionals by conversion, that is, the first is to its excess above the second as the third is to its excess above the fourth.



Let AB be to BE as CD to DF:
then BA is to AE as DC to CF.

Because AB is to BE as CD to DF,
by division [*separando*],

AE is to EB as CF to FD, [v. 17]

and, by inversion,

BE is to EA as DF to FC.

[Simson's Prop. B directly obtained from v. Def. 5]

Wherefore, by composition [*componendo*],

BA is to AE as DC to CF. [v. 18]

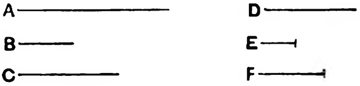
Proposition 20.

If there be three magnitudes, and others equal to them in multitude, which taken two and two are in the same ratio, and if ex aequali *the first be greater than the third, the fourth will also be greater than the sixth; if equal, equal; and, if less, less.*

Let there be three magnitudes A, B, C, and others D, E, F equal to them in multitude, which taken two and two are in the same ratio, so that,

as A is to B, so is D to E,
and as B is to C, so is E to F;

and let A be greater than C *ex aequali*;
I say that D will also be greater than F; if A is equal to C, equal; and, if less, less.



For, since A is greater than C,
and B is some other magnitude,
and the greater has to the same a greater ratio than the less has, [v. 8]
therefore A has to B a greater ratio than C has to B.

But, as A is to B, so is D to E,
and, as C is to B, inversely, so is F to E;
therefore D has also to E a greater ratio than F has to E. [v. 13]

But, of magnitudes which have a ratio to the same, that which has a greater ratio is greater; [v. 10]

therefore D is greater than F.

Similarly we can prove that, if A be equal to C, D will also be equal to F; and if less, less.

Therefore etc.

Q. E. D.

Though, as already remarked, Euclid has not yet given us any definition of *compounded ratios*, Props. 20—23 contain an important part of the theory of such ratios. The term "compounded ratio" is not used, but the propositions connect themselves with the definitions of *ex aequali* in its two forms, the ordinary form defined in Def. 17 and that called *perturbed proportion* in Def. 18. The compounded ratios dealt with in these propositions are those compounded of successive ratios in which the consequent of one is the antecedent of the next, or the antecedent of one is the consequent of the next.

Prop. 22 states the fundamental proposition about the ratio *ex aequali* in its ordinary form, to the effect that,

if a is to b as d is to e,
and b is to c as e is to f,
then a is to c as d is to f,

with the extension to any number of such ratios; Prop. 23 gives the corresponding theorem for the case of *perturbed proportion*, namely that,

if a is to b as e is to f,

and b is to c as d is to e,

then a is to c as d is to f.

Each depends on a preliminary proposition, Prop. 22 on Prop. 20 and Prop. 23 on Prop. 21. The course of the proof will be made most clear by using the algebraic notation.

The preliminary Prop. 20 asserts that,

if $a : b = d : e$,

and $b : c = e : f$,

then, according as $a > = < c$, $d > = < f$.

For, according as a is greater than, equal to, or less than c,

the ratio $a : b$ is greater than, equal to, or less than the ratio $c : b$, [v. 8 or v. 7]

or (since $d : e = a : b$,

and $c : b = f : e$)

the ratio $d : e$ is greater than, equal to, or less than the ratio $f : e$, [by aid of v. 13 and v. 11]

and therefore d is greater than, equal to, or less than f. [v. 10 or v. 9]

It is next proved in Prop. 22 that, by v. 4, the given proportions can be transformed into

$$ma : nb = md : ne,$$

and $nb : pc = ne : pf$,

whence, by v. 20,

according as ma is greater than, equal to, or less than pc,
md is greater than, equal to, or less than pf,

so that, by Def. 5,

$$a : c = d : f.$$

Prop. 23 depends on Prop. 21 in the same way as Prop. 22 on Prop. 20, but the transformation of the ratios in Prop. 23 is to the following:

(1) $ma : mb = ne : nf$

(by a double application of v. 15 and by v. 11),

(2) $mb : nc = md : ne$

(by v. 4, or equivalent steps),

and Prop. 21 is then used.

Simson makes the proof of Prop. 20 slightly more explicit, but the main difference from the text is in the addition of the two other cases which Euclid dismisses with "Similarly we can prove." These cases are:

"Secondly, let A be equal to C; then shall D be equal to F.

Because A and C are equal to one another,

A is to B as C is to B. [v. 7]

But A is to B as D is to E,

and C is to B as F is to E,

wherefore D is to E as F to E; [v. 11]

and therefore D is equal to F. [v. 9]

Next, let A be less than C; then shall D be less than F.

For C is greater than A,
and, as was shown in the first case,

C is to B as F to E,

and, in like manner,

B is to A as E to D;

therefore F is greater than D, by the first case; and therefore D is less than F."

PROPOSITION 21.

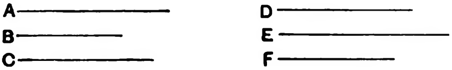
If there be three magnitudes, and others equal to them in multitude, which taken two and two together are in the same ratio, and the proportion of them be perturbed, then, if ex aequali *the first magnitude is greater than the third, the fourth will also be greater than the sixth; if equal, equal; and if less, less.*

Let there be three magnitudes A, B, C, and others D, E, F equal to them in multitude, which taken two and two are in the same ratio, and let the proportion of them be perturbed, so that,

as A is to B, so is E to F,

and, as B is to C, so is D to E,

and let A be greater than C *ex aequali*;
I say that D will also be greater than F; if A is equal to C, equal; and if less, less.



For, since A is greater than C,
and B is some other magnitude,
therefore A has to B a greater ratio than C has to B. [v. 8]

But, as A is to B, so is E to F,
and, as C is to B, inversely, so is E to D.
Therefore also E has to F a greater ratio than E has to D. [v. 13]

But that to which the same has a greater ratio is less; [v. 10]

therefore F is less than D;

therefore D is greater than F.

Similarly we can prove that,

if A be equal to C, D will also be equal to F;

and if less, less.

Therefore etc. Q. E. D.

Algebraically, if $a : b = e : f$,

and $b : c = d : e$,

then, according as $a >=< c$, $d >=< f$.

Simson's alterations correspond to those which he makes in Prop. 20. After the first case he proceeds thus.

"Secondly, let A be equal to C; then shall D be equal to F.

Because A and C are equal,

A is to B as C is to B. [v. 7]

But A is to B as E is to F,

and C is to B as E is to D:

wherefore E is to F as E to D, [v. 11]

and therefore D is equal to F. [v. 9]

Next, let A be less than C; then shall D be less than F.

For C is greater than A,

and, as was shown,

C is to B as E to D,

and, in like manner,

B is to A as F to E;

therefore F is greater than D, by the first case,

and therefore D is less than F."

The proof may be shown thus.

According as $a >=< c$, $a : b >=< c : b$.

But $a : b = e : f$, and, by inversion, $c : b = e : d$.

Therefore, according as $a >=< c$, $e : f >=< e : d$,

and therefore $d >=< f$.

PROPOSITION 22.

If there be any number of magnitudes whatever, and others equal to them in multitude, which taken two and two together are in the same ratio, they will also be in the same ratio ex aequali.

Let there be any number of magnitudes A, B, C, and others D, E, F equal to them in multitude, which taken two and two together are in the same ratio, so that,

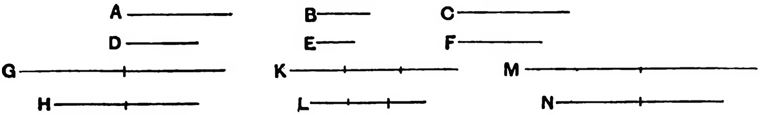
as A is to B, so is D to E,

and, as B is to C, so is E to F;

I say that they will also be in the same ratio *ex aequali*,

< that is, as A is to C, so is D to F >.

For of A, D let equimultiples G, H be taken,
and of B, E other, chance, equimultiples K, L;
and, further, of C, F other, chance, equimultiples M, N.



Then, since, as A is to B, so is D to E,
and of A, D equimultiples G, H have been taken,
and of B, E other, chance, equimultiples K, L,
therefore, as G is to K, so is H to L. [v. 4]

For the same reason also,
as K is to M, so is L to N.

Since, then, there are three magnitudes G, K, M, and others H, L, N equal to them in multitude, which taken two and two together are in the same ratio,
therefore, *ex aequali*, if G is in excess of M, H is also in excess of N;
if equal, equal; and if less, less. [v. 20]

And G, H are equimultiples of A, D,
and M, N other, chance, equimultiples of C, F.

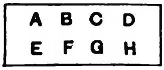
Therefore, as A is to C, so is D to F. [v. Def. 5]

Therefore etc.

Q. E. D.

Euclid enunciates this proposition as true of *any number of magnitudes whatever* forming two sets connected in the manner described, but his proof is confined to the case where each set consists of three magnitudes only. The extension to any number of magnitudes is, however, easy, as shown by Simson.

"Next let there be four magnitudes A, B, C, D, and other four E, F, G, H, which two and two have the same ratio, viz.:



as A is to B, so is E to F,
and as B is to C, so is F to G,
and as C is to D, so is G to H;
A shall be to D as E to H.

Because A, B, C are three magnitudes, and E, F, G other three, which taken two and two have the same ratio,

by the foregoing case,
A is to C as E to G.

But *C* is to *D* as *G* is to *H*;
wherefore again, by the first case,

A is to *D* as *E* to *H*.

And so on, whatever be the number of magnitudes."

PROPOSITION 23.

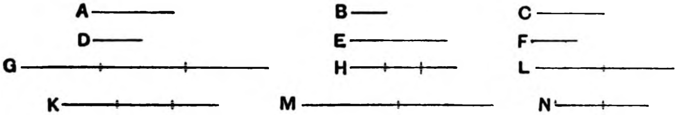
If there be three magnitudes, and others equal to them in multitude, which taken two and two together are in the same ratio, and the proportion of them be perturbed, they will also be in the same ratio ex aequali.

Let there be three magnitudes *A*, *B*, *C*, and others equal to them in multitude, which, taken two and two together, are in the same proportion, namely *D*, *E*, *F*; and let the proportion of them be perturbed, so that,

as *A* is to *B*, so is *E* to *F*,

and, as *B* is to *C*, so is *D* to *E*;

I say that, as *A* is to *C*, so is *D* to *F*.



Of *A*, *B*, *D* let equimultiples *G*, *H*, *K* be taken, and of *C*, *E*, *F* other, chance, equimultiples *L*, *M*, *N*.

Then, since *G*, *H* are equimultiples of *A*, *B*, and parts have the same ratio as the same multiples of them, [v. 15]

therefore, as *A* is to *B*, so is *G* to *H*.

For the same reason also,

as *E* is to *F*, so is *M* to *N*.

And, as *A* is to *B*, so is *E* to *F*;

therefore also, as *G* is to *H*, so is *M* to *N*. [v. 11]

Next, since, as *B* is to *C*, so is *D* to *E*,
alternately, also, as *B* is to *D*, so is *C* to *E*. [v. 16]

And, since *H*, *K* are equimultiples of *B*, *D*,
and parts have the same ratio as their equimultiples,

therefore, as *B* is to *D*, so is *H* to *K*. [v. 15]

But, as *B* is to *D*, so is *C* to *E*;
therefore also, as *H* is to *K*, so is *C* to *E*. [V. 11]

Again, since *L*, *M* are equimultiples of *C*, *E*,
therefore, as *C* is to *E*, so is *L* to *M*. [V. 15]

But, as *C* is to *E*, so is *H* to *K*;
therefore also, as *H* is to *K*, so is *L* to *M*, [V. 11]
and, alternately, as *H* is to *L*, so is *K* to *M*. [V. 16]

But it was also proved that,
as *G* is to *H*, so is *M* to *N*.

Since, then, there are three magnitudes *G*, *H*, *L*, and others equal to them in multitude *K*, *M*, *N*, which taken two and two together are in the same ratio,
and the proportion of them is perturbed,
therefore, *ex aequali*, if *G* is in excess of *L*, *K* is also in excess of *N*;
if equal, equal; and if less, less. [V. 21]

And *G*, *K* are equimultiples of *A*, *D*,
and *L*, *N* of *C*, *F*.

Therefore, as *A* is to *C*, so is *D* to *F*.

Therefore etc.

Q. E. D.

There is an important difference between the version given by Simson of one part of the proof of this proposition and that found in the Greek text of Heiberg. Peyrard's MS. has the version given by Heiberg, but Simson's version has the authority of other MSS. The Basel *editio princeps* gives both versions (Simson's being the first). After it has been proved by means of V. 15 and V. 11 that,

as *G* is to *H*, so is *M* to *N*,

or, with the notation used in the note on Prop. 20,

$$ma : mb = ne : nf,$$

it has to be proved further that,

as *H* is to *L*, so is *K* to *M*,

or $$mb : nc = md : ne,$$

and it is clear that the latter result may be directly inferred from V. 4. The reading translated by Simson makes this inference:

"And because, as *B* is to *C*, so is *D* to *E*,
and *H*, *K* are equimultiples of *B*, *D*,
and *L*, *M* of *C*, *E*,
therefore, as *H* is to *L*, so is *K* to *M*." [V. 4]

The version in Heiberg's text is not only much longer (it adopts the roundabout method of using each of three Propositions V. 11, 15, 16 twice over), but it is open to the objection that it uses v. 16 which is only applicable if the four magnitudes are *of the same kind*; whereas v. 23, the proposition now in question, is not subject to this restriction.

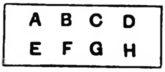
Simson rightly observes that in the last step of the proof it should be stated that "G, K are ***any*** equimultiples ***whatever*** of A, D, and L, N ***any whatever*** of C, F."

He also gives the extension of the proposition to any number of magnitudes, enunciating it thus:

"If there be any number of magnitudes, and as many others, which, taken two and two, in a cross order, have the same ratio; the first shall have to the last of the first magnitudes the same ratio which the first of the others has to the last";

and adding to the proof as follows:

"Next, let there be four magnitudes A, B, C, D, and other four E, F, G, H, which, taken two and two in a cross order, have the same ratio, viz.:



A to B as G to H,
B to C as F to G,
and C to D as E to F;
then A is to D as E to H.

Because A, B, C are three magnitudes, and F, G, H other three which, taken two and two in a cross order, have the same ratio,

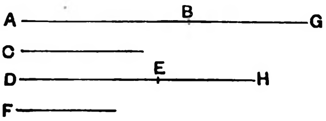
by the first case, A is to C as F to H.
But C is to D as E is to F;
wherefore again, by the first case,
A is to D as E to H.

And so on, whatever be the number of magnitudes."

PROPOSITION 24.

If a first magnitude have to a second the same ratio as a third has to a fourth, and also a fifth have to the second the same ratio as a sixth to the fourth, the first and fifth added together will have to the second the same ratio as the third and sixth have to the fourth.

Let a first magnitude AB have to a second C the same ratio as a third DE has to a fourth F;



and let also a fifth BG have to the second C the same ratio as a sixth EH has to the fourth F;

I say that the first and fifth added together, AG, will have to the second C the same ratio as the third and sixth, DH, has to the fourth F.

For since, as BG is to C, so is EH to F,
inversely, as C is to BG, so is F to EH.

Since, then, as AB is to C, so is DE to F,
and, as C is to BG, so is F to EH,
therefore, *ex aequali*, as AB is to BG, so is DE to EH. [v. 22]

And, since the magnitudes are proportional *separando*, they will also be proportional *componendo*; [v. 18]
therefore, as AG is to GB, so is DH to HE.

But also, as BG is to C, so is EH to F;
therefore, *ex aequali*, as AG is to C, so is DH to F. [v. 22]

Therefore etc. Q. E. D.

Algebraically, if $a : c = d : f$,
and $b : c = e : f$,
then $(a + b) : c = (d + e) : f$.

This proposition is of the same character as those which precede the propositions relating to compounded ratios; but it could not be placed earlier than it is because v. 22 is used in the proof of it.

Inverting the second proportion to

$$c : b = f : e,$$

it follows, by v. 22, that $a : b = d : e$,
whence, by v. 18, $(a + b) : b = (d + e) : e$,
and from this and the second of the two given proportions we obtain, by a fresh application of v. 22,

$$(a + b) : c = (d + e) : f.$$

The first use of v. 22 is important as showing that the opposite process to compounding ratios, or what we should now call *division* of one ratio by another, does not require any new and separate propositions.

Aristotle tacitly uses v. 24 in combination with v. 11 and v. 16, *Meteorologica* III. 5, 376 a 22—26.

Simson adds two corollaries, one of which (Cor. 2) notes the extension to any number of magnitudes.

"The proposition holds true of two ranks of magnitudes whatever be their number, of which each of the first rank has to the second magnitude the same ratio that the corresponding one of the second rank has to a fourth magnitude; as is manifest."

Simson's Cor. 1 states the corresponding proposition to the above with *separando* taking the place of *componendo*, viz., that corresponding to the algebraical form

$$(a - b) : c = (d - e) : f.$$

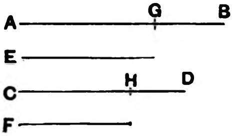
"Cor. 1. If the same hypothesis be made as in the proposition, the excess of the first and fifth shall be to the second as the excess of the third and sixth to the fourth. The demonstration of this is the same with that of the proposition if division be used instead of composition." That is, we use v. 17 instead of v. 18, and conclude that

$$(a - b) : b = (d - e) : e.$$

PROPOSITION 25.

If four magnitudes be proportional, the greatest and the least are greater than the remaining two.

Let the four magnitudes *AB*, *CD*, *E*, *F* be proportional so that, as *AB* is to *CD*, so is *E* to *F*, and let *AB* be the greatest of them and *F* the least;



I say that *AB*, *F* are greater than *CD*, *E*.

For let *AG* be made equal to *E*, and *CH* equal to *F*.

Since, as *AB* is to *CD*, so is *E* to *F*,

and *E* is equal to *AG*, and *F* to *CH*,

therefore, as *AB* is to *CD*, so is *AG* to *CH*.

And since, as the whole *AB* is to the whole *CD*, so is the part *AG* subtracted to the part *CH* subtracted,

the remainder *GB* will also be to the remainder *HD* as the whole *AB* is to the whole *CD*. [v. 19]

But *AB* is greater than *CD*;

therefore *GB* is also greater than *HD*.

And, since *AG* is equal to *E*, and *CH* to *F*,

therefore *AG*, *F* are equal to *CH*, *E*.

And if, *GB*, *HD* being unequal, and *GB* greater, *AG*, *F* be added to *GB* and *CH*, *E* be added to *HD*,

it follows that *AB*, *F* are greater than *CD*, *E*.

Therefore etc.

Q. E. D.

Algebraically, if $a : b = c : d$,

and a is the greatest of the four magnitudes and d the least,

$$a + d > b + c.$$

Simson is right in inserting a word in the setting-out, "let *AB* be the greatest of them and <*consequently*> *F* the least." This follows from the particular case, really included in Def. 5, which Simson makes the subject of his proposition A, the case namely where the equimultiples taken are *once* the several magnitudes.

The proof is as follows.

Since $a : b = c : d$,

$$a - c : b - d = a : b. \qquad \text{[v. 19]}$$

But $a > b$; therefore $(a - c) > (b - d)$. [v. 16 and 14]

Add to each $(c + d)$;

therefore $$(a + d) > (b + c).$$

There is an important particular case of this proposition, which is, however, not mentioned here, viz. the case where $b = c$. The result shows, in this case, that ***the arithmetic mean between two magnitudes is greater than their geometric mean.*** The truth of this is proved for straight lines in VI. 27 by "geometrical algebra," and the theorem forms the *διορισμός* for equations of the second degree.

Simson adds at the end of Book v. four propositions, F, G, H, K, which, however, do not seem to be of sufficient practical use to justify their inclusion here. But he adds at the end of his notes to the Book the following paragraph which deserves quotation word for word.

"The 5th book being thus corrected, I most readily agree to what the learned Dr Barrow says, 'that there is nothing in the whole body of the elements of a more subtile invention, nothing more solidly established, and more accurately handled than the doctrine of proportionals.' And there is some ground to hope that geometers will think that this could not have been said with as good reason, since Theon's time till the present."

Simson's claim herein will readily be admitted by all readers who are competent to form a judgment upon his criticisms and elucidations of Book v.

BOOK VI.

INTRODUCTORY NOTE.

The theory of proportions has been established in Book v. in a perfectly general form applicable to all kinds of magnitudes (although the representation of magnitudes by straight lines gives it a *geometrical* appearance); it is now necessary to apply the theory to the particular case of *geometrical* investigation. The only thing still required in order that this may be done is a proof of the existence of such a magnitude as bears to any given finite magnitude any given finite ratio; and this proof is supplied, so far as regards the subject matter of geometry, by vi. 12 which shows how to construct a fourth proportional to three given straight lines.

A few remarks on the enormous usefulness of the theory of proportions to geometry will not be out of place. We have already in Books i. and ii. made acquaintance with one important part of what has been well called geometrical algebra, the method, namely, of *application of areas*. We have seen that this method, working by the representation of products of two quantities as rectangles, enables us to solve some particular quadratic equations. But the limitations of such a method are obvious. So long as general quantities are represented by *straight lines* only, we cannot, if our geometry is *plane*, deal with products of more than two such quantities; and, even by the use of three dimensions, we cannot work with products of more than three quantities, since no geometrical meaning could be attached to such a product. This limitation disappears so soon as we can represent any general quantity, corresponding to what we denote by a letter in algebra, by a *ratio*; and this we can do because, on the general theory of proportion established in Book v., a ratio may be a ratio of two incommensurable quantities as well as of commensurables. Ratios can be *compounded ad infinitum*, and the division of one ratio by another is equally easy, since it is the same thing as compounding the first ratio with the inverse of the second. Thus e.g. it is seen at once that the *coefficients* in a quadratic of the most general form can be represented by ratios between straight lines, and the solution by means of Books i. and ii. of problems corresponding to quadratic equations with particular coefficients can now be extended to cover *any* quadratic with real roots. As indicated, we can perform, by composition of ratios, the operation corresponding to multiplying algebraical quantities, and this to any extent. We can divide quantities by compounding a ratio with the inverse of the ratio representing the divisor. For the addition and subtraction of quantities we have only to use the geometrical equivalent of bringing to a common denominator, which is effected by means of the fourth proportional.

DEFINITIONS.

1. **Similar rectilineal figures** are such as have their angles severally equal and the sides about the equal angles proportional.

[2. **Reciprocally related** figures. *See note.*]

3. A straight line is said to have been **cut in extreme and mean ratio** when, as the whole line is to the greater segment, so is the greater to the less.

4. The **height** of any figure is the perpendicular drawn from the vertex to the base.

DEFINITION I.

Ὅμοια σχήματα εὐθύγραμμά ἐστιν, ὅσα τάς τε γωνίας ἴσας ἔχει κατὰ μίαν καὶ τὰς περὶ τὰς ἴσας γωνίας πλευρὰς ἀνάλογον.

This definition is quoted by Aristotle, *Anal. post.* II. 17, 99 a 13, where he says that *similarity* (τὸ ὅμοιον) in the case of figures "consists, let us say (ἴσως), in their having their sides proportional and their angles equal." The use of the word ἴσως may suggest that, in Aristotle's time, this definition had not quite established itself in the text-books (Heiberg, *Mathematisches zu Aristoteles*, p. 9).

It was pointed out in Van Swinden's *Elements of Geometry* (Jacobi's edition, 1834, pp. 114—5) that Euclid omits to state an essential part of the definition, namely that "the corresponding sides must be opposite to equal angles," which is necessary in order that the corresponding sides may follow in the same order in both figures.

At the same time the definition states more than is absolutely necessary, for it is true to say that *two polygons are similar when, if the sides and angles are taken in the same order, the angles are equal and the sides about the equal angles are proportional, omitting*

(1) *three consecutive angles,*

or (2) *two consecutive angles and the side common to them,*

or (3) *two consecutive sides and the angle included by them,*

and making no assumption with regard to the omitted sides and angles.

Austin objected to this definition on the ground that it is not obvious that the properties (1) of having their angles respectively equal and (2) of having the sides about the equal angles proportional can *co-exist* in two figures; but, a definition not being concerned to prove the *existence* of the thing defined, the objection falls to the ground. We are properly left to satisfy ourselves as to the existence of similar figures in the course of the exposition in Book VI., where we learn how to construct on any given straight line a rectilineal figure similar to a given one (VI. 18).

DEFINITION 2.

The Greek text gives here a definition of *reciprocally related figures* (ἀντιπεπονθότα σχήματα). "[Two] figures are *reciprocally related* when there are in each of the two figures antecedent and consequent ratios" (Ἀντιπεπονθότα δὲ σχήματά ἐστιν, ὅταν ἐν ἑκατέρῳ τῶν σχημάτων ἡγούμενοί τε καὶ ἑπόμενοι λόγοι ὦσιν). No intelligible meaning can be attached to "antecedent and consequent ratios" here; the sense would require rather "an antecedent and a consequent of (two equal) ratios in each figure." Hence Candalla and Peyrard read λόγων ὅροι ("terms of ratios") instead of λόγοι. Camerer reads λόγων without ὅροι. But the objection to the definition lies deeper. It is never used; when we come, in VI. 14, 15, XI. 34 etc. to parallelograms, triangles etc. having the property indicated, they are not called "reciprocal" parallelograms etc., but parallelograms etc. "*the sides of which are* reciprocally proportional," ὧν ἀντιπεπόνθασιν αἱ πλευραί. Hence Simson appears to be right in condemning the definition; it may have been interpolated from Heron, who has it.

Simson proposes in his note to substitute the following definition. "Two magnitudes are said to be reciprocally proportional to two others when one of the first is to one of the other magnitudes as the remaining one of the last two is to the remaining one of the first." This definition requires that the magnitudes shall be all of the same kind.

DEFINITION 3.

Ἄκρον καὶ μέσον λόγον εὐθεῖα τετμῆσθαι λέγεται, ὅταν ᾖ ὡς ἡ ὅλη πρὸς τὸ μεῖζον τμῆμα, οὕτως τὸ μεῖζον πρὸς τὸ ἔλαττον.

DEFINITION 4.

Ὕψος ἐστὶ παντὸς σχήματος ἡ ἀπὸ τῆς κορυφῆς ἐπὶ τὴν βάσιν κάθετος ἀγομένη.

The definition of "height" is not found in Campanus and is perhaps rightly suspected, since it does not apply in terms to parallelograms, parallelepipeds, cylinders and prisms, though it is used in the *Elements* with reference to these latter figures. Aristotle does not appear to know altitude (ὕψος) in the mathematical sense; he uses κάθετος of triangles (*Meteorologica* III. 3, 373 a 11). The term is however readily understood, and scarcely requires definition.

[DEFINITION 5.

Λόγος ἐκ λόγων συγκεῖσθαι λέγεται, ὅταν αἱ τῶν λόγων πηλικότητες ἐφ' ἑαυτὰς πολλαπλασιασθεῖσαι ποιῶσί τινα.

"A ratio is said to be compounded of ratios when the sizes (πηλικότητες) of the ratios multiplied together make some (? ratio, or size)."]

As already remarked (pp. 116, 132), it is beyond doubt that this definition of ratio is interpolated. It has little MS. authority. The best MS. (P) only has it in the margin; it is omitted altogether in Campanus' translation from the

Arabic; and the other MSS. which contain it do not agree in the position which they give to it. There is no reference to the definition in the place where compound ratio is mentioned for the first time (VI. 23), nor anywhere else in Euclid; neither is it ever referred to by the other great geometers, Archimedes, Apollonius and the rest. It appears to be only twice mentioned at all, (1) in the passage of Eutocius referred to above (p. 116) and (2) by Theon in his commentary on Ptolemy's σύνταξις. Moreover the content of the definition is in itself suspicious. It speaks of the "sizes of ratios being multiplied together (literally, into themselves)," an operation unknown to geometry. There is no wonder that Eutocius, and apparently Theon also, in their efforts to explain it, had to give the word πηλικότης a meaning which has no application except in the case of such ratios as can be expressed by numbers (Eutocius e.g. making it the "number by which the ratio is called"). Nor is it surprising that Wallis should have found it necessary to substitute for the "quantitas" of Commandinus a different translation, "quantuplicity," which he said was represented by the "*exponent of the ratio*" (rationis exponens), what Peletarius had described as "denominatio ipsae proportionis" and Clavius as "denominator." The fact is that the definition is ungeometrical and useless, as was already seen by Savile, in whose view it was one of the two blemishes in the body of geometry (the other being of course Postulate 5).

It is right to add that Hultsch (art. "Eukleides" in Pauly-Wissowa's *Real-Encyclopädie der classischen Altertumswissenschaft*) thought the definition genuine. His grounds are (1) that it stood in the παλαιὰ ἔκδοσις represented by P (though P has it in the margin only) and (2) that some explanation on the subject must have been given by way of preparation for VI. 23, while there is nothing in the definition which is *inconsistent* with the mode of statement of VI. 23. If the definition is after all genuine, I should be inclined to regard it as a mere survival from earlier textbooks, like the first of the two alternative definitions of a solid angle (XI. Def. 11); for its form seems to suit the old theory of proportion, applicable to commensurable magnitudes only, better than the generalised theory of Eudoxus.

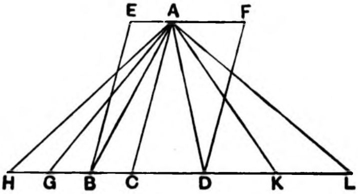
BOOK VI. PROPOSITIONS.

PROPOSITION 1.

Triangles and parallelograms which are under the same height are to one another as their bases.

Let ABC, ACD be triangles and EC, CF parallelograms under the same height;

I say that, as the base BC is to the base CD, so is the triangle ABC to the triangle ACD, and the parallelogram EC to the parallelogram CF.



For let BD be produced in both directions to the points H, L and let [any number of straight lines] BG, GH be made equal to the base BC, and any number of straight lines DK, KL equal to the base CD;

let AG, AH, AK, AL be joined.

Then, since CB, BG, GH are equal to one another,

the triangles ABC, AGB, AHG are also equal to one another. [I. 38]

Therefore, whatever multiple the base HC is of the base BC, that multiple also is the triangle AHC of the triangle ABC.

For the same reason,

whatever multiple the base LC is of the base CD, that multiple also is the triangle ALC of the triangle ACD;

and, if the base HC is equal to the base CL, the triangle AHC is also equal to the triangle ACL, [I. 38] if the base *HC* is in excess of the base *CL*, the triangle *AHC* is also in excess of the triangle *ACL*,

and, if less, less.

Thus, there being four magnitudes, two bases *BC*, *CD* and two triangles *ABC*, *ACD*,

equimultiples have been taken of the base *BC* and the triangle *ABC*, namely the base *HC* and the triangle *AHC*,

and of the base *CD* and the triangle *ADC* other, chance, equimultiples, namely the base *LC* and the triangle *ALC*;

and it has been proved that,

if the base *HC* is in excess of the base *CL*, the triangle *AHC* is also in excess of the triangle *ALC*;

if equal, equal; and, if less, less.

Therefore, as the base *BC* is to the base *CD*, so is the triangle *ABC* to the triangle *ACD*. [V. Def. 5]

Next, since the parallelogram *EC* is double of the triangle *ABC*, [I. 41]

and the parallelogram *FC* is double of the triangle *ACD*,

while parts have the same ratio as the same multiples of them, [V. 15]

therefore, as the triangle *ABC* is to the triangle *ACD*, so is the parallelogram *EC* to the parallelogram *FC*.

Since, then, it was proved that, as the base *BC* is to *CD*, so is the triangle *ABC* to the triangle *ACD*,

and, as the triangle *ABC* is to the triangle *ACD*, so is the parallelogram *EC* to the parallelogram *CF*,

therefore also, as the base *BC* is to the base *CD*, so is the parallelogram *EC* to the parallelogram *FC*. [V. 11]

Therefore etc.

Q. E. D.

4. **Under the same height.** The Greek text has "under the same height *AC*," with a figure in which the side *AC* common to the two triangles is perpendicular to the base and is therefore itself the "height." But, even if the two triangles are placed contiguously so as to have a common side *AC*, it is quite gratuitous to require it to be perpendicular to the base. Theon, on this occasion making an improvement, altered to "which are (ὄντα) under the same height, (namely) the perpendicular drawn from *A* to *BD*." I have ventured to alter so far as to omit "*AC*" and to draw the figure in the usual way.

14. **ABC, AGB, AHG.** Euclid, indifferent to exact order, writes "*AHG*, *AGB*, *ABC*."

46. **Since then it was proved that, as the base BC is to CD, so is the triangle ABC to the triangle ACD.** Here again words have to be supplied in translating the extremely terse Greek ἐπεὶ οὖν ἐδείχθη, ὡς μὲν ἡ βάσις ΒΓ πρὸς τὴν ΓΔ, οὕτως τὸ ΑΒΓ τρίγωνον πρὸς τὸ ΑΓΔ τρίγωνον, literally "since was proved, as the base *BC* to *CD*, so the triangle *ABC* to the triangle *ACD*." Cf. note on V. 16, p. 165.

The proof assumes—what is however an obvious deduction from I. 38—that, of triangles or parallelograms on *unequal* bases and between the same parallels, the greater is that which has the greater base.

It is of course not necessary that the two given triangles should have a common side, as in the figure; the proof is just as easy if they have not. The proposition being equally true of triangles and parallelograms of *equal* heights, Simson states this fact in a corollary thus:

"From this it is plain that triangles and parallelograms that have equal altitudes are to one another as their bases.

Let the figures be so placed as to have their bases in the same straight line; and, if we draw perpendiculars from the vertices of the triangles to the bases, the straight line which joins the vertices is parallel to that in which their bases are, because the perpendiculars are both equal and parallel to one another [I. 33]. Then, if the same construction be made as in the proposition, the demonstration will be the same."

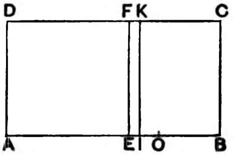
The object of placing the bases in one straight line is to get the triangles and parallelograms *within the same parallels*. Cf. Proclus' remark on I. 38 (p. 405, 17) that having the same height is the same thing as being in the same parallels.

Rectangles, or right-angled triangles, which have one of the sides about the right angle of the same length can be placed so that the equal sides coincide and the others are in a straight line. If then we call the common side the base, the rectangles or the right-angled triangles are to one another as their heights, by VI. 1. Now, instead of each right-angled triangle or rectangle, we can take any other triangle or parallelogram respectively with an equal base and between the same parallels. Thus

Triangles and parallelograms having equal bases are to one another as their heights.

Legendre and those authors of modern text-books who follow him in basing their treatment of proportion on the algebraical definition are obliged to divide their proofs of propositions like this into two parts, the first of which proves the particular theorem in the case where the magnitudes are commensurable, and the second extends it to the case where they are incommensurable.

Legendre (*Éléments de Géométrie*, III. 3) uses for this extension a rigorous method by *reductio ad absurdum* similar to that used by Archimedes in his treatise *On the equilibrium of planes*, I. 7. The following is Legendre's proof of the extension of VI. 1 to incommensurable parallelograms and bases.



The proposition having been proved for commensurable bases, let there be two rectangles $ABCD$, $AEFD$ as in the figure, on bases AB, AE which are incommensurable with one another.

To prove that rect. $ABCD$: rect. $AEFD = AB : AE$.

For, if not, let rect. $ABCD$: rect. $AEFD = AB : AO$, ... (1)

where AO is (for instance) greater than AE.

Divide AB into equal parts each of which is less than EO, and mark off on AO lengths equal to one of the parts; then there will be at least one point of division between E and O.

Let it be I, and draw IK parallel to EF.

Then the rectangles $ABCD$, $AIKD$ are in the ratio of the bases AB, AI, since the latter are commensurable.

Therefore, inverting the proportion,

$$\text{rect. } AIKD : \text{rect. } ABCD = AI : AB \quad\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

From this and (1), *ex aequali*,

$$\text{rect. } AIKD : \text{rect. } AEFD = AI : AO.$$

But $AO > AI$; therefore rect. $AEFD >$ rect. $AIKD$.

But this is impossible, for the rectangle $AEFD$ is less than the rectangle $AIKD$.

Similarly an impossibility can be proved if $AO < AE$.

Therefore rect. $ABCD$: rect. $AEFD = AB : AE$.

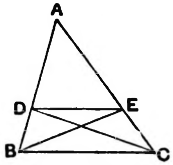
Some modern American and German text-books adopt the less rigorous method of appealing to the theory of *limits*.

PROPOSITION 2.

If a straight line be drawn parallel to one of the sides of a triangle, it will cut the sides of the triangle proportionally; and, if the sides of the triangle be cut proportionally, the line joining the points of section will be parallel to the remaining side of the triangle.

For let DE be drawn parallel to BC, one of the sides of the triangle ABC;

I say that, as BD is to DA, so is CE to EA.



For let BE, CD be joined.

Therefore the triangle BDE is equal to the triangle CDE;

for they are on the same base DE and in the same parallels DE, BC. [I. 38]

And the triangle ADE is another area.

But equals have the same ratio to the same; [v. 7]

therefore, as the triangle BDE is to the triangle ADE, so is the triangle CDE to the triangle ADE.

But, as the triangle BDE is to ADE, so is BD to DA;

for, being under the same height, the perpendicular drawn from E to AB, they are to one another as their bases. [VI. 1]

For the same reason also,

as the triangle CDE is to ADE, so is CE to EA.

Therefore also, as BD is to DA, so is CE to EA. [v. 11]

Again, let the sides *AB*, *AC* of the triangle *ABC* be cut proportionally, so that, as *BD* is to *DA*, so is *CE* to *EA*; and let *DE* be joined.

I say that *DE* is parallel to *BC*.

For, with the same construction,

since, as *BD* is to *DA*, so is *CE* to *EA*,

but, as *BD* is to *DA*, so is the triangle *BDE* to the triangle *ADE*,

and, as *CE* is to *EA*, so is the triangle *CDE* to the triangle *ADE*, [VI. 1]

therefore also,

as the triangle *BDE* is to the triangle *ADE*, so is the triangle *CDE* to the triangle *ADE*. [V. 11]

Therefore each of the triangles *BDE*, *CDE* has the same ratio to *ADE*.

Therefore the triangle *BDE* is equal to the triangle *CDE*; [V. 9]

and they are on the same base *DE*.

But equal triangles which are on the same base are also in the same parallels. [I. 39]

Therefore *DE* is parallel to *BC*.

Therefore etc.

Q. E. D.

Euclid evidently did not think it worth while to distinguish in the enunciation, or in the figure, the cases in which the parallel to the base cuts the other two sides produced (*a*) beyond the point in which they intersect, (*b*) in the other direction. Simson gives the three figures and inserts words in the enunciation, reading "it shall cut the other sides, *or those sides produced*, proportionally" and "if the sides, *or the sides produced*, be cut proportionally."

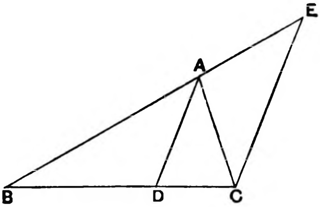
Todhunter observes that the second part of the enunciation ought to make it clear which segments in the proportion correspond to which. Thus e.g., if *AD* were double of *DB*, and *CE* double of *EA*, the sides would be cut proportionally, but *DE* would not be parallel to *BC*. The omission could be supplied by saying "and if the sides of the triangle be cut proportionally *so that the segments adjacent to the third side are corresponding terms in the proportion*."

PROPOSITION 3.

If an angle of a triangle be bisected and the straight line cutting the angle cut the base also, the segments of the base will have the same ratio as the remaining sides of the triangle; and, if the segments of the base have the same ratio as the *remaining sides of the triangle, the straight line joined from the vertex to the point of section will bisect the angle of the triangle.*

Let *ABC* be a triangle, and let the angle *BAC* be bisected by the straight line *AD*;

I say that, as *BD* is to *CD*, so is *BA* to *AC*.



For let *CE* be drawn through *C* parallel to *DA*, and let *BA* be carried through and meet it at *E*.

Then, since the straight line *AC* falls upon the parallels *AD*, *EC*,

the angle *ACE* is equal to the angle *CAD*. [I. 29]

But the angle *CAD* is by hypothesis equal to the angle *BAD*;

therefore the angle *BAD* is also equal to the angle *ACE*.

Again, since the straight line *BAE* falls upon the parallels *AD*, *EC*,

the exterior angle *BAD* is equal to the interior angle *AEC*. [I. 29]

But the angle *ACE* was also proved equal to the angle *BAD*;

therefore the angle *ACE* is also equal to the angle *AEC*,

so that the side *AE* is also equal to the side *AC*. [I. 6]

And, since *AD* has been drawn parallel to *EC*, one of the sides of the triangle *BCE*,

therefore, proportionally, as *BD* is to *DC*, so is *BA* to *AE*.

But *AE* is equal to *AC*; [VI. 2]

therefore, as *BD* is to *DC*, so is *BA* to *AC*.

Again, let *BA* be to *AC* as *BD* to *DC*, and let *AD* be joined;

I say that the angle *BAC* has been bisected by the straight line *AD*.

For, with the same construction,

since, as *BD* is to *DC*, so is *BA* to *AC*, and also, as *BD* is to *DC*, so is *BA* to *AE*: for *AD* has been drawn parallel to *EC*, one of the sides of the triangle *BCE*: [VI. 2]

therefore also, as *BA* is to *AC*, so is *BA* to *AE*. [V. 11]

Therefore *AC* is equal to *AE*, [V. 9]

so that the angle *AEC* is also equal to the angle *ACE*. [I. 5]

But the angle *AEC* is equal to the exterior angle *BAD*, [I. 29]

and the angle *ACE* is equal to the alternate angle *CAD*; [*id.*]

therefore the angle *BAD* is also equal to the angle *CAD*.

Therefore the angle *BAC* has been bisected by the straight line *AD*.

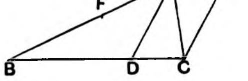
Therefore etc.

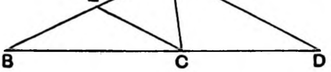
Q. E. D.

The demonstration assumes that *CE will* meet *BA* produced in some point *E*. This is proved in the same way as it is proved in VI. 4 that *BA*, *ED* will meet if produced. The angles *ABD*, *BDA* in the figure of VI. 3 are together less than two right angles, and the angle *BDA* is equal to the angle *BCE*, since *DA*, *CE* are parallel. Therefore the angles *ABC*, *BCE* are together less than two right angles; and *BA*, *CE* must meet, by I. Post. 5.

The corresponding proposition about the segments into which *BC* is divided *externally* by the bisector of the *external angle* at *A* when that bisector meets *BC* produced (i.e. when the sides *AB*, *AC* are not equal) is important. Simson gives it as a separate proposition, A, noting the fact that Pappus assumes the result without proof (Pappus, VII. p. 730, 24).

The best plan is however, as De Morgan says, to combine Props. 3 and A in one proposition, which may be enunciated thus: *If an angle of a triangle be bisected internally or externally by a straight line which cuts the opposite side or the opposite side produced, the segments of that side will have the same ratio as the other sides of the triangle; and, if a side of a triangle be divided internally or externally so that its segments have the same ratio as the other sides of the triangle, the straight line drawn from the point of section to the angular point which is opposite to the first mentioned side will bisect the interior or exterior angle at that angular point.*

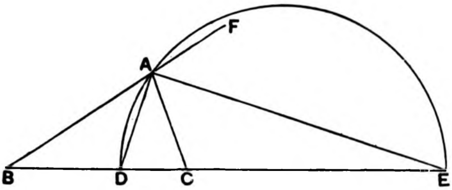




Let *AC* be the smaller of the two sides *AB*, *AC*, so that the bisector *AD* of the exterior angle at *A* may meet *BC* produced beyond *C*. Draw *CE* through *C* parallel to *DA*, meeting *BA* in *E*.

Then, if *FAC* is the exterior angle bisected by *AD* in the case of external bisection, and if a point *F* is taken on *AB* in the figure of VI. 3, the proof of

VI. 3 can be used almost word for word for the other case. We have only to speak of the angle "*FAC*" for the angle "*BAC*," and of the angle "*FAD*" for the angle "*BAD*" wherever they occur, to say "let *BA*, or *BA* produced, meet *CE* in *E*," and to substitute "*BA* or *BA* produced" for "*BAE*" lower down.



If *AD*, *AE* be the internal and external bisectors of the angle *A* in a triangle of which the sides *AB*, *AC* are unequal, *AC* being the smaller, and if *AD*, *AE* meet *BC* and *BC* produced in *D*, *E* respectively,

the ratios of *BD* to *DC* and of *BE* to *EC* are alike equal to the ratio of *BA* to *AC*.

Therefore *BE* is to *EC* as *BD* to *DC*,

that is, *BE* is to *EC* as the difference between *BE* and *ED* is to the difference between *ED* and *EC*,

whence *BE*, *ED*, *EC* are in *harmonic progression*, or *DE* is a *harmonic mean* between *BE* and *EC*, or again *B*, *D*, *C*, *E* is a *harmonic range.*

Since the angle *DAC* is half of the angle *BAC*,

and the angle *CAE* half of the angle *CAF*,

while the angles *BAC*, *CAF* are equal to two right angles,

the angle *DAE* is a right angle.

Hence the circle described on *DE* as diameter passes through *A*.

Now, if the ratio of *BA* to *AC* is given, and if *BC* is given, the points *D*, *E* on *BC* and *BC* produced are given, and therefore so is the circle on *D*, *E* as diameter. Hence *the locus of a point such that its distances from two given points are in a given ratio* (*not being a ratio of equality*) *is a circle.*

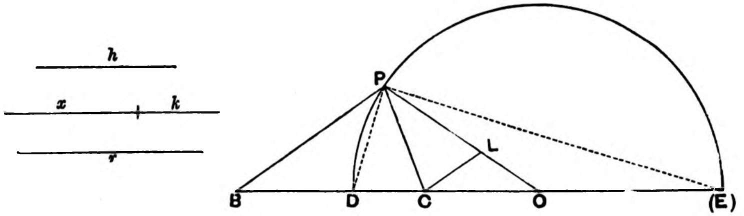
This locus was discussed by Apollonius in his *Plane Loci*, Book II., as we know from Pappus (VII. p. 666), who says that the book contained the theorem that, if from two given points straight lines inflected to another point are in a given ratio, the point in which they meet will lie on either a straight line or a circumference of a circle. The straight line is of course the locus when the ratio is one of equality. The other case is quoted in the following form by Eutocius (Apollonius, ed. Heiberg, II. pp. 180—4).

Given two points in a plane and a proportion between unequal straight lines, it is possible to describe a circle in the plane so that the straight lines inflected from the given points to the circumference of the circle shall have a ratio the same as the given one.

Apollonius' construction, as given by Eutocius, is remarkable because he makes no use of either of the points *D*, *E*. He finds *O*, the centre of the required circle, and the length of its radius directly from the data *BC* and the given ratio which we will call $h : k$. But the construction was not discovered by Apollonius; it belongs to a much earlier date, since it appears in exactly the same form in Aristotle, *Meteorologica* III. 5, 376 a 3 sqq. The analysis leading up to the construction is, as usual, not given either by Aristotle or Eutocius. We are told to take three straight lines x, CO (a length measured along BC produced beyond C, where B, C are the points at which the greater and smaller of the inflected lines respectively terminate), and r, such that, if $h : k$ be the given ratio and $h > k$,

$$k : h = h : k + x, \quad \ldots\ldots\ldots\ldots\ldots\ldots (\alpha)$$

$$x : BC = k : CO = h : r. \quad \ldots\ldots\ldots\ldots\ldots\ldots (\beta)$$



This determines the position of O, and the length of r, the radius of the required circle. The circle is then drawn, *any* point P is taken on it and joined to B, C respectively, and it is proved that

$$PB : PC = h : k.$$

We may conjecture that the analysis proceeded somewhat as follows.

It would be seen that B, C are "conjugate points" with reference to the circle on DE as diameter. (Cf. Apollonius, *Conics*, I. 36, where it is proved, in terms, for a circle as well as for an ellipse and a hyperbola, that, if the polar of B meets the diameter DE in C, then $EC : CD = EB : BD$.)

If O be the middle point of DE, and therefore the centre of the circle, D, E may be eliminated, as in the *Conics*, I. 37, thus.

Since $$EC : CD = EB : BD,$$

it follows that $$EC + CD : EC \sim CD = EB + BD : EB \sim BD,$$

or $$2OD : 2OC = 2OB : 2OD,$$

that is, $$BO \, . \, OC = OD^2 = r^2, \text{ say.}$$

If therefore P be any point on the circle with centre O and radius r,

$$BO : OP = OP : OC,$$

so that BOP, POC are similar triangles.

In addition, $$h : k = BD : DC = BE : EC$$
$$= BD + BE : DE = BO : r.$$

Hence we require that

$$BO : r = r : OC = BP : PC = h : k. \quad \ldots\ldots\ldots\ldots (\delta)$$

Therefore, alternately,

$$k : CO = h : r,$$

which is the second relation in (β) above.

Now assume a length x such that each of the last ratios is equal to $x : BC$, as in (β).

Then $$x : BC = k : CO = h : r.$$

Therefore $$x + k : BO = h : r,$$
and, alternately, $$x + k : h = BO : r$$
$$= h : k, \text{ from } (\delta) \text{ above};$$
and this is the relation (α) which remained to be found.

Apollonius' proof of the construction is given by Eutocius, who begins by saying that it is manifest that r is a mean proportional between BO and OC. This is seen as follows.

From (β) we derive
$$x : BC = k : CO = h : r = (k + x) : BO,$$
whence $$BO : r = (k + x) : h$$
$$= h : k, \text{ by } (\alpha),$$
$$= r : CO, \text{ by } (\beta),$$
and therefore $$r^2 = BO . CO.$$

But the triangles BOP, POC have the angle at O common, and, since $BO : OP = OP : OC$, the triangles are similar and the angles OPC, OBP are equal.

[Up to this point Aristotle's proof is exactly the same; from this point it diverges slightly.]

If now CL be drawn parallel to BP meeting OP in L, the angles BPC LCP are equal also.

Therefore the triangles BPC, PCL are similar, and
$$BP : PC = PC : CL,$$
whence $$BP^2 : PC^2 = BP : CL$$
$$= BO : OC, \text{ by parallels},$$
$$= BO^2 : OP^2 \text{ (since } BO : OP = OP : OC\text{)}.$$
Therefore $$BP : PC = BO : OP$$
$$= h : k \text{ (for } OP = r\text{)}.$$

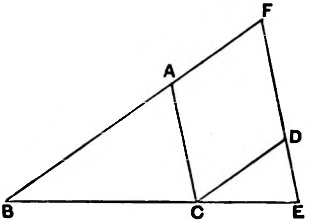
[Aristotle infers this more directly from the similar triangles POB, COP. Since these triangles are similar,
$$OP : CP = OB : BP,$$
whence $$BP : PC = BO : OP$$
$$= h : k.]$$

Apollonius proves lastly, by *reductio ad absurdum*, that the last equation cannot be true with reference to any point P which is not on the circle so described.

PROPOSITION 4.

In equiangular triangles the sides about the equal angles are proportional, and those are corresponding sides which subtend the equal angles.

Let *ABC*, *DCE* be equiangular triangles having the angle *ABC* equal to the angle *DCE*, the angle *BAC* to the angle *CDE*, and further the angle *ACB* to the angle *CED*;



I say that in the triangles *ABC*, *DCE* the sides about the equal angles are proportional, and those are corresponding sides which subtend the equal angles.

For let *BC* be placed in a straight line with *CE*.

Then, since the angles *ABC*, *ACB* are less than two right angles, [I. 17]

and the angle *ACB* is equal to the angle *DEC*,

therefore the angles *ABC*, *DEC* are less than two right angles;

therefore *BA*, *ED*, when produced, will meet. [I. Post. 5]

Let them be produced and meet at *F*.

Now, since the angle *DCE* is equal to the angle *ABC*,

BF is parallel to *CD*. [I. 28]

Again, since the angle *ACB* is equal to the angle *DEC*,

AC is parallel to *FE*. [I. 28]

Therefore *FACD* is a parallelogram;

therefore *FA* is equal to *DC*, and *AC* to *FD*. [I. 34]

And, since *AC* has been drawn parallel to *FE*, one side of the triangle *FBE*,

therefore, as *BA* is to *AF*, so is *BC* to *CE*. [VI. 2]

But *AF* is equal to *CD*;

therefore, as *BA* is to *CD*, so is *BC* to *CE*,

and alternately, as *AB* is to *BC*, so is *DC* to *CE*. [V. 16]

Again, since *CD* is parallel to *BF*,

therefore, as *BC* is to *CE*, so is *FD* to *DE*. [VI. 2]

But *FD* is equal to *AC*;

therefore, as *BC* is to *CE*, so is *AC* to *DE*,

and alternately, as *BC* is to *CA*, so is *CE* to *ED*. [V. 16]

Since then it was proved that,

as AB is to BC, so is DC to CE,

and, as BC is to CA, so is CE to ED;

therefore, *ex aequali*, as BA is to AC, so is CD to DE. [V. 22]

Therefore etc.

Q. E. D.

Todhunter remarks that "the manner in which the two triangles are to be placed is very imperfectly described; their bases are to be in the same straight line and contiguous, their vertices are to be on the same side of the base, and each of the two angles which have a common vertex is to be equal to the remote angle of the other triangle." But surely Euclid's description is sufficient, except for not saying that B and D must be on the same side of BCE.

VI. 4 can be immediately deduced from VI. 2 if we superpose one triangle on the other three times in succession, so that each angle successively coincides with its equal, the triangles being similarly situated, e.g. if (A, B, C and D, E, F being the equal angles respectively) we apply the angle DEF to the angle ABC so that D lies on AB (produced if necessary) and F on BC (produced if necessary). De Morgan prefers this method. "Abandon," he says, "the peculiar mode of construction by which Euclid proves two cases at once; make an angle coincide with its equal, and suppose this process repeated three times, one for each angle."

PROPOSITION 5.

If two triangles have their sides proportional, the triangles will be equiangular and will have those angles equal which the corresponding sides subtend.

Let ABC, DEF be two triangles having their sides proportional, so that,

as AB is to BC, so is DE to EF,

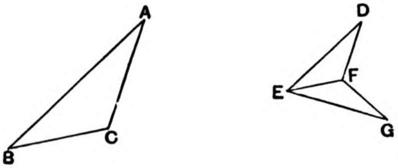
as BC is to CA, so is EF to FD,

and further, as BA is to AC, so is ED to DF;

I say that the triangle ABC is equiangular with the triangle DEF, and they will have those angles equal which the corresponding sides subtend, namely the angle ABC to the angle DEF, the angle BCA to the angle EFD, and further the angle BAC to the angle EDF.

For on the straight line EF, and at the points E, F on it, let there be constructed the angle FEG equal to the angle ABC, and the angle EFG equal to the angle ACB; [I. 23] therefore the remaining angle at A is equal to the remaining angle at G. [I. 32]

Therefore the triangle ABC is equiangular with the triangle GEF.



Therefore in the triangles ABC, GEF the sides about the equal angles are proportional, and those are corresponding sides which subtend the equal angles; [VI. 4]

therefore, as AB is to BC, so is GE to EF.

But, as AB is to BC, so by hypothesis is DE to EF;

therefore, as DE is to EF, so is GE to EF. [V. 11]

Therefore each of the straight lines DE, GE has the same ratio to EF;

therefore DE is equal to GE. [V. 9]

For the same reason

DF is also equal to GF.

Since then DE is equal to EG,

and EF is common,

the two sides DE, EF are equal to the two sides GE, EF;

and the base DF is equal to the base FG;

therefore the angle DEF is equal to the angle GEF, [I. 8]

and the triangle DEF is equal to the triangle GEF,

and the remaining angles are equal to the remaining angles, namely those which the equal sides subtend. [I. 4]

Therefore the angle DFE is also equal to the angle GFE,

and the angle EDF to the angle EGF.

And, since the angle FED is equal to the angle GEF,

while the angle GEF is equal to the angle ABC,

therefore the angle ABC is also equal to the angle DEF.

For the same reason

the angle *ACB* is also equal to the angle *DFE*,

and further, the angle at *A* to the angle at *D*;

therefore the triangle *ABC* is equiangular with the triangle *DEF*.

Therefore etc.

Q. E. D.

This proposition is the complete converse, VI. 6 a partial converse, of VI. 4. Todhunter, after Walker, remarks that the enunciation should make it clear that the sides of the triangles *taken in order* are proportional. It is quite possible that there should be two triangles *ABC*, *DEF* such that

AB is to *BC* as *DE* to *EF*,

and *BC* is to *CA* as *DF* is to *ED* (instead of *EF* to *FD*),

so that *AB* is to *AC* as *DF* to *EF*

(*ex aequali* in *perturbed proportion*);

in this case the sides of the triangles are proportional, but not in the same order, and the triangles are not necessarily equiangular to one another. For a numerical illustration we may suppose the sides of one triangle to be 3, 4 and 5 feet respectively, and those of another to be 12, 15 and 20 feet respectively.

In VI. 5 there is the same *apparent* avoidance of indirect demonstration which has been noticed on I. 48.

PROPOSITION 6.

If two triangles have one angle equal to one angle and the sides about the equal angles proportional, the triangles will be equiangular and will have those angles equal which the corresponding sides subtend.

Let *ABC*, *DEF* be two triangles having one angle *BAC* equal to one angle *EDF* and the sides about the equal angles proportional, so that,

as *BA* is to *AC*, so is *ED* to *DF*;

I say that the triangle *ABC* is equiangular with the triangle *DEF*, and will have the angle *ABC* equal to the angle *DEF*, and the angle *ACB* to the angle *DFE*.

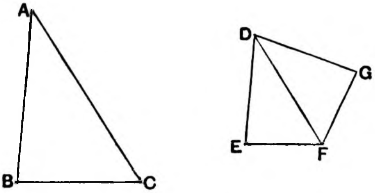
For on the straight line *DF*, and at the points *D*, *F* on it, let there be constructed the angle *FDG* equal to either of the angles *BAC*, *EDF*, and the angle *DFG* equal to the angle *ACB*; [I. 23]

therefore the remaining angle at *B* is equal to the remaining angle at *G*. [I. 32]

Therefore the triangle *ABC* is equiangular with the triangle *DGF*.

Therefore, proportionally, as *BA* is to *AC*, so is *GD* to *DF*. [VI. 4]

But, by hypothesis, as *BA* is to *AC*, so also is *ED* to *DF*; therefore also, as *ED* is to *DF*, so is *GD* to *DF*. [V. 11]



Therefore *ED* is equal to *DG*; [V. 9]
and *DF* is common;

therefore the two sides *ED*, *DF* are equal to the two sides *GD*, *DF*; and the angle *EDF* is equal to the angle *GDF*;

therefore the base *EF* is equal to the base *GF*,

and the triangle *DEF* is equal to the triangle *DGF*,

and the remaining angles will be equal to the remaining angles, namely those which the equal sides subtend. [I. 4]

Therefore the angle *DFG* is equal to the angle *DFE*,

and the angle *DGF* to the angle *DEF*.

But the angle *DFG* is equal to the angle *ACB*;

therefore the angle *ACB* is also equal to the angle *DFE*.

And, by hypothesis, the angle *BAC* is also equal to the angle *EDF*;

therefore the remaining angle at *B* is also equal to the remaining angle at *E*; [I. 32]

therefore the triangle *ABC* is equiangular with the triangle *DEF*.

Therefore etc.

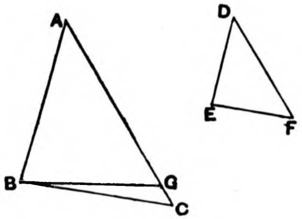
Q. E. D.

PROPOSITION 7.

If two triangles have one angle equal to one angle, the sides about other angles proportional, and the remaining angles either both less or both not less than a right angle, the triangles will be equiangular and will have those angles equal, the sides about which are proportional.

Let ABC, DEF be two triangles having one angle equal to one angle, the angle BAC to the angle EDF, the sides about other angles ABC, DEF proportional, so that, as AB is to BC, so is DE to EF, and, first, each of the remaining angles at C, F less than a right angle;

I say that the triangle ABC is equiangular with the triangle DEF, the angle ABC will be equal to the angle DEF, and the remaining angle, namely the angle at C, equal to the remaining angle, the angle at F.



For, if the angle ABC is unequal to the angle DEF, one of them is greater.

Let the angle ABC be greater;

and on the straight line AB, and at the point B on it, let the angle ABG be constructed equal to the angle DEF. [I. 23]

Then, since the angle A is equal to D,

and the angle ABG to the angle DEF,

therefore the remaining angle AGB is equal to the remaining angle DFE. [I. 32]

Therefore the triangle ABG is equiangular with the triangle DEF.

Therefore, as AB is to BG, so is DE to EF [VI. 4]

But, as DE is to EF, so by hypothesis is AB to BC;

therefore AB has the same ratio to each of the straight lines BC, BG; [V. 11]

therefore BC is equal to BG, [V. 9]

so that the angle at C is also equal to the angle BGC. [I. 5]

But, by hypothesis, the angle at *C* is less than a right angle;

therefore the angle *BGC* is also less than a right angle;

so that the angle *AGB* adjacent to it is greater than a right angle. [I. 13]

And it was proved equal to the angle at *F*;

therefore the angle at *F* is also greater than a right angle.

But it is by hypothesis less than a right angle: which is absurd.

Therefore the angle *ABC* is not unequal to the angle *DEF*;

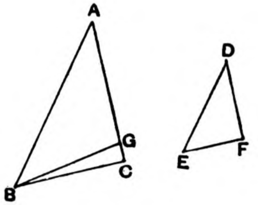
therefore it is equal to it.

But the angle at *A* is also equal to the angle at *D*;

therefore the remaining angle at *C* is equal to the remaining angle at *F*. [I. 32]

Therefore the triangle *ABC* is equiangular with the triangle *DEF*.

But, again, let each of the angles at *C*, *F* be supposed not less than a right angle;



I say again that, in this case too, the triangle *ABC* is equiangular with the triangle *DEF*.

For, with the same construction, we can prove similarly that

BC is equal to *BG*;

so that the angle at *C* is also equal to the angle *BGC*. [I. 5]

But the angle at *C* is not less than a right angle;

therefore neither is the angle *BGC* less than a right angle.

Thus in the triangle *BGC* the two angles are not less than two right angles: which is impossible. [I. 17]

Therefore, once more, the angle *ABC* is not unequal to the angle *DEF*;

therefore it is equal to it.

But the angle at *A* is also equal to the angle at *D*;

therefore the remaining angle at *C* is equal to the remaining angle at *F*. [I. 32]

Therefore the triangle *ABC* is equiangular with the triangle *DEF*.

Therefore etc.

Q. E. D.

Todhunter points out, after Walker, that some more words are necessary to make the enunciation precise: "If two triangles have one angle equal to one angle, the sides about other angles proportional <*so that the sides subtending the equal angles are homologous*>...."

This proposition is the extension to similar triangles of the *ambiguous case* already mentioned as omitted by Euclid in relation to *equality* of triangles in all respects (cf. note following I. 26, Vol. I. p. 306). The enunciation of VI. 7 has suggested the ordinary method of enunciating the *ambiguous case* where *equality* and not similarity is in question. Cf. Todhunter's note on I. 26.

Another possible way of presenting this proposition is given by Todhunter. The essential theorem to prove is:

If two triangles have two sides of the one proportional to two sides of the other, and the angles opposite to one pair of corresponding sides equal, the angles which are opposite to the other pair of corresponding sides shall either be equal or be together equal to two right angles.

For the angles included by the proportional sides must be either equal or unequal.

If they are equal, then, since the triangles have two angles of the one equal to two angles of the other, respectively, they are equiangular to one another.

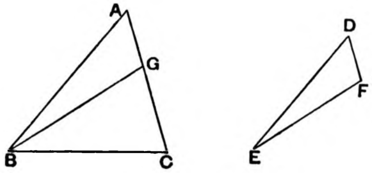
We have therefore only to consider the case in which the angles included by the proportional sides are unequal.

The proof is, except at the end, like that of VI. 7.

Let the triangles *ABC*, *DEF* have the angle at *A* equal to the angle at *D*;

let *AB* be to *BC* as *DE* to *EF*,

but let the angle *ABC* be *not* equal to the angle *DEF*.



The angles *ACB*, *DFE* shall be together equal to two right angles.

For one of the angles *ABC*, *DEF* must be the greater.

Let *ABC* be the greater; and make the angle *ABG* equal to the angle *DEF*.

Then we prove, as in VI. 7, that the triangles *ABG*, *DEF* are equiangular, whence

AB is to *BG* as *DE* is to *EF*.

But *AB* is to *BC* as *DE* is to *EF*, by hypothesis.

Therefore *BG* is equal to *BC*,

and the angle *BGC* is equal to the angle *BCA*.

Now, since the triangles *ABG*, *DEF* are equiangular,

the angle *BGA* is equal to the angle *EFD*.

Add to them respectively the equal angles *BGC*, *BCA*; therefore the angles *BCA*, *EFD* are together equal to the angles *BGA*, *BGC*, i.e. to two right angles.

It follows therefore that the angles *BCA*, *EFD* must be either equal or supplementary.

But (1), if each of them is less than a right angle, they cannot be supplementary, and they must therefore be equal;

(2) if each of them is greater than a right angle, they cannot be supplementary and must therefore be equal;

(3) if one of them is a right angle, they are supplementary and also equal.

Simson distinguishes the last case (3) in his enunciation: "then, if each of the remaining angles be either less or not less than a right angle, *or if one of them be a right angle....*"

The change is right, on the principle of restricting the conditions to the minimum necessary to enable the conclusion to be inferred. Simson adds a separate proof of the case in which one of the remaining angles is a right angle.

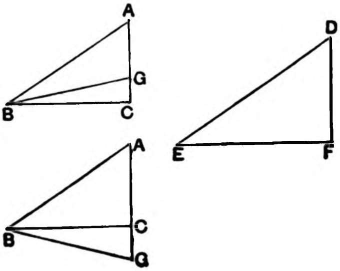
"Lastly, let one of the angles at *C*, *F*, viz. the angle at *C*, be a right angle; in this case likewise the triangle *ABC* is equiangular to the triangle *DEF*.

For, if they be not equiangular, make, at the point *B* of the straight line *AB*, the angle *ABG* equal to the angle *DEF*; then it may be proved, as in the first case, that *BG* is equal to *BC*.

But the angle *BCG* is a right angle;

therefore the angle *BGC* is also a right angle;

whence two of the angles of the triangle *BGC* are together not less than two right angles: which is impossible.



Therefore the triangle *ABC* is equiangular to the triangle *DEF*."

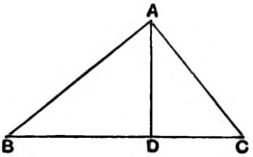
PROPOSITION 8.

If in a right-angled triangle a perpendicular be drawn from the right angle to the base, the triangles adjoining the perpendicular are similar both to the whole and to one another.

Let *ABC* be a right-angled triangle having the angle *BAC* right, and let *AD* be drawn from *A* perpendicular to *BC*;

I say that each of the triangles *ABD*, *ADC* is similar to the whole *ABC* and, further, they are similar to one another.

For, since the angle *BAC* is equal to the angle *ADB*, for each is right,



and the angle at *B* is common to the two triangles *ABC* and *ABD*,

therefore the remaining angle *ACB* is equal to the remaining angle *BAD*; [I. 32]

therefore the triangle *ABC* is equiangular with the triangle *ABD*.

Therefore, as *BC* which subtends the right angle in the triangle *ABC* is to *BA* which subtends the right angle in the triangle *ABD*, so is *AB* itself which subtends the angle at *C* in the triangle *ABC* to *BD* which subtends the equal angle *BAD* in the triangle *ABD*, and so also is *AC* to *AD* which subtends the angle at *B* common to the two triangles. [VI. 4]

Therefore the triangle *ABC* is both equiangular to the triangle *ABD* and has the sides about the equal angles proportional.

Therefore the triangle *ABC* is similar to the triangle *ABD*. [VI. Def. 1]

Similarly we can prove that

the triangle *ABC* is also similar to the triangle *ADC*;

therefore each of the triangles *ABD*, *ADC* is similar to the whole *ABC*.

I say next that the triangles *ABD*, *ADC* are also similar to one another.

For, since the right angle *BDA* is equal to the right angle *ADC*,

and moreover the angle *BAD* was also proved equal to the angle at *C*,

therefore the remaining angle at *B* is also equal to the remaining angle *DAC*; [I. 32]

therefore the triangle *ABD* is equiangular with the triangle *ADC*.

Therefore, as *BD* which subtends the angle *BAD* in the triangle *ABD* is to *DA* which subtends the angle at *C* in the triangle *ADC* equal to the angle *BAD*, so is *AD* itself which subtends the angle at *B* in the triangle *ABD* to *DC* which subtends the angle *DAC* in the triangle *ADC* equal

to the angle at *B*, and so also is *BA* to *AC*, these sides subtending the right angles; [vi. 4]
therefore the triangle *ABD* is similar to the triangle *ADC*. [vi. Def. 1]

Therefore etc.

Porism. From this it is clear that, if in a right-angled triangle a perpendicular be drawn from the right angle to the base, the straight line so drawn is a mean proportional between the segments of the base. Q. E. D.

Simson remarks on this proposition: "It seems plain that some editor has changed the demonstration that Euclid gave of this proposition: For, after he has demonstrated that the triangles are equiangular to one another, he particularly shows that their sides about the equal angles are proportionals, as if this had not been done in the demonstration of prop. 4 of this book: this superfluous part is not found in the translation from the Arabic, and is now left out."

This seems a little hypercritical, for the "particular showing" that the sides about the equal angles are proportionals is really nothing more than a somewhat full citation of vi. 4. Moreover to shorten his proof still more, Simson says, after proving that each of the triangles *ABD*, *ADC* is similar to the whole triangle *ABC*, "And the triangles *ABD*, *ADC* being both equiangular and similar to *ABC* are equiangular and similar to one another," thus assuming a particular case of vi. 21, which might well be proved here, as Euclid proves it, with somewhat more detail.

We observe that, here as generally, Euclid seems to disdain to give the reader such small help as might be afforded by arranging the letters used to denote the triangles so as to show the corresponding angular points in the same order for each pair of triangles; *A* is the first letter throughout, and the other two for each triangle are in the order of the figure from left to right. It may be in compensation for this that he states at such length which side corresponds to which when he comes to the proportions.

In the Greek texts there is an addition to the Porism inserted after "(Being) what it was required to prove," viz. "and further that between the base and any one of the segments the side adjacent to the segment is a mean proportional." Heiberg concludes that these words are an interpolation (1) because they come after the words ὅπερ ἔδει δεῖξαι which as a rule follow the Porism, (2) they are absent from the best Theonine mss., though P and Campanus have them without the ὅπερ ἔδει δεῖξαι. Heiberg's view seems to be confirmed by the fact noted by Austin, that, whereas the first part of the Porism is quoted later in vi. 13, in the lemma before x. 33 and in the lemma after xiii. 13, the second part is *proved* in the former lemma, and elsewhere, as also in Pappus (iii. p. 72, 9—23).

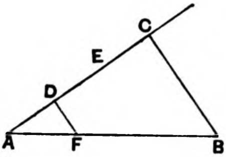
Proposition 9.

From a given straight line to cut off a prescribed part.

Let *AB* be the given straight line;
thus it is required to cut off from *AB* a prescribed part.

Let the third part be that prescribed.

Let a straight line *AC* be drawn through from *A* containing with *AB* any angle;



let a point *D* be taken at random on *AC*, and let *DE*, *EC* be made equal to *AD*. [I. 3]

Let *BC* be joined, and through *D* let *DF* be drawn parallel to it. [I. 31]

Then, since *FD* has been drawn parallel to *BC*, one of the sides of the triangle *ABC*, therefore, proportionally, as *CD* is to *DA*, so is *BF* to *FA*. [VI. 2]

But *CD* is double of *DA*;

therefore *BF* is also double of *FA*;

therefore *BA* is triple of *AF*.

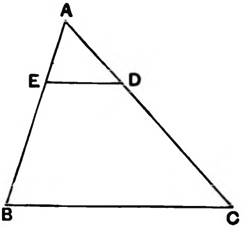
Therefore from the given straight line *AB* the prescribed third part *AF* has been cut off.

Q. E. F.

6. **any angle.** The expression here and in the two following propositions is τυχοῦσα γωνία, corresponding exactly to τυχὸν σημεῖον which I have translated as "a point (taken) *at random*"; but "an angle (taken) at random" would not be so appropriate where it is a question, not of *taking* any angle at all, but of drawing a straight line casually so as to make any angle with another straight line.

Simson observes that "this is demonstrated in a particular case, viz. that in which the third part of a straight line is required to be cut off; which is not at all like Euclid's manner. Besides, the author of that demonstration, from four magnitudes being proportionals, concludes that the third of them is the same multiple of the fourth which the first is of the second; now this is nowhere demonstrated in the 5th book, as we now have it; but the editor assumes it from the confused notion which the vulgar have of proportionals."

The truth of the assumption referred to is proved by Simson in his proposition D given above (p. 128); hence he is able to supply a general and legitimate proof of the present proposition.



"Let *AB* be the given straight line; it is required to cut off any part from it.

From the point *A* draw a straight line *AC* making any angle with *AB*; in *AC* take any point *D*, and take *AC* the same multiple of *AD* that *AB* is of the part which is to be cut off from it;

join *BC*, and draw *DE* parallel to it:

then *AE* is the part required to be cut off.

Because ED is parallel to one of the sides of the triangle ABC, viz. to BC, as CD is to DA, so is BE to EA, [VI. 2]

and, *componendo*,

CA is to AD, as BA to AE. [V. 18]

But CA is a multiple of AD;

therefore BA is the same multiple of AE. [Prop. D]

Whatever part therefore AD is of AC, AE is the same part of AB;

wherefore from the straight line AB the part required is cut off."

The use of Simson's Prop. D can be avoided, as noted by Camerer after Baermann, in the following way. We first prove, as above, that

CA is to AD as BA is to AE.

Then we infer that, alternately,

CA is to BA as AD to AE. [V. 16]

But AD is to AE as $n . AD$ to $n . AE$

(where n is the number of times that AD is contained in AC); [V. 15]

whence AC is to AB as $n . AD$ is to $n . AE$. [V. 11]

In this proportion the first term is equal to the third; therefore [V. 14] the second is equal to the fourth,

so that AB is equal to n times AE.

Prop. 9 is of course only a particular case of Prop. 10.

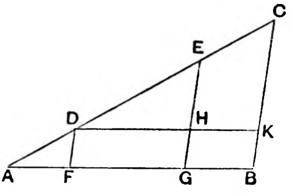
PROPOSITION 10.

To cut a given uncut straight line similarly to a given cut straight line.

Let AB be the given uncut straight line, and AC the straight line cut at the points D, E; and let them be so placed as to contain any angle;

let CB be joined, and through D, E let DF, EG be drawn parallel to BC, and through D let DHK be drawn parallel to AB. [I. 31]

Therefore each of the figures FH, HB is a parallelogram;



therefore DH is equal to FG and HK to GB. [I. 34]

Now, since the straight line HE has been drawn parallel to KC, one of the sides of the triangle DKC,

therefore, proportionally, as CE is to ED, so is KH to HD. [VI. 2]

But KH is equal to BG, and HD to GF;
therefore, as CE is to ED, so is BG to GF.

Again, since FD has been drawn parallel to GE, one of the sides of the triangle AGE,
therefore, proportionally, as ED is to DA, so is GF to FA. [VI. 2]

But it was also proved that,
as CE is to ED, so is BG to GF;
therefore, as CE is to ED, so is BG to GF,
and, as ED is to DA, so is GF to FA.

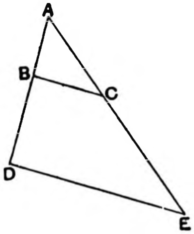
Therefore the given uncut straight line AB has been cut similarly to the given cut straight line AC.

Q. E. F.

PROPOSITION 11.

To two given straight lines to find a third proportional.

Let BA, AC be the two given straight lines, and let them be placed so as to contain any angle;
thus it is required to find a third proportional to BA, AC.



For let them be produced to the points D, E, and let BD be made equal to AC; [I. 3]
let BC be joined, and through D let DE be drawn parallel to it. [I. 31]

Since, then, BC has been drawn parallel to DE, one of the sides of the triangle ADE,
proportionally, as AB is to BD, so is AC to CE. [VI. 2]

But BD is equal to AC;
therefore, as AB is to AC, so is AC to CE.

Therefore to two given straight lines AB, AC a third proportional to them, CE, has been found.

Q. E. F.

1. **to find.** The Greek word, here and in the next two propositions, is προσευρεῖν, literally "to find *in addition*."

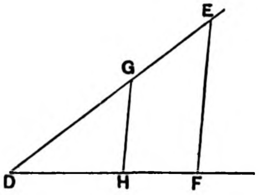
This proposition is again a particular case of the succeeding Prop. 12.

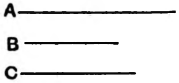
Given a ratio between straight lines, VI. 11 enables us to find the ratio which is its duplicate.

PROPOSITION 12.

To three given straight lines to find a fourth proportional.

Let A, B, C be the three given straight lines; thus it is required to find a fourth proportional to A, B, C.





Let two straight lines DE, DF be set out containing any angle EDF;

let DG be made equal to A, GE equal to B, and further DH equal to C;

let GH be joined, and let EF be drawn through E parallel to it. [I. 31]

Since, then, GH has been drawn parallel to EF, one of the sides of the triangle DEF,

therefore, as DG is to GE, so is DH to HF. [VI. 2]

But DG is equal to A, GE to B, and DH to C;

therefore, as A is to B, so is C to HF.

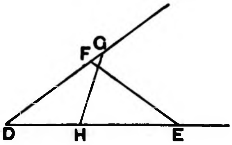
Therefore to the three given straight lines A, B, C a fourth proportional HF has been found.

Q. E. F.

We have here the geometrical equivalent of the "rule of three."

It is of course immaterial whether, as in Euclid's proof, the first and second straight lines are measured on one of the lines forming the angle and the third on the other, or the first and third are measured on one and the second on the other.

If it should be desired that the first and the required fourth be measured on one of the lines, and the second and third on the other, we can use the following construction. Measure DE on one straight line equal to A, and on any other straight line making an angle with the first at the point D measure DF equal to B, and DG equal to C. Join EF, and through G draw GH *anti-parallel* to EF, i.e. make the angle DGH equal to the angle DEF; let GH meet DE (produced if necessary) in H.



DH is then the fourth proportional.

For the triangles *EDF*, *GDH* are similar, and the sides about the equal angles are proportional, so that

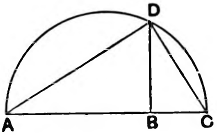
DE is to *DF* as *DG* to *DH*,

or *A* is to *B* as *C* to *DH*.

PROPOSITION 13.

To two given straight lines to find a mean proportional.

Let *AB*, *BC* be the two given straight lines; thus it is required to find a mean proportional to *AB*, *BC*.



Let them be placed in a straight line, and let the semicircle *ADC* be described on *AC*;

let *BD* be drawn from the point *B* at right angles to the straight line *AC*, and let *AD*, *DC* be joined.

Since the angle *ADC* is an angle in a semicircle, it is right. [III. 31]

And, since, in the right-angled triangle *ADC*, *DB* has been drawn from the right angle perpendicular to the base, therefore *DB* is a mean proportional between the segments of the base, *AB*, *BC*. [VI. 8, Por.]

Therefore to the two given straight lines *AB*, *BC* a mean proportional *DB* has been found.

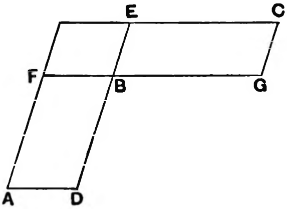
Q. E. F.

This proposition, the Book VI. version of II. 14, is equivalent to the extraction of the square root. It further enables us, given a ratio between straight lines, to find the ratio which is its *sub-duplicate*, or the ratio of which it is duplicate.

PROPOSITION 14.

In equal and equiangular parallelograms the sides about the equal angles are reciprocally proportional; and equiangular parallelograms in which the sides about the equal angles are reciprocally proportional are equal.

Let AB, BC be equal and equiangular parallelograms having the angles at B equal, and let DB, BE be placed in a straight line;



therefore FB, BG are also in a straight line. [I. 14]

I say that, in AB, BC, the sides about the equal angles are reciprocally proportional, that is to say, that, as DB is to BE, so is GB to BF.

For let the parallelogram FE be completed.

Since, then, the parallelogram AB is equal to the parallelogram BC,

and FE is another area,

therefore, as AB is to FE, so is BC to FE. [V. 7]

But, as AB is to FE, so is DB to BE, [VI. 1]

and, as BC is to FE, so is GB to BF. [*id.*]

therefore also, as DB is to BE, so is GB to BF. [V. 11]

Therefore in the parallelograms AB, BC the sides about the equal angles are reciprocally proportional.

Next, let GB be to BF as DB to BE;

I say that the parallelogram AB is equal to the parallelogram BC.

For since, as DB is to BE, so is GB to BF,

while, as DB is to BE, so is the parallelogram AB to the parallelogram FE, [VI. 1]

and, as GB is to BF, so is the parallelogram BC to the parallelogram FE, [VI. 1]

therefore also, as AB is to FE, so is BC to FE; [V. 11]

therefore the parallelogram AB is equal to the parallelogram BC. [V. 9]

Therefore etc.

Q. E. D.

De Morgan says upon this proposition: "Owing to the disjointed manner in which Euclid treats compound ratio, this proposition is strangely out of place. It is a particular case of VI. 23, being that in which the ratio of the sides, compounded, gives a ratio of equality. The proper definition of four magnitudes being reciprocally proportional is that the ratio compounded of their ratios is that of equality."

It is true that VI. 14 is a particular case of VI. 23, but, if either is out of place, it is rather the latter that should be placed before VI. 14, since most of the propositions between VI. 15 and VI. 23 depend upon VI. 14 and 15. But is perfectly consistent with Euclid's manner to give a particular case first and its extension later, and such an arrangement often has great advantages in that it enables the more difficult parts of a subject to be led up to more easily and gradually. Now, if De Morgan's view were here followed, we should, as it seems to me, be committing the mistake of explaining what is relatively easy to understand, viz. two ratios of which one is the inverse of the other, by a more complicated conception, that of compound ratio. In other words, it is easier for a learner to realise the relation indicated by the statement that the sides of equal and equiangular parallelograms are "reciprocally proportional" than to form a conception of parallelograms such that "the ratio compounded of the ratio of their sides is one of equality." For this reason I would adhere to Euclid's arrangement.

The conclusion that, since *DB*, *BE* are placed in a straight line, *FB*, *BG* are also in a straight line is referred to I. 14. The deduction is made clearer by the following steps.

The angle *DBF* is equal to the angle *GBE*;

add to each the angle *FBE*;

therefore the angles *DBF*, *FBE* are together equal to the angles *GBE*, *FBE*. [*C. N.* 2]

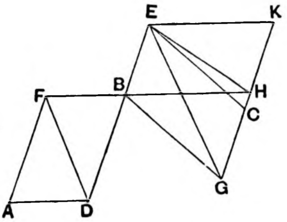
But the angles *DBF*, *FBE* are together equal to two right angles, [I. 13]

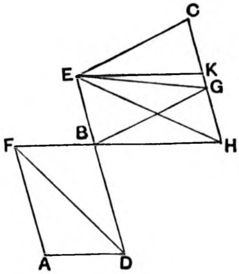
therefore the angles *GBE*, *FBE* are together equal to two right angles, [*C. N.* 1]

and hence *FB*, *BG* are in one straight line. [I. 14]

The result is also obvious from the converse of I. 15 given by Proclus (see note on I. 15).

The proposition VI. 14 contains a theorem and one partial converse of it; so also does VI. 15. To each proposition may be added the other partial converse, which may be enunciated as follows, the words in square brackets applying to the case of triangles (VI. 15).





Equal parallelograms [triangles] which have the sides about one angle in each reciprocally proportional are equiangular [have the angles included by those sides either equal or supplementary.]

Let *AB*, *BC* be equal parallelograms, or let *FBD*, *EBG* be equal triangles, such that the sides about the angles at B are reciprocally proportional, i.e. such that

$$DB : BE = GB : BF.$$

We shall prove that the angles FBD, EBG are either equal or supplementary.

Place the figures so that DB, BE are in one straight line.

Then FB, BG are either in a straight line, or not in a straight line.

(1) If FB, BG are in a straight line, the figure of the proposition (with the diagonals FD, EG drawn) represents the facts, and

the angle FBD is equal to the angle EBG. [I. 15]

(2) If FB, BG are not in a straight line,

produce FB to H so that BH may be equal to BG.

Join EH, and complete the parallelogram $EBHK$.

Now, since $DB : BE = GB : BF$

and $GB = HB$,

$$DB : BE = HB : BF,$$

and therefore, by VI. 14 or 15,

the parallelograms AB, BK are equal, or the triangles FBD, EBH are equal.

But the parallelograms AB, BC are equal, and the triangles FBD, EBG are equal;

therefore the parallelograms BC, BK are equal, and the triangles EBH, EBG are equal.

Therefore these parallelograms or triangles are within the same parallels: that is, G, C, H, K are in a straight line which is parallel to DE. [I. 39]

Now, since BG, BH are equal,

the angles BGH, BHG are equal.

By parallels, it follows that

the angle EBG is equal to the angle DBH,

whence the angle EBG is supplementary to the angle FBD.

PROPOSITION 15.

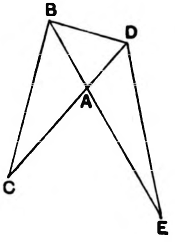
In equal triangles which have one angle equal to one angle the sides about the equal angles are reciprocally proportional; and those triangles which have one angle equal to one angle, and in which the sides about the equal angles are reciprocally proportional, are equal.

Let ABC, ADE be equal triangles having one angle equal to one angle, namely the angle BAC to the angle DAE;

I say that in the triangles ABC, ADE the sides about the equal angles are reciprocally proportional, that is to say, that,

as CA is to AD, so is EA to AB.

For let them be placed so that CA is in a straight line with AD;



therefore EA is also in a straight line with AB. [I. 14]

Let BD be joined.

Since then the triangle ABC is equal to the triangle ADE, and BAD is another area,

therefore, as the triangle CAB is to the triangle BAD, so is the triangle EAD to the triangle BAD. [V. 7]

But, as CAB is to BAD, so is CA to AD, [VI. 1]

and, as EAD is to BAD, so is EA to AB. [*id.*]

Therefore also, as CA is to AD, so is EA to AB. [V. 11]

Therefore in the triangles ABC, ADE the sides about the equal angles are reciprocally proportional.

Next, let the sides of the triangles ABC, ADE be reciprocally proportional, that is to say, let EA be to AB as CA to AD;

I say that the triangle ABC is equal to the triangle ADE.

For, if BD be again joined,

since, as CA is to AD, so is EA to AB,

while, as CA is to AD, so is the triangle ABC to the triangle BAD,

and, as EA is to AB, so is the triangle EAD to the triangle BAD, [VI. 1]

therefore, as the triangle ABC is to the triangle BAD, so is the triangle EAD to the triangle BAD. [V. 11]

Therefore each of the triangles ABC, EAD has the same ratio to BAD.

Therefore the triangle ABC is equal to the triangle EAD. [V. 9]

Therefore etc.

Q. E. D.

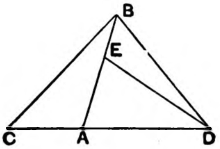
As indicated in the partial converse given in the last note, this proposition is equally true if the angle included by the two sides in one triangle is supplementary, instead of being equal, to the angle included by the two sides in the other.

Let ABC, ADE be two triangles such that the angles BAC, DAE are supplementary, and also

$$CA : AD = EA : AB.$$

In this case we can place the triangles so that CA is in a straight line with AD, and AB lies along AE (since the angle EAC, being supplementary to the angle EAD, is equal to the angle BAC).

If we join BD, the proof given by Euclid applies to this case also.



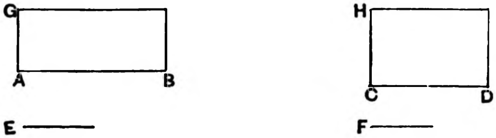
It is true that VI. 15 can be immediately inferred from VI. 14, since a triangle is half of a parallelogram with the same base and height. But, Euclid's object being to give the student a grasp of *methods* rather than results, there seems to be no advantage in deducing one proposition from the other instead of using the same method on each.

PROPOSITION 16.

If four straight lines be proportional, the rectangle contained by the extremes is equal to the rectangle contained by the means; and, if the rectangle contained by the extremes be equal to the rectangle contained by the means, the four straight lines will be proportional.

Let the four straight lines AB, CD, E, F be proportional, so that, as AB is to CD, so is E to F;

I say that the rectangle contained by AB, F is equal to the rectangle contained by CD, E.



Let AG, CH be drawn from the points A, C at right angles to the straight lines AB, CD, and let AG be made equal to F, and CH equal to E.

Let the parallelograms BG, DH be completed.

Then since, as AB is to CD, so is E to F,

while E is equal to CH, and F to AG,

therefore, as AB is to CD, so is CH to AG.

Therefore in the parallelograms BG, DH the sides about the equal angles are reciprocally proportional.

But those equiangular parallelograms in which the sides about the equal angles are reciprocally proportional are equal; [VI. 14]

therefore the parallelogram *BG* is equal to the parallelogram *DH*.

And *BG* is the rectangle *AB*, *F*, for *AG* is equal to *F*; and *DH* is the rectangle *CD*, *E*, for *E* is equal to *CH*; therefore the rectangle contained by *AB*, *F* is equal to the rectangle contained by *CD*, *E*.

Next, let the rectangle contained by *AB*, *F* be equal to the rectangle contained by *CD*, *E*;

I say that the four straight lines will be proportional, so that, as *AB* is to *CD*, so is *E* to *F*.

For, with the same construction,

since the rectangle *AB*, *F* is equal to the rectangle *CD*, *E*,
and the rectangle *AB*, *F* is *BG*, for *AG* is equal to *F*,
and the rectangle *CD*, *E* is *DH*, for *CH* is equal to *E*,
therefore *BG* is equal to *DH*.

And they are equiangular

But in equal and equiangular parallelograms the sides about the equal angles are reciprocally proportional. [VI. 14]

Therefore, as *AB* is to *CD*, so is *CH* to *AG*.

But *CH* is equal to *E*, and *AG* to *F*;

therefore, as *AB* is to *CD*, so is *E* to *F*.

Therefore etc. Q. E. D.

This proposition is a particular case of VI. 14, but one which is on all accounts worth separate statement. It may also be enunciated in the following form:

Rectangles which have their bases reciprocally proportional to their heights are equal in area; and equal rectangles have their bases reciprocally proportional to their heights.

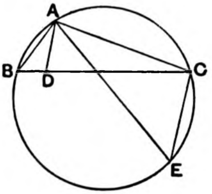
Since any parallelogram is equal to a rectangle of the same height and on the same base, and any triangle with the same height and on the same base is equal to half the parallelogram or rectangle, it follows that *Equal parallelograms or triangles have their bases reciprocally proportional to their heights and* vice versa.

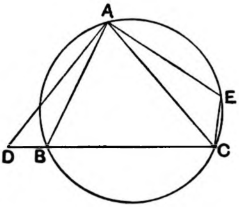
The present place is suitable for giving certain important propositions, including those which Simson adds to Book VI. as Props. B, C and D, which are proved directly by means of VI. 16.

1. Proposition B is a particular case of the following theorem.

If a circle be circumscribed about a triangle ABC *and there be drawn through* A *any two straight lines either both within or both without the angle* BAC, *viz.*

AD *meeting* BC (*produced if necessary*) *in* D *and* AE *meeting the circle again in* E, *such that the angles* DAB, EAC *are equal, then the rectangle* AD, AE *is equal to the rectangle* BA, AC.





Join *CE*.

The angles *BAD*, *EAC* are equal, by hypothesis;

and the angles *ABD*, *AEC* are equal. [III. 21, 22]

Therefore the triangles *ABD*, *AEC* are equiangular.

Hence *BA* is to *AD* as *EA* is to *AC*,

and therefore the rectangle *BA*, *AC* is equal to the rectangle *AD*, *AE*. [VI. 16]

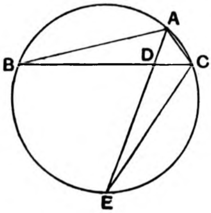
There are now two particular cases to be considered.

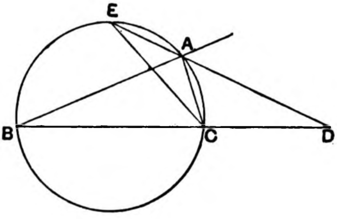
(*a*) Suppose that *AD*, *AE* coincide;

ADE will then bisect the angle *BAC*.

(*b*) Suppose that *AD*, *AE* are in one straight line but that *D*, *E* are on opposite sides of *A*;

AD will then bisect the external angle at *A*.





In the first case (*a*) we have

the rectangle *BA*, *AC* equal to the rectangle *EA*, *AD*;

and the rectangle *EA*, *AD* is equal to the rectangle *ED*, *DA* together with the square on *AD*, [II. 3]

i.e. to the rectangle *BD*, *DC* together with the square on *AD*. [III. 35]

Therefore the rectangle *BA*, *AC* is equal to the rectangle *BD*, *DC* together with the square on *AD*. [This is Simson's Prop. B]

In case (*b*) the rectangle *EA*, *AD* is equal to the excess of the rectangle *ED*, *DA* over the square on *AD*;

therefore the rectangle *BA*, *AC* is equal to the excess of the rectangle *BD*, *DC* over the square on *AD*.

The following converse of Simson's Prop. B may be given: *If a straight line* AD *be drawn from the vertex* A *of a triangle to meet the base, so that the square on* AD *together with the rectangle* BD, DC *is equal to the rectangle* BA, AC, *the line* AD *will bisect the angle* BAC *except when the sides* AB, AC *are equal, in which case every line drawn to the base will have the property mentioned.*

Let the circumscribed circle be drawn, and let AD produced meet it in E; join CE.

The rectangle BD, DC is equal to the rectangle ED, DA. [III. 35]

Add to each the square on AD;

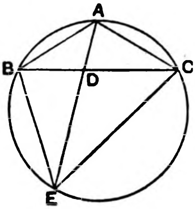
therefore the rectangle BA, AC is equal to the rectangle EA, AD. [hyp. and II. 3]

Hence AB is to AD as AE to AC. [VI. 16]

But the angle ABD is equal to the angle AEC. [III. 21]

Therefore the angles BDA, ECA are either equal or supplementary. [VI. 7 and note]

(*a*) If they are equal, the angles BAD, EAC are also equal, and AD bisects the angle BAC.



(*b*) If they are supplementary, the angle ADC must be equal to the angle ACE.

Therefore the angles BAD, ABD are together equal to the angles ACB, BCE, i.e. to the angles ACD, BAD.

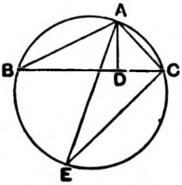
Take away the common angle BAD, and
the angles ABD, ACD are equal, or
AB is equal to AC.

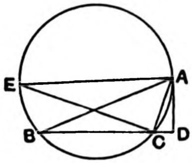
Euclid himself assumes, in Prop. 67 of the *Data*, the result of so much of this proposition as relates to the case where $BA = AC$. He assumes namely, without proof, that, if $BA = AC$, and if D be any point on BC, the rectangle BD, DC together with the square on AD is equal to the square on AB.

PROPOSITION C.

If from any angle of a triangle a straight line be drawn perpendicular to the opposite side, the rectangle contained by the other two sides of the triangle is equal to the rectangle contained by the perpendicular and the diameter of the circle circumscribed about the triangle.

Let ABC be a triangle and AD the perpendicular on AB. Draw the diameter AE of the circle circumscribed about the triangle ABC.





Then shall the rectangle BA, AC be equal to the rectangle EA, AD.

Join EC.

Since the right angle *BDA* is equal to the right angle *ECA* in a semicircle, [III. 31]
and the angles *ABD*, *AEC* in the same segment are equal, [III. 21]
the triangles *ABD*, *AEC* are equiangular.

Therefore, as *BA* is to *AD*, so is *EA* to *AC*, [VI. 4]
whence the rectangle *BA*, *AC* is equal to the rectangle *EA*, *AD*. [VI. 16]

This result corresponds to the trigonometrical formula for R, the radius of the circumscribed circle,

$$R = \frac{abc}{4\Delta}.$$

PROPOSITION D.

This is the highly important lemma given by Ptolemy (ed. Heiberg, Vol. I, pp. 36—7) which is the basis of his calculation of the table of chords in the section of Book I. of the μεγάλη σύνταξις entitled "concerning the size of the straight lines [i.e. chords] in the circle" (περὶ τῆς πηλικότητος τῶν ἐν τῷ κύκλῳ εὐθειῶν).

The theorem may be enunciated thus.

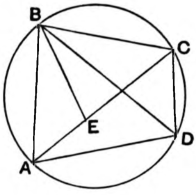
The rectangle contained by the diagonals of any quadrilateral inscribed in a circle is equal to the sum of the rectangles contained by the pairs of opposite sides.

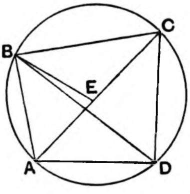
I shall give the proof in Ptolemy's words, with the addition only, in brackets, of two words applying to a second figure not given by Ptolemy.

"Let there be a circle with any quadrilateral *ABCD* inscribed in it, and let *AC*, *BD* be joined.

It is to be proved that the rectangle contained by *AC* and *BD* is equal to the sum of the rectangles *AB*, *DC* and *AD*, *BC*.

For let the angle *ABE* be made equal to the angle contained by *DB*, *BC*.





If then we add [or subtract] the angle *EBD*,
the angle *ABD* will also be equal to the angle *EBC*.

But the angle *BDA* is also equal to the angle *BCE*, [III. 21]
for they subtend the same segment;
therefore the triangle *ABD* is equiangular with the triangle *EBC*.

Hence, proportionally,
as *BC* is to *CE*, so is *BD* to *DA*. [VI. 4]

Therefore the rectangle *BC*, *AD* is equal to the rectangle *BD*, *CE*. [VI. 16]

Again, since the angle *ABE* is equal to the angle *DBC*,
and the angle *BAE* is also equal to the angle *BDC*, [III. 21]
the triangle *ABE* is equiangular with the triangle *DBC*.

Therefore, proportionally,

as BA is to AE, so is BD to DC; [VI. 4]

therefore the rectangle BA, DC is equal to the rectangle BD, AE. [VI. 16]

But it was also proved that

the rectangle BC, AD is equal to the rectangle BD, CE;

therefore the rectangle AC, BD as a whole is equal to the sum of the rectangles AB, DC and AD, BC:

(being) what it was required to prove."

Another proof of this proposition, and of its converse, is indicated by Dr Lachlan (*Elements of Euclid*, pp. 273—4). It depends on two preliminary propositions.

(1) *If two circles be divided, by a chord in each, into segments which are similar respectively, the chords are proportional to the corresponding diameters.*

The proof is instantaneous if we join the ends of each chord to the centre of the circle which it divides, when we obtain two similar triangles.

(2) *If* D *be any point on the circle circumscribed about a triangle* ABC, *and* DX, DY, DZ *be perpendicular to the sides* BC, CA, AB *of the triangle respectively, then* X, Y, Z *lie in one straight line; and*, conversely, *if the feet of the perpendiculars from any point* D *on the sides of a triangle lie in one straight line,* D *lies on the circle circumscribed about the triangle.*

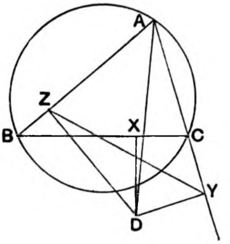
The proof depending on III. 21, 22 is well known.

Now suppose that D is *any* point in the plane of a triangle ABC, and that DX, DY, DZ are perpendicular to the sides BC, CA, AB respectively.

Join YZ, DA.

Then, since the angles at Y, Z are right, A, Y, D, Z lie on a circle of which DA is the diameter.

And YZ divides this circle into segments which are similar respectively to the segments into which BC divides the circle circumscribing ABC, since the angles ZAY, BAC coincide, and their supplements are equal.



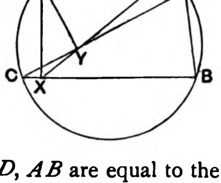
Therefore, if d be the diameter of the circle circumscribing ABC,

BC is to d as YZ is to DA;

and therefore the rectangle AD, BC is equal to the rectangle d, YZ.

Similarly the rectangle BD, CA is equal to the rectangle d, ZX, and the rectangle CD, AB is equal to the rectangle d, XY.

Hence, in a quadrilateral in general, the rectangle contained by the diagonals is less than the sum of the rectangles contained by the pairs of opposite sides.



Next, suppose that D lies on the circle circumscribed about ABC, but so that A, B, C, D follow each other on the circle in this order, as in the figure annexed.

Let DX, DY, DZ be perpendicular to BC, CA, AB respectively, so that X, Y, Z are in a straight line.

Then, since the rectangles AD, BC; BD, CA; CD, AB are equal to the rectangles d, YZ; d, ZX; d, XY respectively, and XZ is equal to the sum of

XY, *YZ*, so that the rectangle *d*, *XZ* is equal to the sum of the rectangles *d*, *XY* and *d*, *YZ*, it follows that

the rectangle *AC*, *BD* is equal to the sum of the rectangles *AD*, *BC* and *AB*, *CD*.

Conversely, if the latter statement is true, while we are supposed to know nothing about the position of *D*, it follows that

XZ must be equal to the sum of *XY*, *YZ*,

so that *X*, *Y*, *Z* must be in a straight line.

Hence, from the theorem (2) above, it follows that *D* must lie on the circle circumscribed about *ABC*, i.e. that *ABCD* is a quadrilateral about which a circle can be described.

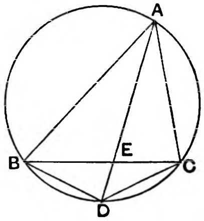
All the above propositions can be proved on the basis of Book III. and without using Book VI., since it is possible by the aid of III. 21 and 35 alone to prove that *in equiangular triangles the rectangles contained by the non-corresponding sides about equal angles are equal to one another* (a result arrived at by combining VI. 4 and VI. 16). This is the method adopted by Casey, H. M. Taylor, and Lachlan; but I fail to see any particular advantage in it.

Lastly, the following proposition may be given which Playfair added as VI. E. It appears in the *Data* of Euclid, Prop. 93, and may be thus enunciated.

If the angle BAC *of a triangle* ABC *be bisected by the straight line* AD *meeting the circle circumscribed about the triangle in D, and if BD be joined, then*

the sum of BA, AC *is to* AD *as* BC *is to* BD.

Join *CD*. Then, since *AD* bisects the angle *BAC*, the subtended arcs *BD*, *DC*, and therefore the chords *BD*, *DC*, are equal.



(1) The result can now be easily deduced from Ptolemy's theorem.

For the rectangle *AD*, *BC* is equal to the sum of the rectangles *AB*, *DC* and *AC*, *BD*, i.e. (since *BD*, *CD* are equal) to the rectangle contained by *BA* + *AC* and *BD*.

Therefore the sum of *BA*, *AC* is to *AD* as *BC* is to *BD*. [VI. 16]

(2) Euclid proves it differently in *Data*, Prop. 93.

Let *AD* meet *BC* in *E*.

Then, since *AE* bisects the angle *BAC*,

BA is to *AC* as *BE* to *EC*, [VI. 3]

or, alternately,

AB is to *BE* as *AC* to *CE*. [V. 16]

Therefore also

BA + *AC* is to *BC* as *AC* to *CE*. [V. 12]

Again, since the angles *BAD*, *EAC* are equal, and the angles *ADB*, *ACE* are also equal, [III. 21]

the triangles *ABD*, *AEC* are equiangular.

Therefore *AC* is to *CE* as *AD* to *BD*. [VI. 4]

Hence $BA + AC$ is to BC as AD to BD, [v. 11]
and, alternately,

$BA + AC$ is to AD as BC is to BD. [v. 16]

Euclid concludes that, if the circle ABC is given in magnitude, and the chord BC cuts off a segment of it containing a given angle (so that, by *Data* Prop. 87, BC and also BD are given in magnitude),

the ratio of $BA + AC$ to AD is given,

and further that (since, by similar triangles, BD is to DE as AC is to CE, while $BA + AC$ is to BC as AC is to CE),

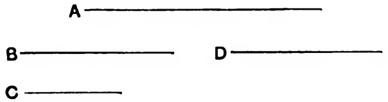
the rectangle $(BA + AC)$, DE, being equal to the rectangle BC, BD, is also given.

PROPOSITION 17

If three straight lines be proportional, the rectangle contained by the extremes is equal to the square on the mean; and, if the rectangle contained by the extremes be equal to the square on the mean, the three straight lines will be proportional.

Let the three straight lines A, B, C be proportional, so that, as A is to B, so is B to C;

I say that the rectangle contained by A, C is equal to the square on B.



Let D be made equal to B.

Then, since, as A is to B, so is B to C,

and B is equal to D,

therefore, as A is to B, so is D to C.

But, if four straight lines be proportional, the rectangle contained by the extremes is equal to the rectangle contained by the means. [VI. 16]

Therefore the rectangle A, C is equal to the rectangle B, D.

But the rectangle B, D is the square on B, for B is equal to D;

therefore the rectangle contained by A, C is equal to the square on B.

Next, let the rectangle A, C be equal to the square on B

I say that, as A is to B, so is B to C.

For, with the same construction,
since the rectangle *A*, *C* is equal to the square on *B*,
while the square on *B* is the rectangle *B*, *D*, for *B* is equal to *D*,
therefore the rectangle *A*, *C* is equal to the rectangle *B*, *D*.

But, if the rectangle contained by the extremes be equal to that contained by the means, the four straight lines are proportional. [VI. 16]

Therefore, as *A* is to *B*, so is *D* to *C*.

But *B* is equal to *D*;

therefore, as *A* is to *B*, so is *B* to *C*.

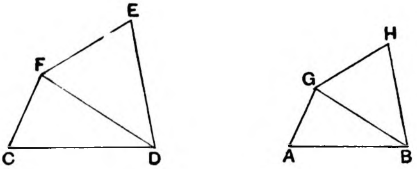
Therefore etc. Q. E. D.

VI. 17 is, of course, a particular case of VI. 16.

PROPOSITION 18.

On a given straight line to describe a rectilineal figure similar and similarly situated to a given rectilineal figure.

Let *AB* be the given straight line and *CE* the given rectilineal figure;
thus it is required to describe on the straight line *AB* a rectilineal figure similar and similarly situated to the rectilineal figure *CE*.



Let *DF* be joined, and on the straight line *AB*, and at the points *A*, *B* on it, let the angle *GAB* be constructed equal to the angle at *C*, and the angle *ABG* equal to the angle *CDF*. [I. 23]

Therefore the remaining angle *CFD* is equal to the angle *AGB*; [I. 32]

therefore the triangle *FCD* is equiangular with the triangle *GAB*.

Therefore, proportionally, as *FD* is to *GB*, so is *FC* to *GA*, and *CD* to *AB*.

Again, on the straight line *BG*, and at the points *B*, *G* on it, let the angle *BGH* be constructed equal to the angle *DFE*, and the angle *GBH* equal to the angle *FDE*. [I. 23]

Therefore the remaining angle at *E* is equal to the remaining angle at *H*; [I. 32]

therefore the triangle *FDE* is equiangular with the triangle *GBH*;

therefore, proportionally, as *FD* is to *GB*, so is *FE* to *GH*, and *ED* to *HB*. [VI. 4]

But it was also proved that, as *FD* is to *GB*, so is *FC* to *GA*, and *CD* to *AB*;

therefore also, as *FC* is to *AG*, so is *CD* to *AB*, and *FE* to *GH*, and further *ED* to *HB*.

And, since the angle *CFD* is equal to the angle *AGB*,

and the angle *DFE* to the angle *BGH*,

therefore the whole angle *CFE* is equal to the whole angle *AGH*.

For the same reason

the angle *CDE* is also equal to the angle *ABH*.

And the angle at *C* is also equal to the angle at *A*,

and the angle at *E* to the angle at *H*.

Therefore *AH* is equiangular with *CE*;

and they have the sides about their equal angles proportional;

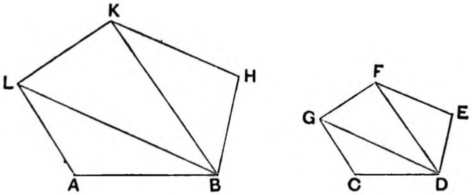
therefore the rectilineal figure *AH* is similar to the rectilineal figure *CE*. [VI. Def. 1]

Therefore on the given straight line *AB* the rectilineal figure *AH* has been described similar and similarly situated to the given rectilineal figure *CE*.

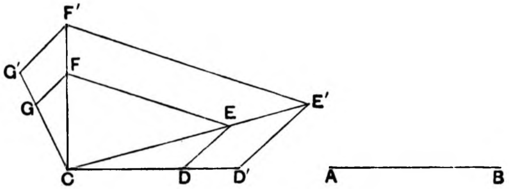
Q. E. F.

Simson thinks the proof of this proposition has been vitiated, his grounds for this view being (1) that it is demonstrated only with reference to quadrilaterals, and does not show how it may be extended to figures of five or more sides, (2) that Euclid infers, from the fact of two triangles being equiangular, that a side of the one is to the corresponding side of the other as another side of the first is to the side corresponding to it in the other, i.e. he permutes, without mentioning the fact that he does so, the proportions obtained in VI. 4, whereas the proof of the very next proposition gives, in a similar case, the intermediate step of permutation. I think this is hypercriticism. As regards (2) it should be noted that the permuted form of the proportion is arrived at first in the *proof* of VI. 4; and the omission of the intermediate step of *alternando*, whether accidental or not, is of no importance. On the other hand, the use of this form of the proportion certainly simplifies the proof of the proposition, since it makes unnecessary the subsequent *ex aequali* steps of Simson's proof, their place being taken by the inference [v. 11] that ratios which are the same with a third ratio are the same with one another.

Nor is the first objection of any importance. We have only to take as the given polygon a polygon of five sides at least, as *CDEFG*, join one extremity of *CD*, say *D*, to each of the angular points other than *C* and *E*, and then use the same mode of construction as Euclid's for any number of successive triangles as *ABL*, *LBK*, etc., that may have to be made. Euclid's construction and proof for a quadrilateral are quite sufficient to show how to deal with the case of a figure of five or any greater number of sides.



Clavius has a construction which, given the power of moving a figure bodily from one position to any other, is easier. *CDEFG* being the given polygon, join *CE*, *CF*. Place *AB* on *CD* so that *A* falls on *C*, and let *B* fall on *D'*, which may either lie on *CD* or on *CD* produced.



Now draw *D'E'* parallel to *DE*, meeting *CE*, produced if necessary, in *E'*, *E'F'* parallel to *EF*, meeting *CF*, produced if necessary, in *F'*, and so on.

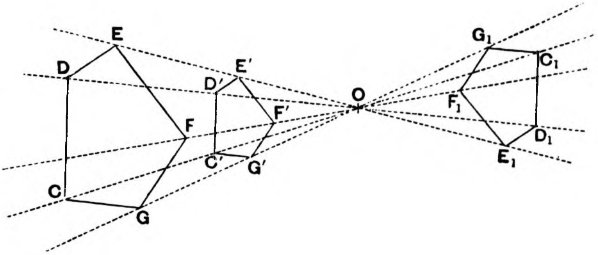
Let the parallel to the last side but one, *FG*, meet *CG*, produced if necessary, in *G'*.

Then *CD'E'F'G'* is similar and similarly situated to *CDEFG*, and it is constructed on *CD'*, a straight line equal to *AB*.

The proof of this is obvious.

A more general construction is indicated in the subjoined figure. If *CDEFG* be the given polygon, suppose its angular points all joined to any point *O* and the connecting straight lines produced both ways. Then, if *C'D'*, a straight line equal to *AB*, be placed so that it is parallel to *CD*, and *C'*, *D'* lie respectively on *OC*, *OD* (this can of course be done by finding fourth proportionals), we have only to draw *D'E'*, *E'F'*, etc., parallel to the corresponding sides of the original polygon in the manner shown.

De Morgan would rearrange Props. 18 and 20 in the following manner. He would combine Prop. 18 and the first part of Prop. 20 into one, with the enunciation:



Pairs of similar triangles, similarly put together, give similar figures; and every pair of similar figures is composed of pairs of similar triangles similarly put together.

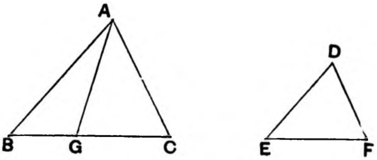
He would then make the *problem* of VI. 18 an application of the first part. In form this would certainly appear to be an improvement; but, provided that the relation of the propositions is understood, the matter of form is perhaps not of great importance.

PROPOSITION 19.

Similar triangles are to one another in the duplicate ratio of the corresponding sides.

Let *ABC*, *DEF* be similar triangles having the angle at *B* equal to the angle at *E*, and such that, as *AB* is to *BC*, so is *DE* to *EF*, so that *BC* corresponds to *EF*; [V. Def. 11]

I say that the triangle *ABC* has to the triangle *DEF* a ratio duplicate of that which *BC* has to *EF*.



For let a third proportional *BG* be taken to *BC*, *EF*, so that, as *BC* is to *EF*, so is *EF* to *BG*; [VI. 11]

and let *AG* be joined.

Since then, as *AB* is to *BC*, so is *DE* to *EF*,

therefore, alternately, as *AB* is to *DE*, so is *BC* to *EF*. [V. 16]

But, as *BC* is to *EF*, so is *EF* to *BG*;
therefore also, as *AB* is to *DE*, so is *EF* to *BG*. [V. 11]

Therefore in the triangles *ABG*, *DEF* the sides about the equal angles are reciprocally proportional.

But those triangles which have one angle equal to one angle, and in which the sides about the equal angles are reciprocally proportional, are equal; [VI. 15]
therefore the triangle *ABG* is equal to the triangle *DEF*.

Now since, as *BC* is to *EF*, so is *EF* to *BG*,
and, if three straight lines be proportional, the first has to the third a ratio duplicate of that which it has to the second, [V. Def. 9]
therefore *BC* has to *BG* a ratio duplicate of that which *CB* has to *EF*.

But, as *CB* is to *BG*, so is the triangle *ABC* to the triangle *ABG*; [VI. 1]
therefore the triangle *ABC* also has to the triangle *ABG* a ratio duplicate of that which *BC* has to *EF*.

But the triangle *ABG* is equal to the triangle *DEF*;
therefore the triangle *ABC* also has to the triangle *DEF* a ratio duplicate of that which *BC* has to *EF*.

Therefore etc.

PORISM. From this it is manifest that, if three straight lines be proportional, then, as the first is to the third, so is the figure described on the first to that which is similar and similarly described on the second.

Q. E. D.

4. **and such that, as AB is to BC, so is DE to EF,** literally "(triangles) having the angle at *B* equal to the angle at *E*, and (*having*), *as AB to BC, so DE to EF*."

Having combined Prop. 18 and the first part of Prop. 20 as just indicated, De Morgan would tack on to Prop. 19 the second part of Prop. 20, which asserts that, if similar polygons be divided into the same number of similar triangles, the triangles are "*homologous* to the wholes" (in the sense that the polygons have the same ratio as the corresponding triangles have), and that the polygons are to one another in the duplicate ratio of corresponding sides. This again, though no doubt an improvement of form, would necessitate the drawing over again of the figure of the altered Proposition 18 and a certain amount of repetition.

Agreeably to his suggestion that Prop. 23 should come before Prop. 14 which is a particular case of it, De Morgan would prove Prop. 19 for *parallelograms* by means of Prop. 23, and thence infer the truth of it for triangles or the halves of the parallelograms. He adds: "The method of Euclid is an elegant application of the *operation* requisite to compound equal ratios, by which the conception of the process is lost sight of." For the general reason given in the note on VI. 14 above, I think that Euclid showed the sounder discretion in the arrangement which he adopted. Moreover it is not easy to see how performing the actual operation of compounding two equal ratios can obscure the process, or the fact that two equal ratios are being compounded. On the definition of compounded ratios and duplicate ratio, De Morgan has himself acutely pointed out that "composition" is here used for the process of detecting the single alteration which produces the effect of two or more, the duplicate ratio being the result of compounding two equal ratios. The proof of VI. 19 does in fact exhibit the single alteration which produces the effect of two. And the *operation* was of the essence of the Greek geometry, because it was the manipulation of ratios in this manner, by simplification and transformation, that gave it so much power, as every one knows who has read, say, Archimedes or Apollonius. Hence the introduction of the necessary *operation*, as well as the theoretical proof, in this proposition seems to me to have been distinctly worth while, and, as it is somewhat simpler in this case than in the more general case of VI. 23, it was in accordance with the plan of enabling the difficulties of Book VI. to be more easily and gradually surmounted to give the simpler case first.

That Euclid wished to emphasise the importance of the *method* adopted, as well as of the result obtained, in VI. 19 seems to me clearly indicated by the Porism which follows the proposition. It is as if he should say: "I have shown you that similar triangles are to one another in the duplicate ratio of corresponding sides; but I have also shown you incidentally how it is possible to work conveniently with duplicate ratios, viz. by transforming them into simple ratios between straight lines. I shall have occasion to illustrate the use of this method in the proof of VI. 22."

The Porism to VI. 19 presents one difficulty. It will be observed that it speaks of the *figure* (εἶδος) described on the first straight line and of that which is similar and similarly described on the second. If "figure" could be regarded as loosely used for the figure *of the proposition*, i.e. for a *triangle*, there would be no difficulty. If on the other hand "the figure" means any rectilineal figure, i.e. any polygon, the Porism is not really established until the next proposition, VI. 20, has been proved, and therefore it is out of place here. Yet the correction τρίγωνον, *triangle*, for εἶδος, *figure*, is due to Theon alone; P and Campanus have "figure," and the reading of Philoponus and Psellus, τετράγωνον, *square*, partly supports εἶδος, since it can be reconciled with εἶδος but not with τρίγωνον. Again the second Porism to VI. 20, in which this Porism is reasserted for any rectilineal figure, and which is omitted by Campanus and only given by P in the margin, was probably interpolated by Theon. Heiberg concludes that Euclid wrote "figure" (εἶδος), and Theon, seeing the difficulty, changed the word into "triangle" here and added Por. 2 to VI. 20 in order to make the matter clear. If one may hazard a guess as to how Euclid made the slip, may it be that he first put it after VI 20 and then, observing that the expression of the duplicate ratio by a single ratio between two straight lines does not come in VI. 20 but in VI. 19, moved the Porism to the end of VI. 19 in order to make the connexion clearer, without noticing that, if this were done, εἶδος would need correction?

The following explanation at the end of the Porism is bracketed by Heiberg, viz. "Since it was proved that, as CB is to BG, so is the triangle

ABC to the triangle *ABG*, that is *DEF*." Such explanations in Porisms are not in Euclid's manner, and the words are not in Campanus, though they date from a time earlier than Theon.

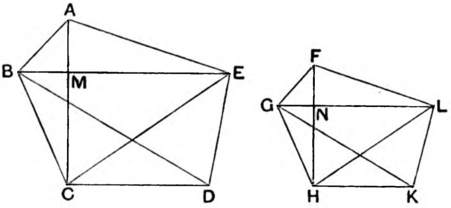
PROPOSITION 20.

Similar polygons are divided into similar triangles, and into triangles equal in multitude and in the same ratio as the wholes, and the polygon has to the polygon a ratio duplicate of that which the corresponding side has to the corresponding side.

Let *ABCDE*, *FGHKL* be similar polygons, and let *AB* correspond to *FG*;

I say that the polygons *ABCDE*, *FGHKL* are divided into similar triangles, and into triangles equal in multitude and in the same ratio as the wholes, and the polygon *ABCDE* has to the polygon *FGHKL* a ratio duplicate of that which *AB* has to *FG*.

Let *BE*, *EC*, *GL*, *LH* be joined.



Now, since the polygon *ABCDE* is similar to the polygon *FGHKL*,

the angle *BAE* is equal to the angle *GFL*;

and, as *BA* is to *AE*, so is *GF* to *FL*. [VI. Def. 1]

Since then *ABE*, *FGL* are two triangles having one angle equal to one angle and the sides about the equal angles proportional,

therefore the triangle *ABE* is equiangular with the triangle *FGL*; [VI. 6]

so that it is also similar; [VI. 4 and Def. 1]

therefore the angle *ABE* is equal to the angle *FGL*.

But the whole angle *ABC* is also equal to the whole angle *FGH* because of the similarity of the polygons;

therefore the remaining angle *EBC* is equal to the angle *LGH*.

And, since, because of the similarity of the triangles *ABE*, *FGL*,

as *EB* is to *BA*, so is *LG* to *GF*,

and moreover also, because of the similarity of the polygons,

as *AB* is to *BC*, so is *FG* to *GH*,

therefore, *ex aequali*, as *EB* is to *BC*, so is *LG* to *GH*; [V. 22]

that is, the sides about the equal angles *EBC*, *LGH* are proportional;

therefore the triangle *EBC* is equiangular with the triangle *LGH*, [VI. 6]

so that the triangle *EBC* is also similar to the triangle *LGH*. [VI. 4 and Def. 1]

For the same reason

the triangle *ECD* is also similar to the triangle *LHK*.

Therefore the similar polygons *ABCDE*, *FGHKL* have been divided into similar triangles, and into triangles equal in multitude.

I say that they are also in the same ratio as the wholes, that is, in such manner that the triangles are proportional, and *ABE*, *EBC*, *ECD* are antecedents, while *FGL*, *LGH*, *LHK* are their consequents, and that the polygon *ABCDE* has to the polygon *FGHKL* a ratio duplicate of that which the corresponding side has to the corresponding side, that is *AB* to *FG*.

For let *AC*, *FH* be joined.

Then since, because of the similarity of the polygons,

the angle *ABC* is equal to the angle *FGH*,

and, as *AB* is to *BC*, so is *FG* to *GH*,

the triangle *ABC* is equiangular with the triangle *FGH*; [VI. 6]

therefore the angle *BAC* is equal to the angle *GFH*,

and the angle *BCA* to the angle *GHF*.

And, since the angle *BAM* is equal to the angle *GFN*,

and the angle *ABM* is also equal to the angle *FGN*, therefore the remaining angle *AMB* is also equal to the remaining angle *FNG*; [I. 32]

therefore the triangle *ABM* is equiangular with the triangle *FGN*.

Similarly we can prove that

the triangle *BMC* is also equiangular with the triangle *GNH*.

Therefore, proportionally, as *AM* is to *MB*, so is *FN* to *NG*,

and, as *BM* is to *MC*, so is *GN* to *NH*;

so that, in addition, *ex aequali*,

as *AM* is to *MC*, so is *FN* to *NH*.

But, as *AM* is to *MC*, so is the triangle *ABM* to *MBC*, and *AME* to *EMC*; for they are to one another as their bases. [VI. 1]

Therefore also, as one of the antecedents is to one of the consequents, so are all the antecedents to all the consequents; [V. 12]

therefore, as the triangle *AMB* is to *BMC*, so is *ABE* to *CBE*.

But, as *AMB* is to *BMC*, so is *AM* to *MC*;

therefore also, as *AM* is to *MC*, so is the triangle *ABE* to the triangle *EBC*.

For the same reason also,

as *FN* is to *NH*, so is the triangle *FGL* to the triangle *GLH*.

And, as *AM* is to *MC*, so is *FN* to *NH*;

therefore also, as the triangle *ABE* is to the triangle *BEC*, so is the triangle *FGL* to the triangle *GLH*;

and, alternately, as the triangle *ABE* is to the triangle *FGL*, so is the triangle *BEC* to the triangle *GLH*.

Similarly we can prove, if *BD*, *GK* be joined, that, as the triangle *BEC* is to the triangle *LGH*, so also is the triangle *ECD* to the triangle *LHK*.

And since, as the triangle *ABE* is to the triangle *FGL*, so is *EBC* to *LGH*, and further *ECD* to *LHK*,

therefore also, as one of the antecedents is to one of the consequents, so are all the antecedents to all the consequents; [V. 12

therefore, as the triangle *ABE* is to the triangle *FGL*,

so is the polygon *ABCDE* to the polygon *FGHKL*.

But the triangle *ABE* has to the triangle *FGL* a ratio duplicate of that which the corresponding side *AB* has to the corresponding side *FG*; for similar triangles are in the duplicate ratio of the corresponding sides. [VI. 19]

Therefore the polygon *ABCDE* also has to the polygon *FGHKL* a ratio duplicate of that which the corresponding side *AB* has to the corresponding side *FG*.

Therefore etc.

Porism. Similarly also it can be proved in the case of quadrilaterals that they are in the duplicate ratio of the corresponding sides. And it was also proved in the case of triangles; therefore also, generally, similar rectilineal figures are to one another in the duplicate ratio of the corresponding sides.

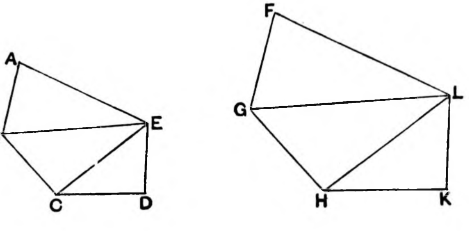
Q. E. D.

2. **in the same ratio as the wholes.** The same word ὁμόλογος is used which I have generally translated by "corresponding." But here it is followed by a dative, ὁμόλογα τοῖς ὅλοις "*homologous with* the wholes," instead of being used absolutely. The meaning can therefore here be nothing else but "in the same ratio with" or "proportional to the wholes"; and Euclid seems to recognise that he is making a special use of the word, because he explains it lower down (l. 46): "the triangles are homologous to the wholes, that is, in such manner that the triangles are proportional, and *ABE*, *EBC*, *ECD* are antecedents, while *FGL*, *LGH*, *LHK* are their consequents."

49. ἑπόμενα αὐτῶν, "*their* consequents," is a little awkward, but may be supposed to indicate which triangles correspond to which as consequent to antecedent.

An alternative proof of the second part of this proposition given after the Porisms is relegated by August and Heiberg to an Appendix as an interpolation. It is shorter than the proof in the text, and is the only one given by many editors, including Clavius, Billingsley, Barrow and Simson. It runs as follows:

"We will now also prove that the triangles are homologous in another and an easier manner.



Again, let the polygons *ABCDE*, *FGHKL* be set out, and let *BE*, *EC*, *GL*, *LH* be joined.

I say that, as the triangle *ABE* is to *FGL*, so is *EBC* to *LGH* and *CDE* to *HKL*.

For, since the triangle *ABE* is similar to the triangle *FGL*, the triangle *ABE* has to the triangle *FGL* a ratio duplicate of that which *BE* has to *GL*.

For the same reason also

the triangle *BEC* has to the triangle *GLH* a ratio duplicate of that which *BE* has to *GL*.

Therefore, as the triangle *ABE* is to the triangle *FGL*, so is *BEC* to *GLH*.

Again, since the triangle *EBC* is similar to the triangle *LGH*,

EBC has to *LGH* a ratio duplicate of that which the straight line *CE* has to *HL*.

For the same reason also

the triangle *ECD* has to the triangle *LHK* a ratio duplicate of that which *CE* has to *HL*.

Therefore, as the triangle *EBC* is to *LGH*, so is *ECD* to *LHK*.

But it was proved that,

as *EBC* is to *LGH*, so also is *ABE* to *FGL*.

Therefore also, as *ABE* is to *FGL*, so is *BEC* to *GLH* and *ECD* to *LHK*.

Q. E. D."

Now Euclid cannot fail to have noticed that the second part of his proposition could be proved in this way. It seems therefore that, in giving the other and longer method, he deliberately wished to avoid using the result of VI. 19, preferring to prove the first two parts of the theorem, as they can be proved, independently of any relation between the areas of similar triangles.

The first part of the Porism, stating that the theorem is true of *quadrilaterals*, would be superfluous but for the fact that technically, according to Book I. Def. 19, the term "polygon" (or figure of many sides, πολύπλευρον) used in the enunciation of the proposition is confined to rectilineal figures of *more than four* sides, so that a quadrilateral might seem to be excluded. The mention of the triangle in addition fills up the tale of "similar rectilineal figures."

The second Porism, Theon's interpolation, given in the text by the editors, but bracketed by Heiberg, is as follows:

"And, if we take *O* a third proportional to *AB*, *FG*, then *BA* has to *O* a ratio duplicate of that which *AB* has to *FG*.

But the polygon has also to the polygon, or the quadrilateral to the quadrilateral, a ratio duplicate of that which the corresponding side has to the corresponding side, that is *AB* to *FG*;

and this was proved in the case of triangles also;

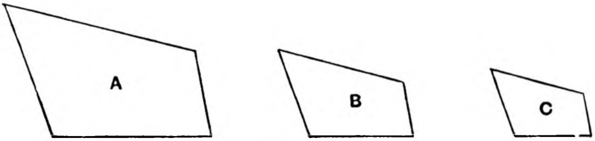
so that it is also manifest generally that, if three straight lines be proportional, as the first is to the third, so will the figure described on the first be to the similar and similarly described figure on the second."

PROPOSITION 21.

Figures which are similar to the same rectilineal figure are also similar to one another.

For let each of the rectilineal figures *A*, *B* be similar to *C*; I say that *A* is also similar to *B*.

For, since *A* is similar to *C*, it is equiangular with it and has the sides about the equal angles proportional. [VI. Def. 1]



Again, since *B* is similar to *C*, it is equiangular with it and has the sides about the equal angles proportional.

Therefore each of the figures *A*, *B* is equiangular with *C* and with *C* has the sides about the equal angles proportional;

therefore *A* is similar to *B*.

Q. E. D.

It will be observed that the text above omits a step which the editions generally have before the final inference "Therefore *A* is similar to *B*." The words omitted are "so that *A* is also equiangular with *B* and [with *B*] has the sides about the equal angles proportional." Heiberg follows P in leaving them out, conjecturing that they may be an addition of Theon's.

PROPOSITION 22.

If four straight lines be proportional, the rectilineal figures similar and similarly described upon them will also be proportional; and, if the rectilineal figures similar and similarly described upon them be proportional, the straight lines will themselves also be proportional.

Let the four straight lines *AB*, *CD*, *EF*, *GH* be proportional,

so that, as *AB* is to *CD*, so is *EF* to *GH*,

and let there be described on *AB*, *CD* the similar and similarly situated rectilineal figures *KAB*, *LCD*,

and on *EF*, *GH* the similar and similarly situated rectilineal figures *MF*, *NH*;

I say that, as *KAB* is to *LCD*, so is *MF* to *NH*.

For let there be taken a third proportional *O* to *AB*, *CD*, and a third proportional *P* to *EF*, *GH*. [VI. 11]

Then since, as *AB* is to *CD*, so is *EF* to *GH*,

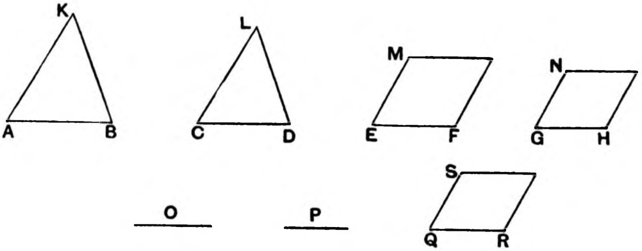
and, as *CD* is to *O*, so is *GH* to *P*,

therefore, *ex aequali*, as *AB* is to *O*, so is *EF* to *P*. [V. 22]

But, as *AB* is to *O*, so is *KAB* to *LCD*,

and, as *EF* is to *P*, so is *MF* to *NH*; [VI. 19, Por.]

therefore also, as *KAB* is to *LCD*, so is *MF* to *NH*. [V. 11]



Next, let *MF* be to *NH* as *KAB* is to *LCD*;

I say also that, as *AB* is to *CD*, so is *EF* to *GH*.

For, if *EF* is not to *GH* as *AB* to *CD*,

let *EF* be to *QR* as *AB* to *CD*, [VI. 12]

and on *QR* let the rectilineal figure *SR* be described similar and similarly situated to either of the two *MF*, *NH*. [VI. 18]

Since then, as *AB* is to *CD*, so is *EF* to *QR*,

and there have been described on *AB*, *CD* the similar and similarly situated figures *KAB*, *LCD*,

and on *EF*, *QR* the similar and similarly situated figures *MF*, *SR*,

therefore, as *KAB* is to *LCD*, so is *MF* to *SR*.

But also, by hypothesis,

as *KAB* is to *LCD*, so is *MF* to *NH*;

therefore also, as *MF* is to *SR*, so is *MF* to *NH*. [V. 11]

Therefore *MF* has the same ratio to each of the figures *NH*, *SR*;

therefore *NH* is equal to *SR*. [V. 9]

But it is also similar and similarly situated to it;

therefore *GH* is equal to *QR*.

And, since, as AB is to CD, so is EF to QR,
while QR is equal to GH,
therefore, as AB is to CD, so is EF to GH.

Therefore etc.

Q. E. D.

The second assumption in the first step of the first part of the proof, viz. that, as CD is to O, so GH to P, should perhaps be explained. It is a deduction [by V. 11] from the facts that

AB is to CD as CD to O,
EF is to GH as GH to P,
and AB is to CD as EF to GH.

The defect in the proof of this proposition is well known, namely the assumption, without proof, that, because the figures NH, SR are equal, besides being similar and similarly situated, their corresponding sides GH, QR are equal. Hence the minimum addition necessary to make the proof complete is a proof of a lemma to the effect that, *if two similar figures are also equal, any pair of corresponding sides are equal.*

To supply this lemma is one alternative; another is to prove, as a preliminary proposition, a much more general theorem, viz. that, *if the duplicate ratios of two ratios are equal, the two ratios are themselves equal.* When this is proved, the second part of VI. 22 is an immediate inference from it, and the effect is, of course, to substitute a new proof instead of supplementing Euclid's.

I. It is to be noticed that the lemma required as a minimum is very like what is needed to supplement VI. 28 and 29, in the proofs of which Euclid assumes that, *if two similar parallelograms are unequal, any side in the greater is greater than the corresponding side in the smaller.* Therefore, on the whole, it seems preferable to adopt the alternative of proving the simpler lemma which will serve to supplement all three proofs, viz. that, *if of two similar rectilineal figures the first is greater than, equal to, or less than, the second, any side of the first is greater than, equal to, or less than, the corresponding side of the second respectively.*

The case of *equality* of the figures is the case required for VI. 22; and the proof of it is given in the Greek text after the proposition. Since to give such a "lemma" after the proposition in which it is required is contrary to Euclid's manner, Heiberg concludes that it is an interpolation, though it is earlier than Theon. The lemma runs thus:

"But that, if rectilineal figures be equal and similar, their corresponding sides are equal to one another we will prove thus.

Let NH, SR be equal and similar rectilineal figures, and suppose that,

as HG is to GN, so is RQ to QS;

I say that RQ is equal to HG.

For, if they are unequal, one of them is greater;

let RQ be greater than HG.

Then, since, as RQ is to QS, so is HG to GN,
alternately also, as RQ is to HG, so is QS to GN;
and QR is greater than HG;
therefore QS is also greater than GN;
so that RS is also greater than HN*.
But it is also equal: which is impossible.
Therefore QR is not unequal to GH;
therefore it is equal to it."

[The step marked * is easy to see if it is remembered that it is only necessary to prove its truth in the case of *triangles* (since similar polygons are divisible into the same number of similar and similarly situated triangles having the same ratio to each other respectively as the polygons have). If the triangles be applied to each other so that the two corresponding sides of each, which are used in the question, and the angles included by them coincide, the truth of the inference is obvious.]

The lemma might also be arrived at by proving that, *if a ratio is greater than a ratio of equality, the ratio which is its duplicate is also greater than a ratio of equality; and if the ratio which is duplicate of another ratio is greater than a ratio of equality, the ratio of which it is the duplicate is also greater than a ratio of equality.* It is not difficult to prove this from the particular case of v. 25 in which the second magnitude is equal to the third, i.e. from the fact that in this case the sum of the extreme terms is greater than double the middle term.

II. We now come to the alternative which substitutes a new proof for the second part of the proposition, making the whole proposition an immediate inference from one to which it is practically equivalent, viz. that

(1) *If two ratios be equal, their duplicate ratios are equal,* and (2) *conversely, if the duplicate ratios of two ratios be equal, the ratios are equal.*

The proof of part (1) is after the manner of Euclid's own proof of the first part of VI. 22.

Let A be to B as C to D,
and let X be a third proportional to A, B, and Y a third proportional to C, D, so that

A is to B as B to X,
and C is to D as D to Y;
whence A is to X in the duplicate ratio of A to B,
and C is to Y in the duplicate ratio of C to D.
Since A is to B as C is to D,
and B is to X as A is to B,
i.e. as C is to D,
i.e. as D is to Y, [v. 11]
therefore, *ex aequali*, A is to X as C is to Y.

Part (2) is much more difficult and is the crux of the whole thing.

Most of the proofs depend on the assumption that, B being any magnitude and P and Q two magnitudes of the same kind, there does exist a magnitude A which is to B in the same ratio as P to Q. It is this same assumption which makes Euclid's proof of V. 18 illegitimate, since it is nowhere proved in Book V. Hence any proof of the proposition now in question which involves this assumption even in the case where B, P, Q are all straight lines should not properly be given as an addition to Book V.; it should at least be postponed until we have learnt, by means of VI. 12, giving the actual construction of a fourth proportional, that such a fourth proportional exists.

Two proofs which are given of the proposition depend upon the following lemma.

If A, B, C *be three magnitudes of one kind, and* D, E, F *three magnitudes of one kind, then, if*

the ratio of A *to* B *is greater than that of* D *to* E,

and *the ratio of* B *to* C *greater than that of* E *to* F,

ex aequali, *the ratio of* A *to* C *is greater than that of* D *to* F.

One proof of this does not depend upon the assumption referred to, and therefore, if this proof is used, the theorem can be added to Book V. The proof is that of Hauber (Camerer's Euclid, p. 358 of Vol. II.) and is reproduced by Mr H. M. Taylor. For brevity we will use symbols.

Take equimultiples mA, mD of A, D and nB, nE of B, E such that

$$mA > nB, \text{ but } mD \not> nE.$$

Also let pB, pE be equimultiples of B, E and qC, qF equimultiples of C, F such that

$$pB > qC, \text{ but } pE \not> qF.$$

Therefore, multiplying the first line by p and the second by n, we have

$$pmA > pnB, \; pmD \not> pnE,$$

and $$npB > nqC, \; npE \not> nqF,$$

whence $$pmA > nqC, \; pmD \not> nqF.$$

Now pmA, pmD are equimultiples of mA, mD,

and nqC, nqF equimultiples of qC, qF.

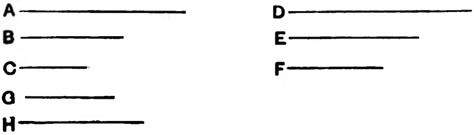
Therefore [V. 3] they are respectively equimultiples of A, D and of C, F.

Hence [V. Def. 7] $A : C > D : F$.

Another proof given by Clavius, though depending on the assumption referred to, is neat.

Take G such that

$$G : C = E : F.$$



Therefore $B : C > G : C$, [V. 13]

and $B > G$. [V. 10]

Therefore $A : G > A : B$. [V. 8]

But $A : B > D : E$.

Therefore, *a fortiori*, $A : G > D : E$.

Suppose H taken such that

$$H : G = D : E.$$

Therefore $A > H$. [v. 13, 10]

Hence $A : C > H : C$. [v. 8]

But $H : G = D : E$,

$G : C = E : F$.

Therefore, *ex aequali*, $H : C = D : F$. [v. 22]

Hence $A : C > D : F$. [v. 13]

Now we can prove that

Ratios of which equal ratios are duplicate are equal.

Suppose that $A : B = B : C$,

and $D : E = E : F$,

and further that $A : C = D : F$,

it is required to prove that

$$A : B = D : E.$$

For, if not, one of the ratios must be greater than the other.

Let $A : B$ be the greater.

Then, since $A : B = B : C$,

and $D : E = E : F$,

while $A : B > D : E$,

it follows that $B : C > E : F$. [v. 13]

Hence, by the lemma, *ex aequali*,

$$A : C > D : F,$$

which contradicts the hypothesis.

Thus the ratios $A : B$ and $D : E$ cannot be unequal; that is, they are equal.

Another proof, given by Dr Lachlan, also assumes the existence of a fourth proportional, but depends upon a simpler lemma to the effect that

It is impossible that two different ratios can have the same duplicate ratio.

For, if possible, let the ratio $A : B$ be duplicate both of $A : X$ and $A : Y$, so that

$$A : X = X : B,$$

and $A : Y = Y : B$.

Let X be greater than Y.

Then $A : X < A : Y$; [v. 8]

that is, $X : B < Y : B$, [v. 11, 13]

or $X < Y$. [v. 10]

But X is greater than Y: which is absurd, etc.

Hence $X = Y$.

Now suppose that $A : B = B : C$,

$D : E = E : F$,

and $A : C = D : F$.

To prove that $A : B = D : E$.

If this is not so, suppose that

$A : B = D : Z$.

Since $A : C = D : F$,

therefore, inversely, $C : A = F : D$.

Therefore, *ex aequali*,

$C : B = F : Z$, [v. 22]

or, inversely, $B : C = Z : F$.

Therefore $A : B = Z : F$. [v. 11]

But $A : B = D : Z$, by hypothesis.

Therefore $D : Z = Z : F$. [v. 11]

Also, by hypothesis, $D : E = E : F$;

whence, by the lemma, $E = Z$.

Therefore $A : B = D : E$.

De Morgan remarks that the best way of remedying the defect in Euclid is to insert the proposition (the lemma to the last proof) that *it is impossible that two different ratios can have the same duplicate ratio*, "which," he says, "immediately proves the second (or defective) case of the theorem." But this seems to be either too much or too little: too much, if we choose to make the *minimum* addition to Euclid (for that addition is a lemma which shall prove that, if a duplicate ratio is a ratio of equality, the ratio of which it is duplicate is also one of equality), and too little if the proof is to be altered in the more fundamental manner explained above.

I think that, if Euclid's attention had been drawn to the defect in his proof of VI. 22 and he had been asked to remedy it, he would have done so by supplying what I have called the minimum lemma and not by making the more fundamental alteration. This I infer from Prop. 24 of the *Data*, where he gives a theorem corresponding to the proposition that *ratios of which equal ratios are duplicate are equal.* The proposition in the *Data* is enunciated thus: *If three straight lines be proportional, and the first have to the third a given ratio, it will also have to the second a given ratio.*

A, B, C being the three straight lines, so that

$$A : B = B : C,$$

and $A : C$ being a given ratio, it is required to prove that $A : B$ is also a given ratio.

Euclid takes any straight line D, and first finds another, F, such that

$$D : F = A : C,$$

whence $D : F$ must be a given ratio, and, as D is given, F is therefore given.

Then he takes E a mean proportional between D, F, so that

$$D : E = E : F.$$

It follows [VI. 17] that

the rectangle D, F is equal to the square on E.

But D, F are both given;

therefore the square on E is given, so that E is also given.

[Observe that De Morgan's lemma is here assumed without proof. It may be proved (1) as it is by De Morgan, whose proof is that given above, p. 245, (2) in the manner of the "minimum lemma," pp. 242—3 above, or (3) as it is by Proclus on I. 46 (see note on that proposition).]

Hence the ratio $D : E$ is given.

Now, since $A : C = D : F$,

and $A : C =$ (square on A): (rect. A, C),

while $D : F =$ (square on D): (rect. D, F), [VI. 1]

therefore (square on A) : (rect. A, C) = (square on D) : (rect. D, F). [V. 11]

But, since $A : B = B : C$, (rect. A, C) = (sq. on B); [VI. 17]

and (rect. D, F) = (sq. on E), from above;

therefore (square on A) : (square on B) = (sq. on D) : (sq. on E).

Therefore, says Euclid,

$$A : B = D : E,$$

that is, *he assumes the truth of* VI. 22 *for squares.*

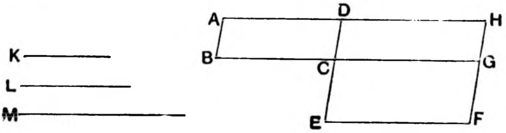
Thus he deduces his proposition from VI. 22, instead of proving VI. 22 by means of it (or the corresponding proposition used by Mr Taylor and Dr Lachlan).

PROPOSITION 23.

Equiangular parallelograms have to one another the ratio compounded of the ratios of their sides.

Let AC, CF be equiangular parallelograms having the angle BCD equal to the angle ECG;

I say that the parallelogram AC has to the parallelogram CF the ratio compounded of the ratios of the sides.



For let them be placed so that BC is in a straight line with CG;

therefore DC is also in a straight line with CE.

Let the parallelogram DG be completed;

let a straight line K be set out, and let it be contrived that,

as BC is to CG, so is K to L,

and, as DC is to CE, so is L to M. [VI. 12]

Then the ratios of *K* to *L* and of *L* to *M* are the same as the ratios of the sides, namely of *BC* to *CG* and of *DC* to *CE*.

But the ratio of *K* to *M* is compounded of the ratio of *K* to *L* and of that of *L* to *M*;

so that *K* has also to *M* the ratio compounded of the ratios of the sides.

Now since, as *BC* is to *CG*, so is the parallelogram *AC* to the parallelogram *CH*, [VI. 1]

while, as *BC* is to *CG*, so is *K* to *L*,

therefore also, as *K* is to *L*, so is *AC* to *CH*. [V. 11]

Again, since, as *DC* is to *CE*, so is the parallelogram *CH* to *CF*, [VI. 1]

while, as *DC* is to *CE*, so is *L* to *M*,

therefore also, as *L* is to *M*, so is the parallelogram *CH* to the parallelogram *CF*. [V. 11]

Since then it was proved that, as *K* is to *L*, so is the parallelogram *AC* to the parallelogram *CH*,

and, as *L* is to *M*, so is the parallelogram *CH* to the parallelogram *CF*,

therefore, *ex aequali*, as *K* is to *M*, so is *AC* to the parallelogram *CF*.

But *K* has to *M* the ratio compounded of the ratios of the sides;

therefore *AC* also has to *CF* the ratio compounded of the ratios of the sides.

Therefore etc.

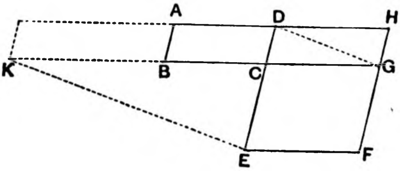
Q. E. D.

1, 6, 19, 36. **the ratio compounded of the ratios of the sides,** *λόγον τὸν συγκείμενον ἐκ τῶν πλευρῶν* which, meaning literally "the ratio compounded *of the sides*," is negligently written here and commonly for *λόγον τὸν συγκείμενον ἐκ τῶν τῶν πλευρῶν* (sc. *λόγων*).

11. **let it be contrived that, as BC is to CG, so is K to L.** The Greek phrase is of the usual terse kind, untranslatable literally: *καὶ γεγονέτω ὡς μὲν ἡ* ΒΓ *πρὸς τὴν* ΓΗ, *οὕτως ἡ* Κ *πρὸς τὸ* Λ, the words meaning "and let (there) be made, as *BC* to *CG*, so *K* to *L*," where *L* is the straight line which has to be constructed.

The second definition of the *Data* says that *A ratio is said to be given if we can find* (πορίσασθαι) [*another ratio that is*] *the same with it.* Accordingly VI. 23 not only proves that equiangular parallelograms have to one another a ratio which is compounded of two others, but shows that that ratio is "given" when its component ratios are given, or that it can be represented as a simple ratio between straight lines.

Just as VI. 23 exhibits the operation necessary for *compounding* two ratios, a proposition (8) of the *Data* indicates the operation by which we may divide one ratio by another. The proposition proves that *Things which have a given ratio to the same thing have also a given ratio to one another.* Euclid's procedure is of course to *compound* one ratio with the *inverse* of the other; but, when this is once done and the result of Prop. 8 obtained, he uses the result in the later propositions as a substitute for the method of composition. Thus he uses the *division* of ratios, instead of composition, in the propositions of the *Data* which deal with the same subject-matter as VI. 23. The effect is to represent the ratio of two equiangular parallelograms as a ratio between straight lines one of which is *one side of one of the parallelograms.* Prop. 56 of the *Data* shows us that, if we want to express the ratio of the parallelogram AC to the parallelogram CF in the figure



of VI. 23 in the form of a ratio in which, for example, the side BC is the antecedent term, the required ratio of the parallelograms is $BC : X$, where

$$DC : CE = CG : X,$$

or X is a fourth proportional to DC and the two sides of the parallelogram CF.

Measure CK along CB, produced if necessary, so that

$$DC : CE = CG : CK$$

(whence CK is equal to X).

[This may be simply done by joining DG and then drawing EK parallel to it meeting CB in K.]

Complete the parallelogram AK.

Then, since $DC : CE = CG : CK$,

the parallelograms DK, CF are equal. [VI. 14]

Therefore $(AC) : (CF) = (AC) : (DK)$ [V. 7]

$= BC : CK$ [VI. 1]

$= BC : X$.

Prop. 68 of the *Data* uses the same construction to prove that, *If two equiangular parallelograms have to one another a given ratio, and one side have to one side a given ratio, the remaining side will also have to the remaining side a given ratio.*

I do not use the figure of the *Data* but, for convenience' sake, I adhere to the figure given above. Suppose that the ratio of the parallelograms is given, and also that of CD to CE.

Apply to CD the parallelogram DK equal to CF and such that CK, CB coincide in direction. [I. 45]

Then the ratio of AC to KD is given, being equal to that of AC to CF.

And $(AC) : (KD) = CB : CK$;

therefore the ratio of CB to CK is given.

But, since $KD = CF$,

$$CD : CE = CG : CK. \qquad [\text{VI. } 14]$$

Hence $CG : CK$ is a given ratio.

And $CB : CK$ was proved to be a given ratio.

Therefore the ratio of CB to CG is given. [*Data*, Prop. 8]

Lastly we may refer to Prop. 70 of the *Data*, the first part of which proves what corresponds exactly to VI. 23, namely that, *If in two equiangular parallelograms the sides containing the equal angles have a given ratio to one another* [i.e. one side in one to one side in the other], *the parallelograms themselves will also have a given ratio to one another.* [Here the ratios of BC to CG and of CD to CE are given.]

The construction is the same as in the last case, and we have KD equal to CF, so that

$$CD : CE = CG : CK. \qquad [\text{VI. } 14]$$

But the ratio of CD to CE is given;

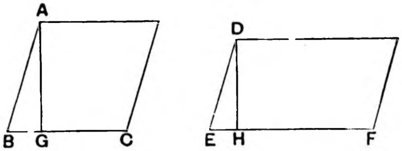
therefore the ratio of CG to CK is given.

And, by hypothesis, the ratio of CG to CB is given.

Therefore, by *dividing* the ratios [*Data*, Prop. 8], we see that the ratio of CB to CK, and therefore [VI. 1] the ratio of AC to DK, or of AC to CF, is given.

Euclid extends these propositions to the case of two parallelograms which have *given* but not equal angles.

Pappus (VII. p. 928) exhibits the result of VI. 23 in a different way, which throws new light on compounded ratios. He proves, namely, that *a parallelogram is to an equiangular parallelogram as the rectangle contained by the adjacent sides of the first is to the rectangle contained by the adjacent sides of the second.*



Let AC, DF be equiangular parallelograms on the bases BC, EF, and let the angles at B, E be equal.

Draw perpendiculars AG, DH to BC, EF respectively.

Since the angles at B, G are equal to those at E, H,

the triangles ABG, DEH are equiangular.

Therefore $$BA : AG = ED : DH. \qquad [\text{VI. } 4]$$

But $$BA : AG = (\text{rect. } BA, BC) : (\text{rect. } AG, BC),$$

and $$ED : DH = (\text{rect. } ED, EF) : (\text{rect. } DH, EF). \qquad [\text{VI. } 1]$$

Therefore [V. 11 and V. 16]

$$(\text{rect. } AB, BC) : (\text{rect. } DE, EF) = (\text{rect. } AG, BC) : (\text{rect. } DH, EF)$$
$$= (AC) : (DF).$$

Thus it is proved that the ratio compounded of the ratios $AB : DE$ and $BC : EF$ is equal to the ratio of the rectangle AB, BC to the rectangle DE, EF.

Since each parallelogram in the figure of the proposition can be divided into pairs of equal triangles, and all the triangles which are the halves of either parallelogram have two sides respectively equal and the angles included by them equal or supplementary, it can be at once deduced from VI. 23 (or it can be independently proved by the same method) that *triangles which have one angle of the one equal or supplementary to one angle of the other are in the ratio compounded of the ratios of the sides about the equal or supplementary angles.* Cf. Pappus VII. pp. 894—6.

VI. 23 also shows that *rectangles, and therefore parallelograms or triangles, are to one another in the ratio compounded of the ratios of their bases and heights.*

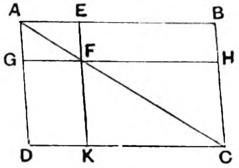
The converse of VI. 23 is also true, as is easily proved by *reductio ad absurdum.* More generally, *if two parallelograms or triangles are in the ratio compounded of the ratios of two adjacent sides, the angles included by those sides are either equal or supplementary.*

PROPOSITION 24.

In any parallelogram the parallelograms about the diameter are similar both to the whole and to one another.

Let $ABCD$ be a parallelogram, and AC its diameter, and let EG, HK be parallelograms about AC;

I say that each of the parallelograms EG, HK is similar both to the whole $ABCD$ and to the other.



For, since EF has been drawn parallel to BC, one of the sides of the triangle ABC,

proportionally, as BE is to EA, so is CF to FA. [VI. 2]

Again, since FG has been drawn parallel to CD, one of the sides of the triangle ACD,

proportionally, as CF is to FA, so is DG to GA. [VI. 2].

But it was proved that,

as CF is to FA, so also is BE to EA;

therefore also, as BE is to EA, so is DG to GA,

and therefore, *componendo*,

as BA is to AE, so is DA to AG, [V. 18]

and, alternately,

as BA is to AD, so is EA to AG. [V. 16]

Therefore in the parallelograms $ABCD$, EG, the sides about the common angle BAD are proportional.

And, since GF is parallel to DC,

the angle AFG is equal to the angle DCA;
and the angle DAC is common to the two triangles ADC, AGF;

therefore the triangle ADC is equiangular with the triangle AGF.

For the same reason

the triangle ACB is also equiangular with the triangle AFE,
and the whole parallelogram $ABCD$ is equiangular with the parallelogram EG.

Therefore, proportionally,

as AD is to DC, so is AG to GF,
as DC is to CA, so is GF to FA,
as AC is to CB, so is AF to FE,
and further, as CB is to BA, so is FE to EA.

And, since it was proved that,

as DC is to CA, so is GF to FA,
and, as AC is to CB, so is AF to FE,
therefore, *ex aequali*, as DC is to CB, so is GF to FE. [V. 22]

Therefore in the parallelograms $ABCD$, EG the sides about the equal angles are proportional;
therefore the parallelogram $ABCD$ is similar to the parallelogram EG. [VI. Def. 1]

For the same reason

the parallelogram $ABCD$ is also similar to the parallelogram KH;
therefore each of the parallelograms EG, HK is similar to $ABCD$.

But figures similar to the same rectilineal figure are also similar to one another; [VI. 21]
therefore the parallelogram EG is also similar to the parallelogram HK.

Therefore etc.

Q. E. D.

Simson was of opinion that this proof was made up by some unskilful editor out of two others, the first of which proved by parallels (VI. 2) that the sides about the common angle in the parallelograms are proportional, while the other used the similarity of triangles (VI. 4). It is of course true that, when we have proved by VI. 2 the fact that the sides about the common angle are proportional, we can infer the proportionality of the other sides directly from I. 34 combined with V. 7. But it does not seem to me unnatural that Euclid should (1) deliberately refrain from making any use of I. 34 and (2) determine beforehand that he would prove the sides proportional *in a definite order* beginning with the sides *EA*, *AG* and *BA*, *AD* about the common angle and then taking the remaining sides in the order indicated by the order of the letters *A*, *G*, *F*, *E*. Given that Euclid started the proof with such a fixed intention in his mind, the course taken presents no difficulty, nor is the proof unsystematic or unduly drawn out. And its genuineness seems to me supported by the fact that the proof, when once the first two sides about the common angle have been disposed of, follows closely the order and method of VI. 18. Moreover, it could readily be adapted to the more general case of two polygons having a common angle and the other corresponding sides respectively parallel.

The parallelograms in the proposition are of course similarly situated as well as similar; and those "about the diameter" may be "about" the diameter *produced* as well as about the diameter itself.

From the first part of the proof it follows that parallelograms which have one angle equal to one angle and the sides about those angles proportional are similar.

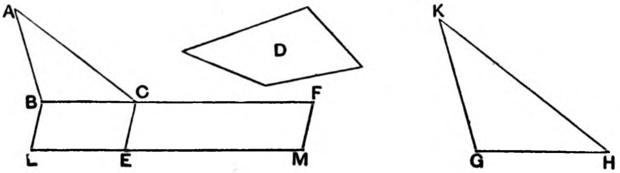
Prop. 26 is the converse of Prop. 24, and there seems to be no reason why they should be separated as they are in the text by the interposition of VI. 25. Campanus has VI. 24 and 26 as VI. 22 and 23 respectively, VI. 23 as VI. 24, and VI. 25 as we have it.

PROPOSITION 25.

To construct one and the same figure similar to a given rectilineal figure and equal to another given rectilineal figure.

Let *ABC* be the given rectilineal figure to which the figure to be constructed must be similar, and *D* that to which it must be equal;

thus it is required to construct one and the same figure similar to *ABC* and equal to *D*.



Let there be applied to *BC* the parallelogram *BE* equal to the triangle *ABC* [I. 44], and to *CE* the parallelogram *CM* equal to *D* in the angle *FCE* which is equal to the angle *CBL*. [I. 45]

Therefore *BC* is in a straight line with *CF*, and *LE* with *EM*.

Now let *GH* be taken a mean proportional to *BC*, *CF* [VI. 13], and on *GH* let *KGH* be described similar and similarly situated to *ABC*. [VI. 18]

Then, since, as *BC* is to *GH*, so is *GH* to *CF*, and, if three straight lines be proportional, as the first is to the third, so is the figure on the first to the similar and similarly situated figure described on the second, [VI. 19, Por.] therefore, as *BC* is to *CF*, so is the triangle *ABC* to the triangle *KGH*.

But, as *BC* is to *CF*, so also is the parallelogram *BE* to the parallelogram *EF*. [VI. 1]

Therefore also, as the triangle *ABC* is to the triangle *KGH*, so is the parallelogram *BE* to the parallelogram *EF*; therefore, alternately, as the triangle *ABC* is to the parallelogram *BE*, so is the triangle *KGH* to the parallelogram *EF*. [V. 16]

But the triangle *ABC* is equal to the parallelogram *BE*; therefore the triangle *KGH* is also equal to the parallelogram *EF*.

But the parallelogram *EF* is equal to *D*; therefore *KGH* is also equal to *D*.

And *KGH* is also similar to *ABC*.

Therefore one and the same figure *KGH* has been constructed similar to the given rectilineal figure *ABC* and equal to the other given figure *D*.

Q. E. D.

3. **to which the figure to be constructed must be similar,** literally "to which it is required to construct (one) similar," ᾧ δεῖ ὅμοιον συστήσασθαι.

This is the highly important problem which Pythagoras is credited with having solved. Compare the passage from Plutarch (*Symp.* VIII. 2, 4) quoted in the note on I. 44 above, Vol. I. pp. 343—4.

We are bidden to construct a rectilineal figure which shall have the *form* of one and the *size* of another rectilineal figure. The corresponding proposition of the *Data*, Prop. 55, asserts that, "if an area (χωρίον) be given in form (εἴδει) and in magnitude, its sides will also be given in magnitude."

Simson sees signs of corruption in the text of this proposition also. In the first place, the proof speaks of the *triangle ABC*, though, according to the enunciation, the figure for which *ABC* is taken may be *any* rectilineal figure, εὐθύγραμμον "rectilineal figure" would be more correct, or εἶδος, "figure"; the mistake, however, of using τρίγωνον is not one of great importance, being no doubt due to the accident by which the figure was drawn as a triangle in the diagram.

The other observation is more important. After Euclid has proved that

$$(\text{fig. } ABC) : (\text{fig. } KGH) = (BE) : (EF),$$

he might have inferred *directly* from v. 14 that, since ABC is equal to BE, KGH is equal to EF. For v. 14 includes the proof of the fact that, if A is to B as C is to D, and A is equal to C, then B is equal to D, or that of four proportional magnitudes, if the first is equal to the third, the second is equal to the fourth. Instead of proceeding in this way, Euclid first permutes the proportion by v. 16 into

$$(\text{fig. } ABC) : (BE) = (\text{fig. } KGH) : (EF),$$

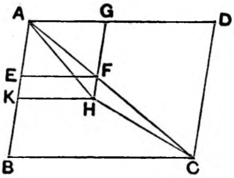
and then infers, as if the inference were easier in this form, that, since the *first* is equal to the *second*, the *third* is equal to the *fourth*. Yet there is no proposition to this effect in Euclid. The same unnecessary step of permutation is also found in the Greek text of XI. 23 and XII. 2, 5, 11, 12 and 18. In reproducing the proofs we may simply leave out the steps and refer to v. 14.

PROPOSITION 26.

If from a parallelogram there be taken away a parallelogram similar and similarly situated to the whole and having a common angle with it, it is about the same diameter with the whole

For from the parallelogram $ABCD$ let there be taken away the parallelogram AF similar and similarly situated to $ABCD$, and having the angle DAB common with it;

I say that $ABCD$ is about the same diameter with AF.



For suppose it is not, but, if possible, let AHC be the diameter < of $ABCD$ >, let GF be produced and carried through to H, and let HK be drawn through H parallel to either of the straight lines AD, BC. [I. 31]

Since, then, $ABCD$ is about the same diameter with KG, therefore, as DA is to AB, so is GA to AK. [VI. 24]

But also, because of the similarity of $ABCD$, EG,

as DA is to AB, so is GA to AE;

therefore also, as GA is to AK, so is GA to AE. [V. 11]

Therefore GA has the same ratio to each of the straight lines AK, AE.

Therefore AE is equal to AK [v. 9], the less to the greater: which is impossible.

Therefore $ABCD$ cannot but be about the same diameter with AF;

therefore the parallelogram $ABCD$ is about the same diameter with the parallelogram AF.

Therefore etc.

Q. E. D.

"For suppose it is not, but, if possible, let AHC be the diameter." What is meant is "For, if AFC is not the diameter of the parallelogram AC, let AHC be its diameter." The Greek text has ἔστω αὐτῶν διάμετρος ἡ ΑΘΓ; but clearly αὐτῶν is wrong, as we cannot assume that one straight line is the diameter of both parallelograms, which is just what we have to prove. F and V omit the αὐτῶν, and Heiberg prefers this correction to substituting αὐτοῦ after Peyrard. I have inserted "< of $ABCD$ >" to make the meaning clear.

If the straight line AHC does not pass through F, it must meet either GF or GF produced in some point H. The reading in the text "and let GF be *produced and carried through to* H" (καὶ ἐκβληθεῖσα ἡ HZ διήχθω ἐπὶ τὸ Θ) corresponds to the supposition that H is on GF produced. The words were left out by Theon, evidently because in the figure of the MSS. the letters E, Z and K, Θ were interchanged. Heiberg therefore, following August, has preferred to retain the words and to correct the figure, as well as the passage in the text where AE, AK were interchanged to be in accord with the MS. figure.

It is of course possible to prove the proposition directly, as is done by Dr Lachlan. Let AF, AC be the diagonals, and let us make no assumption as to how they fall.

Then, since EF is parallel to AG and therefore to BC,

the angles AEF, ABC are equal.

And, since the parallelograms are similar,

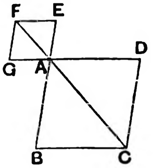
$$AE : EF = AB : BC. \qquad \text{[VI. Def. 1]}$$

Hence the triangles AEF, ABC are similar, [VI. 6]

and therefore the angle FAE is equal to the angle CAB.

Therefore AF falls on AC.

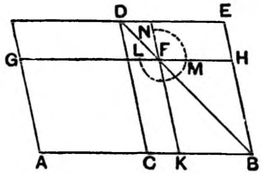
The proposition is equally true if the parallelogram which is similar and similarly situated to the given parallelogram is not "taken away" from it, but is so placed that it is entirely outside the other, while two sides form an angle vertically opposite to an angle of the other. In this case the diameters are not "the same," in the words of the enunciation, but are in a straight line with one another. This extension of the proposition is, as will be seen, necessary for obtaining, according to the method adopted by Euclid in his solution of the problem in VI. 28, the second solution of that problem.



PROPOSITION 27.

Of all the parallelograms applied to the same straight line and deficient by parallelogrammic figures similar and similarly situated to that described on the half of the straight line, that parallelogram is greatest which is applied to the half of the straight line and is similar to the defect.

Let *AB* be a straight line and let it be bisected at *C*; let there be applied to the straight line *AB* the parallelogram *AD* deficient by the parallelogrammic figure *DB* described on the half of *AB*, that is, *CB*;



I say that, of all the parallelograms applied to *AB* and deficient by parallelogrammic figures similar and similarly situated to *DB*, *AD* is greatest.

For let there be applied to the straight line *AB* the parallelogram *AF* deficient by the parallelogrammic figure *FB* similar and similarly situated to *DB*;

I say that *AD* is greater than *AF*.

For, since the parallelogram *DB* is similar to the parallelogram *FB*,

they are about the same diameter. [VI. 26]

Let their diameter *DB* be drawn, and let the figure be described.

Then, since *CF* is equal to *FE*, [I. 43]
and *FB* is common,

therefore the whole *CH* is equal to the whole *KE*.

But *CH* is equal to *CG*, since *AC* is also equal to *CB*. [I. 36]

Therefore *GC* is also equal to *EK*.

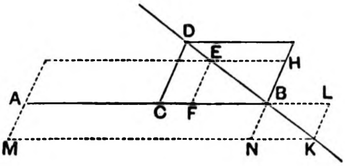
Let *CF* be added to each;

therefore the whole *AF* is equal to the gnomon *LMN*;

so that the parallelogram *DB*, that is, *AD*, is greater than the parallelogram *AF*.

Therefore etc.

We have already (note on I. 44) seen the significance, in Greek geometry, of the theory of "the application of areas, their exceeding and their falling-short." In I. 44 it was a question of "applying to a given straight line (exactly, without 'excess' or 'defect') a parallelogram equal to a given rectilineal figure, in a given angle." Here, in VI. 27—29, it is a question of parallelograms applied to a straight line but "*deficient* (or *exceeding*) *by parallelograms similar and similarly situated to a given parallelogram.*" Apart from size, it is easy to construct any number of parallelograms "deficient" or "exceeding" in the manner described. Given the straight line AB to which the parallelogram has to be applied, we describe on the base CB, where C is on AB, or on BA produced beyond A, any parallelogram "similarly situated" and either equal or similar to the given parallelogram (Euclid takes the similar and similarly situated parallelogram on half the line), draw the diagonal BD, take on it (produced if necessary) any points as E, K, draw EF, or KL, parallel to CD to meet AB or AB produced and complete the parallelograms, as AH, ML.



If the point E is taken on BD or BD produced beyond D, it must be so taken that EF meets AB between A and B. Otherwise the parallelogram AE would not be applied to AB itself, as it is required to be.

The parallelograms BD, BE, being about the same diameter, are similar [VI. 24], and BE is the defect of the parallelogram AE relatively to AB. AE is then a parallelogram applied to AB but deficient by a parallelogram similar and similarly situated to BD.

If K is on DB produced, the parallelogram BK is similar to BD, but it is the *excess* of the parallelogram AK relatively to the base AB. AK is a parallelogram applied to AB but exceeding by a parallelogram similar and similarly situated to BD.

Thus it is seen that BD produced both ways is the *locus* of points, such as E or K, which determine, with the direction of CD, the position of A, and the direction of AB, parallelograms applied to AB and *deficient* or *exceeding* by parallelograms similar and similarly situated to the given parallelogram.

The importance of VI. 27—29 from a historical point of view cannot be overrated. They give the geometrical equivalent of the algebraical solution of the most general form of quadratic equation when that equation has a real and positive root. It will also enable us to find a real *negative* root of a quadratic equation; for such an equation can, by altering the sign of x, be turned into another with a real *positive* root, when the geometrical method again becomes applicable. It will also, as we shall see, enable us to represent *both* roots when both are real and positive, and therefore to represent both roots when both are real but either positive or negative.

The method of these propositions was constantly used by the Greek geometers in the solution of problems, and they constitute the foundation of Book X. of the *Elements* and of Apollonius' treatment of the conic sections. Simson's observation on the subject is entirely justified. He says namely on VI. 28, 29: "These two problems, to the first of which the 27th Prop. is necessary, are the most general and useful of all in the Elements, and are most frequently made use of by the ancient geometers in the solution of other problems; and therefore are very ignorantly left out by Tacquet and

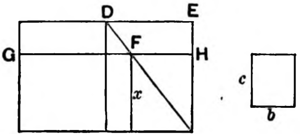
Dechales in their editions of the Elements, who pretend that they are scarce of any use."

It is strange that, with this observation before him, even Todhunter should have written as follows. "We have omitted in the sixth Book Propositions 27, 28, 29 and the first solution which Euclid gives of Proposition 30, as they appear now to be never required, and have been condemned as useless by various modern commentators; see Austin, Walker and Lardner."

VI. 27 contains the διορισμός, the condition for a real solution, of the problem contained in the proposition following it. The maximum of all the parallelograms having the given property which can be applied to a given straight line is that which is described upon half the line (τὸ ἀπὸ τῆς ἡμισείας ἀναγραφόμενον). This corresponds to the condition that an equation of the form

$$ax - px^2 = A$$

may have a real root. The correctness of the result may be seen by taking the case in which the parallelograms are rectangles, which enables us to leave out of account the *sine of the angle* of the parallelograms without any real loss of generality. Suppose the sides of the rectangle to which the *defect* is to be similar to be as b to c, b corresponding to the side of the defect which lies along AB. Suppose that $AKFG$ is any parallelogram applied to AB having the given property, that $AB = a$, and that $FK = x$. Then



$$KB = \frac{b}{c}x, \text{ and therefore } AK = a - \frac{b}{c}x.$$

Hence $\left(a - \frac{b}{c}x\right)x = S$, where S is the area of the rectangle $AKFG$.

Thus, given the equation

$$ax - \frac{b}{c}x^2 = S,$$

where S is undetermined, VI. 27 tells us that, if x is to have a real value, S cannot be greater than the rectangle CE.

Now $CB = \frac{a}{2}$, and therefore $CD = \frac{c}{b} \cdot \frac{a}{2}$;

whence
$$S \not> \frac{c}{b} \cdot \frac{a^2}{4},$$

which is just the same result as we obtain by the algebraical method.

In the particular case where the *defect* of the parallelogram is to be a *square*, the condition becomes the statement of the fact that, *if a straight line be divided into two parts, the rectangle contained by the parts cannot exceed the square on half the line.*

Now suppose that, instead of taking F on BD as in the figure of the proposition, we take F on BD produced beyond D but so that DF is less than BD.

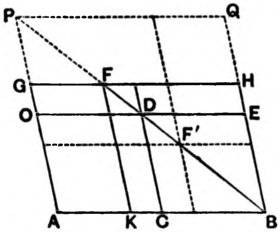
Complete the figure, as shown, after the manner of the construction in the proposition.

Then the parallelogram *FKBH* is similar to the given parallelogram to which the defect is to be similar. Hence the parallelogram *GAKF* is also a parallelogram applied to *AB* and satisfying the given condition.

We can now prove that *GAKF* is less than *CE* or *AD*.

Let *ED* produced meet *AG* in *O*.

Now, since *BF* is the diagonal of the parallelogram *KH*, the complements *KD*, *DH* are equal.



But

$DH = DG$, and DG is greater than OF.

Therefore $KD > OF$.

Add *OK* to each;

and AD, or CE, $> AF$.

This other "case" of the proposition is found in all the MSS., but Heiberg relegates it to the Appendix as being very obviously interpolated. The reasons for this course are that it is not in Euclid's manner to give a separate demonstration of such a "case"; it is rather his habit to give one case only and to leave the student to satisfy himself about any others (cf. I. 7). Internal evidence is also against the genuineness of the separate proof. It is put after the *conclusion* of the proposition instead of before it, and, if Euclid had intended to discuss two cases, he would have distinguished them at the beginning of the proposition, as it was his invariable practice to do. Moreover the second "case" is the less worth giving because it can be so easily reduced to the first. For suppose F' to be taken on *BD* so that $FD = F'D$. Produce *BF* to meet *AG* produced in *P*. Complete the parallelogram *BAPQ*, and draw through F' straight lines parallel to and meeting its opposite sides.

Then the complement $F'Q$ is equal to the complement AF'.

And it is at once seen that AF, $F'Q$ are equal and similar. Hence the solution of the problem represented by *AF* or $F'Q$ gives a parallelogram of the same size as AF' arrived at as in the first "case."

It is worth noting that the actual difference between the parallelogram *AF* and the maximum area *AD* that it can possibly have is represented in the figure. The difference is the small parallelogram *DF*.

PROPOSITION 28.

To a given straight line to apply a parallelogram equal to a given rectilineal figure and deficient by a parallelogrammic figure similar to a given one: thus the given rectilineal figure must not be greater than the parallelogram described on the half of the straight line and similar to the defect.

Let *AB* be the given straight line, *C* the given rectilineal figure to which the figure to be applied to *AB* is required to be equal, not being greater than the parallelogram described on the half of *AB* and similar to the defect, and *D* the parallelogram to which the defect is required to be similar;

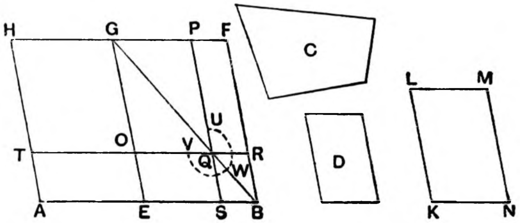
thus it is required to apply to the given straight line *AB* a parallelogram equal to the given rectilineal figure *C* and deficient by a parallelogrammic figure which is similar to *D*.

Let *AB* be bisected at the point *E*, and on *EB* let *EBFG* be described similar and similarly situated to *D*; [VI. 18]

let the parallelogram *AG* be completed.

If then *AG* is equal to *C*, that which was enjoined will have been done;

for there has been applied to the given straight line *AB* the parallelogram *AG* equal to the given rectilineal figure *C* and deficient by a parallelogrammic figure *GB* which is similar to *D*.



But, if not, let *HE* be greater than *C*.

Now *HE* is equal to *GB*;

therefore *GB* is also greater than *C*.

Let *KLMN* be constructed at once equal to the excess by which *GB* is greater than *C* and similar and similarly situated to *D*. [VI. 25]

But *D* is similar to *GB*;

therefore *KM* is also similar to *GB*. [VI. 21]

Let, then, *KL* correspond to *GE*, and *LM* to *GF*.

Now, since *GB* is equal to *C*, *KM*,

therefore *GB* is greater than *KM*;

therefore also *GE* is greater than *KL*, and *GF* than *LM*.

Let *GO* be made equal to *KL*, and *GP* equal to *LM*; and let the parallelogram *OGPQ* be completed;

therefore it is equal and similar to *KM*.

Therefore *GQ* is also similar to *GB*; [VI. 21]

therefore *GQ* is about the same diameter with *GB*. [VI. 26]

Let *GQB* be their diameter, and let the figure be described.

Then, since *BG* is equal to *C*, *KM*,

and in them *GQ* is equal to *KM*,

therefore the remainder, the gnomon *UWV*, is equal to the remainder *C*.

And, since *PR* is equal to *OS*,

let *QB* be added to each;

therefore the whole *PB* is equal to the whole *OB*.

But *OB* is equal to *TE*, since the side *AE* is also equal to the side *EB*; [I. 36]

therefore *TE* is also equal to *PB*.

Let *OS* be added to each;

therefore the whole *TS* is equal to the whole, the gnomon *VWU*.

But the gnomon *VWU* was proved equal to *C*;

therefore *TS* is also equal to *C*.

Therefore to the given straight line *AB* there has been applied the parallelogram *ST* equal to the given rectilineal figure *C* and deficient by a parallelogrammic figure *QB* which is similar to *D*.

Q. E. F.

The second part of the enunciation of this proposition which states the διορισμός appears to have been considerably amplified, but not improved in the process, by Theon. His version would read as follows. "But the given rectilineal figure, that namely to which the applied parallelogram must be equal (ᾧ δεῖ ἴσον παραβαλεῖν), must not be greater than that applied to the half (παραβαλλομένου instead of ἀναγραφομένου), the defects being similar, (namely) that (of the parallelogram applied) to the half and that (of the required parallelogram) which must have a similar defect" (ὁμοίων ὄντων τῶν ἐλλειμμάτων τοῦ τε ἀπὸ τῆς ἡμισείας καὶ ᾧ δεῖ ὅμοιον ἐλλείπειν). The first amplification "that to which the applied parallelogram must be equal" is quite unnecessary, since "the given rectilineal figure" could mean nothing else. The above attempt at a translation will show how difficult it is to make sense of the words at the end; they speak of *two* defects apparently and, while one may well be the "defect on the half," the other can hardly be *the given parallelogram* "to which the defect (of the required parallelogram) must be similar." Clearly the reading given above (from P) is by far the better.

In this proposition and the next there occurs the tacit assumption (already alluded to in the note on VI. 22) that *if, of two similar parallelograms, one is greater than the other, either side of the greater is greater than the corresponding side of the less.*

As already remarked, VI. 28 is the geometrical equivalent of the solution of the quadratic equation

$$ax - \frac{b}{c} x^2 = S,$$

subject to the condition necessary to admit of a real solution, namely that

$$S \ngtr \frac{c}{b} \cdot \frac{a^2}{4}.$$

The corresponding proposition in the *Data* is (Prop. 58), *If a given* (*area*) *be applied* (i.e. in the form of a parallelogram) *to a given straight line and be deficient by a figure* (i.e. a parallelogram) *given in species, the breadths of the defect are given.*

To exhibit the exact correspondence between Euclid's geometrical and the ordinary algebraical method of solving the equation we will, as before (in order to avoid bringing in a constant dependent on the sine of the angle of the parallelograms), suppose the parallelograms to be rectangles. To solve the equation algebraically we change the signs and write it

$$\frac{b}{c} x^2 - ax = - S.$$

We may now complete the square by adding $\frac{c}{b} \cdot \frac{a^2}{4}$.

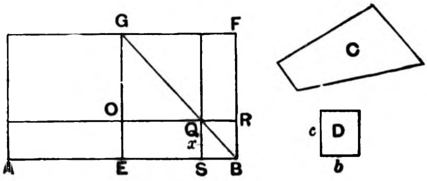
Thus
$$\frac{b}{c} x^2 - ax + \frac{c}{b} \cdot \frac{a^2}{4} = \frac{c}{b} \cdot \frac{a^2}{4} - S;$$

and, extracting the square root, we have

$$\sqrt{\frac{b}{c}} \cdot x - \sqrt{\frac{c}{b}} \cdot \frac{a}{2} = \pm \sqrt{\frac{c}{b} \cdot \frac{a^2}{4} - S},$$

and
$$x = \frac{c}{b} \cdot \frac{a}{2} \pm \sqrt{\frac{c}{b}\left(\frac{c}{b} \cdot \frac{a^2}{4} - S\right)}.$$

Now let us observe Euclid's method.



He first describes *GEBF* on *EB* (half of *AB*) similar to the given parallelogram *D*.

He then places in one angle *FGE* of *GEBF* a similar and similarly situated parallelogram *GQ*, equal to the difference between the parallelogram *GB* and the area *C*.

With our notation, $GO : OQ = c : b$,

whence
$$OQ = GO \cdot \frac{b}{c}.$$

Similarly $$\frac{a}{2} = EB = GE \cdot \frac{b}{c},$$

so that $$GE = \frac{c}{b} \cdot \frac{a}{2}.$$

Therefore the parallelogram $GQ = GO^2 \cdot \frac{b}{c}$,

and the parallelogram $GB = \frac{c}{b} \cdot \frac{a^2}{4}$.

Thus, in taking the parallelogram GQ equal to $(GB - S)$, Euclid really finds GO from the equation

$$GO^2 \cdot \frac{b}{c} = \frac{c}{b} \cdot \frac{a^2}{4} - S.$$

The value which he finds is

$$GO = \sqrt{\frac{c}{b}\left(\frac{c}{b} \cdot \frac{a^2}{4} - S\right)},$$

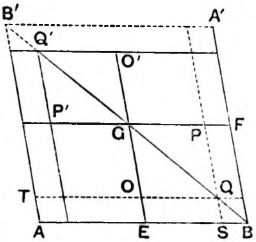
and he finds QS (or x) by *subtracting* GO from GE; whence

$$x = \frac{c}{b} \cdot \frac{a}{2} - \sqrt{\frac{c}{b}\left(\frac{c}{b} \cdot \frac{a^2}{4} - S\right)}.$$

It will be observed that Euclid only gives one solution, that corresponding to the *negative* sign before the radical. But the reason must be the same as that for which he only gives one "case" in VI. 27. He cannot have failed to see how to *add* GO to GE would give another solution. As shown under the last proposition, the other solution can be arrived at

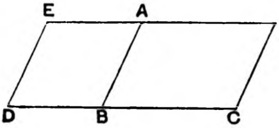
(1) by placing the parallelogram $GOQP$ in the angle vertically opposite to FGE so that GQ' lies along BG *produced*. The parallelogram AQ' then gives the second solution. The side of this parallelogram lying along AB is equal to SB. The other side is what we have called x, and in this case

$$x = EG + GO$$
$$= \frac{c}{b} \cdot \frac{a}{2} + \sqrt{\frac{c}{b}\left(\frac{c}{b} \cdot \frac{a^2}{4} - S\right)}.$$



(2) A parallelogram similar and equal to AQ' can also be obtained by producing BG till it meets AT produced and completing the parallelogram $B'ABA'$, whence it is seen that the complement QA' is equal to the complement AQ, besides being equal and similar and similarly situated to AQ'.

A particular case of this proposition, indicated in Prop. 85 of the *Data*, is that in which the sides of the defect are equal, so that the defect is a *rhombus* with a given angle. Prop. 85 proves that, *If two straight lines contain a given area in a given angle, and the sum of the straight lines be given, each of them will be given also.* AB, BC being the given straight lines "containing a given area AC in a given angle ABC," one side CB is produced to D so that BD is equal to AB, and the parallelograms are completed. Then, by hypothesis, CD is of given length, and AC is a parallelo-



gram applied to *CD* falling short by a rhombus (*AD*) with a given angle *EDB*. The case is thus a particular case of Prop. 58 of the *Data* quoted above (p. 263) as corresponding to VI. 28.

A particular case of the last, that namely in which the defect is a *square*, corresponding to the equation

$$ax - x^2 = b^2,$$

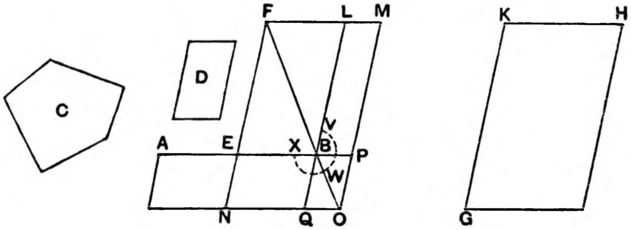
is important. This is the problem of *applying to a given straight line a rectangle equal to a given area and falling short by a square*; and it can be solved, without the aid of Book VI., as shown above under II. 5 (Vol. I. pp. 383—4).

PROPOSITION 29.

To a given straight line to apply a parallelogram equal to a given rectilineal figure and exceeding by a parallelogrammic figure similar to a given one.

Let *AB* be the given straight line, *C* the given rectilineal figure to which the figure to be applied to *AB* is required to be equal, and *D* that to which the excess is required to be similar;

thus it is required to apply to the straight line *AB* a parallelogram equal to the rectilineal figure *C* and exceeding by a parallelogrammic figure similar to *D*.



Let *AB* be bisected at *E*;

let there be described on *EB* the parallelogram *BF* similar and similarly situated to *D*;

and let *GH* be constructed at once equal to the sum of *BF*, *C* and similar and similarly situated to *D*. [VI. 25]

Let *KH* correspond to *FL* and *KG* to *FE*.

Now, since *GH* is greater than *FB*,

therefore *KH* is also greater than *FL*, and *KG* than *FE*.

Let *FL*, *FE* be produced,
let *FLM* be equal to *KH*, and *FEN* to *KG*,
and let *MN* be completed;
therefore *MN* is both equal and similar to *GH*.
But *GH* is similar to *EL*;
therefore *MN* is also similar to *EL*; [VI. 21]
therefore *EL* is about the same diameter with *MN*. [VI. 26]

Let their diameter *FO* be drawn, and let the figure be described.

Since *GH* is equal to *EL*, *C*,
while *GH* is equal to *MN*,
therefore *MN* is also equal to *EL*, *C*.
Let *EL* be subtracted from each;
therefore the remainder, the gnomon *XWV*, is equal to *C*.
Now, since *AE* is equal to *EB*,
AN is also equal to *NB* [I. 36], that is, to *LP* [I. 43].
Let *EO* be added to each;
therefore the whole *AO* is equal to the gnomon *VWX*.
But the gnomon *VWX* is equal to *C*;
therefore *AO* is also equal to *C*.

Therefore to the given straight line *AB* there has been applied the parallelogram *AO* equal to the given rectilineal figure *C* and exceeding by a parallelogrammic figure *QP* which is similar to *D*, since *PQ* is also similar to *EL* [VI. 24].

Q. E. F.

The corresponding proposition in the *Data* is (Prop. 59), *If a given (area) be applied* (i.e. in the form of a parallelogram) *to a given straight line exceeding by a figure given in species, the breadths of the excess are given.*

The problem of VI. 29 corresponds of course to the solution of the quadratic equation

$$ax + \frac{b}{c}x^2 = S.$$

The algebraical solution of this equation gives

$$x = -\frac{c}{b}\cdot\frac{a}{2} \pm \sqrt{\frac{c}{b}\left(\frac{c}{b}\cdot\frac{a^2}{4} + S\right)}.$$

The exact correspondence of Euclid's method to the algebraical solution may be seen, as in the case of VI. 28, by supposing the parallelograms to be *rectangles*. In this case Euclid's construction on *EB* of the parallelogram *EL* similar to *D* is equivalent to finding that

$$FE = \frac{c}{b}\cdot\frac{a}{2}, \text{ and } EL = \frac{c}{b}\cdot\frac{a^2}{4}.$$

His determination of the similar parallelogram MN equal to the sum of EL and S corresponds to proving that

$$FN^2 \cdot \frac{b}{c} = \frac{c}{b} \cdot \frac{a^2}{4} + S,$$

or

$$FN = \sqrt{\frac{c}{b}\left(\frac{c}{b} \cdot \frac{a^2}{4} + S\right)},$$

whence x is found as

$$x = FN - FE = \sqrt{\frac{c}{b}\left(\frac{c}{b} \cdot \frac{a^2}{4} + S\right)} - \frac{c}{b} \cdot \frac{a}{2}.$$

Euclid takes, in this case, the solution corresponding to the *positive* sign before the radical because, from his point of view, that would be the *only* solution.

No διορισμός is necessary because a real geometrical solution is always possible whatever be the size of S.

Again the *Data* has a proposition indicating the particular case in which the *excess* is a *rhombus* with a given angle. Prop. 84 proves that, *If two straight lines contain a given area in a given angle, and one of the straight lines is greater than the other by a given straight line, each of the two straight lines is given also.* The proof reduces the proposition to a particular case of *Data*, Prop. 59, quoted above as corresponding to VI. 29.

Again there is an important particular case which can be solved by means of Book II. only, as shown under II. 6 above (Vol. I. pp. 386—8), the case namely in which the excess is a *square*, corresponding to the solution of the equation

$$ax + x^2 = b^2.$$

This is the problem of *applying to a given straight line a rectangle equal to a given area and exceeding by a square.*

PROPOSITION 30.

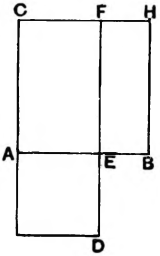
To cut a given finite straight line in extreme and mean ratio.

Let AB be the given finite straight line;

thus it is required to cut AB in extreme and mean ratio.

On AB let the square BC be described;

and let there be applied to AC the parallelogram CD equal to BC and exceeding by the figure AD similar to BC. [VI. 29]



Now BC is a square;

therefore AD is also a square.

And, since BC is equal to CD,

let CE be subtracted from each;

therefore the remainder BF is equal to the remainder AD.

But it is also equiangular with it;

therefore in *BF*, *AD* the sides about the equal angles are reciprocally proportional; [VI. 14]

therefore, as *FE* is to *ED*, so is *AE* to *EB*.

But *FE* is equal to *AB*, and *ED* to *AE*.

Therefore, as *BA* is to *AE*, so is *AE* to *EB*.

And *AB* is greater than *AE*;

therefore *AE* is also greater than *EB*.

Therefore the straight line *AB* has been cut in extreme and mean ratio at *E*, and the greater segment of it is *AE*.

Q. E. F.

It will be observed that the construction in the text is a direct application of the preceding Prop. 29 in the particular case where the *excess* of the parallelogram which is applied is a *square*. This fact coupled with the position of VI. 30 is a sufficient indication that the construction is Euclid's.

In one place Theon appears to have amplified the argument. The text above says "But *FE* is equal to *AB*," while the MSS. B, F, V and p have "But *FE* is equal to *AC*, that is, to *AB*."

The MSS. give after ὅπερ ἔδει ποιῆσαι an alternative construction which Heiberg relegates to the Appendix. The text-books give this construction alone and leave out the other. It will be remembered that the alternative proof does no more than refer to the equivalent construction in II. 11.

"Let *AB* be cut at *C* so that the rectangle *AB*, *BC* is equal to the square on *CA*. [II. 11]

Since then the rectangle *AB*, *BC* is equal to the square on *CA*,

therefore, as *BA* is to *AC*, so is *AC* to *CB*. [VI. 17]

Therefore *AB* has been cut in extreme and mean ratio at *C*."

It is intrinsically improbable that this alternative construction was added to the other by Euclid himself. It is however just the kind of interpolation that might be expected from an editor. If Euclid had preferred the alternative construction, he would have been more likely to give it alone.

PROPOSITION 31.

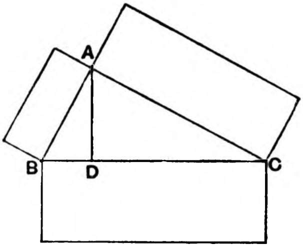
In right-angled triangles the figure on the side subtending the right angle is equal to the similar and similarly described figures on the sides containing the right angle.

Let *ABC* be a right-angled triangle having the angle *BAC* right;

I say that the figure on *BC* is equal to the similar and similarly described figures on *BA*, *AC*.

Let *AD* be drawn perpendicular.

Then since, in the right-angled triangle *ABC*, *AD* has been drawn from the right angle at A perpendicular to the base BC,



the triangles ABD, ADC adjoining the perpendicular are similar both to the whole ABC and to one another. [VI. 8]

And, since ABC is similar to ABD,

therefore, as CB is to BA, so is AB to BD. [VI. Def. 1]

And, since three straight lines are proportional,

as the first is to the third, so is the figure on the first to the similar and similarly described figure on the second. [VI. 19, Por.]

Therefore, as CB is to BD, so is the figure on CB to the similar and similarly described figure on BA.

For the same reason also,

as BC is to CD, so is the figure on BC to that on CA;

so that, in addition,

as BC is to BD, DC, so is the figure on BC to the similar and similarly described figures on BA, AC.

But BC is equal to BD, DC;

therefore the figure on BC is also equal to the similar and similarly described figures on BA, AC.

Therefore etc.

Q. E. D.

As we have seen (note on I. 47), this extension of I. 47 is credited by Proclus to Euclid personally.

There is one inference in the proof which requires examination. Euclid proves that

$$CB : BD = (\text{figure on } CB) : (\text{figure on } BA),$$

and that

$$BC : CD = (\text{figure on } BC) : (\text{figure on } CA),$$

and then infers directly that

$$BC : (BD + CD) = (\text{fig. on } BC) : (\text{sum of figs. on } BA \text{ and } AC).$$

Apparently V. 24 must be relied on as justifying this inference. But it is not directly applicable; for what it proves is that, if

$$a : b = c : d,$$

and

$$e : b = f : d,$$

then

$$(a + e) : b = (c + f) : d.$$

Thus we should *invert* the first two proportions given above (by Simson's

Prop. B which, as we have seen, is a direct consequence of the definition of proportion), and thence infer by V. 24 that

$$(BD+CD) : BC = (\text{sum of figs. on } BA, AC) : (\text{fig. on } BC).$$

But $BD + CD$ is equal to BC;

therefore (by Simson's Prop. A, which again is an immediate consequence of the definition of proportion) the sum of the figures on BA, AC is equal to the figure on BC.

The MSS. again give an alternative proof which Heiberg places in the Appendix. It first shows that the similar figures on the three sides have the same ratios to one another as the *squares* on the sides respectively. Whence, by using I. 47 and the same argument based on V. 24 as that explained above, the result is obtained.

If it is considered essential to have a proof which does not use Simson's Props. B and A or any proposition but those actually given by Euclid, no method occurs to me except the following.

Eucl. V. 22 proves that, if a, b, c are three magnitudes, and d, e, f three others, such that

$$a : b = d : e,$$
$$b : c = e : f,$$

then, *ex aequali*, $a : c = d : f$.

If now in addition $a : b = b : c$,

so that, also, $d : e = e : f$,

the ratio $a : c$ is duplicate of the ratio $a : b$, and the ratio $d : f$ duplicate of the ratio $d : e$, whence *the ratios which are duplicate of equal ratios are equal.*

Now (fig. on AC) : (fig. on AB) = the ratio duplicate of $AC : AB$
= the ratio duplicate of $CD : DA$
= $CD : BD$.

Hence (sum of figs. on AC, AB) : (fig. on AB) = $BC : BD$. [V. 18]

But (fig. on BC) : (fig. on AB) = $BC : BD$
(as in Euclid's proof).

Therefore the sum of the figures on AC, AB has to the figure on AB the same ratio as the figure on BC has to the figure on AB, whence

the figures on AC, AB are together equal to the figure on BC. [V. 9]

PROPOSITION 32.

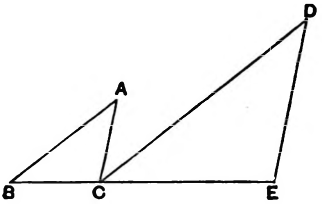
If two triangles having two sides proportional to two sides be placed together at one angle so that their corresponding sides are also parallel, the remaining sides of the triangles will be in a straight line.

Let ABC, DCE be two triangles having the two sides BA, AC proportional to the two sides DC, DE, so that, as AB is to AC, so is DC to DE, and AB parallel to DC, and AC to DE;

I say that BC is in a straight line with CE.

For, since AB is parallel to DC, and the straight line AC has fallen upon them, the alternate angles BAC, ACD are equal to one another. [I. 29]

For the same reason the angle CDE is also equal to the angle ACD; so that the angle BAC is equal to the angle CDE.



And, since ABC, DCE are two triangles having one angle, the angle at A, equal to one angle, the angle at D,

and the sides about the equal angles proportional,

so that, as BA is to AC, so is CD to DE,

therefore the triangle ABC is equiangular with the triangle DCE; [VI. 6]

therefore the angle ABC is equal to the angle DCE.

But the angle ACD was also proved equal to the angle BAC;

therefore the whole angle ACE is equal to the two angles ABC, BAC.

Let the angle ACB be added to each;

therefore the angles ACE, ACB are equal to the angles BAC, ACB, CBA.

But the angles BAC, ABC, ACB are equal to two right angles; [I. 32]

therefore the angles ACE, ACB are also equal to two right angles.

Therefore with a straight line AC, and at the point C on it, the two straight lines BC, CE not lying on the same side make the adjacent angles ACE, ACB equal to two right angles;

therefore BC is in a straight line with CE. [I. 14]

Therefore etc.

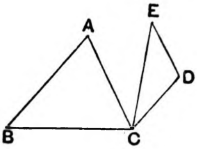
Q. E. D.

It has often been pointed out (e.g. by Clavius, Lardner and Todhunter) that the enunciation of this proposition is not precise enough. Suppose that

ABC is a triangle. From C draw CD parallel to BA and of any length. From D draw DE parallel to CA and of such length that

$$CD : DE = BA : AC.$$

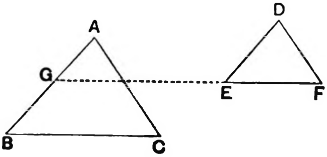
Then the triangles ABC, ECD, which have the angular point C common literally satisfy Euclid's enunciation; but by no possibility can CE be in a straight line with CB if, as in the case supposed, the angles included by the corresponding sides are supplementary (unless both are right angles). Hence the included angles must be *equal*, so that the triangles must be *similar*. That being so, if they are to have nothing more than one angular point common, and two pairs of corresponding sides are to be ***parallel*** as distinguished from one or both being in the ***same straight line***, the triangles can only be placed so that the corresponding sides in both are on the same side of the third side of either, and the sides (other than the third sides) which meet at the common angular point are not corresponding sides.



Todhunter remarks that the proposition seems of no use. Presumably he did not know that it *is* used by Euclid himself in xiii. 17. This is so however, and therefore it was not necessary, as several writers have thought, to do away with the proposition and find a substitute which should be more useful.

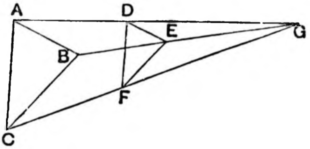
1. De Morgan proposes this theorem: "If two similar triangles be placed with their bases parallel, and the equal angles at the bases towards the same parts, the other sides are parallel, each to each; or one pair of sides are in the same straight line and the other pair are parallel."

2. Dr Lachlan substitutes the somewhat similar theorem, "If two similar triangles be placed so that two sides of the one are parallel to the corresponding sides of the other, the third sides are parallel."

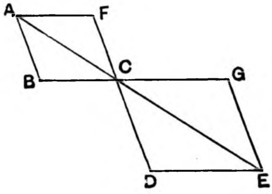


But it is to be observed that these propositions can be proved without using Book vi. at all; they can be proved from Book i., and the triangles may as well be called "equiangular" simply. It is true that Book vi. is no more than formally necessary to Euclid's proposition. He merely uses vi. 6 because his enunciation does not *say* that the triangles are similar; and he only proves them to be similar in order to conclude that they are equiangular. From this point of view Mr Taylor's substitute seems the best, viz.

3. "If two triangles have sides parallel in pairs, the straight lines joining the corresponding vertices meet in a point, or are parallel."



Simson has a theory (unnecessary in the circumstances) as to the possible object of vi. 32 as it stands. He points out that the enunciation of vi. 26 might be more general so as to cover the case of similar and similarly situated parallelograms with equal angles not coincident but vertically opposite. It can then be proved that the diagonals drawn through the common angular point are in one straight line. If *ABCF*, *CDEG* be similar and similarly situated parallelograms, so that *BCG*, *DCF* are straight lines, and if the diagonals *AC*, *CE* be drawn, the triangles *ABC*, *CDE* are similar and *are placed exactly as described in* VI. 32, so that *AC*, *CE* are in a straight line. Hence Simson suggests that there may have been, in addition to the indirect demonstration in VI. 26, a *direct* proof covering the case just given which may have used the result of VI. 32. I think however that the place given to the latter proposition in Book VI. is against this view.

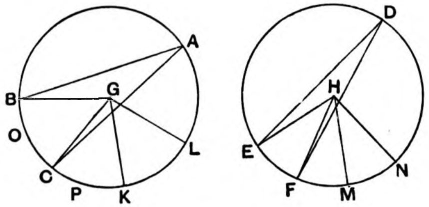


PROPOSITION 33.

In equal circles angles have the same ratio as the circumferences on which they stand, whether they stand at the centres or at the circumferences.

Let *ABC*, *DEF* be equal circles, and let the angles *BGC*, *EHF* be angles at their centres *G*, *H*, and the angles *BAC*, *EDF* angles at the circumferences;

I say that, as the circumference *BC* is to the circumference *EF*, so is the angle *BGC* to the angle *EHF*, and the angle *BAC* to the angle *EDF*.



For let any number of consecutive circumferences *CK*, *KL* be made equal to the circumference *BC*,

and any number of consecutive circumferences *FM*, *MN* equal to the circumference *EF*;

and let *GK*, *GL*, *HM*, *HN* be joined.

Then, since the circumferences *BC*, *CK*, *KL* are equal to one another,

the angles *BGC*, *CGK*, *KGL* are also equal to one another; [III. 27] therefore, whatever multiple the circumference *BL* is of *BC*, that multiple also is the angle *BGL* of the angle *BGC*.

For the same reason also,

whatever multiple the circumference *NE* is of *EF*, that multiple also is the angle *NHE* of the angle *EHF*.

If then the circumference *BL* is equal to the circumference *EN*, the angle *BGL* is also equal to the angle *EHN*; [III. 27] if the circumference *BL* is greater than the circumference *EN*, the angle *BGL* is also greater than the angle *EHN*; and, if less, less.

There being then four magnitudes, two circumferences *BC*, *EF*, and two angles *BGC*, *EHF*,

there have been taken, of the circumference *BC* and the angle *BGC* equimultiples, namely the circumference *BL* and the angle *BGL*,

and of the circumference *EF* and the angle *EHF* equimultiples, namely the circumference *EN* and the angle *EHN*.

And it has been proved that,

if the circumference *BL* is in excess of the circumference *EN*, the angle *BGL* is also in excess of the angle *EHN*;

if equal, equal;

and if less, less.

Therefore, as the circumference *BC* is to *EF*, so is the angle *BGC* to the angle *EHF*. [V. Def. 5]

But, as the angle *BGC* is to the angle *EHF*, so is the angle *BAC* to the angle *EDF*; for they are doubles respectively.

Therefore also, as the circumference *BC* is to the circumference *EF*, so is the angle *BGC* to the angle *EHF*, and the angle *BAC* to the angle *EDF*.

Therefore etc.

Q. E. D.

This proposition as generally given includes a second part relating to *sectors* of circles, corresponding to the following words added to the enunciation: "and further the sectors, as constructed at the centres" (ἔτι δὲ καὶ οἱ τομεῖς ἅτε [or οἵτε] πρὸς τοῖς κέντροις συνιστάμενοι). There is of course a corresponding addition to the "definition" or "particular statement," "and further the sector *GBOC* to the sector *HEQF*." These additions are clearly due to Theon, as may be gathered from his own statement in his commentary on the μαθηματικὴ σύνταξις of Ptolemy, "But that sectors in equal circles are to one another as the angles on which they stand, has been proved by me in my edition of the

Elements at the end of the sixth book." Campanus omits them, and P has them only in a later hand in the margin or between the lines. Theon's proof scarcely needs to be given here in full, as it can easily be supplied. From the equality of the arcs *BC*, *CK* he infers [III. 29] the equality of the chords *BC*, *CK*. Hence, the radii being equal, the triangles *GBC*, *GCK* are equal in all respects [I. 8, 4]. Next, since the arcs *BC*, *CK* are equal, so are the arcs *BAC*, *CAK*. Therefore the angles at the circumference subtended by the latter, i.e. the angles in the segments *BOC*, *CPK*, are equal [III. 27], and the segments are therefore similar [III. Def. 11] and equal [III. 24].

Adding to the equal segments the equal triangles *GBC*, *GCK* respectively, we see that

the sectors *GBC*, *GCK* are equal.

Thus, in equal circles, sectors standing on equal arcs are equal; and the rest of the proof proceeds as in Euclid's proposition.

As regards Euclid's proposition itself, it will be noted that (1), besides quoting the theorem in III. 27 that in equal circles angles which stand on equal arcs are equal, the proof assumes that the angle standing on a greater arc is greater and that standing on a less arc is less. This is indeed a sufficiently obvious deduction from III. 27.

(2) Any equimultiples *whatever* are taken of the angle *BGC* and the arc *BC*, and any equimultiples *whatever* of the angle *EHF* and the arc *EF*. (Accordingly the words "*any equimultiples whatever*" should have been used in the step immediately preceding the inference that the angles are proportional to the arcs, where the text merely states that there have been taken of the circumference *BC* and the angle *BGC equimultiples BL* and *BGL*.) But, if *any multiple* of an angle is regarded as being itself an angle, it follows that the restriction in I. Deff. 8, 10, 11, 12 of the term *angle* to an angle *less than two right angles* is implicitly given up; as De Morgan says, "the angle breaks prison." Mr Dodgson (*Euclid and his Modern Rivals*, p. 193) argues that Euclid conceived of the multiple of an angle as so many separate angles not added together into one, and that, when it is inferred that, where two such multiples of an angle are equal, the arcs subtended are also equal, the argument is that the sum total of the first set of angles is equal to the sum total of the second set, and hence the second set can be broken up and put together again in such amounts as to make a set equal, each to each, to the first set, and then the sum total of the arcs will evidently be equal also. If on the other hand the multiples of the angles are regarded as single angular magnitudes, the equality of the subtending arcs is not inferrible directly from Euclid, because *his* proof of III. 26 only applies to cases where the angle is less than the sum of two right angles. (As a matter of fact, it is a question of inferring equality of angles or multiples of angles from equality of arcs, and not the converse, so that the reference should have been to III. 27, but this does not affect the question at issue.) Of course it is against this view of Mr Dodgson that Euclid speaks throughout of "*the angle* BGL" and "*the angle* EHN" (*ἡ ὑπὸ* ΒΗΛ *γωνία*, *ἡ ὑπὸ* ΕΘΝ *γωνία*). I think the probable explanation is that here, as in III. 20, 21, 26 and 27, Euclid deliberately took no cognisance of the case in which the multiples of the angles in question would be greater than two right angles. If his attention had been called to the fact that III. 20 takes no account of the case where the segment is less than a semicircle, so that the angle in the segment is obtuse, and therefore the "angle at the centre" in that case (if the term were still applicable) would be

greater than two right angles, Euclid would no doubt have refused to regard the latter as an angle, and would have represented it otherwise, e.g. as the sum of two angles or as what is left when an *angle* in the true sense is subtracted from four right angles. Here then, if Euclid had been asked what course he would take if the multiples of the angles in question should be greater than two right angles, he would probably have represented them, I think, as being *equal to so many right angles plus an angle less than a right angle*, or *so many times two right angles plus an angle, acute or obtuse*. Then the equality of the arcs would be the equality of the sums of so many circumferences, semi-circumferences or quadrants plus arcs less than a semicircle or a quadrant. Hence I agree with Mr Dodgson that VI. 33 affords no evidence of a recognition by Euclid of "angles" greater than two right angles

Theon adds to his theorem about sectors the Porism that, *As the sector is to the sector, so also is the angle to the angle*. This corollary was used by Zenodorus in his tract περὶ ἰσομέτρων σχημάτων preserved by Theon in his commentary on Ptolemy's σύνταξις, unless indeed Theon himself interpolated the words (ὡς δ' ὁ τομεὺς πρὸς τὸν τομέα, ἡ ὑπὸ ΕΘΛ γωνία πρὸς τὴν ὑπὸ ΜΘΛ).

BOOK VII.

DEFINITIONS.

1. An **unit** is that by virtue of which each of the things that exist is called one.

2. A **number** is a multitude composed of units.

3. A number is **a part** of a number, the less of the greater, when it measures the greater;

4. but **parts** when it does not measure it.

5. The greater number is a **multiple** of the less when it is measured by the less.

6. An **even number** is that which is divisible into two equal parts.

7. An **odd number** is that which is not divisible into two equal parts, or that which differs by an unit from an even number.

8. An **even-times even number** is that which is measured by an even number according to an even number.

9. An **even-times odd number** is that which is measured by an even number according to an odd number.

10. An **odd-times odd number** is that which is measured by an odd number according to an odd number.

11. A **prime number** is that which is measured by an unit alone.

12. Numbers **prime to one another** are those which are measured by an unit alone as a common measure.

13. A **composite number** is that which is measured by some number.

14. Numbers **composite to one another** are those which are measured by some number as a common measure.

15. A number is said to **multiply** a number when that which is multiplied is added to itself as many times as there are units in the other, and thus some number is produced.

16. And, when two numbers having multiplied one another make some number, the number so produced is called **plane**, and its **sides** are the numbers which have multiplied one another.

17. And, when three numbers having multiplied one another make some number, the number so produced is **solid**, and its **sides** are the numbers which have multiplied one another.

18. A **square number** is equal multiplied by equal, or a number which is contained by two equal numbers.

19. And a **cube** is equal multiplied by equal and again by equal, or a number which is contained by three equal numbers.

20. Numbers are **proportional** when the first is the same multiple, or the same part, or the same parts, of the second that the third is of the fourth.

21. **Similar plane** and **solid** numbers are those which have their sides proportional.

22. A **perfect number** is that which is equal to its own parts.

DEFINITION I.

Μονάς ἐστιν, καθ' ἣν ἕκαστον τῶν ὄντων ἓν λέγεται.

Iamblichus (fl. *circa* 300 A.D.) tells us (*Comm. on Nicomachus*, ed. Pistelli, p. 11, 5) that the Euclidean definition of an *unit* or a *monad* was the definition given by "more recent" writers (οἱ νεώτεροι), and that it lacked the words "even though it be collective" (κἂν συστηματικὸν ᾖ). He also gives (*ibid.* p. 11) a number of other definitions. (1) According to "some of the Pythagoreans," "an unit is the boundary between number and parts" (μονάς ἐστιν ἀριθμοῦ καὶ μορίων μεθόριον), "because from it, as from a seed and eternal root, ratios increase reciprocally on either side," i.e. on one side we have multiple ratios continually increasing and on the other (if the unit be subdivided) submultiple ratios with denominators continually increasing. (2) A somewhat similar definition is that of Thymaridas, an ancient Pythagorean, who defined a monad as "limiting quantity" (περαίνουσα ποσότης), the beginning and the end of a thing being equally an extremity (πέρας). Perhaps the words together with their explanation may best be expressed by "limit of fewness." Theon of Smyrna (p. 18, 6, ed. Hiller) adds the explanation that the monad is "that which, when the multitude is diminished by way of continued subtraction, is deprived of all number and takes an abiding position (μονήν) and rest." If, after arriving at an unit in this way, we proceed to divide the unit itself into parts, we straightway have multitude again. (3) Some, according to Iamblichus (p. 11, 16), defined it as the "form of forms" (εἰδῶν εἶδος) because it potentially comprehends all forms of number, e.g. it is a polygonal number of any number of sides from three upwards, a solid number in all forms, and so on. (We are forcibly reminded of the latest theories of number as a "Gattung" of "Mengen" or as a "class of classes.") (4) Again an unit, says Iamblichus, is the first, or smallest, in the category of *how many* (ποσόν), the common part or beginning of *how many*. Aristotle defines it as "the indivisible in the (category of) quantity," τὸ κατὰ τὸ ποσὸν ἀδιαίρετον (*Metaph.* 1089 b 35), ποσόν including in Aristotle continuous as well as discrete quantity; hence it is distinguished from a point by the fact that it has not position: "Of the indivisible in the category of, and *quâ*, quantity, that which is every way (indivisible) and destitute of position is called an *unit*, and that which is every way indivisible and has position is a *point*" (*Metaph.* 1016 b 25). (5) In accordance with the last distinction, Aristotle calls the unit "a point without position," στιγμὴ ἄθετος (*Metaph.* 1084 b 26). (6) Lastly, Iamblichus says that the school of Chrysippus defined it in a confused manner (συγκεχυμένως) as "multitude *one* (πλῆθος ἕν)," whereas it is alone contrasted with multitude. On a comparison of these definitions, it would seem that Euclid intended his to be a more *popular* one than those of his predecessors, δημώδης, as Nicomachus called Euclid's definition of an *even number*.

The etymological signification of the word μονάς is supposed by Theon of Smyrna (p. 19, 7—13) to be either (1) that it remains unaltered if it be multiplied by itself any number of times, or (2) that it is separated and *isolated* (μεμονῶσθαι) from the rest of the multitude of numbers. Nicomachus also observes (I. 8, 2) that, while any number is half the sum (1) of the adjacent numbers on each side, (2) of numbers equidistant on each side, the unit is *most solitary* (μονωτάτη) in that it has not a number on each side but only on one side, and it is half of the latter alone, i.e. of 2.

DEFINITION 2.

Ἀριθμὸς δὲ τὸ ἐκ μονάδων συγκείμενον πλῆθος.

The definition of a *number* is again only one out of many that are on record. Nicomachus (I. 7, 1) combines several into one, saying that it is "a defined multitude (πλῆθος ὡρισμένον), or a collection of units (μονάδων σύστημα), or a flow of quantity made up of units" (ποσότητος χύμα ἐκ μονάδων συγκείμενον). Theon, in words almost identical with those attributed by Stobaeus (*Eclogae*, I. 1, 8) to Moderatus, a Pythagorean, says (p. 18, 3—5): "A number is a collection of units, or a progression (προποδισμός) of multitude beginning from an unit and a retrogression (ἀναποδισμός) ceasing at an unit." According to Iamblichus (p. 10) the description "collection of units" (μονάδων σύστημα) was applied to the *how many*, i.e. to number, by Thales, following the Egyptian view (κατὰ τὸ Αἰγυπτιακὸν ἀρέσκον), while it was Eudoxus the Pythagorean who said that a number was "a defined multitude" (πλῆθος ὡρισμένον). Aristotle has a number of definitions which come to the same thing: "limited multitude" (πλῆθος τὸ πεπερασμένον, *Metaph.* 1020 a 13), "multitude" (or "combination") "of units" or "multitude of indivisibles" (*ibid.* 1053 a 30, 1039 a 12, 1085 b 22), "several *ones*" (ἕνα πλείω, *Phys.* III. 7, 207 b 7), "multitude measurable by one" (*Metaph.* 1057 a 3) and "multitude measured and multitude of measures," the "measure" being unity, τὸ ἕν (*ibid.* 1088 a 5).

DEFINITION 3.

Μέρος ἐστὶν ἀριθμὸς ἀριθμοῦ ὁ ἐλάσσων τοῦ μείζονος, ὅταν καταμετρῇ τὸν μείζονα.

By *a part* Euclid means a submultiple, as he does in V. Def. 1, with which definition this one is identical except for the substitution of *number* (ἀριθμός) for *magnitude* (μέγεθος); cf. note on V. Def. 1. Nicomachus uses the word "submultiple" (ὑποπολλαπλάσιος) also. He defines it in a way corresponding to his definition of multiple (see note on Def. 5 below) as follows (I. 18, 2): "The submultiple, which is by nature first in the division of inequality (called) less, is the number which, when compared with a greater, can measure it more times than once so as to fill it exactly (πληρούντως)." Similarly *sub-double* (ὑποδιπλάσιος) is found in Nicomachus meaning *half*, and so on.

DEFINITION 4.

Μέρη δέ, ὅταν μὴ καταμετρῇ.

By the expression *parts* (μέρη, the plural of μέρος) Euclid denotes what we should call a *proper fraction*. That is, *a part* being a submultiple, the rather inconvenient term *parts* means any number of such submultiples making up a fraction less than unity. I have not found the word used in this special sense elsewhere, e.g. in Nicomachus, Theon of Smyrna or Iamblichus, except in one place of Theon (p. 79, 26) where it is used of a proper fraction, of which $\frac{2}{3}$ is an illustration.

DEFINITION 5.

Πολλαπλάσιος δὲ ὁ μείζων τοῦ ἐλάσσονος, ὅταν καταμετρῆται ὑπὸ τοῦ ἐλάσσονος.

The definition of a *multiple* is identical with that in v. Def. 2, except that the masculine of the adjectives is used agreeing with *ἀριθμός* understood instead of the neuter agreeing with *μέγεθος* understood. Nicomachus (I. 18, 1) defines a multiple as being "a species of the greater which is naturally first in order and origin, being the number which, when considered in comparison with another, contains it in itself completely more than once."

DEFINITIONS 6, 7.

6. Ἄρτιος ἀριθμός ἐστιν ὁ δίχα διαιρούμενος.

7. Περισσὸς δὲ ὁ μὴ διαιρούμενος δίχα ἢ [ὁ] μονάδι διαφέρων ἀρτίου ἀριθμοῦ.

Nicomachus (I. 7, 2) somewhat amplifies these definitions of *even* and *odd* numbers thus. "That is *even* which is capable of being divided into two equal parts without an unit falling in the middle, and that is *odd* which cannot be divided into two equal parts because of the aforesaid intervention (*μεσιτείαν*) of the unit." He adds that this definition is derived "from the popular conception" (*ἐκ τῆς δημώδους ὑπολήψεως*). In contrast to this, he gives (I. 7, 3) the Pythagorean definition, which is, as usual, interesting. "An *even* number is that which admits of being divided, by one and the same operation, into the greatest and the least (parts), greatest in size (*πηλικότητι*) but least in quantity (*ποσότητι*)...while an *odd* number is that which cannot be so treated, but is divided into two unequal parts." That is, as Iamblichus says (p. 12, 2—9), an even number is divided into parts which are the *greatest* possible "parts," namely halves, and into the *fewest* possible, namely two, two being the first "number" or "collection of units." According to another ancient definition quoted by Nicomachus (I. 7, 4), an even number is that which can be divided both into two equal parts and into two unequal parts (except the first one, the number 2, which is only susceptible of division into equals), but, however it is divided, must have its two parts *of the same kind*, i.e. both even or both odd; while an odd number is that which can only be divided into two unequal parts, and those parts always of *different* kinds, i.e. one odd and one even. Lastly, the definition of odd and even "by means of each other" says that an odd number is that which differs by an unit from an even number on both sides of it, and an even number that which differs by an unit from an odd number on each side. This alternative definition of an odd number is the same thing as the second half of Euclid's definition, "the number which differs by an unit from an even number." This evidently pre-Euclidean definition is condemned by Aristotle as unscientific, because odd and even are coordinate, both being *differentiae* of number, so that one should not be defined by means of the other (*Topics* VI. 4, 142 b 7—10).

DEFINITION 8.

Ἀρτιάκις ἄρτιος ἀριθμός ἐστιν ὁ ὑπὸ ἀρτίου ἀριθμοῦ μετρούμενος κατὰ ἄρτιον ἀριθμόν.

Euclid's definition of an *even-times even* number differs from that given by the later writers, Nicomachus, Theon of Smyrna and Iamblichus; and the inconvenience of it is shown when we come to IX. 34, where it is proved that a certain sort of number is *both* "even-times even" and "even-times odd." According to the more precise classification of the three other authorities, the "even-times even" and the "even-times odd" are mutually exclusive and are two of three subdivisions into which even numbers fall. Of these three subdivisions the "even-times even" and the "even-times odd" form the extremes, and the "odd-times even" is as it were intermediate, showing the character of both extremes (cf. note on the following definition). The *even-times even* is then the number which has its halves even, the halves of the halves even, and so on, until unity is reached. In short the *even-times even* number is always of the form 2^n. Hence Iamblichus (pp. 20, 21) says Euclid's definition of it as that which is measured by an even number an even number of times is erroneous. In support of this he quotes the number 24 which is four times 6, or six times 4, but yet is not "even-times even" according to Euclid himself (οὐδὲ κατ' αὐτόν), by which he must apparently mean that 24 is also 8 times 3, which does not satisfy Euclid's definition. There can however be no doubt that Euclid meant what he said in his definition as we have it; otherwise IX. 32, which proves that a number of the form 2^n is *even-times even only*, would be quite superfluous and a mere repetition of the definition, while, as already stated, IX. 34 clearly indicates Euclid's view that a number might at the same time be both even-times even and even-times odd. Hence the μόνως which some editor of the commentary of Philoponus on Nicomachus found in some copies, making the definition say that the even-times even number is *only* measured by even numbers an even number of times, is evidently an interpolation by some one who wished to reconcile Euclid's definition with the Pythagorean (cf. Heiberg, *Euklid-studien*, p. 200).

A consequential characteristic of the series of even-times even numbers noted by Nicomachus brings in a curious use of the word δύναμις (generally *power* in the sense of square, or square root). He says (I. 8, 6—7) that any part, i.e. any submultiple, of an even-times even number is called by an even-times even designation, while it also has an even-times even *value* (it is ἀρτιάκις ἀρτιοδύναμοι) when expressed as so many actual units. That is, the $\frac{1}{2^m}$th part of 2^n (where m is less than n) is called after the even-times even number 2^m, while its actual *value* (δύναμις) in units is 2^{n-m}, which is also an even-times even number. Thus all the parts, or submultiples, of even-times even numbers, as well as the even-times even numbers themselves, are connected with one kind of number only, the *even*.

DEFINITION 9.

Ἀρτιάκις δὲ περισσός ἐστιν ὁ ὑπὸ ἀρτίου ἀριθμοῦ μετρούμενος κατὰ περισσὸν ἀριθμόν.

Euclid uses the term *even-times odd* (ἀρτιάκις περισσός), whereas Nicomachus and the others make it one word, *even-odd* (ἀρτιοπέριττος). According to the stricter definition given by the latter (I. 9, 1), the *even-odd* number is related to the *even-times even* as the other extreme. It is such a number as, when once halved, leaves as quotient an odd number; that is, it is of the form $2(2m+1)$. Nicomachus sets the even-odd numbers out as follows,

6, 10, 14, 18, 22, 26, 30, etc.

In this case, as Nicomachus observes, any part, or submultiple, is called by a name *not* corresponding in kind to its actual value (δύναμις) in units. Thus, in the case of 18, the $\frac{1}{2}$ part is called after the even number 2, but its *value* is the odd number 9, and the $\frac{1}{3}$rd part is called after the odd number 3, while its *value* is the even number 6, and so on.

The third class of even numbers according to the strict subdivision is the *odd-even* (περισσάρτιος). Numbers are of this class when they can be halved twice or more times successively, but the quotient left when they can no longer be halved is an odd number and not unity. They are therefore of the form $2^{n+1}(2m+1)$, where n, m are integers. They are, so to say, intermediate between, or a mixture of, the extreme classes *even-times even* and *even-odd*, for the following reasons. (1) Their subdivision by 2 proceeds for some way like that of the even-times even, but ends in the way that the division of the even-odd by 2 ends. (2) The numbers after which submultiples are called and their *value* (δύναμις) in units may be both of one kind, i.e. both odd or both even (as in the case of the even-times even), or again may be one odd and one even as in the case of the even-odd. For example 24 is an odd-even number; the $\frac{1}{4}$th, $\frac{1}{12}$th, $\frac{1}{6}$th or $\frac{1}{2}$ parts of it are even, but the $\frac{1}{3}$rd part of it, or 8, is even, and the $\frac{1}{8}$th part of it, or 3, is odd. (3) Nicomachus shows (I. 10, 6—9) how to form all the numbers of the odd-even class. Set out two lines (*a*) of odd numbers beginning with 3, (*b*) of even-times even numbers beginning with 4, thus:

(*a*) 3, 5, 7, 9, 11, 13, 15 etc.
(*b*) 4, 8, 16, 32, 64, 128, 256 etc.

Now multiply each of the first numbers into each of the second row. Let the products of one of the first into all the second set make horizontal rows; we then get the rows

12, 24, 48, 96, 192, 384, 768 etc.
20, 40, 80, 160, 320, 640, 1280 etc.
28, 56, 112, 224, 448, 896, 1792 etc.
36, 72, 144, 288, 576, 1152, 2304 etc.

and so on.

Now, says Nicomachus, you will be surprised to see (φανήσεταί σοι θαυμαστῶς) that (*a*) the *vertical* rows have the property of the *even-odd* series, 6, 10, 14, 18, 22 etc., viz. that, if an odd number of successive numbers be taken, the middle number is half the sum of the extremes, and if an even number, the two middle numbers together are equal to the sum of the extremes, (*b*) the *horizontal* rows have the property of the *even-times even* series 4, 8, 16 etc., viz. that the product of the extremes of any number of successive terms is equal, if their number be odd, to the square of the middle term, or, if their number be even, to the product of the two middle terms.

Let us now return to Euclid. His 9th definition states that an *even-times odd* number is a number which, when divided by an even number, gives an odd number as quotient. Following this definition in our text comes a 10th definition which defines an *odd-times even* number; this is stated to be a number which, when divided by an odd number, gives an even number as quotient. According to these definitions any *even-times odd* number would also be *odd-times even*, and, from the fact that Iamblichus notes this, we may fairly conclude that he found Def. 10 as well as Def. 9 in the text of Euclid which he used. But, if both definitions are genuine, the enunciations of IX. 33 and IX. 34 as we have them present difficulties. IX. 33 says that "If a number have its half odd, it is even-times odd *only*"; but, on the assumption that both definitions are genuine, this would not be true, for the number would be *odd-times even* as well. IX. 34 says that "If a number neither be one of those which are continually doubled from 2, nor have its half odd, it is both even-times even and even-times odd." The term *odd-times even* (περισσάκις ἄρτιος) not occurring in these propositions, nor anywhere else after the definition, that definition becomes superfluous. Iamblichus however (p. 24, 7—14) quotes these enunciations differently. In the first he has instead of "even-times odd only" the words "*both even-times odd and odd-times even*"; and, in the second, for "both even-times even and even-times odd" he has "is both even-times even and at the same time even-times odd and odd-times even." In both cases therefore "odd-times even" is added to the enunciation as Iamblichus had it; the words cannot have been added by Iamblichus himself because he himself does not use the term *odd-times even*, but the one word *odd-even* (περισσάρτιος). In order to get over the difficulties involved by Def. 10 and these differences of reading we have practically to choose between (1) accepting Iamblichus' reading in all three places and (2) adhering to the reading of our MSS. in IX. 33, 34 and rejecting Def. 10 altogether as an interpolation. Now the readings of our text of IX. 33, 34 are those of the Vatican MS. and the Theonine MSS. as well; hence they must go back to a time before Theon, and must therefore be almost as old as those of Iamblichus. Heiberg considers it improbable that Euclid would wish to maintain a pointless distinction between *even-times odd* and *odd-times even*, and on the whole concludes that Def. 10 was first interpolated by some ignorant person who did not notice the difference between the Euclidean and Pythagorean classification, but merely noticed the absence of a definition of *odd-times even* and fabricated one as a companion to the other. When this was done, it would be easy to see that the statement in IX. 33 that the number referred to is "even-times odd *only*" was not strictly true, and that the addition of the words "and odd-times even" was necessary in IX. 33 and IX. 34 as well.

DEFINITION 10.

Περισσάκις δὲ περισσὸς ἀριθμός ἐστιν ὁ ὑπὸ περισσοῦ ἀριθμοῦ μετρούμενος κατὰ περισσὸν ἀριθμόν.

The *odd-times odd* number is not defined as such by Nicomachus and Iamblichus; for them these numbers would apparently belong to the *composite* subdivision of *odd* numbers. Theon of Smyrna on the other hand says (p. 23, 21) that *odd-times odd* was one of the names applied to *prime* numbers (excluding 2), for these have two odd factors, namely 1 and the number itself. This is certainly a curious use of the term.

DEFINITION 11.

Πρῶτος ἀριθμός ἐστιν ὁ μονάδι μόνῃ μετρούμενος.

A *prime* number (πρῶτος ἀριθμὸς) is called by Nicomachus, Theon, and Iamblichus a "prime *and incomposite* (ἀσύνθετος) number." Theon (p. 23, 9) defines it practically as Euclid does, viz. as a number "measured by no number, but by an unit only." Aristotle too says that a prime number is not measured by any number (*Anal. post.* II. 13, 96 a 36), an unit not being a number (*Metaph.* 1088 a 6), but only the beginning of number (Theon of Smyrna says the same thing, p. 24, 23). According to Nicomachus (I. 11, 2) the prime number is a subdivision, not of numbers, but of *odd* numbers; it is "an odd number which admits of no other part except that which is called after its own name (παρώνυμον ἑαυτῷ)." The prime numbers are 3, 5, 7 etc., and there is no submultiple of 3 except $\frac{1}{3}$rd, no submultiple of 11 except $\frac{1}{11}$th, and so on. In all these cases the only submultiple is an unit. According to Nicomachus 3 is the first prime number, whereas Aristotle (*Topics* VIII. 2, 157 a 39) regards 2 as a prime number: "as the dyad is the only even number which is prime," showing that this divergence from the Pythagorean doctrine was earlier than Euclid. The number 2 also satisfies Euclid's definition of a prime number. Iamblichus (p. 30, 27 sqq.) makes this the ground of another attack upon Euclid. His argument (the text of which, however, leaves much to be desired) appears to be that 2 is the *only* even number which has no other part except an unit, while the subdivisions of the even, as previously explained by him (the *even-times even*, the *even-odd*, and *odd-even*), all exclude primeness, and he has previously explained that 2 is *potentially* even-odd, being obtained by multiplying by 2 the *potentially* odd, i.e. the unit; hence 2 is regarded by him as bound up with the subdivisions of even, which exclude primeness. Theon seems to hold the same view as regards 2, but supports it by an apparent circle. A prime number, he says (p. 23, 14—23), is also called *odd-times odd*; therefore only odd numbers are prime and incomposite. Even numbers are not measured by the unit alone, except 2, which therefore (p. 24, 7) is odd-*like* (περισσοειδής) without being prime.

A variety of other names were applied to prime numbers. We have already noted the curious designation of them as *odd-times odd*. According to Iamblichus (p. 27, 3—5) some called them *euthymetric* (εὐθυμετρικός), and Thymaridas *rectilinear* (εὐθυγραμμικός), the ground being that they can only be set out in one dimension with no breadth (ἀπλατὴς γὰρ ἐν τῇ ἐκθέσει ἐφ' ἓν μόνον διιστάμενος). The same aspect of a prime number is also expressed by Aristotle, who (*Metaph.* 1020 b 3) contrasts the composite number with that which is only in one dimension (μόνον ἐφ' ἓν ὤν). Theon of Smyrna (p. 23, 12) gives γραμμικός (*linear*) as the alternative name instead of εὐθυγραμμικός. In either case, to make the word a proper description of a prime number we have to understand the word *only*; a prime number is that which is *linear*, or *rectilinear*, *only*. For Nicomachus, who uses the form *linear*, expressly says (II. 13, 6) that *all* numbers are so, i.e. all can be represented as linear by dots to the required amount placed in a line.

A prime number was called *prime* or *first*, according to Nicomachus (I. 11, 3), because it can only be arrived at by putting together a certain number of units, and the unit is the beginning of number (cf. Aristotle's second sense of πρῶτος "as not being composed of *numbers*," ὡς μὴ συγκεῖσθαι ἐξ ἀριθμῶν, *Anal. Post.* II. 13, 96 a 37), and also, according to Iamblichus, because there is no number before it, being a collection of units (μονάδων σύστημα), of which it is a multiple, and it appears *first* as a basis for other numbers to be multiples of.

DEFINITION 12.

Πρῶτοι πρὸς ἀλλήλους ἀριθμοί εἰσιν οἱ μονάδι μόνῃ μετρούμενοι κοινῷ μέτρῳ.

By way of further emphasising the distinction between "prime" and "prime to one another," Theon of Smyrna (p. 23, 6—8) calls the former "prime *absolutely*" (ἁπλῶς), and the latter "prime to one another and *not* *absolutely*" or "*not in themselves*" (οὐ καθ' αὑτούς). The latter (p. 24, 8—10) are "measured by the unit [sc. only] as common measure, even though, taken by themselves (ὡς πρὸς ἑαυτούς), they be measured by some other numbers." From Theon's illustrations it is clear that with him as with Euclid a number prime to another may be even as well as odd. In Nicomachus (I. 11, 1) and Iamblichus (p. 26, 19), on the other hand, the number which is "in itself secondary (δεύτερος) and composite (σύνθετος), but in relation to another prime and incomposite," is a subdivision of *odd*. I shall call more particular attention to this difference of classification when we have reached the definitions of "composite" and "composite to one another"; for the present it is to be noted that Nicomachus (I. 13, 1) defines a number prime *to another* after the same manner as the absolutely prime; it is a number which "is measured not only by the unit as the common measure but also by some other measure, and for this reason can also admit of a part or parts called by a different name besides that called by the same name (as itself), but, when examined in comparison with another number of similar character, is found not to be capable of being measured by a common measure in relation to the other, nor to have the same part, called by the same name as (any of) those simply (ἁπλῶς) contained in the other; e.g. 9 in relation to 25, for each of these is in itself secondary and composite, but, in comparison with one another, they have an unit alone as a common measure and no part is called by the same name in both, but the *third* in one is not in the other, nor is the *fifth* in the other found in the first."

DEFINITION 13.

Σύνθετος ἀριθμός ἐστιν ὁ ἀριθμῷ τινι μετρούμενος.

Euclid's definition of *composite* is again the same as Theon's definition of numbers "composite in relation to themselves," which (p. 24, 16) are "numbers measured by any less number," the unit being, as usual, not regarded as a number. Theon proceeds to say that "of composite numbers they call those which are contained by two numbers *plane*, as being investigated in two dimensions and, as it were, contained by a length and a breadth, while (they call) those (which are contained) by three (numbers) *solid*, as having the third dimension added to them." To a similar effect is the remark of Aristotle (*Metaph.* 1020 b 3) that certain numbers are "composite and are not only in one dimension but such as the plane and the solid (figure) are representations of (μίμημα), these numbers being so many times so many (ποσάκις ποσοί), or so many times so many times so many (ποσάκις ποσάκις ποσοί) respectively." These subdivisions of composite numbers are, of course, the subject of Euclid's definitions 17, 18 respectively. Euclid's composite numbers may be either even or odd, like those of Theon, who gives 6 as an instance, 6 being measured by both 2 and 3.

DEFINITION 14.

Σύνθετοι δὲ πρὸς ἀλλήλους ἀριθμοί εἰσιν οἱ ἀριθμῷ τινι μετρούμενοι κοινῷ μέτρῳ.

Theon (p. 24, 18), like Euclid, defines numbers *composite to one another* as "those which are measured by any common measure whatever" (excluding unity, as usual). Theon instances 8 and 6, with 2 as common measure, and 6 and 9, with 3 as common measure.

As hinted above, there is a great difference between Euclid's classification of prime and composite numbers, and of numbers prime and composite to one another, and the classification found in Nicomachus (I. 11—13) and Iamblichus. According to the latter, all these kinds of numbers are subdivisions of the class of *odd* numbers only. As the class of *even* numbers is divided into three kinds, (1) the even-times even, (2) the even-odd, which form the extremes, and (3) the odd-even, which is, as it were, intermediate to the other two, so the class of *odd* numbers is divided into three, of which the third is again a mean between two extremes. The three are:

(1) the *prime and incomposite*, which is like Euclid's prime number except that it excludes 2;

(2) the *secondary and composite*, which is "odd because it is a distinct part of one and the same genus (διὰ τὸ ἐξ ἑνὸς καὶ τοῦ αὐτοῦ γένους διακεκρίσθαι) but has in it nothing of the nature of a first principle (ἀρχοειδές); for it arises from adding some other number (to itself), so that, besides having a part called by the same name as itself, it possesses a part or parts called by another name." Nicomachus cites 9, 15, 21, 25, 27, 33, 35, 39. It is made clear that not only must the factors be both odd, but they must all be prime numbers. This is obviously a very inconvenient restriction of the use of the word *composite*, a word of general signification.

(3) is that which is "*secondary and composite in itself but prime and incomposite to another.*" The actual words in which this is defined have been given above in the note on Def. 12. Here again all the factors must be odd and prime.

Besides the inconvenience of restricting the term *composite* to *odd* numbers which are composite, there is in this classification the further serious defect, pointed out by Nesselmann (*Die Algebra der Griechen*, 1842, p. 194), that subdivisions (2) and (3) overlap, subdivision (2) including the whole of subdivision (3). The origin of this confusion is no doubt to be found in Nicomachus' perverse anxiety to be symmetrical; by hook or by crook he must divide *odd* numbers into three kinds as he had divided the *even*. Iamblichus (p. 28, 13) carries his desire to be logical so far as to point out why there cannot be a fourth kind of number contrary in character to (3), namely a number which should be "prime and incomposite in itself, but secondary and composite to another"!

DEFINITION 15.

Ἀριθμὸς ἀριθμὸν πολλαπλασιάζειν λέγεται, ὅταν, ὅσαι εἰσὶν ἐν αὐτῷ μονάδες, τοσαυτάκις συντεθῇ ὁ πολλαπλασιαζόμενος, καὶ γένηταί τις.

This is the well known primary definition of multiplication as an abbreviation of addition.

DEFINITION 16.

Ὅταν δὲ δύο ἀριθμοὶ πολλαπλασιάσαντες ἀλλήλους ποιῶσί τινα, ὁ γενόμενος ἐπίπεδος καλεῖται, πλευραὶ δὲ αὐτοῦ οἱ πολλαπλασιάσαντες ἀλλήλους ἀριθμοί.

The words *plane* and *solid* applied to numbers are of course adapted from their use with reference to geometrical figures. A number is therefore called *linear* (γραμμικός) when it is regarded as in one dimension, as being a *length* (μῆκος). When it takes another dimension in addition, namely *breadth* (πλάτος), it is in two dimensions and becomes *plane* (ἐπίπεδος). The distinction between a *plane* and a *plane number* is marked by the use of the neuter in the former case, and the masculine, agreeing with ἀριθμός, in the latter case. So with a *square* and a *square number*, and so on. The most obvious form of a plane number is clearly that corresponding to a rectangle in geometry; the number is the product of two linear numbers regarded as *sides* (πλευραί) forming the length and breadth respectively. Such a number is, as Aristotle says, "so many times so many," and a plane is its counterpart (μίμημα). So Plato, in the *Theaetetus* (147 E—148 B), says: "We divided all numbers into two kinds, (1) that which can be expressed as equal multiplied by equal (τὸν δυνάμενον ἴσον ἰσάκις γίγνεσθαι), and which, likening its form to the square, we called *square* and equilateral; (2) that which is intermediate, and includes 3 and 5 and every number which cannot be expressed as equal multiplied by equal, but is either less times more or more times less, being always contained by a greater and a less side, which number we likened to the oblong figure (προμήκει σχήματι) and called an *oblong* number.... Such *lines* therefore as *square the equilateral and plane number* [i.e. which can form a plane number with equal sides, or a square] we defined as *length* (μῆκος); but such as square the oblong (here ἑτερομήκης) [i.e. the square of which is equal to the oblong] we called *roots* (δυνάμεις) as not being commensurable with the others in length, but only in the plane areas (ἐπιπέδοις), to which the squares on them are equal (ἃ δύνανται)." This passage seems to make it clear that Plato would have represented numbers as Euclid does, by straight lines proportional in length to the numbers they represent (so far as practicable); for, since 3 and 5 are with Plato oblong numbers, and *lines* with him represent the sides of oblong numbers (since a line represents the "root," the square on which is equal to the oblong), it follows that the *unit* representing the smaller side must have been represented as a line, and 3, the larger side, as a line of three times the length. But there is another possible way of representing numbers, not by lines of a certain length, but by *points* disposed in various ways, in straight lines or otherwise. Iamblichus tells us (p. 56, 27) that "in old days they represented the quantuplicities of number in a more natural way (φυσικώτερον) by splitting them up into units, and not, as in our day, by symbols" (συμβολικῶς). Aristotle too (*Metaph.* 1092 b 10) mentions one Eurytus as having settled what number belonged to what, such a number to a man, such a number to a horse, and so on, "copying their shapes" (reading τούτων, with Zeller) "*with pebbles* (ταῖς ψήφοις), *just as those do who arrange numbers in the forms of triangles or squares.*" We accordingly find numbers represented in Nicomachus and Theon of Smyrna by a number of α's ranged like points according to geometrical figures. According to this system, any number could be represented by points in a straight line, in which case, says Iamblichus (p. 56, 26), we shall call it rectilinear because it is *without breadth* and only advances in length (ἀπλατῶς ἐπὶ μόνον τὸ μῆκος πρόεισιν). The prime number was called by Thymaridas rectilinear *par excellence*, because it was without breadth and in one dimension *only* (ἐφ' ἓν μόνον διιστάμενος). By this must be meant the impossibility of representing, say, 3 as a plane number, in Plato's sense, i.e. as a product of two numbers corresponding to a rectangle in geometry; and this view would appear to rest simply upon the representation of a number by *points*, as distinct from lines. Three dots in a straight line would have *no* breadth; and if breadth were introduced in the sense of producing a rectangle, i.e. by placing the same number of dots in a second line below the first line, the first *plane* number would be 4, and 3 would not be a plane number at all, as Plato says it is. It seems therefore to have been the alternative representation of a number by points, and not lines, which gave rise to the different view of a plane number which we find in Nicomachus and the rest. By means of separate points we can represent numbers in geometrical forms other than rectangles and squares. One dot with two others symmetrically arranged below it shows a *triangle*, which is a figure *in two dimensions* as much as a rectangle or parallelogram is. Similarly we can arrange certain numbers in the form of regular *pentagons* or other polygons. According therefore to this mode of representation, 3 is the first *plane* number, being a *triangular* number. The method of formation of triangular, square, pentagonal and other polygonal numbers is minutely described in Nicomachus (II. 8—11), who distinguishes the separate series of *gnomons* belonging to each, i.e. gives the law determining the number which has to be added to a polygonal number with n in a side, in order to make it into a number of the same form but with $n + 1$ in a side (the addend being of course the gnomon). Thus the gnomonic series for triangular numbers is 1, 2, 3, 4, 5... ; that for squares 1, 3, 5, 7... ; that for pentagonal numbers 1, 4, 7, 10..., and so on. The subject need not detain us longer here, as we are at present only concerned with the different views of what constitutes a *plane* number.

Of *plane* numbers in the Platonic and Euclidean sense we have seen that Plato recognises *two* kinds, the *square* and the *oblong* (προμήκης or ἑτερομήκης). Here again Euclid's successors, at all events, subdivided the class more elaborately. Nicomachus, Theon of Smyrna, and Iamblichus divide *plane* numbers with unequal *sides* into (1) ἑτερομήκεις, the nearest thing to squares, viz. numbers in which the greater side exceeds the less side by 1 only, or numbers of the form $n(n+1)$, e.g. $1 . 2$, $2 . 3$, $3 . 4$, etc. (according to Nicomachus), and (2) προμήκεις, or those whose sides differ by 2 or more, i.e. are of the form $n(n+m)$, where m is not less than 2 (Nicomachus illustrates by $2 . 4$, $3 . 6$, etc.). Theon of Smyrna (p. 30, 8—14) makes προμήκεις include ἑτερομήκεις, saying that their sides may differ by 1 or more; he also speaks of *parallelogram*-numbers as those which have one side different from the other by 2 or more; I do not find this latter term in Nicomachus or Iamblichus, and indeed it seems superfluous, as *parallelogram* is here only another name for oblong. Iamblichus (p. 74, 23 sqq.), always critical of Euclid, attacks him again here for confusing the subject by supposing that the ἑτερομήκης number is the product of *any* two different numbers multiplied together, and by not distinguishing the oblong (προμήκης) from it: "for his definition declares the same number to be square and also ἑτερομήκης, as for example 36, 16 and many others: which would be equivalent to the odd number being the same thing as the even." No importance need be attached to this exaggerated statement; it is in any case merely a matter of words, and it is curious that Euclid does not in fact use the word ἑτερομήκης of *numbers* at all, but only of geometrical oblong figures as opposed to squares, so that Iamblichus can apparently only have inferred that he used it in an unorthodox manner from the *geometrical* use of the term in the definitions of Book I. and from the fact that he does not give the two subdivisions of *plane numbers* which are not square, but seems only to divide *plane numbers* into square and not-square. The argument that ἑτερομήκεις numbers are a *natural*, and therefore essential, subdivision Iamblichus appears to found on the method of successive addition by which they can be evolved; as square numbers are obtained by successively adding *odd* numbers as gnomons, so ἑτερομήκεις are obtained by adding *even* numbers as gnomons. Thus $1 . 2 = 2$, $2 . 3 = 2 + 4$, $3 \cdot 4 = 2 + 4 + 6$, and so on.

DEFINITION 17.

Ὅταν δὲ τρεῖς ἀριθμοὶ πολλαπλασιάσαντες ἀλλήλους ποιῶσί τινα, ὁ γενόμενος στερεός ἐστιν, πλευραὶ δὲ αὐτοῦ οἱ πολλαπλασιάσαντες ἀλλήλους ἀριθμοί.

What has been said of the two apparently different ways of regarding a *plane* number seems to apply equally, *mutatis mutandis*, to the definitions of a *solid* number. Aristotle regards it as a number which is so many times so many times so many (ποσάκις ποσάκις ποσόν). Plato finishes the passage about lines which represent the sides of *square* numbers and lines which are *roots* (δυνάμεις), i.e. the squares on which are equal to the rectangle representing a number which is oblong and not square, by adding the words, "And another similar property belongs to solids" (καὶ περὶ τὰ στερεὰ ἄλλο τοιοῦτον). That is, apparently, there would be a corresponding term to *root* (δύναμις)—practically representing a surd—to denote the side of a cube equal to a parallelepiped representing a solid number which is the product of three factors but not a cube. Such is a solid number when numbers are represented by *straight lines*: it corresponds in general to a parallelepiped and, when all the factors are equal, to a cube.

But again, if numbers be represented by *points*, we may have solid numbers (i.e. numbers in three dimensions) in the form of *pyramids* as well. The first number of this kind is 4, since we may have three points forming an equilateral triangle in one plane and a fourth point placed in another plane. The length of the sides can be increased by 1 successively; and we can have a series of pyramidal numbers, with triangles, squares or polygons as bases, made up of layers of triangles, squares or similar polygons respectively, each of which layers has one less in the side than the layer below it, until the top of the pyramid is reached, which of course is one point representing unity. Nicomachus (II. 13—16), Theon of Smyrna (p. 41—2), and Iamblichus (p. 95, 15 sqq.), all give the different kinds of *pyramidal* solid numbers in addition to the other kinds.

These three writers make the following further distinctions between solid numbers which are the product of three factors.

1. First there is the equal by equal by equal (ἰσάκις ἰσάκις ἴσος), which is, of course, the cube.

2. The other extreme is the unequal by unequal by unequal (ἀνισάκις ἀνισάκις ἄνισος), or that in which all the dimensions are different, e.g. the product of 2, 3, 4 or 2, 4, 8 or 3, 5, 12. These were, according to Nicomachus (II. 16), called *scalene*, while some called them σφηνίσκοι (*wedge-shaped*), others σφηκίσκοι (from σφήξ, a *wasp*), and others βωμίσκοι (*altar-shaped*). Theon appears to use the last term only, while Iamblichus of course gives all three names.

3. Intermediate to these, as it were, come the numbers "whose *planes* form ἑτερομήκεις numbers" (i.e. numbers of the form $n(n + 1)$). These, says Nicomachus, are called *parallelepipedal*.

Lastly come two classes of such numbers each of which has two equal dimensions but not more.

4. If the third dimension is less than the others, the number is *equal by equal by less* (ἰσάκις ἴσος ἐλαττονάκις) and is called a *plinth* (πλινθίς), e.g. 8 . 8 . 3.

5. If the third dimension is greater than the others, the number is *equal by equal by greater* (ἰσάκις ἴσος μειζονάκις) and is called a *beam* (δοκίς), e.g. 3 . 3 . 7. Another name for this latter kind of number (according to Iamblichus) was στηλίς (diminutive of στήλη).

Lastly, in connexion with pyramidal numbers, Nicomachus (II. 14, 5) distinguishes numbers corresponding to *frusta* of pyramids. These are *truncated* (κόλουροι), *twice-truncated* (δικόλουροι), *thrice-truncated* (τρικόλουροι) pyramids, and so on, the term being used mostly in theoretic treatises (ἐν συγγράμμασι μάλιστα τοῖς θεωρηματικοῖς). The *truncated* pyramid was formed by cutting off the point forming the vertex. The *twice-truncated* was that which lacked the vertex and the next plane, and so on. Theon of Smyrna (p. 42, 4) only mentions the *truncated* pyramid as "that with its vertex cut off" (ἡ τὴν κορυφὴν ἀποτετμημένη), saying that some also called it a trapezium, after the similitude of a plane trapezium formed by cutting the top off a triangle by a straight line parallel to the base.

DEFINITION 18.

Τετράγωνος ἀριθμός ἐστιν ὁ ἰσάκις ἴσος ἢ [ὁ] ὑπὸ δύο ἴσων ἀριθμῶν περιεχόμενος.

A particular kind of square distinguished by Nicomachus and the rest was the square number which ended (in the decimal notation) with the same number as its side, e.g. 1, 25, 36, which are the squares of 1, 5 and 6. These square numbers were called *cyclic* (κυκλικοί) on the analogy of circles in geometry which return again to the point from which they started.

DEFINITION 19.

Κύβος δὲ ὁ ἰσάκις ἴσος ἰσάκις ἢ [ὁ] ὑπὸ τριῶν ἴσων ἀριθμῶν περιεχόμενος.

Similarly cube numbers which ended with the same number as their sides, *and the squares of those sides also*, were called *spherical* (σφαιρικοί) or *recurrent* (ἀποκαταστατικοί). One might have expected that the term *spherical* would be applicable also to the cubes of numbers which ended with the same digit as the *side* but not necessarily with the same digit as the *square* of the side also. E.g. the cube of 4, i.e. 64, ends with the same digit as 4, but not with the same digit as 16. But apparently 64 was not called a spherical number, the only instances given by Nicomachus and the rest being those cubed from numbers ending with 5 or 6, which end with the same digit if *squared*. A *spherical* number is in fact derived from a *circular* number only, and that by adding another equal dimension. Obviously, as Nesselmann says, the names *cyclic* and *spherical* applied to numbers appeal to an entirely different principle from that on which the figured numbers so far dealt with were formed.

DEFINITION 20.

Ἀριθμοὶ ἀνάλογόν εἰσιν, ὅταν ὁ πρῶτος τοῦ δευτέρου καὶ ὁ τρίτος τοῦ τετάρτου ἰσάκις ᾖ πολλαπλάσιος ἢ τὸ αὐτὸ μέρος ἢ τὰ αὐτὰ μέρη ὦσιν.

Euclid does not give in this Book any definition of ratio, doubtless because it could only be the same as that given at the beginning of Book v., with numbers substituted for "homogeneous magnitudes" and "in respect of *size*" (πηλικότητα) omitted or altered. We do not find that Nicomachus and the rest give any substantially different definition of a *ratio* between numbers. Theon of Smyrna says, in fact (p. 73, 16), that "ratio in the sense of proportion (λόγος ὁ κατ' ἀνάλογον) is a sort of relation of two homogeneous terms to one another, as for example, double, triple." Similarly Nicomachus says (II. 21, 3) that "a ratio is a relation of two terms to one another," the word for "relation" being in both cases the same as Euclid's (σχέσις). Theon of Smyrna goes on to classify ratios as greater, less, or equal, i.e. as ratios of greater inequality, less inequality, or equality, and then to specify certain arithmetical ratios which had special names, for which he quotes the authority of Adrastus. The names were πολλαπλάσιος, ἐπιμόριος, ἐπιμερής, πολλαπλασιεπιμόριος, πολλαπλασιεπιμερής (the first of which is, of course, a multiple, while the rest are the equivalent of certain types of improper fractions as we should call them), and the reciprocals of each of these described by prefixing ὑπό or *sub*. After describing these particular classes of arithmetical ratios, Theon goes on to say that numbers still have ratios to one another even if they are different from all those previously described. We need not therefore concern ourselves with the various types; it is sufficient to observe that any ratio between numbers can be expressed in the manner indicated in Euclid's definition of arithmetical proportion, for the greater is, in relation to the less, either one or a combination of more than one of the three things, (1) a multiple, (2) a submultiple, (3) a proper fraction.

It is when we come to the definition of *proportion* that we begin to find differences between Euclid, Nicomachus, Theon and Iamblichus. "Proportion," says Theon (p. 82, 6), "is similarity or sameness of more ratios than one," which is of course unobjectionable if it is previously understood what a *ratio* is; but confusion was brought in by those (like Thrasyllus) who said that there were *three proportions* (ἀναλογίαι), the arithmetic, geometric, and harmonic, where of course the reference is to arithmetic, geometric and harmonic *means* (μεσότητες). Hence it was necessary to explain, as Adrastus did (Theon, p. 106, 15), that of the several *means* "the geometric was called both proportion *par excellence* and primary...though the other means were also commonly called *proportions* by some writers." Accordingly we have Nicomachus trying to extend the term "proportion" to cover the various *means* as well as a proportion in three or four terms in the ordinary sense. He says (II. 21, 2): "Proportion, *par excellence* (κυρίως), is the bringing together (σύλληψις) to the same (point) of two or more *ratios*; or, more generally, (the bringing together) of two or more *relations* (σχέσεων), even though they be subjected not to the same *ratio* but to a difference or some other (law)." Iamblichus keeps the senses of the word more distinct. He says, like Theon, that "proportion is similarity or sameness of several ratios" (p. 98, 14), and that "it is to be premised that it was the geometrical (proportion) which the ancients called proportion *par excellence*, though it is now common to apply the name generally to all the remaining means as well" (p. 100, 15). Pappus remarks (III. p. 70, 17), "A mean differs from a proportion in this respect that, if anything is a proportion, it is also a mean, but not conversely. For there are three means, of which one is arithmetic, one geometric and one harmonic." The last remark implies plainly enough that there is only one *proportion* (ἀναλογία) in the proper sense. So, too, says Iamblichus in another place (p. 104, 19): "the second, the geometric, mean has been called *proportion par excellence* because the terms contain the same ratio, being separated according to the same proportion (ἀνὰ τὸν αὐτὸν λόγον διεστῶτες)." The natural conclusion is that of Nesselmann, that originally the geometric proportion was called ἀναλογία, the others, the arithmetic, the harmonic, etc., *means*; but later usage had obliterated the distinction.

Of proportions in the ancient and Euclidean sense Theon (p. 82, 10) distinguished the *continuous* (συνεχής) and the *separated* (διῃρημένη), using the same terms as Aristotle (*Eth. Nic.* 1131 a 32). The meaning is of course clear: in the *continuous* proportion the consequent of one ratio is the antecedent of the next; in the *separated* proportion this is not so. Nicomachus (II. 21, 5—6) uses the words *connected* (συνημμένη) and *disjoined* (διεζευγμένη) respectively. Euclid regularly speaks of numbers in continuous proportion as "proportional in order, or successively" (ἑξῆς ἀνάλογον).

DEFINITION 21.

Ὅμοιοι ἐπίπεδοι καὶ στερεοὶ ἀριθμοί εἰσιν οἱ ἀνάλογον ἔχοντες τὰς πλευράς.

Theon of Smyrna remarks (p. 36, 12) that, among plane numbers, *all* squares are similar, while of ἑτερομήκεις those are similar "whose sides, that is, the numbers containing them, are proportional." Here ἑτερομήκης must evidently be used, not in the sense of a number of the form $n(n+1)$, but as synonymous with προμήκης, *any* oblong number; so that on this occasion Theon follows the terminology of Plato and (according to Iamblichus) of Euclid. Obviously, if the strict sense of ἑτερομήκης is adhered to, no two numbers of that form can be similar unless they are also *equal*. We may compare Iamblichus' elaborate contrast of the square and the ἑτερομήκης. Since the two sides of the square are equal, a square number might, as he says (p. 82, 9), be fitly called ἰδιομήκης (Nicomachus uses ταὐτομήκης) in contrast to ἑτερομήκης; and the ancients, according to him, called square numbers "the same" and "similar" (ταὐτούς τε καὶ ὁμοίους), but ἑτερομήκεις numbers "dissimilar and other" (ἀνομοίους καὶ θατέρους).

With regard to solid numbers, Theon remarks in like manner (p. 37, 2) that *all* cube numbers are similar, while of the others those are similar whose sides are proportional, i.e. in which, as length is to length, so is breadth to breadth and height to height.

DEFINITION 22.

Τέλειος ἀριθμός ἐστιν ὁ τοῖς ἑαυτοῦ μέρεσιν ἴσος ὤν.

Theon of Smyrna (p. 45, 9 sqq.) and Nicomachus (I. 16) both give the same definition of a *perfect* number, as well as the law of formation of such numbers which Euclid proves in the later proposition, IX. 36. They add however definitions of two other kinds of numbers in contrast with it, (1) the *over-perfect* (ὑπερτελής in Nicomachus, ὑπερτέλειος in Theon), the sum of whose parts, i.e. submultiples, is greater than the number itself, e.g. 12, 24 etc., the sum of the parts of 12 being $6+4+3+2+1=16$, and the sum of the parts of 24 being $12+8+6+4+3+2+1=36$, (2) the *defective* (ἐλλιπής), the sum of whose parts is less than the whole, e.g. 8 or 14, the parts in the first case adding up to $4+2+1$, or 7, and in the second case to $7+2+1$, or 10. All three classes are however made by Theon subdivisions of numbers in general, but by Nicomachus subdivisions of *even* numbers.

The term *perfect* was used by the Pythagoreans, but in another sense, of 10; while Theon tells us (p. 46, 14) that 3 was also called perfect "because it is the first number that has beginning, middle and extremity; it is also both a *line* and a *plane* (for it is an equilateral triangle having each side made up of two units), and it is the first link and potentiality of the solid (for a solid must be conceived of in three dimensions)."

There are certain unexpressed axioms used in Book VII. as there are in earlier Books.

The following may be noted.

1. If A measures B, and B measures C, A will measure C.
2. If A measures B, and also measures C, A will measure the difference between B and C when they are unequal.
3. If A measures B, and also measures C, A will measure the sum of B and C.

It is clear, from what we know of the Pythagorean theory of numbers, of musical intervals expressed by numbers, of different kinds of *means* etc., that the substance of Euclid Books VII.—IX. was no new thing but goes back, at least, to the Pythagoreans. It is well known that the mathematics of Plato's *Timaeus* is essentially Pythagorean. It is therefore *a priori* probable (if not perhaps quite certain) that Plato πυθαγορίζει even in the passage (32 A, B) where he speaks of numbers "whether solid or square" in continued proportion, and proceeds to say that between *planes* one mean suffices, but to connect two *solids* two means are necessary. This passage has been much discussed, but I think that by "planes" and "solids" Plato certainly meant *square* and *solid numbers* respectively, so that the allusion must be to the theorems established in Eucl. VIII. 11, 12, that between two square numbers there is one mean proportional number, and between two cube numbers there are two mean proportional numbers[1].

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[1] It is true that *similar* plane and solid numbers have the same property (Eucl. VIII. 18, 19); but, if Plato had meant similar plane and solid numbers generally, I think it would have been necessary to specify that they were "similar," whereas, seeing that the *Timaeus* is as a whole concerned with regular figures, there is nothing unnatural in allowing *regular* or *equilateral* to be understood. Further Plato speaks first of δυνάμεις and ὄγκοι and then of "planes" (ἐπίπεδα) and "solids" (στερεά) in such a way as to suggest that δυνάμεις correspond to ἐπίπεδα and ὄγκοι to στερεά. Now the regular meaning of δύναμις is *square* (or sometimes *square root*), and I think it is here used in the sense of *square*, notwithstanding that Plato seems to speak of *three* squares in continued proportion, whereas, in general, the mean between two squares as extremes would not be square but oblong. And, if δυνάμεις are squares, it is reasonable to suppose that the ὄγκοι are also *equilateral*, i.e. the "solids" are cubes. I am aware that Th. Häbler (*Bibliotheca Mathematica*, VIII$_3$, 1908, pp. 173—4) thinks that the passage is to be explained by reference to the problem of the duplication of the cube, and does not refer to numbers at all. Against this we have to put the evidence of Nicomachus (II. 24, 6) who, in speaking of "a certain Platonic theorem," quotes the very same results of Eucl. VIII. 11, 12. Secondly, it is worth noting that Häbler's explanation is distinctly ruled out by Democritus the Platonist (3rd cent. A.D.) who, according to Proclus

It is no less clear that, in his method and line of argument, Euclid was following earlier models, though no doubt making improvements in the exposition. The tract on the *Sectio Canonis*, κατατομη κανόνος (as to the genuineness of which see above, Vol. I., p. 17) is in style and in the form of the propositions generally akin to the *Elements*. In one proposition (2) the author says "*we learned* (ἐμάθομεν) that, if as many numbers as we please be in (continued) proportion, and the first measures the last, the first will also measure the intermediate numbers"; here he practically quotes *Elem.* VIII. 7. In the 3rd proposition he proves that no number can be a mean between two numbers in the ratio known as ἐπιμόριος, the ratio, that is, of $n + 1$ to n, where n is any integer greater than unity. Now, fortunately, Boethius, *De institutione musica*, III. 11 (pp. 285—6, ed. Friedlein), has preserved a proof by Archytas of this same proposition; and the proof is substantially identical with that of Euclid. The two proofs are placed side by side in an article by Tannery (*Bibliotheca Mathematica*, VI$_3$, 1905/6, p. 227). Archytas writes the smaller term of the proportion first (instead of the greater, as Euclid does). Let, he says, A, B be the "superparticularis proportio" (ἐπιμόριον διάστημα in Euclid). Take C, DE the smallest numbers which are in the ratio of A to B. [Here DE means $D + E$: and in this respect the notation is different from that of Euclid who, as usual, takes a line DF divided into two parts at G, GF corresponding to E, and DG to D, in Archytas' notation. The step of taking C, DE, the smallest numbers in the ratio of A to B, presupposes Eucl. VII. 33.] Then DE exceeds C by an aliquot part of itself and of C [cf. the definition of ἐπιμόριος ἀριθμός in Nicomachus, I. 19, 1]. Let D be the excess [i.e. E is supposed equal to C]. "I say that D is not a number but an unit."

For, if D is a number and a part of DE, it measures DE; hence it measures E, that is, C. Thus D measures both C and DE, which is impossible; for the smallest numbers which are in the same ratio as any numbers are prime to one another. [This presupposes Eucl. VII. 22.] Therefore D is an unit; that is, DE exceeds C by an unit. Hence no number can be found which is a mean between two numbers C, DE. Therefore neither can any number be a mean between the original numbers A, B which are in the same ratio [this implies Eucl. VII. 20].

We have then here a clear indication of the existence at least as early as the date of Archytas (about 430—365 B.C.) of an *Elements of Arithmetic* in the form which we call Euclidean; and no doubt text-books of the sort existed even before Archytas, which probably Archytas himself and Eudoxus improved and developed in their turn.

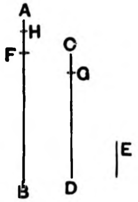
(*In Platonis Timaeum commentaria*, 149 C), said that the difficulties of the passage of the *Timaeus* had *misled* some people into connecting it with the duplication of the cube, whereas it really referred to *similar* planes and solids with sides in *rational numbers*. Thirdly, I do not think that, under the supposition that the Delian problem is referred to, we get the required sense. The problem in that case is not that of finding two mean proportionals *between two cubes* but that of finding a second cube the content of which shall be equal to twice, or k times (where k is any number not a complete cube), the content of a given cube (a^3). Two mean proportionals are found, not between cubes, but between two *straight lines* in the ratio of 1 to k, or between a and ka. Unless k is a cube, there would be no point in saying that two means are necessary to connect 1 and k, and not one mean; for $\sqrt[3]{k}$ is no more natural than $\sqrt{k}$, and would be less natural in the case where k happened to be square. On the other hand, if k is a cube, so that it is a question of finding means between *cube numbers*, the dictum of Plato is perfectly intelligible; nor is any real difficulty caused by the generality of the statement that two means are *always* necessary to connect them, because any property enunciated generally of two cube numbers should obviously be true of cubes *as such*, that is, it must hold in the extreme case of two cubes which are *prime to one another*.

BOOK VII. PROPOSITIONS.

PROPOSITION 1.

Two unequal numbers being set out, and the less being continually subtracted in turn from the greater, if the number which is left never measures the one before it until an unit is left, the original numbers will be prime to one another.

For, the less of two unequal numbers *AB*, *CD* being continually subtracted from the greater, let the number which is left never measure the one before it until an unit is left;



I say that *AB*, *CD* are prime to one another, that is, that an unit alone measures *AB*, *CD*.

For, if *AB*, *CD* are not prime to one another, some number will measure them.

Let a number measure them, and let it be *E*; let *CD*, measuring *BF*, leave *FA* less than itself,

let *AF*, measuring *DG*, leave *GC* less than itself,

and let *GC*, measuring *FH*, leave an unit *HA*.

Since, then, *E* measures *CD*, and *CD* measures *BF*, therefore *E* also measures *BF*.

But it also measures the whole *BA*;

therefore it will also measure the remainder *AF*.

But *AF* measures *DG*;

therefore *E* also measures *DG*.

But it also measures the whole *DC*·

therefore it will also measure the remainder *CG*.

But *CG* measures *FH*;

therefore *E* also measures *FH*.

But it also measures the whole *FA*;

therefore it will also measure the remainder, the unit *AH*, though it is a number: which is impossible.

Therefore no number will measure the numbers *AB*, *CD*; therefore *AB*, *CD* are prime to one another. [VII. Def. 12]

Q. E. D.

It is proper to remark here that the representation in Books VII. to IX. of numbers by straight lines is adopted by Heiberg from the MSS. The method of those editors who substitute *points* for lines is open to objection because it practically necessitates, in many cases, the use of specific numbers, which is contrary to Euclid's manner.

"Let *CD*, measuring *BF*, leave *FA* less than itself." This is a neat abbreviation for saying, measure along *BA* successive lengths equal to *CD* until a point *F* is reached such that the length *FA* remaining is less than *CD*; in other words, let *BF* be the largest exact multiple of *CD* contained in *BA*.

Euclid's method in this proposition is an application to the particular case of prime numbers of the method of finding the greatest common measure of two numbers not prime to one another, which we shall find in the next proposition. With our notation, the method may be shown thus. Supposing the two numbers to be a, b, we have, say,

$$
\begin{array}{llll}
b\,)\,a\,(\,p & & & \\
\;\;\underline{pb} & & & \\
\;\;\;c\,)\,b\,(\,q & & & \\
\;\;\;\;\;\;\underline{qc} & & & \\
\;\;\;\;\;\;\;d\,)\,c\,(\,r & & & \\
\;\;\;\;\;\;\;\;\;\;\underline{rd} & & & \\
\;\;\;\;\;\;\;\;\;\;\;1 & & &
\end{array}
$$

If now a, b are not prime to one another, they must have a common measure e, where e is some integer, not unity.

And since e measures a, b, it measures $a - pb$, i.e. c.

Again, since e measures b, c, it measures $b - qc$, i.e. d,

and lastly, since e measures c, d, it measures $c - rd$, i.e. 1:

which is impossible.

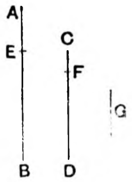
Therefore there is no integer, except unity, that measures a, b, which are accordingly prime to one another.

Observe that Euclid assumes as an axiom that, if a, b are both divisible by c, so is $a - pb$. In the next proposition he assumes as an axiom that c will in the case supposed divide $a + pb$.

PROPOSITION 2.

Given two numbers not prime to one another, to find their greatest common measure.

Let *AB*, *CD* be the two given numbers not prime to one another.



Thus it is required to find the greatest common measure of *AB*, *CD*.

If now *CD* measures *AB*—and it also measures itself—*CD* is a common measure of *CD*, *AB*.

And it is manifest that it is also the greatest; for no greater number than *CD* will measure *CD*.

But, if *CD* does not measure *AB*, then, the less of the numbers *AB*, *CD* being continually subtracted from the greater, some number will be left which will measure the one before it.

For an unit will not be left; otherwise *AB*, *CD* will be prime to one another [VII. 1], which is contrary to the hypothesis.

Therefore some number will be left which will measure the one before it.

Now let *CD*, measuring *BE*, leave *EA* less than itself, let *EA*, measuring *DF*, leave *FC* less than itself,

and let *CF* measure *AE*.

Since then, *CF* measures *AE*, and *AE* measures *DF*,

therefore *CF* will also measure *DF*.

But it also measures itself;

therefore it will also measure the whole *CD*.

But *CD* measures *BE*;

therefore *CF* also measures *BE*.

But it also measures *EA*;

therefore it will also measure the whole *BA*.

But it also measures *CD*;

therefore *CF* measures *AB*, *CD*.

Therefore *CF* is a common measure of *AB*, *CD*.

I say next that it is also the greatest.

For, if CF is not the greatest common measure of AB, CD, some number which is greater than CF will measure the numbers AB, CD.

Let such a number measure them, and let it be G.

Now, since G measures CD, while CD measures BE, G also measures BE.

But it also measures the whole BA;

therefore it will also measure the remainder AE.

But AE measures DF;

therefore G will also measure DF.

But it also measures the whole DC;

therefore it will also measure the remainder CF, that is, the greater will measure the less: which is impossible.

Therefore no number which is greater than CF will measure the numbers AB, CD;

therefore CF is the greatest common measure of AB, CD.

PORISM. From this it is manifest that, if a number measure two numbers, it will also measure their greatest common measure. Q. E. D..

Here we have the exact method of finding the greatest common measure given in the text-books of algebra, including the *reductio ad absurdum* proof that the number arrived at is not only a common measure but the *greatest* common measure. The process of finding the greatest common measure is simply shown thus:

$$\begin{array}{l}
b\,)\,a\,(\,p\\
\quad\;\underline{pb}\\
\quad\;\; c\,)\,b\,(\,q\\
\qquad\;\;\underline{qc}\\
\qquad\;\;\; d\,)\,c\,(\,r\\
\qquad\qquad\underline{rd}
\end{array}$$

We shall arrive, says Euclid, at some number, say d, which measures the one before it, i.e. such that $c = rd$. Otherwise the process would go on until we arrived at unity. This is impossible because in that case a, b would be prime to one another, which is contrary to the hypothesis.

Next, like the text-books of algebra, he goes on to show that d will be *some* common measure of a, b. For d measures c;
therefore it measures $qc + d$, that is, b,
and hence it measures $pb + c$, that is, a.

Lastly, he proves that d is the *greatest* common measure of a, b as follows.

Suppose that e is a common measure greater than d.

Then e, measuring a, b, must measure $a - pb$, or c.

Similarly e must measure $b-qc$, that is, d: which is impossible, since e is by hypothesis greater than d.

Therefore etc.

Euclid's proposition is thus *identical* with the algebraical proposition as generally given, e.g. in Todhunter's algebra, except that of course Euclid's numbers are integers.

Nicomachus gives the same rule (though without proving it) when he shows how to determine whether two given *odd* numbers are prime or not prime to one another, and, if they are not prime to one another, what is their common measure. We are, he says, to compare the numbers in turn by continually taking the less from the greater as many times as possible, then taking the remainder as many times as possible from the less of the original numbers, and so on; this process "will finish either at an unit or at some one and the same number," by which it is implied that the division of a greater number by a less is done by *separate subtractions* of the less. Thus, with regard to 21 and 49, Nicomachus says, "I subtract the less from the greater; 28 is left; then again I subtract from this the same 21 (for this is possible); 7 is left; I subtract this from 21, 14 is left; from which I again subtract 7 (for this is possible); 7 will be left, but 7 cannot be subtracted from 7." The last phrase is curious, but the meaning of it is obvious enough, as also the meaning of the phrase about ending "at one and the same number."

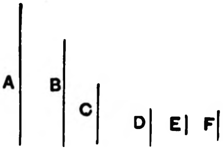
The proof of the Porism is of course contained in that part of the proposition which proves that G, a common measure different from CF, must measure CF. The supposition, thereby proved to be false, that G is greater than CF does not affect the validity of the proof that G measures CF in any case.

PROPOSITION 3.

Given three numbers not prime to one another, to find their greatest common measure.

Let A, B, C be the three given numbers not prime to one another;

thus it is required to find the greatest common measure of A, B, C.



For let the greatest common measure, D, of the two numbers A, B be taken; [VII. 2]

then D either measures, or does not measure, C.

First, let it measure it.

But it measures A, B also;

therefore D measures A, B, C;

therefore D is a common measure of A, B, C.

I say that it is also the greatest.

For, if *D* is not the greatest common measure of *A*, *B*, *C*, some number which is greater than *D* will measure the numbers *A*, *B*, *C*.

Let such a number measure them, and let it be *E*.

Since then *E* measures *A*, *B*, *C*,

it will also measure *A*, *B*;

therefore it will also measure the greatest common measure of *A*, *B*. [VII. 2, Por.]

But the greatest common measure of *A*, *B* is *D*;

therefore *E* measures *D*, the greater the less: which is impossible.

Therefore no number which is greater than *D* will measure the numbers *A*, *B*, *C*;

therefore *D* is the greatest common measure of *A*, *B*, *C*.

Next, let *D* not measure *C*;

I say first that *C*, *D* are not prime to one another.

For, since *A*, *B*, *C* are not prime to one another, some number will measure them.

Now that which measures *A*, *B*, *C* will also measure *A*, *B*, and will measure *D*, the greatest common measure of *A*, *B*. [VII. 2, Por.]

But it measures *C* also;

therefore some number will measure the numbers *D*, *C*;

therefore *D*, *C* are not prime to one another.

Let then their greatest common measure *E* be taken. [VII. 2]

Then, since *E* measures *D*,

and *D* measures *A*, *B*,

therefore *E* also measures *A*, *B*.

But it measures *C* also;

therefore *E* measures *A*, *B*, *C*;

therefore *E* is a common measure of *A*, *B*, *C*.

I say next that it is also the greatest.

For, if *E* is not the greatest common measure of *A*, *B*, *C*, some number which is greater than *E* will measure the numbers *A*, *B*, *C*.

Let such a number measure them, and let it be *F*.

Now, since *F* measures *A*, *B*, *C*,
it also measures *A*, *B*;
therefore it will also measure the greatest common measure of *A*, *B*. [VII. 2, Por.]

But the greatest common measure of *A*, *B* is *D*;
therefore *F* measures *D*.

And it measures *C* also;
therefore *F* measures *D*, *C*;
therefore it will also measure the greatest common measure of *D*, *C*. [VII. 2, Por.]

But the greatest common measure of *D*, *C* is *E*;
therefore *F* measures *E*, the greater the less: which is impossible.

Therefore no number which is greater than *E* will measure the numbers *A*, *B*, *C*;
therefore *E* is the greatest common measure of *A*, *B*, *C*.

Q. E. D.

Euclid's proof is here longer than we should make it because he distinguishes two cases, the simpler of which is really included in the other.

Having taken the greatest common measure, say d, of a, b, two of the three given numbers a, b, c, he distinguishes the cases

(1) in which d measures c,

(2) in which d does not measure c.

In the first case the greatest common measure of d, c is d itself; in the second case it has to be found by a repetition of the process of VII. 2. In either case the greatest common measure of a, b, c is the greatest common measure of d, c.

But, after disposing of the simpler case, Euclid thinks it necessary to prove that, if d does not measure c, d and c must necessarily *have* a greatest common measure. This he does by means of the original hypothesis that a, b, c are not prime to one another. Since they are not prime to one another, they must have a common measure; any common measure of a, b is a measure of d, and therefore any common measure of a, b, c is a common measure of d, c; hence d, c must have a common measure, and are therefore not prime to one another.

The proofs of cases (1) and (2) repeat exactly the same argument as we saw in VII. 2, and it is proved separately for d in case (1) and e in case (2), where e is the greatest common measure of d, c,

(α) that it is a common measure of a, b, c,

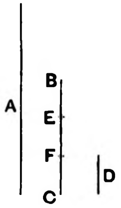
(β) that it is the *greatest* common measure.

Heron remarks (an-Nairīzī, ed. Curtze, p. 191) that the method does not only enable us to find the greatest common measure of *three* numbers; it can be used to find the greatest common measure of as many numbers as we please. This is because any number measuring two numbers also measures their greatest common measure; and hence we can find the G.C.M. of pairs, then the G.C.M. of pairs of these, and so on, until only two numbers are left and we find the G.C.M. of these. Euclid tacitly assumes this extension in VII. 33, where he takes the greatest common measure of *as many numbers as we please.*

PROPOSITION 4.

Any number is either a part or parts of any number, the less of the greater.

Let A, BC be two numbers, and let BC be the less; I say that BC is either a part, or parts, of A.



For A, BC are either prime to one another or not.

First, let A, BC be prime to one another.

Then, if BC be divided into the units in it, each unit of those in BC will be some part of A; so that BC is parts of A.

Next let A, BC not be prime to one another; then BC either measures, or does not measure, A.

If now BC measures A, BC is a part of A.

But, if not, let the greatest common measure D of A, BC be taken; [VII. 2]

and let BC be divided into the numbers equal to D, namely BE, EF, FC.

Now, since D measures A, D is a part of A.

But D is equal to each of the numbers BE, EF, FC; therefore each of the numbers BE, EF, FC is also a part of A; so that BC is parts of A.

Therefore etc.

Q. E. D.

The meaning of the enunciation is of course that, if a, b be two numbers of which b is the less, then b is either a *submultiple* or *some proper fraction* of a.

(1) If a, b are prime to one another, divide each into its units; then b contains b of the same parts of which a contains a. Therefore b is "parts" or a *proper fraction* of a.

(2) If a, b be not prime to one another, either b measures a, in which case b is a submultiple or "part" of a, or, if g be the greatest common measure of a, b, we may put $a = mg$ and $b = ng$, and b will contain n of the same parts (g) of which a contains m, so that b is again "parts," or a *proper fraction*, of a.

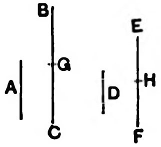
PROPOSITION 5.

If a number be a part of a number, and another be the same part of another, the sum will also be the same part of the sum that the one is of the one.

For let the number A be a part of BC,

and another, D, the same part of another EF that A is of BC;

I say that the sum of A, D is also the same part of the sum of BC, EF that A is of BC.



For since, whatever part A is of BC, D is also the same part of EF,

therefore, as many numbers as there are in BC equal to A, so many numbers are there also in EF equal to D.

Let BC be divided into the numbers equal to A, namely BG, GC,

and EF into the numbers equal to D, namely EH, HF;

then the multitude of BG, GC will be equal to the multitude of EH, HF.

And, since BG is equal to A, and EH to D,

therefore BG, EH are also equal to A, D.

For the same reason

GC, HF are also equal to A, D.

Therefore, as many numbers as there are in BC equal to A, so many are there also in BC, EF equal to A, D.

Therefore, whatever multiple BC is of A, the same multiple also is the sum of BC, EF of the sum of A, D.

Therefore, whatever part A is of BC, the same part also is the sum of A, D of the sum of BC, EF.

Q. E. D.

If $a = \frac{1}{n} b$, and $c = \frac{1}{n} d$, then

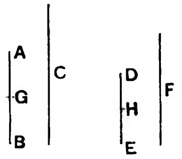
$$a + c = \frac{1}{n}(b + d).$$

The proposition is of course true for any quantity of pairs of numbers similarly related, as is the next proposition also; and both propositions are used in the extended form in VII. 9, 10.

PROPOSITION 6.

If a number be parts of a number, and another be the same parts of another, the sum will also be the same parts of the sum that the one is of the one.

For let the number AB be parts of the number C, and another, DE, the same parts of another, F, that AB is of C;



I say that the sum of AB, DE is also the same parts of the sum of C, F that AB is of C.

For since, whatever parts AB is of C, DE is also the same parts of F, therefore, as many parts of C as there are in AB, so many parts of F are there also in DE.

Let AB be divided into the parts of C, namely AG, GB, and DE into the parts of F, namely DH, HE; thus the multitude of AG, GB will be equal to the multitude of DH, HE.

And since, whatever part AG is of C, the same part is DH of F also, therefore, whatever part AG is of C, the same part also is the sum of AG, DH of the sum of C, F. [VII. 5]

For the same reason, whatever part GB is of C, the same part also is the sum of GB, HE of the sum of C, F.

Therefore, whatever parts AB is of C, the same parts also is the sum of AB, DE of the sum of C, F.

Q. E. D.

If $$a = \frac{m}{n} b, \text{ and } c = \frac{m}{n} d,$$

then $$a + c = \frac{m}{n} (b + d).$$

More generally, if

$$a = \frac{m}{n} b, \; c = \frac{m}{n} d, \; e = \frac{m}{n} f, \; \dots$$

then $$(a + c + e + g + \dots) = \frac{m}{n} (b + d + f + h + \dots).$$

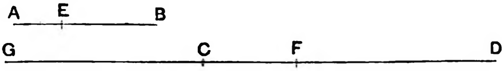
In Euclid's proposition $m < n$, but the generality of the result is of course not affected. This proposition and the last are complementary to V. 1, which proves the corresponding result with *multiple* substituted for "*part*" or "*parts*."

PROPOSITION 7.

If a number be that part of a number, which a number subtracted is of a number subtracted, the remainder will also be the same part of the remainder that the whole is of the whole.

For let the number *AB* be that part of the number *CD* which *AE* subtracted is of *CF* subtracted;

I say that the remainder *EB* is also the same part of the remainder *FD* that the whole *AB* is of the whole *CD*.



For, whatever part *AE* is of *CF*, the same part also let *EB* be of *CG*.

Now since, whatever part *AE* is of *CF*, the same part also is *EB* of *CG*,

therefore, whatever part *AE* is of *CF*, the same part also is *AB* of *GF*. [VII. 5]

But, whatever part *AE* is of *CF*, the same part also, by hypothesis, is *AB* of *CD*;

therefore, whatever part *AB* is of *GF*, the same part is it of *CD* also;

therefore *GF* is equal to *CD*.

Let *CF* be subtracted from each;

therefore the remainder *GC* is equal to the remainder *FD*.

Now since, whatever part *AE* is of *CF*, the same part also is *EB* of *GC*,

while *GC* is equal to *FD*,

therefore, whatever part *AE* is of *CF*, the same part also is *EB* of *FD*.

But, whatever part *AE* is of *CF*, the same part also is *AB* of *CD*;

therefore also the remainder *EB* is the same part of the remainder *FD* that the whole *AB* is of the whole *CD*.

Q. E. D.

If $a = \frac{1}{n} b$ and $c = \frac{1}{n} d$, we are to prove that

$$a - c = \frac{1}{n}(b - d),$$

a result differing from that of VII. 5 in that *minus* is substituted for *plus*. Euclid's method is as follows.

Suppose that e is taken such that

$$a - c = \frac{1}{n} e. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Now $$c = \frac{1}{n} d.$$

Therefore $$a = \frac{1}{n}(d + e), \qquad [\text{VII. } 5]$$

whence, from the hypothesis, $d + e = b$,

so that $e = b - d$,

and, substituting this value of e in (1), we have

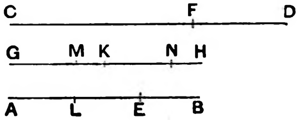
$$a - c = \frac{1}{n}(b - d).$$

PROPOSITION 8.

If a number be the same parts of a number that a number subtracted is of a number subtracted, the remainder will also be the same parts of the remainder that the whole is of the whole.

For let the number *AB* be the same parts of the number *CD* that *AE* subtracted is of *CF* subtracted;

I say that the remainder *EB* is also the same parts of the remainder *FD* that the whole *AB* is of the whole *CD*.



For let *GH* be made equal to *AB*.

Therefore, whatever parts *GH* is of *CD*, the same parts also is *AE* of *CF*.

Let *GH* be divided into the parts of *CD*, namely *GK*, *KH*, and *AE* into the parts of *CF*, namely *AL*, *LE*;

thus the multitude of *GK*, *KH* will be equal to the multitude of *AL*, *LE*.

Now since, whatever part *GK* is of *CD*, the same part also is *AL* of *CF*,

while *CD* is greater than *CF*,

therefore *GK* is also greater than *AL*.

Let *GM* be made equal to *AL*.

Therefore, whatever part *GK* is of *CD*, the same part also is *GM* of *CF*;

therefore also the remainder *MK* is the same part of the remainder *FD* that the whole *GK* is of the whole *CD*. [VII. 7]

Again, since, whatever part *KH* is of *CD*, the same part also is *EL* of *CF*,

while *CD* is greater than *CF*,

therefore *HK* is also greater than *EL*.

Let *KN* be made equal to *EL*.

Therefore, whatever part *KH* is of *CD*, the same part also is *KN* of *CF*;

therefore also the remainder *NH* is the same part of the remainder *FD* that the whole *KH* is of the whole *CD*. [VII. 7]

But the remainder *MK* was also proved to be the same part of the remainder *FD* that the whole *GK* is of the whole *CD*;

therefore also the sum of *MK*, *NH* is the same parts of *DF* that the whole *HG* is of the whole *CD*.

But the sum of *MK*, *NH* is equal to *EB*,

and *HG* is equal to *BA*;

therefore the remainder *EB* is the same parts of the remainder *FD* that the whole *AB* is of the whole *CD*.

Q. E. D.

If $$a = \frac{m}{n} b \text{ and } c = \frac{m}{n} d, \qquad (m < n)$$

then $$a - c = \frac{m}{n}(b - d).$$

Euclid's proof amounts to the following.

Take e equal to $\frac{1}{n} b$, and f equal to $\frac{1}{n} d$.

Then since, by hypothesis, $b > d$,

$$e > f,$$

and, by VII. 7, $$e - f = \frac{1}{n}(b - d).$$

Repeat this for all the parts equal to e and f that there are in a, b respectively, and we have, by addition (a, b containing m of such parts respectively),

$$m(e-f)=\frac{m}{n}(b-d).$$

But $$m(e-f)=a-c.$$

Therefore $$a-c=\frac{m}{n}(b-d).$$

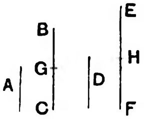
The propositions VII. 7, 8 are complementary to V. 5 which gives the corresponding result with *multiple* in the place of "part" or "parts."

PROPOSITION 9.

If a number be a part of a number, and another be the same part of another, alternately also, whatever part or parts the first is of the third, the same part, or the same parts, will the second also be of the fourth.

For let the number *A* be a part of the number *BC*, and another, *D*, the same part of another, *EF*, that *A* is of *BC*;

I say that, alternately also, whatever part or parts *A* is of *D*, the same part or parts is *BC* of *EF* also.



For since, whatever part *A* is of *BC*, the same part also is *D* of *EF*,

therefore, as many numbers as there are in *BC* equal to *A*, so many also are there in *EF* equal to *D*.

Let *BC* be divided into the numbers equal to *A*, namely *BG*, *GC*,

and *EF* into those equal to *D*, namely *EH*, *HF*;

thus the multitude of *BG*, *GC* will be equal to the multitude of *EH*, *HF*.

Now, since the numbers *BG*, *GC* are equal to one another, and the numbers *EH*, *HF* are also equal to one another,

while the multitude of *BG*, *GC* is equal to the multitude of *EH*, *HF*,

therefore, whatever part or parts *BG* is of *EH*, the same part or the same parts is *GC* of *HF* also;

so that, in addition, whatever part or parts *BG* is of *EH*, the same part also, or the same parts, is the sum *BC* of the sum *EF*. [VII. 5, 6]

But *BG* is equal to *A*, and *EH* to *D*;
therefore, whatever part or parts *A* is of *D*, the same part or the same parts is *BC* of *EF* also.

Q. E. D.

If $a = \frac{1}{n} b$ and $c = \frac{1}{n} d$, then, whatever fraction ("part" or "parts") a is of c, the same fraction will b be of d.

Dividing b into each of its parts equal to a, and d into each of its parts equal to c, it is clear that, whatever fraction one of the parts a is of one of the parts c, the same fraction is any other of the parts a of any other of the parts c.

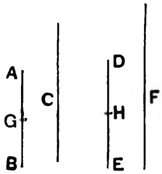
And the number of the parts a is equal to the number of the parts c, viz. n.

Therefore, by VII. 5, 6, na is the same fraction of nc that a is of c, i.e. b is the same fraction of d that a is of c.

PROPOSITION 10.

If a number be parts of a number, and another be the same parts of another, alternately also, whatever parts or part the first is of the third, the same parts or the same part will the second also be of the fourth.

For let the number *AB* be parts of the number *C*, and another, *DE*, the same parts of another, *F*;



I say that, alternately also, whatever parts or part *AB* is of *DE*, the same parts or the same part is *C* of *F* also.

For since, whatever parts *AB* is of *C*, the same parts also is *DE* of *F*,
therefore, as many parts of *C* as there are in *AB*, so many parts also of *F* are there in *DE*.

Let *AB* be divided into the parts of *C*, namely *AG*, *GB*, and *DE* into the parts of *F*, namely *DH*, *HE*;
thus the multitude of *AG*, *GB* will be equal to the multitude of *DH*, *HE*.

Now since, whatever part *AG* is of *C*, the same part also is *DH* of *F*,
alternately also, whatever part or parts *AG* is of *DH*,
the same part or the same parts is *C* of *F* also. [VII. 9]

For the same reason also,
whatever part or parts *GB* is of *HE*, the same part or the same parts is *C* of *F* also;

so that, in addition, whatever parts or part *AB* is of *DE*, the same parts also, or the same part, is *C* of *F*. [VII. 5, 6]

Q. E. D.

If $a = \frac{m}{n} b$ and $c = \frac{m}{n} d$, then, whatever fraction a is of c, the same fraction is b of d.

To prove this, a is divided into its m parts equal to b/n, and c into its m parts equal to d/n.

Then, by VII. 9, whatever fraction one of the m parts of a is of one of the m parts of c, the same fraction is b of d.

And, by VII. 5, 6, whatever fraction one of the m parts of a is of one of the m parts of c, the same fraction is the sum of the parts of a (that is, a) of the sum of the parts of c (that is, c).

Whence the result follows.

In the Greek text, after the words "so that, in addition" in the last line but one, is an additional explanation making the reference to VII. 5, 6 clearer, as follows: "whatever part or parts *AG* is of *DH*, the same part or the same parts is *GB* of *HE* also;

therefore also, whatever part or parts *AG* is of *DH*, the same part or the same parts is *AB* of *DE* also. [VII. 5, 6]

But it was proved that, whatever part or parts *AG* is of *DH*, the same part or the same parts is *C* of *F* also;

therefore also" etc. as in the last two lines of the text.

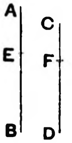
Heiberg concludes, on the authority of P, which only has the words in the margin in a later hand, that they may be attributed to Theon.

PROPOSITION 11.

If, as whole is to whole, so is a number subtracted to a number subtracted, the remainder will also be to the remainder as whole to whole.

As the whole *AB* is to the whole *CD*, so let *AE* subtracted be to *CF* subtracted;

I say that the remainder *EB* is also to the remainder *FD* as the whole *AB* to the whole *CD*.



Since, as *AB* is to *CD*, so is *AE* to *CF*, whatever part or parts *AB* is of *CD*, the same part or the same parts is *AE* of *CF* also; [VII. Def. 20]

Therefore also the remainder *EB* is the same part or parts of *FD* that *AB* is of *CD*. [VII. 7, 8]

Therefore, as *EB* is to *FD*, so is *AB* to *CD*. [VII. Def. 20]

Q. E. D.

It will be observed that, in dealing with the proportions in Props. 11—13, Euclid only contemplates the case where the first number is "a part" or "parts" of the second, while in Prop. 13 he assumes the first to be "a part" or "parts" of the third also; that is, the first number is in all three propositions assumed to be less than the second, and in Prop. 13 less than the third also. Yet the figures in Props. 11 and 13 are inconsistent with these assumptions. If the facts are taken to correspond to the figures in these propositions, it is necessary to take account of the other possibilities involved in the definition of proportion (VII. Def. 20), that the first number may also be a multiple, or a multiple *plus* "a part" or "parts" (including *once* as a multiple in this case), of each number with which it is compared. Thus a number of different cases would have to be considered. The remedy is to make the ratio which is in the lower terms the first ratio, and to invert the ratios, if necessary, in order to make "a part" or "parts" literally apply.

If $a : b = c : d$, $(a > c, b > d)$

then $(a - c) : (b - d) = a : b$.

This proposition for numbers corresponds to V. 19 for magnitudes. The enunciation is the same except that the masculine (agreeing with ἀριθμός) takes the place of the neuter (agreeing with μέγεθος).

The proof is no more than a combination of the arithmetical definition of proportion (VII. Def. 20) with the results of VII. 7, 8. The language of proportions is turned into the language of fractions by Def. 20; the results of VII. 7, 8 are then used and the language retransformed by Def. 20 into the language of proportions.

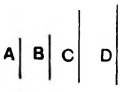
PROPOSITION 12.

If there be as many numbers as we please in proportion, then, as one of the antecedents is to one of the consequents, so are all the antecedents to all the consequents.

Let A, B, C, D be as many numbers as we please in proportion, so that,

as A is to B, so is C to D;

I say that, as A is to B, so are A, C to B, D.



For since, as A is to B, so is C to D, whatever part or parts A is of B, the same part or parts is C of D also. [VII. Def. 20]

Therefore also the sum of A, C is the same part or the same parts of the sum of B, D that A is of B. [VII. 5, 6]

Therefore, as A is to B, so are A, C to B, D. [VII. Def. 20]

If $a : a' = b : b' = c : c' = \ldots$,

then each ratio is equal to $(a + b + c + \ldots) : (a' + b' + c' + \ldots)$.

The proposition corresponds to V. 12, and the enunciation is word for word the same with that of V. 12 except that ἀριθμός takes the place of μέγεθος.

Again the proof merely connects the arithmetical definition of proportion (VII. Def. 20) with the results of VII. 5, 6, which are quoted as true for any number of numbers, and not merely for two numbers as in the enunciations of VII. 5, 6.

Proposition 13.

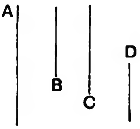
If four numbers be proportional, they will also be proportional alternately.

Let the four numbers *A*, *B*, *C*, *D* be proportional, so that,

as *A* is to *B*, so is *C* to *D*;

I say that they will also be proportional alternately, so that,

as *A* is to *C*, so will *B* be to *D*.



For since, as *A* is to *B*, so is *C* to *D*, therefore, whatever part or parts *A* is of *B*, the same part or the same parts is *C* of *D* also. [vii. Def. 20]

Therefore, alternately, whatever part or parts *A* is of *C*, the same part or the same parts is *B* of *D* also. [vii. 10]

Therefore, as *A* is to *C*, so is *B* to *D*. [vii. Def. 20]

Q. E. D.

If $a : b = c : d$,

then, alternately, $a : c = b : d$.

The proposition corresponds to v. 16 for magnitudes, and the proof consists in connecting vii. Def. 20 with the result of vii. 10.

Proposition 14.

If there be as many numbers as we please, and others equal to them in multitude, which taken two and two are in the same ratio, they will also be in the same ratio ex aequali.

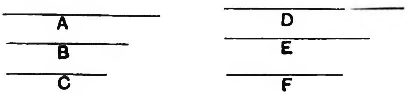
Let there be as many numbers as we please *A*, *B*, *C*, and others equal to them in multitude *D*, *E*, *F*, which taken two and two are in the same ratio, so that,

as *A* is to *B*, so is *D* to *E*,

and, as *B* is to *C*, so is *E* to *F*;

I say that, *ex aequali*,

as *A* is to *C*, so also is *D* to *F*.



For, since, as *A* is to *B*, so is *D* to *E*, therefore, alternately,

as *A* is to *D*, so is *B* to *E*. [vii. 13]

Again, since, as *B* is to *C*, so is *E* to *F*,
therefore, alternately,
as *B* is to *E*, so is *C* to *F*. [VII. 13]

But, as *B* is to *E*, so is *A* to *D*;
therefore also, as *A* is to *D*, so is *C* to *F*.

Therefore, alternately,
as *A* is to *C*, so is *D* to *F*. [*id.*]

If $a : b = d : e$,
and $b : c = e : f$,
then, *ex aequali*, $a : c = d : f$;
and the same is true however many successive numbers are so related.

The proof is simplicity itself.

By VII. 13, alternately, $a : d = b : e$,
and $b : e = c : f$.
Therefore $a : d = c : f$,
and, again alternately, $a : c = d : f$.

Observe that this simple method cannot be used to prove the corresponding proposition for magnitudes, V. 22, although V. 22 has been preceded by the two propositions in that Book corresponding to the propositions used here, viz. V. 16 and V. 11. The reason of this is that this method would only prove V. 22 for six magnitudes *all of the same kind*, whereas the magnitudes in V. 22 are not subject to this limitation.

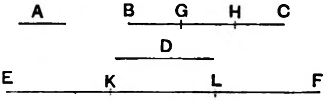
Heiberg remarks in a note on VII. 19 that, while Euclid has proved several propositions of Book V. over again, by a separate proof, for numbers, he has neglected to do so in certain cases; e.g., he often uses V. 11 in these propositions of Book VII., V. 9 in VII. 19, V. 7 in the same proposition, and so on. Thus Heiberg would apparently suppose Euclid to use V. 11 in the last step of the present proof (*Ratios which are the same with the same ratio are also the same with one another*). I think it preferable to suppose that Euclid regarded the last step as axiomatic; since, by the definition of proportion, the first number is the same multiple or the same part or the same parts of the second that the third is of the fourth: the assumption is no more than an assumption that the numbers or proper fractions which are respectively equal to the same number or proper fraction are equal to one another.

Though the proposition is only proved of six numbers, the extension to as many as we please (as expressed in the enunciation) is obvious.

PROPOSITION 15.

If an unit measure any number, and another number measure any other number the same number of times, alternately also, the unit will measure the third number the same number of times that the second measures the fourth.

For let the unit A measure any number BC, and let another number D measure any other number EF the same number of times;



I say that, alternately also, the unit A measures the number D the same number of times that BC measures EF.

For, since the unit A measures the number BC the same number of times that D measures EF,

therefore, as many units as there are in BC, so many numbers equal to D are there in EF also.

Let BC be divided into the units in it, BG, GH, HC,

and EF into the numbers EK, KL, LF equal to D.

Thus the multitude of BG, GH, HC will be equal to the multitude of EK, KL, LF.

And, since the units BG, GH, HC are equal to one another,

and the numbers EK, KL, LF are also equal to one another,

while the multitude of the units BG, GH, HC is equal to the multitude of the numbers EK, KL, LF,

therefore, as the unit BG is to the number EK, so will the unit GH be to the number KL, and the unit HC to the number LF.

Therefore also, as one of the antecedents is to one of the consequents, so will all the antecedents be to all the consequents; [VII. 12]

therefore, as the unit BG is to the number EK, so is BC to EF.

But the unit BG is equal to the unit A,

and the number EK to the number D.

Therefore, as the unit A is to the number D, so is BC to EF.

Therefore the unit A measures the number D the same number of times that BC measures EF. Q. E. D.

If there be four numbers 1, m, a, ma (such that 1 measures m the same number of times that a measures ma), 1 measures a the same number of times that m measures ma.

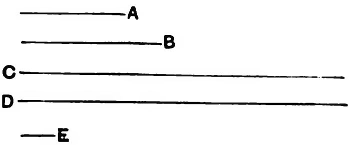
Except that the first number is unity and the numbers are said to *measure* instead of being a *part* of others, this proposition and its proof do not differ from VII. 9; in fact this proposition is a particular case of the other.

PROPOSITION 16.

If two numbers by multiplying one another make certain numbers, the numbers so produced will be equal to one another.

Let A, B be two numbers, and let A by multiplying B make C, and B by multiplying A make D;

I say that C is equal to D.



For, since A by multiplying B has made C,

therefore B measures C according to the units in A.

But the unit E also measures the number A according to the units in it;

therefore the unit E measures A the same number of times that B measures C.

Therefore, alternately, the unit E measures the number B the same number of times that A measures C. [VII. 15]

Again, since B by multiplying A has made D,

therefore A measures D according to the units in B.

But the unit E also measures B according to the units in it;

therefore the unit E measures the number B the same number of times that A measures D.

But the unit E measured the number B the same number of times that A measures C;

therefore A measures each of the numbers C, D the same number of times.

Therefore C is equal to D. Q. E. D.

2. **The numbers so produced.** The Greek has οἱ γενόμενοι ἐξ αὐτῶν, "the (numbers) produced *from them*." By "from them" Euclid means "from the original numbers," though this is not very clear even in the Greek. I think ambiguity is best avoided by leaving out the words.

This proposition proves that, *if any numbers be multiplied together, the order of multiplication is indifferent*, or $ab = ba$.

It is important to get a clear understanding of what Euclid means when he speaks of *one number multiplying another*. VII. Def. 15 states that the effect of "a multiplying b" is taking a times b. We shall always represent "a times b" by ab and "b times a" by ba. This being premised, the proof that $ab = ba$ may be represented as follows in the language of proportions.

By VII. Def. 20, $1 : a = b : ab$.

Therefore, alternately, $1 : b = a : ab$. [VII. 13]

Again, by VII. Def. 20, $1 : b = a : ba$.

Therefore $a : ab = a : ba$,

or $ab = ba$.

Euclid does not use the language of proportions but that of fractions or their equivalent measures, quoting VII. 15, a particular case of VII. 13 differently expressed, instead of VII. 13 itself.

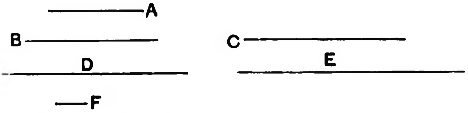
PROPOSITION 17.

If a number by multiplying two numbers make certain numbers, the numbers so produced will have the same ratio as the numbers multiplied.

For let the number A by multiplying the two numbers B, C make D, E;

I say that, as B is to C, so is D to E.

For, since A by multiplying B has made D, therefore B measures D according to the units in A.



But the unit F also measures the number A according to the units in it;

therefore the unit F measures the number A the same number of times that B measures D.

Therefore, as the unit F is to the number A, so is B to D. [VII. Def. 20]

For the same reason,

as the unit F is to the number A, so also is C to E;

therefore also, as B is to D, so is C to E.

Therefore, alternately, as B is to C, so is D to E. [VII. 13]

Q. E. D.

$$b : c = ab : ac.$$

In this case Euclid translates the language of measures into that of proportions, and the proof is exactly like that set out in the last note.

By VII. Def. 20, $1 : a = b : ab$,

and $1 : a = c : ac$.

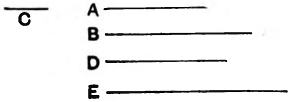
Therefore $b : ab = c : ac$,

and, alternately, $b : c = ab : ac$. [VII. 13]

PROPOSITION 18.

If two numbers by multiplying any number make certain numbers, the numbers so produced will have the same ratio as the multipliers.

For let two numbers *A*, *B* by multiplying any number *C* make *D*, *E*;



I say that, as *A* is to *B*, so is *D* to *E*.

For, since *A* by multiplying *C* has made *D*,
therefore also *C* by multiplying *A* has made *D*. [VII. 16]
For the same reason also
C by multiplying *B* has made *E*.

Therefore the number *C* by multiplying the two numbers *A*, *B* has made *D*, *E*.

Therefore, as *A* is to *B*, so is *D* to *E*. [VII. 17]

It is here proved that $a : b = ac : bc$.
The argument is as follows.
$ac = ca$. [VII. 16]
Similarly $bc = cb$.
And $a : b = ca : cb$; [VII. 17]
therefore $a : b = ac : bc$.

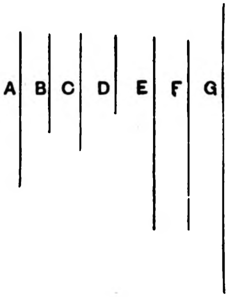
PROPOSITION 19.

If four numbers be proportional, the number produced from the first and fourth will be equal to the number produced from the second and third; and, if the number produced from the first and fourth be equal to that produced from the second and third, the four numbers will be proportional.

Let *A*, *B*, *C*, *D* be four numbers in proportion, so that,
as *A* is to *B*, so is *C* to *D*;
and let *A* by multiplying *D* make *E*, and let *B* by multiplying *C* make *F*;
I say that *E* is equal to *F*.

For let *A* by multiplying *C* make *G*.

Since, then, A by multiplying C has made G, and by multiplying D has made E,
the number A by multiplying the two numbers C, D has made G, E.



Therefore, as C is to D, so is G to E. [VII. 17]

But, as C is to D, so is A to B;
therefore also, as A is to B, so is G to E.

Again, since A by multiplying C has made G,
but, further, B has also by multiplying C made F,
the two numbers A, B by multiplying a certain number C have made G, F.

Therefore, as A is to B, so is G to F. [VII. 18]

But further, as A is to B, so is G to E also;
therefore also, as G is to E, so is G to F.

Therefore G has to each of the numbers E, F the same ratio;
therefore E is equal to F. [cf. V. 9]

Again, let E be equal to F;
I say that, as A is to B, so is C to D.

For, with the same construction,
since E is equal to F,
therefore, as G is to E, so is G to F. [cf. V. 7]

But, as G is to E, so is C to D, [VII. 17]
and, as G is to F, so is A to B. [VII. 18]

Therefore also, as A is to B, so is C to D.

Q. E. D.

If $a : b = c : d$,
then $ad = bc$; and conversely.

The proof is equivalent to the following.

(1) $ac : ad = c : d$ [VII. 17]
$= a : b$.

But $a : b = ac : bc$. [VII. 18]

Therefore $ac : ad = ac : bc$,

or $ad = bc$.

(2) Since $ad = bc$,

$ac : ad = ac : bc$.

But $ac : ad = c : d$, [VII. 17]

and $ac : bc = a : b$. [VII. 18]

Therefore $a : b = c : d$.

As indicated in the note on VII. 14 above, Heiberg regards Euclid as basing the inferences contained in the last step of part (1) of this proof and in the first step of part (2) on the propositions V. 9 and V. 7 respectively, since he has not proved those propositions separately for numbers in this Book. I prefer to suppose that he regarded the inferences as obvious and not needing proof, in view of the definition of numbers which are in proportion. E.g., if ac is the same fraction ("part" or "parts") of ad that ac is of bc, it is obvious that ad must be equal to bc.

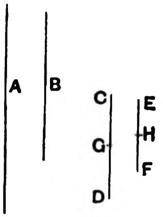
Heiberg omits from his text here, and relegates to an Appendix, a proposition appearing in the manuscripts V, p, ϕ to the effect that, if *three* numbers be proportional, the product of the extremes is equal to the square of the mean, and conversely. It does not appear in P in the first hand, B has it in the margin only, and Campanus omits it, remarking that Euclid does not give the proposition about *three* proportionals as he does in VI. 17, since it is easily proved by the proposition just given. Moreover an-Nairīzī quotes the proposition about three proportionals *as an observation on* VII. 19 probably due to Heron (who is mentioned by name in the preceding paragraph).

PROPOSITION 20.

The least numbers of those which have the same ratio with them measure those which have the same ratio the same number of times, the greater the greater and the less the less.

For let *CD*, *EF* be the least numbers of those which have the same ratio with *A*, *B*;

I say that *CD* measures *A* the same number of times that *EF* measures *B*.



Now *CD* is not parts of *A*.

For, if possible, let it be so;

therefore *EF* is also the same parts of *B* that *CD* is of *A*. [VII. 13 and Def. 20]

Therefore, as many parts of *A* as there are in *CD*, so many parts of *B* are there also in *EF*.

Let *CD* be divided into the parts of *A*, namely *CG*, *GD*, and *EF* into the parts of *B*, namely *EH*, *HF*;

thus the multitude of *CG*, *GD* will be equal to the multitude of *EH*, *HF*.

Now, since the numbers *CG*, *GD* are equal to one another, and the numbers *EH*, *HF* are also equal to one another,

while the multitude of *CG*, *GD* is equal to the multitude of *EH*, *HF*,

therefore, as *CG* is to *EH*, so is *GD* to *HF*.

Therefore also, as one of the antecedents is to one of the consequents, so will all the antecedents be to all the consequents. [VII. 12]

Therefore, as *CG* is to *EH*, so is *CD* to *EF*.

Therefore *CG*, *EH* are in the same ratio with *CD*, *EF*, being less than they:

which is impossible, for by hypothesis *CD*, *EF* are the least numbers of those which have the same ratio with them.

Therefore *CD* is not parts of *A*;

therefore it is a part of it. [VII. 4]

And *EF* is the same part of *B* that *CD* is of *A*; [VII. 13 and Def. 20]

therefore *CD* measures *A* the same number of times that *EF* measures *B*.

Q. E. D.

If a, b are the least numbers among those which have the same ratio (i.e. if a/b is a fraction in its lowest terms), and c, d are any others in the same ratio, i.e. if

$$a : b = c : d,$$

then $a = \frac{1}{n} c$ and $b = \frac{1}{n} d$, where n is some integer.

The proof is by *reductio ad absurdum*, thus.

[Since $a < c$, a is some proper fraction ("part" or "parts") of c, by VII. 4.]

Now a cannot be equal to $\frac{m}{n} c$, where m is an integer less than n but greater than 1.

For, if $a = \frac{m}{n} c$, $b = \frac{m}{n} d$ also. [VII. 13 and Def. 20]

Take each of the m parts of a with each of the m parts of b, two and two; the ratio of the members of all pairs is the same ratio $\frac{1}{m} a : \frac{1}{m} b$.

Therefore

$$\frac{1}{m} a : \frac{1}{m} b = a : b. \qquad \text{[VII. 12]}$$

But $\frac{1}{m} a$ and $\frac{1}{m} b$ are respectively less than a, b and they are in the same ratio: which contradicts the hypothesis.

Hence a can only be "a part" of c, or

$$a \text{ is of the form } \frac{1}{n}c,$$

and therefore $$b \text{ is of the form } \frac{1}{n}d.$$

Here also Heiberg omits a proposition which was no doubt interpolated by Theon (B, V, p, φ have it as VII. 22, but P only has it in the margin and in a later hand; Campanus also omits it) proving for numbers the *ex aequali* proposition when "the proportion is perturbed," i.e. (cf. enunciation of V. 22) if

$$a : b = e : f, \quad \text{..............................(1)}$$

and $$b : c = d : e, \quad \text{..............................(2)}$$

then $$a : c = d : f.$$

The proof (see Heiberg's Appendix) depends on VII. 19.

From (1) we have $af = be$,

and from (2) $be = cd$. [VII. 19]

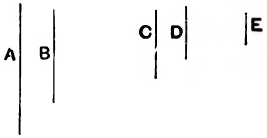
Therefore $af = cd$,

and accordingly $a : c = d : f$. [VII. 19]

PROPOSITION 21.

Numbers prime to one another are the least of those which have the same ratio with them.

Let A, B be numbers prime to one another; I say that A, B are the least of those which have the same ratio with them.



For, if not, there will be some numbers less than A, B which are in the same ratio with A, B.

Let them be C, D.

Since, then, the least numbers of those which have the same ratio measure those which have the same ratio the same number of times, the greater the greater and the less the less, that is, the antecedent the antecedent and the consequent the consequent, [VII. 20]

therefore C measures A the same number of times that D measures B.

Now, as many times as C measures A, so many units let there be in E.

Therefore D also measures B according to the units in E.

And, since C measures A according to the units in E, therefore E also measures A according to the units in C. [vii. 16]

For the same reason

E also measures B according to the units in D. [vii. 16]

Therefore E measures A, B which are prime to one another: which is impossible. [vii. Def. 12]

Therefore there will be no numbers less than A, B which are in the same ratio with A, B.

Therefore A, B are the least of those which have the same ratio with them.

Q. E. D.

In other words, if a, b are prime to one another, the ratio $a : b$ is "in its lowest terms."

The proof is equivalent to the following.

If not, suppose that c, d are the *least* numbers for which

$$a : b = c : d.$$

[Euclid only supposes *some* numbers c, d in the ratio of a to b such that $c < a$, and (consequently) $d < b$. It is however necessary to suppose that c, d are the *least* numbers in that ratio in order to enable vii. 20 to be used in the proof.]

Then [vii. 20] $a = mc$, and $b = md$, where m is some integer.

Therefore $$a = cm, \quad b = dm,$$ [vii. 16]

and m is a common measure of a, b, though these are prime to one another. which is impossible. [vii. Def. 12]

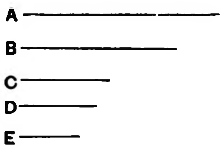
Thus the least numbers in the ratio of a to b cannot be less than a, b themselves.

Where I have quoted vii. 16 Heiberg regards the reference as being to vii. 15. I think the phraseology of the text combined with that of Def. 15 suggests the former rather than the latter.

Proposition 22.

The least numbers of those which have the same ratio with them are prime to one another.

Let A, B be the least numbers of those which have the same ratio with them;



I say that A, B are prime to one another.

For, if they are not prime to one another, some number will measure them.

Let some number measure them, and let it be C.

And, as many times as C measures A, so many units let there be in D,

and, as many times as C measures B, so many units let there be in E

Since C measures A according to the units in D, therefore C by multiplying D has made A. [VII. Def. 15]

For the same reason also

C by multiplying E has made B.

Thus the number C by multiplying the two numbers D, E has made A, B;

therefore, as D is to E, so is A to B; [VII. 17]

therefore D, E are in the same ratio with A, B, being less than they: which is impossible.

Therefore no number will measure the numbers A, B.

Therefore A, B are prime to one another.

Q. E. D.

If $a : b$ is "in its lowest terms," a, b are prime to one another.

Again the proof is indirect.

If a, b are not prime to one another, they have some common measure c, and

$$a = mc, \quad b = nc.$$

Therefore $m : n = a : b$. [VII. 17 or 18]

But m, n are less than a, b respectively, so that $a : b$ is not in its lowest terms: which is contrary to the hypothesis.

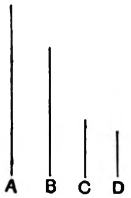
Therefore etc.

PROPOSITION 23.

If two numbers be prime to one another, the number which measures the one of them will be prime to the remaining number.

Let A, B be two numbers prime to one another, and let any number C measure A;

I say that C, B are also prime to one another.



For, if C, B are not prime to one another, some number will measure C, B.

Let a number measure them, and let it be D.

Since D measures C, and C measures A, therefore D also measures A.

But it also measures B;

therefore D measures A, B which are prime to one another: which is impossible. [VII. Def. 12]

Therefore no number will measure the numbers C, B.

Therefore C, B are prime to one another.

Q. E. D.

If a, mb are prime to one another, b is prime to a. For, if not, some number d will measure both a and b, and therefore both a and mb: which is contrary to the hypothesis.

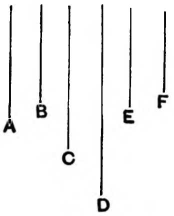
Therefore etc.

PROPOSITION 24.

If two numbers be prime to any number, their product also will be prime to the same.

For let the two numbers A, B be prime to any number C, and let A by multiplying B make D;

I say that C, D are prime to one another.



For, if C, D are not prime to one another, some number will measure C, D.

Let a number measure them, and let it be E.

Now, since C, A are prime to one another, and a certain number E measures C, therefore A, E are prime to one another. [VII. 23]

As many times, then, as E measures D, so many units let there be in F;

therefore F also measures D according to the units in E. [VII. 16]

Therefore E by multiplying F has made D. [VII. Def. 15]

But, further, A by multiplying B has also made D;

therefore the product of E, F is equal to the product of A, B.

But, if the product of the extremes be equal to that of the means, the four numbers are proportional; [VII. 19]

therefore, as E is to A, so is B to F.

But A, E are prime to one another,

numbers which are prime to one another are also the least of those which have the same ratio, [VII. 21]

and the least numbers of those which have the same ratio with them measure those which have the same ratio the same number of times, the greater the greater and the less the less, that is, the antecedent the antecedent and the consequent the consequent; [VII. 20]

therefore E measures B.

But it also measures C;

therefore E measures B, C which are prime to one another: which is impossible. [VII. Def. 12]

Therefore no number will measure the numbers C, D.

Therefore C, D are prime to one another.

Q. E. D.

1. **their product.** ὁ ἐξ αὐτῶν γενόμενος, literally "the (number) produced from them," will henceforth be translated as "their product."

If a, b are both prime to c, then ab, c are prime to one another.

The proof is again by *reductio ad absurdum*.

If ab, c are not prime to one another, let them be measured by a and be equal to md, nd, say, respectively.

Now, since a, c are prime to one another and d measures c,

a, d are prime to one another. [VII. 23]

But, since $ab = md$,

$d : a = b : m$. [VII. 19]

Therefore [VII. 20] d measures b,

or $b = pd$, say.

But $c = nd$.

Therefore d measures both b and c, which are therefore not prime to one another: which is impossible.

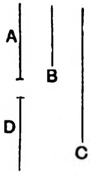
Therefore etc.

PROPOSITION 25.

If two numbers be prime to one another, the product of one of them into itself will be prime to the remaining one.

Let A, B be two numbers prime to one another, and let A by multiplying itself make C:

I say that B, C are prime to one another.



For let D be made equal to A.

Since A, B are prime to one another, and A is equal to D,

therefore D, B are also prime to one another.

Therefore each of the two numbers D, A is prime to B;

therefore the product of D, A will also be prime to B. [VII. 24]

But the number which is the product of *D*, *A* is *C*.

Therefore *C*, *B* are prime to one another. Q. E. D.

1. **the product of one of them into itself.** The Greek, ὁ ἐκ τοῦ ἑνὸς αὐτῶν γενόμενος, literally "the number produced from the one of them," leaves "multiplied into itself" to be understood.

If a, b are prime to one another,

$$a^2 \text{ is prime to } b.$$

Euclid takes d equal to a, so that d, a are both prime to b.

Hence, by VII. 24, da, i.e. a^2, is prime to b.

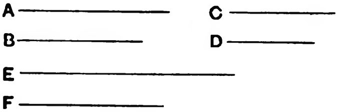
The proposition is a particular case of the preceding proposition; and the method of proof is by substitution of different numbers in the result of that proposition.

PROPOSITION 26.

If two numbers be prime to two numbers, both to each, their products also will be prime to one another.

For let the two numbers *A*, *B* be prime to the two numbers *C*, *D*, both to each, and let *A* by multiplying *B* make *E*, and let *C* by multiplying *D* make *F*;

I say that *E*, *F* are prime to one another.



For, since each of the numbers *A*, *B* is prime to *C*, therefore the product of *A*, *B* will also be prime to *C*. [VII. 24]

But the product of *A*, *B* is *E*;

therefore *E*, *C* are prime to one another.

For the same reason

E, *D* are also prime to one another.

Therefore each of the numbers *C*, *D* is prime to *E*.

Therefore the product of *C*, *D* will also be prime to *E*. [VII. 24]

But the product of *C*, *D* is *F*.

Therefore *E*, *F* are prime to one another. Q. E. D.

If both a and b are prime to each of two numbers c, d, then ab, cd will be prime to one another.

Since a, b are both prime to c,

ab, c are prime to one another. [VII. 24]

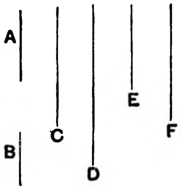
Similarly ab, d are prime to one another.

Therefore c, d are both prime to ab,

and so therefore is cd. [VII. 24]

PROPOSITION 27.

If two numbers be prime to one another, and each by multiplying itself make a certain number, the products will be prime to one another; and, if the original numbers by multiplying the products make certain numbers, the latter will also be prime to one another [and this is always the case with the extremes].



Let *A*, *B* be two numbers prime to one another, let *A* by multiplying itself make *C*, and by multiplying *C* make *D*,

and let *B* by multiplying itself make *E*, and by multiplying *E* make *F*;

I say that both *C*, *E* and *D*, *F* are prime to one another.

For, since *A*, *B* are prime to one another,
and *A* by multiplying itself has made *C*,
therefore *C*, *B* are prime to one another. [VII. 25]

Since then *C*, *B* are prime to one another,
and *B* by multiplying itself has made *E*,
therefore *C*, *E* are prime to one another. [*id.*]

Again, since *A*, *B* are prime to one another,
and *B* by multiplying itself has made *E*,
therefore *A*, *E* are prime to one another. [*id.*]

Since then the two numbers *A*, *C* are prime to the two numbers *B*, *E*, both to each,
therefore also the product of *A*, *C* is prime to the product of *B*, *E*. [VII. 26]

And the product of *A*, *C* is *D*, and the product of *B*, *E* is *F*.

Therefore *D*, *F* are prime to one another.

Q. E. D.

If a, b are prime to one another, so are a^2, b^2 and so are a^3, b^3; and, generally, a^n, b^n are prime to one another.

The words in the enunciation which assert the truth of the proposition for *any* powers are suspected and bracketed by Heiberg because (1) in περὶ τοὺς ἄκρους the use of ἄκροι is peculiar, for it can only mean "the last products," and (2) the words have nothing corresponding to them in the proof, much less is the generalisation proved. Campanus omits the words in the enunciation, though he adds to the proof a remark that the proposition is true of any, the same or different, powers of a, b. Heiberg concludes that the words are an interpolation of date earlier than Theon.

Euclid's proof amounts to this.

Since a, b are prime to one another, so are a^2, b [VII. 25], and therefore also a^2, b^2. [VII. 25]

Similarly [VII. 25] a, b^2 are prime to one another.

Therefore a, a^2 and b, b^2 satisfy the description in the enunciation of VII. 26.

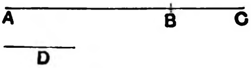
Hence a^3, b^3 are prime to one another.

PROPOSITION 28.

If two numbers be prime to one another, the sum will also be prime to each of them; and, if the sum of two numbers be prime to any one of them, the original numbers will also be prime to one another.

For let two numbers AB, BC prime to one another be added;

I say that the sum AC is also prime to each of the numbers AB, BC.



For, if CA, AB are not prime to one another,

some number will measure CA, AB.

Let a number measure them, and let it be D.

Since then D measures CA, AB,

therefore it will also measure the remainder BC.

But it also measures BA;

therefore D measures AB, BC which are prime to one another: which is impossible. [VII. Def. 12]

Therefore no number will measure the numbers CA, AB; therefore CA, AB are prime to one another.

For the same reason

AC, CB are also prime to one another.

Therefore CA is prime to each of the numbers AB, BC.

Again, let CA, AB be prime to one another;

I say that AB, BC are also prime to one another.

For, if AB, BC are not prime to one another,

some number will measure AB, BC.

Let a number measure them, and let it be D.

Now, since D measures each of the numbers AB, BC, it will also measure the whole CA.

But it also measures AB;

therefore D measures CA, AB which are prime to one another: which is impossible. [VII. Def. 12]

Therefore no number will measure the numbers AB, BC.

Therefore AB, BC are prime to one another.

Q. E. D.

If a, b are prime to one another, $a+b$ will be prime to both a and b; and conversely.

For suppose $(a+b)$, a are not prime to one another. They must then have some common measure d.

Therefore d also divides the difference $(a+b)-a$, or b, as well as a; and therefore a, b are not prime to one another: which is contrary to the hypothesis.

Therefore $a+b$ is prime to a.

Similarly $a+b$ is prime to b.

The converse is proved in the same way.

Heiberg remarks on Euclid's assumption that, if c measures both a and b, it also measures $a \pm b$. But it has already (VII. 1, 2) been assumed, more generally, as an axiom that, in the case supposed, c measures $a \pm pb$.

PROPOSITION 29.

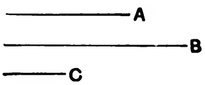
Any prime number is prime to any number which it does not measure.

Let A be a prime number, and let it not measure B;

I say that B, A are prime to one another.

For, if B, A are not prime to one another,

some number will measure them.



Let C measure them.

Since C measures B,

and A does not measure B,

therefore C is not the same with A.

Now, since C measures B, A,

therefore it also measures A which is prime, though it is not the same with it:

which is impossible.

Therefore no number will measure *B*, *A*.

Therefore *A*, *B* are prime to one another.

Q. E. D.

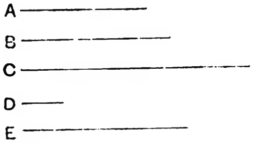
If *a* is prime and does not measure *b*, then *a*, *b* are prime to one another. The proof is self-evident.

PROPOSITION 30.

If two numbers by multiplying one another make some number, and any prime number measure the product, it will also measure one of the original numbers.

For let the two numbers *A*, *B* by multiplying one another make *C*, and let any prime number *D* measure *C*;

I say that *D* measures one of the numbers *A*, *B*.



For let it not measure *A*.

Now *D* is prime;

therefore *A*, *D* are prime to one another. [VII. 29]

And, as many times as *D* measures *C*, so many units let there be in *E*.

Since then *D* measures *C* according to the units in *E*, therefore *D* by multiplying *E* has made *C*. [VII. Def. 15]

Further, *A* by multiplying *B* has also made *C*;

therefore the product of *D*, *E* is equal to the product of *A*, *B*.

Therefore, as *D* is to *A*, so is *B* to *E*. [VII. 19]

But *D*, *A* are prime to one another,

primes are also least, [VII. 21]

and the least measure the numbers which have the same ratio the same number of times, the greater the greater and the less the less, that is, the antecedent the antecedent and the consequent the consequent; [VII. 20]

therefore *D* measures *B*.

Similarly we can also show that, if *D* do not measure *B*, it will measure *A*.

Therefore *D* measures one of the numbers *A*, *B*.

Q. E. D.

If c, a prime number, measure ab, c will measure either a or b.
Suppose c does not measure a.
Therefore c, a are prime to one another. [VII. 29]
Suppose $ab = mc$.
Therefore $c : a = b : m$. [VII. 19]
Hence [VII. 20, 21] c measures b.
Similarly, if c does not measure b, it measures a.
Therefore it measures one or other of the two numbers a, b.

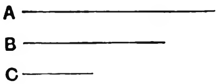
PROPOSITION 31.

Any composite number is measured by some prime number.

Let A be a composite number;

I say that A is measured by some prime number.

For, since A is composite, some number will measure it.



Let a number measure it, and let it be B.

Now, if B is prime, what was enjoined will have been done.

But if it is composite, some number will measure it.

Let a number measure it, and let it be C.

Then, since C measures B,

and B measures A,

therefore C also measures A.

And, if C is prime, what was enjoined will have been done.

But if it is composite, some number will measure it.

Thus, if the investigation be continued in this way, some prime number will be found which will measure the number before it, which will also measure A.

For, if it is not found, an infinite series of numbers will measure the number A, each of which is less than the other: which is impossible in numbers.

Therefore some prime number will be found which will measure the one before it, which will also measure A.

Therefore any composite number is measured by some prime number.

8. **if B is prime, what was enjoined will have been done,** i.e. the implied *problem* of finding a prime number which measures *A*.

18. **some prime number will be found which will measure.** In the Greek the sentence stops here, but it is necessary to add the words "the number before it, which will also measure *A*," which are found a few lines further down. It is possible that the words may have fallen out of P here by a simple mistake due to ὁμοιοτέλευτον (Heiberg).

Heiberg relegates to the Appendix an alternative proof of this proposition, to the following effect. Since *A* is composite, some number will measure it. Let *B* be the *least* such number. I say that *B* is prime. For, if not, *B* is composite, and some number will measure it, say *C*; so that *C* is less than *B*. But, since *C* measures *B*, and *B* measures *A*, *C* must measure *A*. And *C* is less than *B*: which is contrary to the hypothesis.

PROPOSITION 32.

Any number either is prime or is measured by some prime number.

Let *A* be a number;

I say that *A* either is prime or is measured by some prime number.

If now *A* is prime, that which was enjoined will have been done.

But if it is composite, some prime number will measure it. [VII. 31]

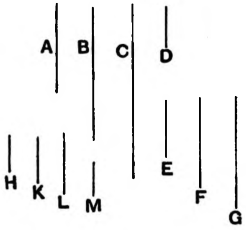
Therefore any number either is prime or is measured by some prime number.

Q. E. D.

PROPOSITION 33.

Given as many numbers as we please, to find the least of those which have the same ratio with them.

Let *A*, *B*, *C* be the given numbers, as many as we please; thus it is required to find the least of those which have the same ratio with *A*, *B*, *C*.



A, *B*, *C* are either prime to one another or not.

Now, if *A*, *B*, *C* are prime to one another, they are the least of those which have the same ratio with them. [VII. 21]

But, if not, let *D* the greatest common measure of *A*, *B*, *C* be taken, [VII. 3] and, as many times as D measures the numbers A, B, C respectively, so many units let there be in the numbers E, F, G respectively.

Therefore the numbers E, F, G measure the numbers A, B, C respectively according to the units in D. [VII. 16]

Therefore E, F, G measure A, B, C the same number of times;

therefore E, F, G are in the same ratio with A, B, C. [VII. Def. 20]

I say next that they are the least that are in that ratio.

For, if E, F, G are not the least of those which have the same ratio with A, B, C,

there will be numbers less than E, F, G which are in the same ratio with A, B, C.

Let them be H, K, L;

therefore H measures A the same number of times that the numbers K, L measure the numbers B, C respectively.

Now, as many times as H measures A, so many units let there be in M;

therefore the numbers K, L also measure the numbers B, C respectively according to the units in M.

And, since H measures A according to the units in M,

therefore M also measures A according to the units in H. [VII. 16]

For the same reason

M also measures the numbers B, C according to the units in the numbers K, L respectively;

Therefore M measures A, B, C.

Now, since H measures A according to the units in M,

therefore H by multiplying M has made A. [VII. Def. 15]

For the same reason also

E by multiplying D has made A.

Therefore the product of E, D is equal to the product of H, M.

Therefore, as E is to H, so is M to D. [VII. 19]

But E is greater than H;

therefore M is also greater than D.

And it measures A, B, C:

which is impossible, for by hypothesis D is the greatest common measure of A, B, C.

Therefore there cannot be any numbers less than E, F, G which are in the same ratio with A, B, C.

Therefore E, F, G are the least of those which have the same ratio with A, B, C.

Q. E. D.

17. **the numbers E, F, G measure the numbers A, B, C respectively,** literally (as usual) "each of the numbers E, F, G measures each of the numbers A, B, C."

Given any numbers $a, b, c, \ldots$, to find the least numbers that are in the same ratio.

Euclid's method is the obvious one, and the result is verified by *reductio ad absurdum*.

We will, like Euclid, take three numbers only, a, b, c.

Let g, their greatest common measure, be found [VII. 3], and suppose that

$$a = mg, \text{ i.e. } gm, \qquad [\text{VII. } 16]$$
$$b = ng, \text{ i.e. } gn,$$
$$c = pg, \text{ i.e. } gp.$$

It follows, by VII. Def. 20, that

$$m : n : p = a : b : c.$$

m, n, p shall be the numbers required.

For, if not, let x, y, z be the least numbers in the same ratio as a, b, c, being less than m, n, p.

Therefore
$$a = kx \text{ (or } xk\text{, VII. 16)},$$
$$b = ky \text{ (or } yk\text{)},$$
$$c = kz \text{ (or } zk\text{)},$$
where k is some integer. [VII. 20]

Thus
$$mg = a = xk.$$

Therefore
$$m : x = k : g. \qquad [\text{VII. } 19]$$

And $m > x$; therefore $k > g$.

Since then k measures a, b, c, it follows that g is not the greatest common measure: which contradicts the hypothesis.

Therefore etc.

It is to be observed that Euclid merely supposes that x, y, z are smaller numbers than m, n, p in the ratio of a, b, c; but, in order to justify the next inference, which apparently can only depend on VII. 20, x, y, z must also be assumed to be the *least* numbers in the ratio of a, b, c.

The inference from the last proportion that, since $m > x$, $k > g$ is supposed by Heiberg to depend upon VII. 13 and V. 14 together. I prefer to regard Euclid as making the inference quite independently of Book V. E.g., the proportion could just as well be written

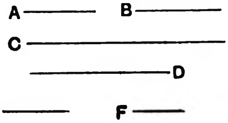
$$x : m = g : k,$$

when the definition of proportion in Book VII. (Def. 20) gives all that we want, since, whatever proper fraction x is of m, the same proper fraction is g of k.

PROPOSITION 34.

Given two numbers, to find the least number which they measure.

Let A, B be the two given numbers;
thus it is required to find the least number which they measure.



Now A, B are either prime to one another or not.

First, let A, B be prime to one another, and let A by multiplying B make C;
therefore also B by multiplying A has made C. [VII. 16]

Therefore A, B measure C

I say next that it is also the least number they measure.

For, if not, A, B will measure some number which is less than C.

Let them measure D.

Then, as many times as A measures D, so many units let there be in E,
and, as many times as B measures D, so many units let there be in F;
therefore A by multiplying E has made D,
and B by multiplying F has made D; [VII. Def. 15]
therefore the product of A, E is equal to the product of B, F.

Therefore, as A is to B, so is F to E. [VII. 19]

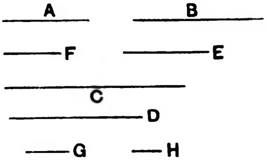
But A, B are prime,
primes are also least, [VII. 21]
and the least measure the numbers which have the same ratio the same number of times, the greater the greater and the less the less; [VII. 20]
therefore B measures E, as consequent consequent.

And, since A by multiplying B, E has made C, D,
therefore, as B is to E, so is C to D. [VII. 17]

But B measures E;
therefore C also measures D, the greater the less:
which is impossible.

Therefore A, B do not measure any number less than C; therefore C is the least that is measured by A, B.

Next, let A, B not be prime to one another, and let F, E, the least numbers of those which have the same ratio with A, B, be taken; [VII. 33]
therefore the product of A, E is equal to the product of B, F. [VII. 19]



And let A by multiplying E make C;
therefore also B by multiplying F has made C;
therefore A, B measure C.

I say next that it is also the least number that they measure.

For, if not, A, B will measure some number which is less than C.

Let them measure D.

And, as many times as A measures D, so many units let there be in G,
and, as many times as B measures D, so many units let there be in H.

Therefore A by multiplying G has made D,
and B by multiplying H has made D.

Therefore the product of A, G is equal to the product of B, H;
therefore, as A is to B, so is H to G. [VII. 19]

But, as A is to B, so is F to E.

Therefore also, as F is to E, so is H to G.

But F, E are least,
and the least measure the numbers which have the same ratio the same number of times, the greater the greater and the less the less; [VII. 20]
therefore E measures G.

And, since A by multiplying E, G has made C, D,
therefore, as E is to G, so is C to D. [VII. 17]

But E measures G;
therefore C also measures D, the greater the less:
which is impossible.

Therefore A, B will not measure any number which is less than C.

Therefore C is the least that is measured by A, B.

Q. E. D.

This is the problem of finding the *least common multiple* of two numbers, as a, b.

I. If a, b be prime to one another, the L.C.M. is ab.

For, if not, let it be d, some number less than ab.

Then $d = ma = nb$, where m, n are integers.

Therefore $a : b = n : m$, [VII. 19]

and hence, a, b being prime to one another,

b measures m. [VII. 20, 21]

But $b : m = ab : am$ [VII. 17]

$= ab : d$.

Therefore ab measures d: which is impossible.

II. If a, b be not prime to one another, find the numbers which are the least of those having the ratio of a to b, say m, n; [VII. 33]

then $a : b = m : n$,

and $an = bm$ $(= c$, say); [VII. 19]

c is then the L.C.M.

For, if not, let it be d $(< c)$, so that

$ap = bq = d$, where p, q are integers.

Then $a : b = q : p$, [VII. 19]

whence $m : n = q : p$,

so that n measures p. [VII. 20, 21]

And $n : p = an : ap = c : d$,

so that c measures d:

which is impossible.

Therefore etc.

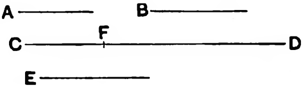
By VII. 33, $\left.\begin{array}{l} m = \dfrac{a}{g} \\ n = \dfrac{b}{g} \end{array}\right\}$, where g is the G.C.M. of a, b.

Hence the L.C.M. is $\dfrac{ab}{g}$.

PROPOSITION 35.

If two numbers measure any number, the least number measured by them will also measure the same.

For let the two numbers A, B measure any number CD, and let E be the least that they measure;



I say that E also measures CD.

For, if E does not measure CD, let E, measuring DF, leave CF less than itself.

Now, since A, B measure E,
and E measures DF,
therefore A, B will also measure DF.

But they also measure the whole CD;
therefore they will also measure the remainder CF which is less than E:
which is impossible.

Therefore E cannot fail to measure CD;
therefore it measures it.

Q. E. D.

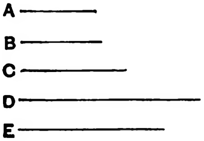
The *least* common multiple of any two numbers must measure any other common multiple.

The proof is obvious, depending on the fact that, if any number divides a and b, it also divides $a-pb$.

PROPOSITION 36.

Given three numbers, to find the least number which they measure.

Let A, B, C be the three given numbers;
thus it is required to find the least number which they measure.



Let D, the least number measured by the two numbers A, B, be taken. [VII. 34]

Then C either measures, or does not measure, D.

First, let it measure it.

But A, B also measure D;
therefore A, B, C measure D.

I say next that it is also the least that they measure.

For, if not, A, B, C will measure some number which is less than D.

Let them measure E.

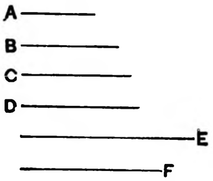
Since A, B, C measure E,
therefore also A, B measure E.

Therefore the least number measured by A, B will also measure E. [VII. 35]

But D is the least number measured by A, B;
therefore D will measure E, the greater the less:
which is impossible.

Therefore A, B, C will not measure any number which is less than D;

therefore D is the least that A, B, C measure.



Again, let C not measure D,
and let E, the least number measured by C, D, be taken. [VII. 34]

Since A, B measure D,
and D measures E,
therefore also A, B measure E.

But C also measures E;
therefore also A, B, C measure E.

I say next that it is also the least that they measure.

For, if not, A, B, C will measure some number which is less than E.

Let them measure F.

Since A, B, C measure F,
therefore also A, B measure F;
therefore the least number measured by A, B will also measure F. [VII. 35]

But D is the least number measured by A, B;
therefore D measures F.

But C also measures F;
therefore D, C measure F,
so that the least number measured by D, C will also measure F.

But E is the least number measured by C, D;
therefore E measures F, the greater the less:
which is impossible.

Therefore A, B, C will not measure any number which is less than E.

Therefore E is the least that is measured by A, B, C.

Q. E. D.

Euclid's rule for finding the L.C.M. of *three* numbers a, b, c is the rule with which we are familiar. The L.C.M. of a, b is first found, say d, and then the L.C.M. of d and c is found.

Euclid distinguishes the cases (1) in which c measures d, (2) in which c does not measure d. We need only reproduce the proof of the general case (2). The method is that of *reductio ad absurdum*.

Let e be the L.C.M. of d, c.

Since a, b both measure d, and d measures e,

a, b both measure e.

So does c.

Therefore e is *some* common multiple of a, b, c.

If it is not the *least*, let f be the L.C.M.

Now a, b both measure f;
therefore d, their L.C.M., also measures f. [VII. 35]

Thus d, c both measure f;
therefore e, their L.C.M., measures f: [VII. 35]
which is impossible, since $f < e$.

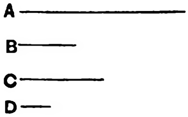
Therefore etc.

The process can be continued *ad libitum*, so that we can find the L.C.M., not only of three, but of as many numbers as we please.

PROPOSITION 37.

If a number be measured by any number, the number which is measured will have a part called by the same name as the measuring number.

For let the number A be measured by any number B;
I say that A has a part called by the same name as B.



For, as many times as B measures A, so many units let there be in C.

Since B measures A according to the units in C,
and the unit D also measures the number C according to the units in it,

therefore the unit *D* measures the number *C* the same number of times as *B* measures *A*.

Therefore, alternately, the unit *D* measures the number *B* the same number of times as *C* measures *A*; [VII. 15]
therefore, whatever part the unit *D* is of the number *B*, the same part is *C* of *A* also.

But the unit *D* is a part of the number *B* called by the same name as it;
therefore *C* is also a part of *A* called by the same name as *B*, so that *A* has a part *C* which is called by the same name as *B*.

Q. E. D.

If b measures a, then $\frac{1}{b}$th of a is a whole number.

Let $a = m \cdot b$.

Now $m = m \cdot 1$.

Thus $1, m, b, a$ satisfy the enunciation of VII. 15;
therefore m measures a the same number of times that 1 measures b.

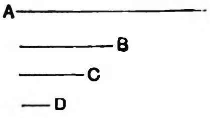
But 1 is $\frac{1}{b}$th part of b;

therefore m is $\frac{1}{b}$th part of a.

PROPOSITION 38.

If a number have any part whatever, it will be measured by a number called by the same name as the part.

For let the number *A* have any part whatever, *B*, and let *C* be a number called by the same name as the part *B*;
I say that *C* measures *A*.



For, since *B* is a part of *A* called by the same name as *C*,
and the unit *D* is also a part of *C* called by the same name as it,
therefore, whatever part the unit *D* is of the number *C*, the same part is *B* of *A* also;
therefore the unit *D* measures the number *C* the same number of times that *B* measures *A*.

Therefore, alternately, the unit *D* measures the number *B* the same number of times that *C* measures *A*. [VII. 15]

Therefore *C* measures *A*.

Q. E. D.

This proposition is practically a restatement of the preceding proposition. It asserts that, if b is $\frac{1}{m}$th part of a,

i.e., if $b = \frac{1}{m} a$,

then m measures a.

We have $b = \frac{1}{m} a$,

and $1 = \frac{1}{m} m$.

Therefore $1, m, b, a$, satisfy the enunciation of VII. 15, and therefore m measures a the same number of times as 1 measures b, or

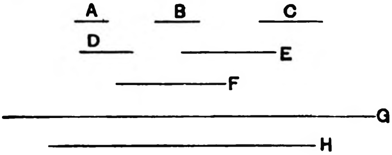
$$m = \frac{1}{b} a.$$

PROPOSITION 39.

To find the number which is the least that will have given parts.

Let *A*, *B*, *C* be the given parts;

thus it is required to find the number which is the least that will have the parts *A*, *B*, *C*.



Let *D*, *E*, *F* be numbers called by the same name as the parts *A*, *B*, *C*,

and let *G*, the least number measured by *D*, *E*, *F*, be taken. [VII. 36]

Therefore *G* has parts called by the same name as *D*, *E*, *F*. [VII. 37]

But *A*, *B*, *C* are parts called by the same name as *D*, *E*, *F*; therefore *G* has the parts *A*, *B*, *C*.

I say next that it is also the least number that has.

For, if not, there will be some number less than *G* which will have the parts *A*, *B*, *C*.

Let it be *H*.

Since *H* has the parts *A*, *B*, *C*,

therefore *H* will be measured by numbers called by the same name as the parts *A*, *B*, *C*. [VII. 38]

But *D*, *E*, *F* are numbers called by the same name as the parts *A*, *B*, *C*;

therefore *H* is measured by *D*, *E*, *F*.

And it is less than *G*: which is impossible.

Therefore there will be no number less than *G* that will have the parts *A*, *B*, *C*.

Q. E. D.

This again is practically a restatement in another form of the problem of finding the L.C.M.

To find a number which has $\frac{1}{a}$th, $\frac{1}{b}$th and $\frac{1}{c}$th parts.

Let d be the L.C.M. of a, b, c.

Thus d has $\frac{1}{a}$th, $\frac{1}{b}$th and $\frac{1}{c}$th parts. [VII. 37]

If it is not the least number which has, let the least such number be e.

Then, since e has those parts,

e is measured by a, b, c; and $e < d$:

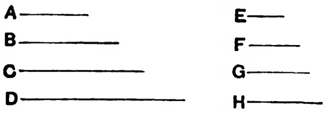
which is impossible.

BOOK VIII.

Proposition 1.

If there be as many numbers as we please in continued proportion, and the extremes of them be prime to one another, the numbers are the least of those which have the same ratio with them.

Let there be as many numbers as we please, A, B, C, D, in continued proportion, and let the extremes of them A, D be prime to one another; I say that A, B, C, D are the least of those which have the same ratio with them.



For, if not, let E, F, G, H be less than A, B, C, D, and in the same ratio with them.

Now, since A, B, C, D are in the same ratio with E, F, G, H,

and the multitude of the numbers A, B, C, D is equal to the multitude of the numbers E, F, G, H,

therefore, *ex aequali*,

as A is to D, so is E to H. [vii. 14]

But A, D are prime,

primes are also least, [vii. 21]

and the least numbers measure those which have the same ratio the same number of times, the greater the greater and the less the less, that is, the antecedent the antecedent and the consequent the consequent. [vii. 20]

Therefore A measures E, the greater the less: which is impossible.

Therefore E, F, G, H which are less than A, B, C, D are not in the same ratio with them.

Therefore A, B, C, D are the least of those which have the same ratio with them.

Q. E. D.

What we call a *geometrical progression* is with Euclid a series of terms "in continued proportion" (ἑξῆς ἀνάλογον).

This proposition proves that, if a, b, c, ... k are a series of numbers in geometrical progression, and if a, k are prime to one another, the series is in the lowest terms possible with the same common ratio.

The proof is in *form* by *reductio ad absurdum*. We should no doubt desert this *form* while retaining the substance. If a', b', c', ... k' be *any* other series of numbers in G.P. with the same common ratio as before, we have, *ex aequali*,

$$a : k = a' : k', \qquad \text{[VII. 14]}$$

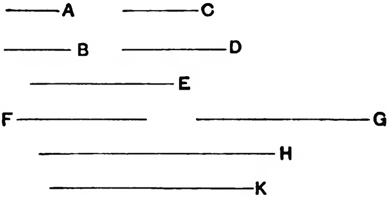
whence, since a, k are prime to one another, a, k measure a', k' respectively, so that a', k' are *greater* than a, k respectively.

PROPOSITION 2.

To find numbers in continued proportion, as many as may be prescribed, and the least that are in a given ratio.

Let the ratio of A to B be the given ratio in least numbers;

thus it is required to find numbers in continued proportion, as many as may be prescribed, and the least that are in the ratio of A to B.



Let four be prescribed;

let A by multiplying itself make C, and by multiplying B let it make D;

let B by multiplying itself make E;

further, let A by multiplying C, D, E make F, G, H,

and let B by multiplying E make K.

Now, since A by multiplying itself has made C,
and by multiplying B has made D,
therefore, as A is to B, so is C to D. [VII. 17]

Again, since A by multiplying B has made D,
and B by multiplying itself has made E,
therefore the numbers A, B by multiplying B have made the numbers D, E respectively.

Therefore, as A is to B, so is D to E. [VII. 18]

But, as A is to B, so is C to D;
therefore also, as C is to D, so is D to E.

And, since A by multiplying C, D has made F, G,
therefore, as C is to D, so is F to G. [VII. 17]

But, as C is to D, so was A to B;
therefore also, as A is to B, so is F to G.

Again, since A by multiplying D, E has made G, H,
therefore, as D is to E, so is G to H. [VII. 17]

But, as D is to E, so is A to B.

Therefore also, as A is to B, so is G to H.

And, since A, B by multiplying E have made H, K,
therefore, as A is to B, so is H to K. [VII. 18]

But, as A is to B, so is F to G, and G to H.

Therefore also, as F is to G, so is G to H, and H to K;
therefore C, D, E, and F, G, H, K are proportional in the ratio of A to B.

I say next that they are the least numbers that are so.

For, since A, B are the least of those which have the same ratio with them,
and the least of those which have the same ratio are prime to one another, [VII. 22]
therefore A, B are prime to one another.

And the numbers A, B by multiplying themselves respectively have made the numbers C, E, and by multiplying the numbers C, E respectively have made the numbers F, K;
therefore C, E and F, K are prime to one another respectively. [VII. 27]

But, if there be as many numbers as we please in continued proportion, and the extremes of them be prime to one another, they are the least of those which have the same ratio with them. [VIII. 1]

Therefore *C*, *D*, *E* and *F*, *G*, *H*, *K* are the least of those which have the same ratio with *A*, *B*. Q. E. D.

PORISM. From this it is manifest that, if three numbers in continued proportion be the least of those which have the same ratio with them, the extremes of them are squares, and, if four numbers, cubes.

To find a series of numbers in geometrical progression and in the least terms which have a given common ratio (understanding by that term *the ratio of one term to the next*).

Reduce the given ratio to its lowest terms, say, $a : b$. (This can be done by VII. 33.)

Then $$a^n, a^{n-1}b, a^{n-2}b^2, \ldots a^2b^{n-2}, ab^{n-1}, b^n$$
is the required series of numbers if $(n+1)$ terms are required.

That this is a series of terms with the given common ratio is clear from VII. 17, 18.

That the G.P. is in the smallest terms possible is proved thus.

a, b are prime to one another, since the ratio $a : b$ is in its lowest terms. [VII. 22]

Therefore a^2, b^2 are prime to one another; so are a^3, b^3 and, generally, a^n, b^n. [VII. 27]

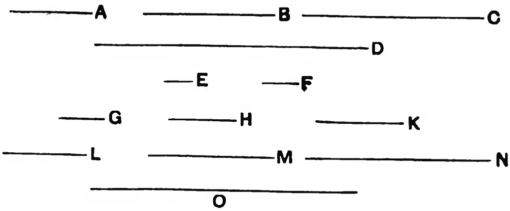
Whence the G.P. is in the smallest possible terms, by VIII. 1.

The Porism observes that, if there are n terms in the series, the extremes are $(n-1)$th powers.

PROPOSITION 3.

If as many numbers as we please in continued proportion be the least of those which have the same ratio with them, the extremes of them are prime to one another.

Let as many numbers as we please, *A*, *B*, *C*, *D*, in continued proportion be the least of those which have the same ratio with them;



I say that the extremes of them *A*, *D* are prime to one another.

For let two numbers *E*, *F*, the least that are in the ratio of *A*, *B*, *C*, *D*, be taken, [VII. 33]

then three others *G*, *H*, *K* with the same property;

and others, more by one continually, [VIII. 2]

until the multitude taken becomes equal to the multitude of the numbers *A*, *B*, *C*, *D*.

Let them be taken, and let them be *L*, *M*, *N*, *O*.

Now, since *E*, *F* are the least of those which have the same ratio with them, they are prime to one another. [VII. 22]

And, since the numbers *E*, *F* by multiplying themselves respectively have made the numbers *G*, *K*, and by multiplying the numbers *G*, *K* respectively have made the numbers *L*, *O*, [VIII. 2, Por.]

therefore both *G*, *K* and *L*, *O* are prime to one another. [VII. 27]

And, since *A*, *B*, *C*, *D* are the least of those which have the same ratio with them,

while *L*, *M*, *N*, *O* are the least that are in the same ratio with *A*, *B*, *C*, *D*,

and the multitude of the numbers *A*, *B*, *C*, *D* is equal to the multitude of the numbers *L*, *M*, *N*, *O*,

therefore the numbers *A*, *B*, *C*, *D* are equal to the numbers *L*, *M*, *N*, *O* respectively;

therefore *A* is equal to *L*, and *D* to *O*.

And *L*, *O* are prime to one another.

Therefore *A*, *D* are also prime to one another.

Q. E. D.

The proof consists in merely equating the given numbers to the terms of a series found in the manner of VIII. 2.

If a, b, c, ... k (n terms) be a geometrical progression in the lowest terms having a given common ratio, the terms must respectively be of the form

$$\alpha^{n-1},\quad \alpha^{n-2}\beta,\ \dots\ \alpha^2\beta^{n-3},\quad \alpha\beta^{n-2},\quad \beta^{n-1}$$

found by VIII. 2, where $\alpha : \beta$ is the ratio $a : b$ expressed in its lowest terms, so that α, β are prime to one another [VII. 22], and hence α^{n-1}, β^{n-1} are prime to one another [VII. 27].

But the two series must be the same, so that

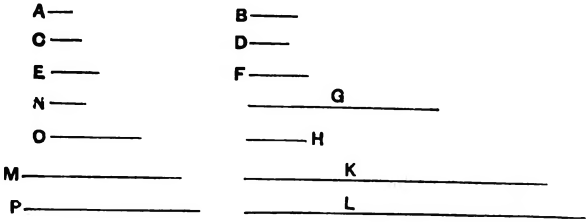
$$a = \alpha^{n-1},\quad b = \beta^{n-1}$$

PROPOSITION 4.

Given as many ratios as we please in least numbers, to find numbers in continued proportion which are the least in the given ratios.

Let the given ratios in least numbers be that of A to B, that of C to D, and that of E to F;

thus it is required to find numbers in continued proportion which are the least that are in the ratio of A to B, in the ratio of C to D, and in the ratio of E to F.



Let G, the least number measured by B, C, be taken. [VII. 34]

And, as many times as B measures G, so many times also let A measure H,

and, as many times as C measures G, so many times also let D measure K.

Now E either measures or does not measure K.

First, let it measure it.

And, as many times as E measures K, so many times let F measure L also.

Now, since A measures H the same number of times that B measures G,

therefore, as A is to B, so is H to G. [VII. Def. 20, VII. 13]

For the same reason also,

as C is to D, so is G to K,

and further, as E is to F, so is K to L;

therefore H, G, K, L are continuously proportional in the ratio of A to B, in the ratio of C to D, and in the ratio of E to F.

I say next that they are also the least that have this property.

For, if *H*, *G*, *K*, *L* are not the least numbers continuously proportional in the ratios of *A* to *B*, of *C* to *D*, and of *E* to *F*, let them be *N*, *O*, *M*, *P*.

Then since, as *A* is to *B*, so is *N* to *O*,

while *A*, *B* are least,

and the least numbers measure those which have the same ratio the same number of times, the greater the greater and the less the less, that is, the antecedent the antecedent and the consequent the consequent;

therefore *B* measures *O*. [VII. 20]

For the same reason

C also measures *O*;

therefore *B*, *C* measure *O*;

therefore the least number measured by *B*, *C* will also measure *O*. [VII. 35]

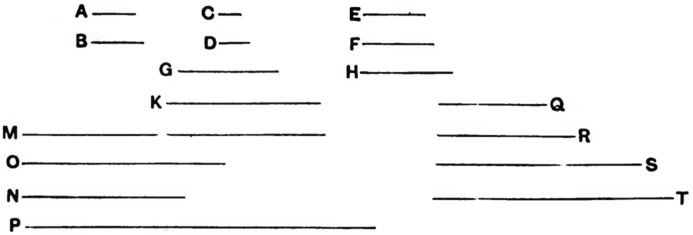
But *G* is the least number measured by *B*, *C*;

therefore *G* measures *O*, the greater the less:

which is impossible.

Therefore there will be no numbers less than *H*, *G*, *K*, *L* which are continuously in the ratio of *A* to *B*, of *C* to *D*, and of *E* to *F*.

Next, let *E* not measure *K*.



Let *M*, the least number measured by *E*, *K*, be taken.

And, as many times as *K* measures *M*, so many times let *H*, *G* measure *N*, *O* respectively,

and, as many times as *E* measures *M*, so many times let *F* measure *P* also.

Since *H* measures *N* the same number of times that *G* measures *O*,

therefore, as *H* is to *G*, so is *N* to *O*. [VII. 13 and Def. 20]

But, as H is to G, so is A to B;
therefore also, as A is to B, so is N to O.

For the same reason also,

as C is to D, so is O to M.

Again, since E measures M the same number of times that F measures P,
therefore, as E is to F, so is M to P; [VII. 13 and Def. 20]
therefore N, O, M, P are continuously proportional in the ratios of A to B, of C to D, and of E to F.

I say next that they are also the least that are in the ratios $A:B$, $C:D$, $E:F$.

For, if not, there will be some numbers less than N, O, M, P continuously proportional in the ratios $A:B$, $C:D$, $E:F$.

Let them be Q, R, S, T.

Now since, as Q is to R, so is A to B,
while A, B are least,
and the least numbers measure those which have the same ratio with them the same number of times, the antecedent the antecedent and the consequent the consequent, [VII. 20]
therefore B measures R.

For the same reason C also measures R;
therefore B, C measure R.

Therefore the least number measured by B, C will also measure R. [VII. 35]

But G is the least number measured by B, C;
therefore G measures R.

And, as G is to R, so is K to S: [VII. 13]
therefore K also measures S.

But E also measures S;
therefore E, K measure S.

Therefore the least number measured by E, K will also measure S. [VII. 35]

But M is the least number measured by E, K;
therefore M measures S, the greater the less:
which is impossible.

Therefore there will not be any numbers less than N, O, M, P continuously proportional in the ratios of A to B, of C to D, and of E to F;

therefore *N*, *O*, *M*, *P* are the least numbers continuously proportional in the ratios $A:B$, $C:D$, $E:F$. Q. E. D.

69, 71, 99. **the ratios A : B, C : D, E : F.** This abbreviated expression is in the Greek οἱ ΑΒ, ΓΔ, ΕΖ λόγοι.

The term "in continued proportion" is here not used in its proper sense, since a geometrical progression is not meant, but a series of terms each of which bears to the succeeding term a *given*, but not the *same*, ratio.

The proposition furnishes a good example of the cumbrousness of the Greek method of dealing with non-determinate numbers. The proof in fact is not easy to follow without the help of modern symbolical notation. If this be used, the reasoning can be made clear enough.

Euclid takes *three* given ratios and therefore requires to find *four* numbers. We will leave out the simpler particular case which he puts first, that namely in which *E* accidentally measures *K*, the multiple of *D* found in the first few lines; and we will reproduce the general case with *three* ratios.

Let the ratios in their lowest terms be

$$a:b,\quad c:d,\quad e:f$$

Take l_1, the L.C.M. of b, c, and suppose that

$$l_1 = mb = nc.$$

Form the numbers

$$ma,\ \left.\begin{matrix} mb \\ = nc \end{matrix}\right\},\ nd.$$

These are in the ratios of a to b and of c to d respectively.

Next, let l_2 be the L.C.M. of nd, e, and let

$$l_2 = pnd = qe.$$

Now form the numbers

$$pma,\quad \left.\begin{matrix} pmb \\ = pnc \end{matrix}\right\},\quad \left.\begin{matrix} pnd \\ = qe \end{matrix}\right\},\quad qf,$$

and these are the four numbers required.

If they are *not* the least in the given ratios, let

$$x,\quad y,\quad z,\quad u$$

be less numbers in the given ratios.

Since $a:b$ is in its lowest terms, and

$$a:b = x:y,$$

b measures y.

Similarly, since $c:d = y:z$,

c measures y.

Therefore l_1, the L.C.M. of b, c, measures y.

But $l_1 : nd\,[= c:d] = y:z$.

Therefore nd measures z.

And, since $e:f = z:u$,

e measures z.

Therefore l_2, the L.C.M. of nd, e, measures z: which is impossible, since $z < l_2$ or pnd.

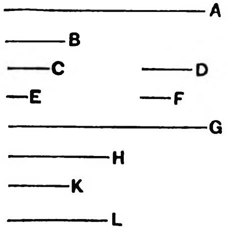
The step (line 86) inferring that $G:R = K:S$ is of course *alternando* from $G:K\,[= C:D] = R:S$.

It will be observed that VIII. 4 corresponds to the portion of VI. 23 which shows how to *compound* two ratios between straight lines.

PROPOSITION 5.

Plane numbers have to one another the ratio compounded of the ratios of their sides.

Let A, B be plane numbers, and let the numbers C, D be the sides of A, and E, F of B;



I say that A has to B the ratio compounded of the ratios of the sides.

For, the ratios being given which C has to E and D to F, let the least numbers G, H, K that are continuously in the ratios $C : E$, $D : F$ be taken, so that,

as C is to E, so is G to H,

and, as D is to F, so is H to K. [VIII. 4]

And let D by multiplying E make L.

Now, since D by multiplying C has made A, and by multiplying E has made L,

therefore, as C is to E, so is A to L. [VII. 17]

But, as C is to E, so is G to H;

therefore also, as G is to H, so is A to L.

Again, since E by multiplying D has made L, and further by multiplying F has made B,

therefore, as D is to F, so is L to B. [VII. 17]

But, as D is to F, so is H to K;

therefore also, as H is to K, so is L to B.

But it was also proved that,

as G is to H, so is A to L;

therefore, *ex aequali*,

as G is to K, so is A to B. [VII. 14]

But G has to K the ratio compounded of the ratios of the sides;

therefore A also has to B the ratio compounded of the ratios of the sides.

Q. E. D.

1, 5, 29, 31. **compounded of the ratios of their sides.** As in VI. 23, the Greek has the less exact phrase, "compounded of their sides."

If $$a = cd, \quad b = ef,$$
then a has to b the ratio compounded of $c : e$ and $d : f$.

Take three numbers the least which are continuously in the given ratios.

If l is the L.C.M. of e, d and $l = me = nd$, the three numbers are

$$mc,\quad \left.\begin{matrix} me \\ = nd \end{matrix}\right\},\quad nf. \qquad \text{[VIII. 4]}$$

Now $dc : de = c : e$ [VII. 17]

$= mc : me = mc : nd.$

Also $ed : ef = d : f$ [VII. 17]

$= nd : nf.$

Therefore, *ex aequali*, $cd : ef = mc : nf$

$=$ (ratio compounded of $c : e$ and $d : f$).

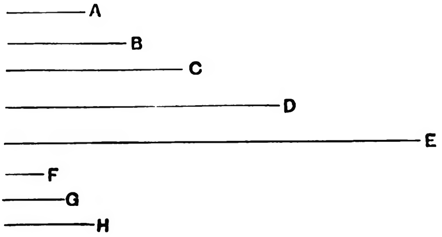
It will be seen that this proof follows exactly the method of VI. 23 for parallelograms.

PROPOSITION 6.

If there be as many numbers as we please in continued proportion, and the first do not measure the second, neither will any other measure any other.

Let there be as many numbers as we please, A, B, C, D, E, in continued proportion, and let A not measure B;

I say that neither will any other measure any other.



Now it is manifest that A, B, C, D, E do not measure one another in order; for A does not even measure B.

I say, then, that neither will any other measure any other.

For, if possible, let A measure C.

And, however many A, B, C are, let as many numbers F, G, H, the least of those which have the same ratio with A, B, C, be taken. [VII. 33]

Now, since F, G, H are in the same ratio with A, B, C, and the multitude of the numbers A, B, C is equal to the multitude of the numbers F, G, H,

therefore, *ex aequali*, as A is to C, so is F to H. [VII. 14]

And since, as A is to B, so is F to G,
while A does not measure B,
therefore neither does F measure G; [VII. Def. 20]
therefore F is not an unit, for the unit measures any number.

Now F, H are prime to one another. [VIII. 3]

And, as F is to H, so is A to C;
therefore neither does A measure C.

Similarly we can prove that neither will any other measure any other.

Q. E. D.

Let $a, b, c \dots k$ be a geometrical progression in which a does not measure b.

Suppose, if possible, that a measures some term of the series, as f.

Take x, y, z, u, v, w the *least* numbers in the ratio a, b, c, d, e, f.

Since $x : y = a : b$,
and a does not measure b,
x does not measure y; therefore x cannot be unity.

And, *ex aequali*, $x : w = a : f$.
Now x, w are prime to one another. [VIII. 3]

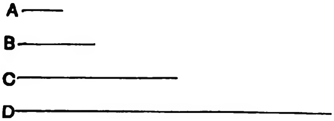
Therefore a does not measure f.

We can of course prove that an intermediate term, as b, does not measure a later term f by using the series b, c, d, e, f and remembering that, since $b : c = a : b$, b does not measure c.

PROPOSITION 7.

If there be as many numbers as we please in continued proportion, and the first measure the last, it will measure the second also.

Let there be as many numbers as we please, A, B, C, D, in continued proportion; and let A measure D;
I say that A also measures B.



For, if A does not measure B, neither will any other of the numbers measure any other. [VIII. 6]

But A measures D.

Therefore A also measures B.

Q. E. D.

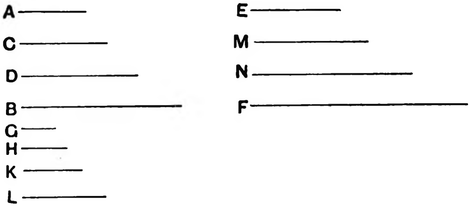
An obvious proof by *reductio ad absurdum* from VIII. 6.

PROPOSITION 8.

If between two numbers there fall numbers in continued proportion with them, then, however many numbers fall between them in continued proportion, so many will also fall in continued proportion between the numbers which have the same ratio with the original numbers.

Let the numbers C, D fall between the two numbers A, B in continued proportion with them, and let E be made in the same ratio to F as A is to B;

I say that, as many numbers as have fallen between A, B in continued proportion, so many will also fall between E, F in continued proportion.



For, as many as A, B, C, D are in multitude, let so many numbers G, H, K, L, the least of those which have the same ratio with A, C, D, B, be taken; [VII. 33]

therefore the extremes of them G, L are prime to one another. [VIII. 3]

Now, since A, C, D, B are in the same ratio with G, H, K, L,

and the multitude of the numbers A, C, D, B is equal to the multitude of the numbers G, H, K, L,

therefore, *ex aequali*, as A is to B, so is G to L. [VII. 14]

But, as A is to B, so is E to F;

therefore also, as G is to L, so is E to F.

But G, L are prime,

primes are also least, [VII. 21]

and the least numbers measure those which have the same ratio the same number of times, the greater the greater and the less the less, that is, the antecedent the antecedent and the consequent the consequent. [VII. 20]

Therefore G measures E the same number of times as L measures F.

Next, as many times as G measures E, so many times let H, K also measure M, N respectively;
therefore G, H, K, L measure E, M, N, F the same number of times.

Therefore G, H, K, L are in the same ratio with E, M, N, F. [VII. Def. 20]

But G, H, K, L are in the same ratio with A, C, D, B;
therefore A, C, D, B are also in the same ratio with E, M, N, F.

But A, C, D, B are in continued proportion;
therefore E, M, N, F are also in continued proportion.

Therefore, as many numbers as have fallen between A, B in continued proportion with them, so many numbers have also fallen between E, F in continued proportion.

Q. E. D.

1. **fall.** The Greek word is ἐμπίπτειν, "fall *in*" = "can be interpolated."

If $a : b = e : f$, and between a, b there are any number of geometric means c, d, there will be as many such means between e, f.

Let $\alpha, \beta, \gamma, \ldots, \delta$ be the least possible terms in the same ratio as $a, c, d, \ldots b$.

Then α, δ are prime to one another, [VIII. 3]
and, *ex aequali*,

$$\alpha : \delta = a : b$$
$$= e : f.$$

Therefore $e = m\alpha$, $f = m\delta$, where m is some integer. [VII. 20]

Take the numbers $m\alpha, \quad m\beta, \quad m\gamma, \ \ldots \ m\delta$.

This is a series in the given ratio, and we have the same number of geometric means between $m\alpha$, $m\delta$, or e, f, that there are between a, b.

PROPOSITION 9.

If two numbers be prime to one another, and numbers fall between them in continued proportion, then, however many numbers fall between them in continued proportion, so many will also fall between each of them and an unit in continued proportion.

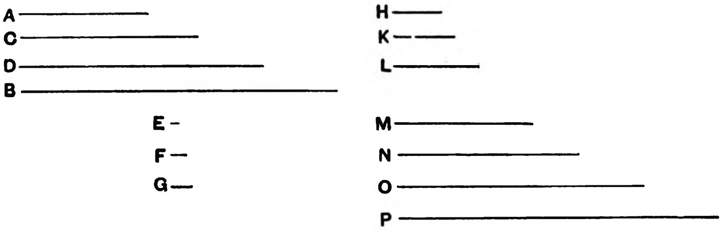
Let A, B be two numbers prime to one another, and let C, D fall between them in continued proportion,
and let the unit E be set out;
I say that, as many numbers as fall between A, B in con-

tinued proportion, so many will also fall between either of the numbers *A*, *B* and the unit in continued proportion.

For let two numbers *F*, *G*, the least that are in the ratio of *A*, *C*, *D*, *B*, be taken,

three numbers *H*, *K*, *L* with the same property,

and others more by one continually, until their multitude is equal to the multitude of *A*, *C*, *D*, *B*. [VIII. 2]



Let them be taken, and let them be *M*, *N*, *O*, *P*.

It is now manifest that *F* by multiplying itself has made *H* and by multiplying *H* has made *M*, while *G* by multiplying itself has made *L* and by multiplying *L* has made *P*. [VIII. 2, Por.]

And, since *M*, *N*, *O*, *P* are the least of those which have the same ratio with *F*, *G*,

and *A*, *C*, *D*, *B* are also the least of those which have the same ratio with *F*, *G*, [VIII. 1]

while the multitude of the numbers *M*, *N*, *O*, *P* is equal to the multitude of the numbers *A*, *C*, *D*, *B*,

therefore *M*, *N*, *O*, *P* are equal to *A*, *C*, *D*, *B* respectively;

therefore *M* is equal to *A*, and *P* to *B*.

Now, since *F* by multiplying itself has made *H*,

therefore *F* measures *H* according to the units in *F*.

But the unit *E* also measures *F* according to the units in it;

therefore the unit *E* measures the number *F* the same number of times as *F* measures *H*.

Therefore, as the unit *E* is to the number *F*, so is *F* to *H*. [VII. Def. 20]

Again, since *F* by multiplying *H* has made *M*,

therefore *H* measures *M* according to the units in *F*.

But the unit E also measures the number F according to the units in it;
therefore the unit E measures the number F the same number of times as H measures M.

Therefore, as the unit E is to the number F, so is H to M.

But it was also proved that, as the unit E is to the number F, so is F to H;
therefore also, as the unit E is to the number F, so is F to H, and H to M.

But M is equal to A;
therefore, as the unit E is to the number F, so is F to H, and H to A.

For the same reason also,
as the unit E is to the number G, so is G to L and L to B.

Therefore, as many numbers as have fallen between A, B in continued proportion, so many numbers also have fallen between each of the numbers A, B and the unit E in continued proportion.

Q. E. D.

Suppose there are n geometric means between a, b, two numbers prime to one another; there are the same number (n) of geometric means between 1 and a and between 1 and b.

If c, d... are the n means between a, b,

$$a, \quad c, \quad d \ldots b$$

are the least numbers in that ratio, since a, b are prime to one another. [VIII. 1]

The terms are therefore respectively identical with

$$\alpha^{n+1}, \quad \alpha^n\beta, \quad \alpha^{n-1}\beta^2 \ldots \alpha\beta^n, \quad \beta^{n+1},$$

where α, β is the common ratio in its lowest terms. [VIII. 2, Por.]

Thus $$a = \alpha^{n+1}, \quad b = \beta^{n+1}.$$

Now $$1 : \alpha = \alpha : \alpha^2 = \alpha^2 : \alpha^3 \ldots = \alpha^n : \alpha^{n+1},$$

and $$1 : \beta = \beta : \beta^2 = \beta^2 : \beta^3 \ldots = \beta^n : \beta^{n+1};$$

whence there are n geometric means between 1, α, and between 1, b.

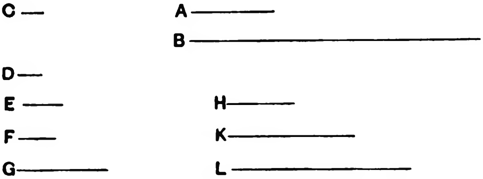
PROPOSITION 10.

If numbers fall between each of two numbers and an unit in continued proportion, however many numbers fall between each of them and an unit in continued proportion, so many also will fall between the numbers themselves in continued proportion.

For let the numbers D, E and F, G respectively fall between the two numbers A, B and the unit C in continued proportion;

I say that, as many numbers as have fallen between each of the numbers A, B and the unit C in continued proportion, so many numbers will also fall between A, B in continued proportion.

For let D by multiplying F make H, and let the numbers D, F by multiplying H make K, L respectively.



Now, since, as the unit C is to the number D, so is D to E, therefore the unit C measures the number D the same number of times as D measures E. [VII. Def. 20]

But the unit C measures the number D according to the units in D;

therefore the number D also measures E according to the units in D;

therefore D by multiplying itself has made E.

Again, since, as C is to the number D, so is E to A, therefore the unit C measures the number D the same number of times as E measures A.

But the unit C measures the number D according to the units in D;

therefore E also measures A according to the units in D;

therefore D by multiplying E has made A.

For the same reason also

F by multiplying itself has made G, and by multiplying G has made B.

And, since D by multiplying itself has made E and by multiplying F has made H,

therefore, as D is to F, so is E to H. [VII. 17]

For the same reason also,

as *D* is to *F*, so is *H* to *G*. [VII. 18]

Therefore also, as *E* is to *H*, so is *H* to *G*.

Again, since *D* by multiplying the numbers *E*, *H* has made *A*, *K* respectively,

therefore, as *E* is to *H*, so is *A* to *K*. [VII. 17]

But, as *E* is to *H*, so is *D* to *F*;

therefore also, as *D* is to *F*, so is *A* to *K*.

Again, since the numbers *D*, *F* by multiplying *H* have made *K*, *L* respectively,

therefore, as *D* is to *F*, so is *K* to *L*. [VII. 18]

But, as *D* is to *F*, so is *A* to *K*;

therefore also, as *A* is to *K*, so is *K* to *L*.

Further, since *F* by multiplying the numbers *H*, *G* has made *L*, *B* respectively,

therefore, as *H* is to *G*, so is *L* to *B*. [VII. 17]

But, as *H* is to *G*, so is *D* to *F*;

therefore also, as *D* is to *F*, so is *L* to *B*.

But it was also proved that,

as *D* is to *F*, so is *A* to *K* and *K* to *L*;

therefore also, as *A* is to *K*, so is *K* to *L* and *L* to *B*.

Therefore *A*, *K*, *L*, *B* are in continued proportion.

Therefore, as many numbers as fall between each of the numbers *A*, *B* and the unit *C* in continued proportion, so many also will fall between *A*, *B* in continued proportion.

Q. E. D.

If there be n geometric means between 1 and a, and also between 1 and b, there will be n geometric means between a and b.

The proposition is the converse of the preceding.

The n means with the extremes form two geometric series of the form

$$1, \quad \alpha, \quad \alpha^2 \ldots \alpha^n, \quad \alpha^{n+1},$$

$$1, \quad \beta, \quad \beta^2 \ldots \beta^n, \quad \beta^{n+1},$$

where

$$\alpha^{n+1} = a, \quad \beta^{n+1} = b.$$

By multiplying the last term in the first line by the first in the second, the last but one in the first line by the second in the second, and so on, we get the series

$$\alpha^{n+1}, \quad \alpha^n\beta, \quad \alpha^{n-1}\beta^2 \ldots \alpha^2\beta^{n-1}, \quad \alpha\beta^n, \quad \beta^{n+1}$$

and we have the n means between a and b.

It will be observed that, when Euclid says "*For the same reason* also, as *D* is to *F*, so is *H* to *G*," the reference is really to VII. 18 instead of VII. 17.

He infers namely that $D \times F : F \times F = D : F$. But since, by VII. 16, the order of multiplication is indifferent, he is practically justified in saying "for the same reason." The same thing occurs in later propositions.

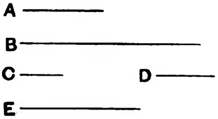
PROPOSITION 11.

Between two square numbers there is one mean proportional number, and the square has to the square the ratio duplicate of that which the side has to the side.

Let A, B be square numbers,

and let C be the side of A, and D of B;

I say that between A, B there is one mean proportional number, and A has to B the ratio duplicate of that which C has to D.



For let C by multiplying D make E.

Now, since A is a square and C is its side,

therefore C by multiplying itself has made A.

For the same reason also

D by multiplying itself has made B.

Since then C by multiplying the numbers C, D has made A, E respectively,

therefore, as C is to D, so is A to E. [VII. 17]

For the same reason also,

as C is to D, so is E to B. [VII. 18]

Therefore also, as A is to E, so is E to B.

Therefore between A, B there is one mean proportional number.

I say next that A also has to B the ratio duplicate of that which C has to D.

For, since A, E, B are three numbers in proportion,

therefore A has to B the ratio duplicate of that which A has to E. [V. Def. 9]

But, as A is to E, so is C to D.

Therefore A has to B the ratio duplicate of that which the side C has to D. Q. E. D.

According to Nicomachus the theorems in this proposition and the next, that two squares have *one* geometric mean, and two cubes *two* geometric means, between them are Platonic. Cf. *Timaeus*, 32 A sqq. and the note thereon, p. 294 above.

a^2, b^2 being two squares, it is only necessary to form the product ab and to prove that

$$a^2,\quad ab,\quad b^2$$

are in geometrical progression. Euclid proves that

$$a^2 : ab = ab : b^2$$

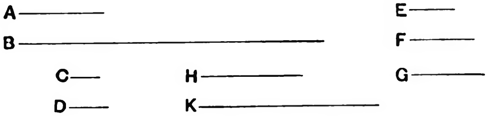
by means of VII. 17, 18, as usual.

In assuming that, since a^2 is to b^2 in the duplicate ratio of a^2 to ab, a^2 is to b^2 in the duplicate ratio of a to b, Euclid assumes that ratios which are the duplicates of equal ratios are equal. This, an obvious inference from V. 22, can be inferred just as easily for numbers from VII. 14.

PROPOSITION 12.

Between two cube numbers there are two mean proportional numbers, and the cube has to the cube the ratio triplicate of that which the side has to the side.

Let A, B be cube numbers,
and let C be the side of A, and D of B;
I say that between A, B there are two mean proportional numbers, and A has to B the ratio triplicate of that which C has to D.



For let C by multiplying itself make E, and by multiplying D let it make F;
let D by multiplying itself make G,
and let the numbers C, D by multiplying F make H, K respectively.

Now, since A is a cube, and C its side,
and C by multiplying itself has made E,
therefore C by multiplying itself has made E and by multiplying E has made A.

For the same reason also
D by multiplying itself has made G and by multiplying G has made B.

And, since C by multiplying the numbers C, D has made E, F respectively,
therefore, as C is to D, so is E to F. [VII. 17]

For the same reason also,

as C is to D, so is F to G. [VII. 18]

Again, since C by multiplying the numbers E, F has made A, H respectively,
therefore, as E is to F, so is A to H. [VII. 17]

But, as E is to F, so is C to D.

Therefore also, as C is to D, so is A to H.

Again, since the numbers C, D by multiplying F have made H, K respectively,
therefore, as C is to D, so is H to K. [VII. 18]

Again, since D by multiplying each of the numbers F, G has made K, B respectively,
therefore, as F is to G, so is K to B. [VII. 17]

But, as F is to G, so is C to D;
therefore also, as C is to D, so is A to H, H to K, and K to B.

Therefore H, K are two mean proportionals between A, B.

I say next that A also has to B the ratio triplicate of that which C has to D.

For, since A, H, K, B are four numbers in proportion, therefore A has to B the ratio triplicate of that which A has to H. [V. Def. 10]

But, as A is to H, so is C to D;
therefore A also has to B the ratio triplicate of that which C has to D.

Q. E. D.

The cube numbers a^3, b^3 being given, Euclid forms the products a^2b, ab^2 and then proves, as usual, by means of VII. 17, 18 that

$$a^3,\quad a^2b,\quad ab^2,\quad b^3$$

are in continued proportion.

He assumes that, since a^3 has to b^3 the ratio triplicate of $a^3 : a^2b$, the ratio $a^3 : b^3$ is triplicate of the ratio $a : b$ which is equal to $a^3 : a^2b$. This is again an obvious inference from VII. 14.

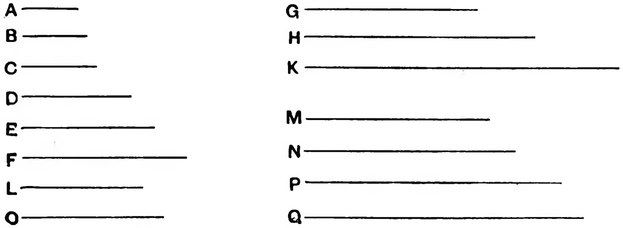
PROPOSITION 13.

If there be as many numbers as we please in continued proportion, and each by multiplying itself make some number, the products will be proportional; and, if the original numbers by multiplying the products make certain numbers, the latter will also be proportional.

Let there be as many numbers as we please, *A*, *B*, *C*, in continued proportion, so that, as *A* is to *B*, so is *B* to *C*;

let *A*, *B*, *C* by multiplying themselves make *D*, *E*, *F*, and by multiplying *D*, *E*, *F* let them make *G*, *H*, *K*;

I say that *D*, *E*, *F* and *G*, *H*, *K* are in continued proportion.



For let *A* by multiplying *B* make *L*,

and let the numbers *A*, *B* by multiplying *L* make *M*. *N* respectively.

And again let *B* by multiplying *C* make *O*,

and let the numbers *B*, *C* by multiplying *O* make *P*, *Q* respectively.

Then, in manner similar to the foregoing, we can prove that

D, *L*, *E* and *G*, *M*, *N*, *H* are continuously proportional in the ratio of *A* to *B*,

and further *E*, *O*, *F* and *H*, *P*, *Q*, *K* are continuously proportional in the ratio of *B* to *C*.

Now, as *A* is to *B*, so is *B* to *C*;

therefore *D*, *L*, *E* are also in the same ratio with *E*, *O*, *F*,

and further *G*, *M*, *N*, *H* in the same ratio with *H*, *P*, *Q*, *K*.

And the multitude of *D*, *L*, *E* is equal to the multitude of *E*, *O*, *F*, and that of *G*, *M*, *N*, *H* to that of *H*, *P*, *Q*, *K*;

therefore, *ex aequali*,

as *D* is to *E*, so is *E* to *F*,

and, as *G* is to *H*, so is *H* to *K*. [VII. 14]

If $a, b, c \dots$ be a series in geometrical progression, then

$$\begin{array}{rlll} & a^2, & b^2, & c^2 \dots \\ \text{and} & a^3, & b^3, & c^3 \dots \end{array}\Bigg\} \text{ are also in geometrical progression.}$$

Heiberg brackets the words added to the enunciation which extend the theorem to any powers. The words are "and this always occurs with the extremes" (καὶ ἀεὶ περὶ τοὺς ἄκρους τοῦτο συμβαίνει). They seem to be rightly suspected on the same grounds as the same words added to the enunciation of VII. 27. There is no allusion to them in the proof, much less any proof of the extension.

Euclid forms, besides the squares and cubes of the given numbers, the products ab, a^2b, ab^2, bc, b^2c, bc^2. When he says that "we prove in manner similar to the foregoing," he indicates successive uses of VII. 17, 18 as in VIII. 12.

With our notation the proof is as easy to see for *any* powers as for squares and cubes.

To prove that a^n, b^n, $c^n \dots$ are in geometrical progression.

Form all the means between a^n, b^n, and set out the series

$$a^n, \quad a^{n-1}b, \quad a^{n-2}b^2 \dots ab^{n-1}, \quad b^n.$$

The common ratio of one term to the next is $a : b$.

Next take the geometrical progression

$$b^n, \quad b^{n-1}c, \quad b^{n-2}c^2 \dots bc^{n-1}, \quad c^n,$$

the common ratio of which is $b : c$.

Proceed thus for all pairs of consecutive terms.

Now $$a : b = b : c = \dots$$

Therefore any pair of succeeding terms in one series are in the same ratio as any pair of succeeding terms in any other of the series.

And the number of terms in each is the same, namely $(n + 1)$.

Therefore, *ex aequali*,

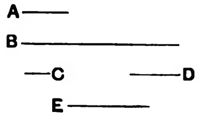
$$a^n : b^n = b^n : c^n = c^n : d^n = \dots$$

PROPOSITION 14.

If a square measure a square, the side will also measure the side; and, if the side measure the side, the square will also measure the square.

Let A, B be square numbers, let C, D be their sides, and let A measure B;

I say that C also measures D.



For let C by multiplying D make E; therefore A, E, B are continuously proportional in the ratio of C to D. [VIII. 11]

And, since A, E, B are continuously proportional, and A measures B, therefore A also measures E. [VIII. 7]

And, as A is to E, so is C to D;
therefore also C measures D. [VII. Def. 20]

Again, let C measure D;
I say that A also measures B.

For, with the same construction, we can in a similar manner prove that A, E, B are continuously proportional in the ratio of C to D.

And since, as C is to D, so is A to E,
and C measures D,
therefore A also measures E. [VII. Def. 20]

And A, E, B are continuously proportional;
therefore A also measures B.

Therefore etc.

Q. E. D.

If a^2 measures b^2, a measures b; and, if a measures b, a^2 measures b^2.

(1) a^2, ab, b^2 are in continued proportion in the ratio of a to b.

Therefore, since a^2 measures b^2,
a^2 measures ab. [VIII. 7]
But $a^2 : ab = a : b$.
Therefore a measures b.

(2) Since a measures b, a^2 measures ab.
And a^2, ab, b^2 are continuously proportional.
Thus ab measures b^2.
And a^2 measures ab.
Therefore a^2 measures b^2.

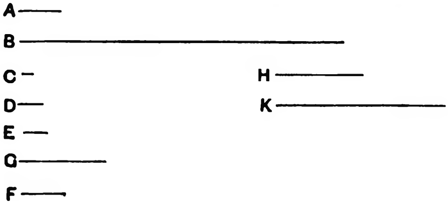
It will be seen that Euclid puts the last step shortly, saying that, since a^2 measures ab, and a^2, ab, b^2 are in continued proportion, a^2 measures b^2. The same thing happens in VIII. 15, where the series of terms is one more than here.

PROPOSITION 15.

If a cube number measure a cube number, the side will also measure the side; and, if the side measure the side, the cube will also measure the cube.

For let the cube number A measure the cube B,
and let C be the side of A and D of B;
I say that C measures D.

For let C by multiplying itself make E,
and let D by multiplying itself make G;
further, let C by multiplying D make F,
and let C, D by multiplying F make H, K respectively.



Now it is manifest that E, F, G and A, H, K, B are continuously proportional in the ratio of C to D. [VIII. 11, 12]

And, since A, H, K, B are continuously proportional,
and A measures B,
therefore it also measures H. [VIII. 7]

And, as A is to H, so is C to D;
therefore C also measures D. [VII. Def. 20]

Next, let C measure D;
I say that A will also measure B.

For, with the same construction, we can prove in a similar manner that A, H, K, B are continuously proportional in the ratio of C to D.

And, since C measures D,
and, as C is to D, so is A to H,
therefore A also measures H, [VII. Def. 20]
so that A measures B also.

Q. E. D.

If a^3 measures b^3, a measures b; and *vice versa*. The proof is, *mutatis mutandis*, the same as for squares.

(1) a^3, a^2b, ab^2, b^3 are continuously proportional in the ratio of a to b; and a^3 measures b^3.

Therefore a^3 measures a^2b; [VIII. 7]
and hence a measures b.

(2) Since a measures b, a^3 measures a^2b.

And, a^3, a^2b, ab^2, b^3 being continuously proportional, each term measures the succeeding term;
therefore a^3 measures b^3.

PROPOSITION 16.

If a square number do not measure a square number, neither will the side measure the side; and, if the side do not measure the side, neither will the square measure the square.

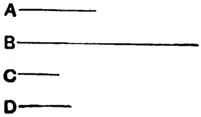
Let A, B be square numbers, and let C, D be their sides; and let A not measure B;

I say that neither does C measure D.

For, if C measures D, A will also measure B. [VIII. 14]

But A does not measure B;

therefore neither will C measure D.



Again, let C not measure D;

I say that neither will A measure B.

For, if A measures B, C will also measure D. [VIII. 14]

But C does not measure D;

therefore neither will A measure B.

Q. E. D.

If a^2 does not measure b^2, a will not measure b; and, if a does not measure b, a^2 will not measure b^2.

The proof is a mere *reductio ad absurdum* using VIII. 14.

PROPOSITION 17.

If a cube number do not measure a cube number, neither will the side measure the side; and, if the side do not measure the side, neither will the cube measure the cube.

For let the cube number A not measure the cube number B,

and let C be the side of A, and D of B;

I say that C will not measure D.

For if C measures D, A will also measure B. [VIII. 15]

But A does not measure B;

therefore neither does C measure D.

Again, let C not measure D;

I say that neither will A measure B.

For, if A measures B, C will also measure D. [VIII. 15]

But C does not measure D;

therefore neither will A measure B.

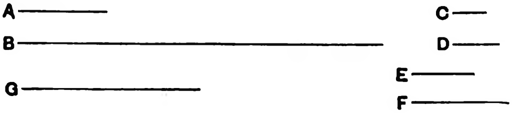
Q. E. D.

If a^3 does not measure b^3, a will not measure b; and *vice versa*. Proved by *reductio ad absurdum* employing VIII. 15.

PROPOSITION 18.

Between two similar plane numbers there is one mean proportional number; and the plane number has to the plane number the ratio duplicate of that which the corresponding side has to the corresponding side.

Let A, B be two similar plane numbers, and let the numbers C, D be the sides of A, and E, F of B.



Now, since similar plane numbers are those which have their sides proportional, [VII. Def. 21]

therefore, as C is to D, so is E to F.

I say then that between A, B there is one mean proportional number, and A has to B the ratio duplicate of that which C has to E, or D to F, that is, of that which the corresponding side has to the corresponding side.

Now since, as C is to D, so is E to F,

therefore, alternately, as C is to E, so is D to F. [VII. 13]

And, since A is plane, and C, D are its sides,

therefore D by multiplying C has made A.

For the same reason also

E by multiplying F has made B.

Now let D by multiplying E make G.

Then, since D by multiplying C has made A, and by multiplying E has made G,

therefore, as C is to E, so is A to G. [VII. 17]

But, as C is to E, so is D to F;

therefore also, as D is to F, so is A to G.

Again, since E by multiplying D has made G, and by multiplying F has made B,

therefore, as D is to F, so is G to B. [VII. 17]

But it was also proved that,

as D is to F, so is A to G;

therefore also, as A is to G, so is G to B.

Therefore A, G, B are in continued proportion.

Therefore between A, B there is one mean proportional number.

I say next that A also has to B the ratio duplicate of that which the corresponding side has to the corresponding side, that is, of that which C has to E or D to F.

For, since A, G, B are in continued proportion,

A has to B the ratio duplicate of that which it has to G. [V. Def. 9]

And, as A is to G, so is C to E, and so is D to F.

Therefore A also has to B the ratio duplicate of that which C has to E or D to F.

Q. E. D.

If ab, cd be "similar plane numbers," i.e. products of factors such that

$$a : b = c : d,$$

there is one mean proportional between ab and cd; and ab is to cd in the duplicate ratio of a to c or of b to d.

Form the product bc (or ad, which is equal to it, by VII. 19).

Then $$ab,\ \left.\begin{matrix} bc \\ = ad \end{matrix}\right\},\ cd$$

is a series of terms in geometrical progression.

For $$a : b = c : d.$$

Therefore $$a : c = b : d.$$ [VII. 13]

Therefore $$ab : bc = bc : cd.$$ [VII. 17 and 16]

Thus bc (or ad) is a geometric mean between ab, cd.

And ab is to cd in the duplicate ratio of ab to bc or of bc to cd, that is, of a to c or of b to d.

PROPOSITION 19.

Between two similar solid numbers there fall two mean proportional numbers; and the solid number has to the similar solid number the ratio triplicate of that which the corresponding side has to the corresponding side.

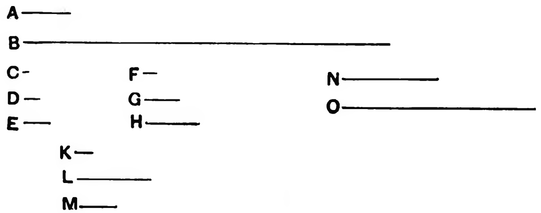
Let *A*, *B* be two similar solid numbers, and let *C*, *D*, *E* be the sides of *A*, and *F*, *G*, *H* of *B*.

Now, since similar solid numbers are those which have their sides proportional, [VII. Def. 21]

therefore, as *C* is to *D*, so is *F* to *G*,

and, as *D* is to *E*, so is *G* to *H*.

I say that between *A*, *B* there fall two mean proportional numbers, and *A* has to *B* the ratio triplicate of that which *C* has to *F*, *D* to *G*, and also *E* to *H*.



For let *C* by multiplying *D* make *K*, and let *F* by multiplying *G* make *L*.

Now, since *C*, *D* are in the same ratio with *F*, *G*,

and *K* is the product of *C*, *D*, and *L* the product of *F*, *G*, *K*, *L* are similar plane numbers; [VII. Def. 21]

therefore between *K*, *L* there is one mean proportional number. [VIII. 18]

Let it be *M*

Therefore *M* is the product of *D*, *F*, as was proved in the theorem preceding this. [VIII. 18]

Now, since *D* by multiplying *C* has made *K*, and by multiplying *F* has made *M*,

therefore, as *C* is to *F*, so is *K* to *M*. [VII. 17]

But, as *K* is to *M*, so is *M* to *L*.

Therefore *K*, *M*, *L* are continuously proportional in the ratio of *C* to *F*.

And since, as *C* is to *D*, so is *F* to *G*,
alternately therefore, as *C* is to *F*, so is *D* to *G*. [VII. 13]

For the same reason also,

as *D* is to *G*, so is *E* to *H*.

Therefore *K*, *M*, *L* are continuously proportional in the ratio of *C* to *F*, in the ratio of *D* to *G*, and also in the ratio of *E* to *H*.

Next, let *E*, *H* by multiplying *M* make *N*, *O* respectively.

Now, since *A* is a solid number, and *C*, *D*, *E* are its sides, therefore *E* by multiplying the product of *C*, *D* has made *A*.

But the product of *C*, *D* is *K*;
therefore *E* by multiplying *K* has made *A*.

For the same reason also

H by multiplying *L* has made *B*.

Now, since *E* by multiplying *K* has made *A*, and further also by multiplying *M* has made *N*,
therefore, as *K* is to *M*, so is *A* to *N*. [VII. 17]

But, as *K* is to *M*, so is *C* to *F*, *D* to *G*, and also *E* to *H*;
therefore also, as *C* is to *F*, *D* to *G*, and *E* to *H*, so is *A* to *N*.

Again, since *E*, *H* by multiplying *M* have made *N*, *O* respectively,
therefore, as *E* is to *H*, so is *N* to *O*. [VII. 18]

But, as *E* is to *H*, so is *C* to *F* and *D* to *G*;
therefore also, as *C* is to *F*, *D* to *G*, and *E* to *H*, so is *A* to *N* and *N* to *O*.

Again, since *H* by multiplying *M* has made *O*, and further also by multiplying *L* has made *B*,
therefore, as *M* is to *L*, so is *O* to *B*. [VII. 17]

But, as *M* is to *L*, so is *C* to *F*, *D* to *G*, and *E* to *H*.

Therefore also, as *C* is to *F*, *D* to *G*, and *E* to *H*, so not only is *O* to *B*, but also *A* to *N* and *N* to *O*.

Therefore *A*, *N*, *O*, *B* are continuously proportional in the aforesaid ratios of the sides.

I say that *A* also has to *B* the ratio triplicate of that which the corresponding side has to the corresponding side, that is, of the ratio which the number *C* has to *F*, or *D* to *G*, and also *E* to *H*.

For, since *A*, *N*, *O*, *B* are four numbers in continued proportion,

therefore *A* has to *B* the ratio triplicate of that which *A* has to *N*. [v. Def. 10]

But, as *A* is to *N*, so it was proved that *C* is to *F*, *D* to *G*, and also *E* to *H*.

Therefore *A* also has to *B* the ratio triplicate of that which the corresponding side has to the corresponding side, that is, of the ratio which the number *C* has to *F*, *D* to *G*, and also *E* to *H*. Q. E. D.

In other words, if $a : b : c = d : e : f$, then there are two geometric means between abc, def; and abc is to def in the triplicate ratio of a to d, or b to e, or c to f.

Euclid first takes the plane numbers ab, de (leaving out c, f) and forms the product bd. Thus, as in VIII. 18,

$$ab,\ \left.\begin{matrix} bd \\ = ea \end{matrix}\right\},\ de$$

are three terms in geometrical progression in the ratio of a to d, or of b to e.

He next forms the products of c, f respectively into the mean bd.

Then $$abc,\quad cbd,\quad fbd,\quad def$$

are in geometrical progression in the ratio of a to d etc.

For $$\left.\begin{aligned} abc : cbd &= ab : bd = a : d \\ bd : fbd &= c : f \\ fbd : def &= bd : de = b : e \end{aligned}\right\}. \qquad \text{[VII. 17]}$$

And $$a : d = b : e = c : f.$$

The ratio of abc to def is the ratio triplicate of that of abc to cbd, i.e. of that of a to d etc.

PROPOSITION 20.

If one mean proportional number fall between two numbers, the numbers will be similar plane numbers.

For let one mean proportional number *C* fall between the two numbers *A*, *B*;

I say that *A*, *B* are similar plane numbers.

Let *D*, *E*, the least numbers of those which have the same ratio with *A*, *C*, be taken; [VII. 33]

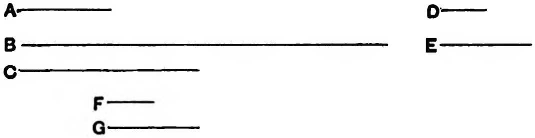
therefore *D* measures *A* the same number of times that *E* measures *C*. [VII. 20]

Now, as many times as *D* measures *A*, so many units let there be in *F*;

therefore *F* by multiplying *D* has made *A*,

so that *A* is plane, and *D*, *F* are its sides.

Again, since *D*, *E* are the least of the numbers which have the same ratio with *C*, *B*,
therefore *D* measures *C* the same number of times that *E* measures *B*. [VII. 20]



As many times, then, as *E* measures *B*, so many units let there be in *G*;
therefore *E* measures *B* according to the units in *G*;
therefore *G* by multiplying *E* has made *B*.

Therefore *B* is plane, and *E*, *G* are its sides.

Therefore *A*, *B* are plane numbers.

I say next that they are also similar.

For, †since *F* by multiplying *D* has made *A*, and by multiplying *E* has made *C*,
therefore, as *D* is to *E*, so is *A* to *C*, that is, *C* to *B*. [VII. 17]

Again,† since *E* by multiplying *F*, *G* has made *C*, *B* respectively,
therefore, as *F* is to *G*, so is *C* to *B*. [VII. 17]

But, as *C* is to *B*, so is *D* to *E*;
therefore also, as *D* is to *E*, so is *F* to *G*.

And alternately, as *D* is to *F*, so is *E* to *G*. [VII. 13]

Therefore *A*, *B* are similar plane numbers; for their sides are proportional. Q. E. D.

25. **For, since F......27. C to B.** The text has clearly suffered corruption here. It is not necessary to *infer* from other facts that, as *D* is to *E*, so is *A* to *C*; for this is part of the hypotheses (ll. 6, 7). Again, there is no explanation of the statement (l. 25) that *F* by multiplying *E* has made *C*. It is the statement and explanation of this latter fact which are alone wanted; after which the proof proceeds as in l. 28. We might therefore substitute for ll. 25—28 the following.

"For, since *E* measures *C* the same number of times that *D* measures *A* [l. 8], that is, according to the units in *F* [l. 10], therefore *F* by multiplying *E* has made *C*.

And, since *E* by multiplying *F*, *G*," etc. etc.

This proposition is the converse of VIII. 18. If a, c, b are in geometrical progression, a, b are "similar plane numbers."

Let $\alpha : \beta$ be the ratio $a : c$ (and therefore also the ratio $c : b$) in its lowest terms.

Then [VII. 20]

$$a = m\alpha, \quad c = m\beta, \quad \text{where } m \text{ is some integer,}$$
$$c = n\alpha, \quad b = n\beta, \quad \text{where } n \text{ is some integer.}$$

Thus a, b are both products of two factors, i.e. plane.

Again, $a : \beta = a : c = c : b$

$= m : n.$ [VII. 18]

Therefore, alternately, $a : m = \beta : n,$ [VII. 13]

and hence ma, $n\beta$ are *similar* plane numbers.

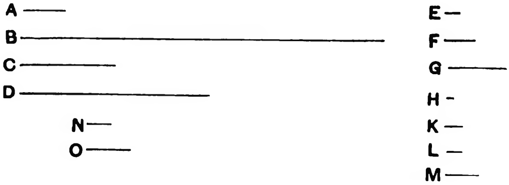
[Our notation makes the second part still more obvious, for $c = m\beta = na$.]

PROPOSITION 21.

If two mean proportional numbers fall between two numbers, the numbers are similar solid numbers.

For let two mean proportional numbers C, D fall between the two numbers A, B;

I say that A, B are similar solid numbers.



For let three numbers E, F, G, the least of those which have the same ratio with A, C, D, be taken; [VII. 33 or VIII. 2]

therefore the extremes of them E, G are prime to one another. [VIII. 3]

Now, since one mean proportional number F has fallen between E, G,

therefore E, G are similar plane numbers. [VIII. 20]

Let, then, H, K be the sides of E, and L, M of G.

Therefore it is manifest from the theorem before this that E, F, G are continuously proportional in the ratio of H to L and that of K to M.

Now, since E, F, G are the least of the numbers which have the same ratio with A, C, D,

and the multitude of the numbers E, F, G is equal to the multitude of the numbers A, C, D,

therefore, *ex aequali*, as E is to G, so is A to D. [VII. 14]

But E, G are prime,

primes are also least, [VII. 21]

and the least measure those which have the same ratio with them the same number of times, the greater the greater and the less the less, that is, the antecedent the antecedent and the consequent the consequent; [VII. 20]

therefore *E* measures *A* the same number of times that *G* measures *D*.

Now, as many times as *E* measures *A*, so many units let there be in *N*.

Therefore *N* by multiplying *E* has made *A*.

But *E* is the product of *H*, *K*;

therefore *N* by multiplying the product of *H*, *K* has made *A*.

Therefore *A* is solid, and *H*, *K*, *N* are its sides.

Again, since *E*, *F*, *G* are the least of the numbers which have the same ratio as *C*, *D*, *B*,

therefore *E* measures *C* the same number of times that *G* measures *B*.

Now, as many times as *E* measures *C*, so many units let there be in *O*.

Therefore *G* measures *B* according to the units in *O*;

therefore *O* by multiplying *G* has made *B*.

But *G* is the product of *L*, *M*;

therefore *O* by multiplying the product of *L*, *M* has made *B*.

Therefore *B* is solid, and *L*, *M*, *O* are its sides;

therefore *A*, *B* are solid.

I say that they are also similar.

For since *N*, *O* by multiplying *E* have made *A*, *C*,

therefore, as *N* is to *O*, so is *A* to *C*, that is, *E* to *F*. [VII. 18]

But, as *E* is to *F*, so is *H* to *L* and *K* to *M*;

therefore also, as *H* is to *L*, so is *K* to *M* and *N* to *O*.

And *H*, *K*, *N* are the sides of *A*, and *O*, *L*, *M* the sides of *B*.

Therefore *A*, *B* are similar solid numbers. Q. E. D.

The converse of VIII. 19. If a, c, d, b are in geometrical progression, a, b are "similar solid numbers."

Let α, β, γ be the least numbers in the ratio of a, c, d (and therefore also of c, d, b). [VII. 33 or VIII. 2]

Therefore α, γ are prime to one another. [VIII. 3]

They are also "similar plane numbers." [VIII. 20]

Let $$\alpha = mn, \quad \gamma = pq,$$

where $$m : n = p : q.$$

Then, by the proof of VIII. 20,

$$\alpha : \beta = m : p = n : q.$$

Now, *ex aequali*, $a : d = \alpha : \gamma$, [VII. 14]

and, since α, γ are prime to one another,

$$a = r\alpha, \quad d = r\gamma, \quad \text{where } r \text{ is an integer.}$$

But $\alpha = mn$:

therefore $a = rmn$, and therefore a is "solid."

Again, *ex aequali*, $c : b = \alpha : \gamma$,

and therefore $c = s\alpha$, $b = s\gamma$, where s is an integer.

Thus $b = spq$, and b is therefore "solid."

Now $\alpha : \beta = a : c = r\alpha : s\alpha$

$= r : s$. [VII. 18]

And, from above, $\alpha : \beta = m : p = n : q$.

Therefore $r : s = m : p = n : q$,

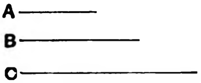
and hence a, b are *similar* solid numbers.

PROPOSITION 22.

If three numbers be in continued proportion, and the first be square, the third will also be square.

Let A, B, C be three numbers in continued proportion, and let A the first be square;

I say that C the third is also square.



For, since between A, C there is one mean proportional number, B,

therefore A, C are similar plane numbers. [VIII. 20]

But A is square;

therefore C is also square. Q. E. D.

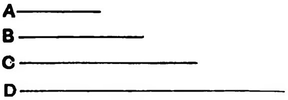
A mere application of VIII. 20 to the particular case where one of the "similar plane numbers" is square.

PROPOSITION 23.

If four numbers be in continued proportion, and the first be cube, the fourth will also be cube.

Let A, B, C, D be four numbers in continued proportion, and let A be cube;

I say that D is also cube.



For, since between A, D there are two mean proportional numbers B, C,

therefore A, D are similar solid numbers. [VIII. 21]

But A is cube;
therefore D is also cube.

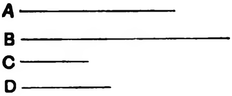
Q. E. D.

A mere application of VIII. 21 to the case where one of the "similar solid numbers" is a cube.

PROPOSITION 24.

If two numbers have to one another the ratio which a square number has to a square number, and the first be square, the second will also be square.

For let the two numbers A, B have to one another the ratio which the square number C has to the square number D, and let A be square;



I say that B is also square.

For, since C, D are square, C, D are similar plane numbers.

Therefore one mean proportional number falls between C, D. [VIII. 18]

And, as C is to D, so is A to B;
therefore one mean proportional number falls between A, B also. [VIII. 8]

And A is square;
therefore B is also square. [VIII. 22]

Q. E. D.

If $a : b = c^2 : d^2$, and a is a square, then b is also a square.
For c^2, d^2 have one mean proportional cd. [VIII. 18]
Therefore a, b, which are in the same ratio, have one mean proportional. [VIII. 8]
And, since a is square, b must also be a square. [VIII. 22]

PROPOSITION 25.

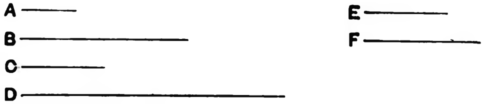
If two numbers have to one another the ratio which a cube number has to a cube number, and the first be cube, the second will also be cube.

For let the two numbers A, B have to one another the ratio which the cube number C has to the cube number D, and let A be cube;

I say that B is also cube.

For, since *C*, *D* are cube,
C, *D* are similar solid numbers.

Therefore two mean proportional numbers fall between *C*, *D*. [VIII. 19]



And, as many numbers as fall between *C*, *D* in continued proportion, so many will also fall between those which have the same ratio with them; [VIII. 8]
so that two mean proportional numbers fall between *A*, *B* also.

Let *E*, *F* so fall.

Since, then, the four numbers *A*, *E*, *F*, *B* are in continued proportion,
and *A* is cube,
therefore *B* is also cube. [VIII. 23]

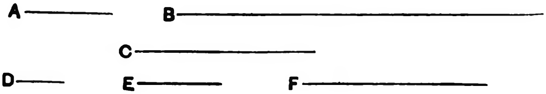
Q. E. D.

If $a : b = c^3 : d^3$, and a is a cube, then b is also a cube.
For c^3, d^3 have two mean proportionals. [VIII. 19]
Therefore a, b also have two mean proportionals. [VIII. 8]
And a is a cube:
therefore b is a cube. [VIII. 23]

PROPOSITION 26.

Similar plane numbers have to one another the ratio which a square number has to a square number.

Let *A*, *B* be similar plane numbers;
I say that *A* has to *B* the ratio which a square number has to a square number.



For, since *A*, *B* are similar plane numbers,
therefore one mean proportional number falls between *A*, *B*. [VIII. 18]

Let it so fall, and let it be C;

and let D, E, F, the least numbers of those which have the same ratio with A, C, B, be taken; [VII. 33 or VIII. 2]

therefore the extremes of them D, F are square. [VIII. 2, Por.]

And since, as D is to F, so is A to B,

and D, F are square,

therefore A has to B the ratio which a square number has to a square number.

Q. E. D.

If a, b are similar "plane numbers," let c be the mean proportional between them. [VIII. 18]

Take α, β, γ the smallest numbers in the ratio of a, c, b. [VII. 33 or VIII. 2]

Then α, γ are squares. [VIII. 2, Por.]

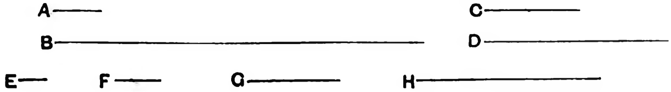
Therefore a, b are in the ratio of a square to a square.

PROPOSITION 27.

Similar solid numbers have to one another the ratio which a cube number has to a cube number.

Let A, B be similar solid numbers;

I say that A has to B the ratio which a cube number has to a cube number.



For, since A, B are similar solid numbers,

therefore two mean proportional numbers fall between A, B. [VIII. 19]

Let C, D so fall,

and let E, F, G, H, the least numbers of those which have the same ratio with A, C, D, B, and equal with them in multitude, be taken; [VII. 33 or VIII. 2]

therefore the extremes of them E, H are cube. [VIII. 2, Por.]

And, as E is to H, so is A to B;

therefore A also has to B the ratio which a cube number has to a cube number.

Q. E. D.

The same thing as VIII. 26 with cubes. It is proved in the same way except that VIII. 19 is used instead of VIII. 18.

The last note of an-Nairīzī in which the name of Heron is mentioned is on this proposition. Heron is there stated (p. 194—5, ed. Curtze) to have added the two propositions that,

1. *If two numbers have to one another the ratio of a square to a square, the numbers are similar plane numbers;*

2. *If two numbers have to one another the ratio of a cube to a cube, the numbers are similar solid numbers.*

The propositions are of course the converses of VIII. 26, 27 respectively. They are easily proved.

(1) If $a : b = c^2 : d^2$,

then, since there is one mean proportional (cd) between c^2, d^2, [VIII. 11 or 18]

there is also one mean proportional between a, b. [VIII. 8]

Therefore a, b are similar plane numbers. [VIII. 20]

(2) is similarly proved by the use of VIII. 12 or 19, VIII. 8, VIII. 21.

The insertion by Heron of the first of the two propositions, the converse of VIII. 26, is perhaps an argument in favour of the correctness of the text of IX. 10, though (as remarked in the note on that proposition) it does not give the easiest proof. Cf. Heron's extension of VII. 3 tacitly assumed by Euclid in VII. 33.

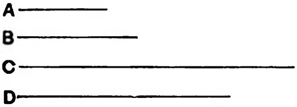
BOOK IX.

PROPOSITION 1.

If two similar plane numbers by multiplying one another make some number, the product will be square.

Let *A*, *B* be two similar plane numbers, and let *A* by multiplying *B* make *C*;

I say that *C* is square.



For let *A* by multiplying itself make *D*.

Therefore *D* is square.

Since then *A* by multiplying itself has made *D*, and by multiplying *B* has made *C*,

therefore, as *A* is to *B*, so is *D* to *C*. [VII. 17]

And, since *A*, *B* are similar plane numbers,

therefore one mean proportional number falls between *A*, *B*. [VIII. 18]

But, if numbers fall between two numbers in continued proportion, as many as fall between them, so many also fall between those which have the same ratio; [VIII. 8]

so that one mean proportional number falls between *D*, *C* also.

And *D* is square;

therefore *C* is also square. [VIII. 22]

Q. E. D.

The product of two similar plane numbers is a square.

Let a, b be two similar plane numbers.

Now $a : b = a^2 : ab$. [VII. 17]

And between a, b there is one mean proportional. [VIII. 18]

Therefore between $a^2 : ab$ there is one mean proportional. [VIII. 8]

And a^2 is square;

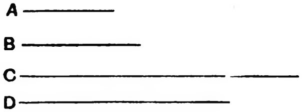
therefore ab is square. [VIII. 22]

PROPOSITION 2.

If two numbers by multiplying one another make a square number, they are similar plane numbers.

Let A, B be two numbers, and let A by multiplying B make the square number C;

I say that A, B are similar plane numbers.



For let A by multiplying itself make D;

therefore D is square.

Now, since A by multiplying itself has made D, and by multiplying B has made C,

therefore, as A is to B, so is D to C. [VII. 17]

And, since D is square, and C is so also,

therefore D, C are similar plane numbers.

Therefore one mean proportional number falls between D, C. [VIII. 18]

And, as D is to C, so is A to B;

therefore one mean proportional number falls between A, B also. [VIII. 8]

But, if one mean proportional number fall between two numbers, they are similar plane numbers; [VIII. 20]

therefore A, B are similar plane numbers.

Q. E. D.

If ab is a square number, a, b are similar plane numbers. (The converse of IX. 1.)

For $a : b = a^2 : ab$. [VII. 17]

And a^2, ab being square numbers, and therefore similar plane numbers, they have one mean proportional. [VIII. 18]

Therefore a, b also have one mean proportional. [VIII. 8]

whence a, b are similar plane numbers. [VIII. 20]

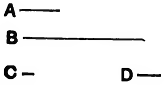
PROPOSITION 3.

If a cube number by multiplying itself make some number, the product will be cube.

For let the cube number A by multiplying itself make B;

I say that B is cube.

For let *C*, the side of *A*, be taken, and let *C* by multiplying itself make *D*.



It is then manifest that *C* by multiplying *D* has made *A*.

Now, since *C* by multiplying itself has made *D*,
therefore *C* measures *D* according to the units in itself.

But further the unit also measures *C* according to the units in it;
therefore, as the unit is to *C*, so is *C* to *D*. [VII. Def. 20]

Again, since *C* by multiplying *D* has made *A*,
therefore *D* measures *A* according to the units in *C*.

But the unit also measures *C* according to the units in it;
therefore, as the unit is to *C*, so is *D* to *A*.

But, as the unit is to *C*, so is *C* to *D*;
therefore also, as the unit is to *C*, so is *C* to *D*, and *D* to *A*.

Therefore between the unit and the number *A* two mean proportional numbers *C*, *D* have fallen in continued proportion.

Again, since *A* by multiplying itself has made *B*,
therefore *A* measures *B* according to the units in itself.

But the unit also measures *A* according to the units in it;
therefore, as the unit is to *A*, so is *A* to *B*. [VII. Def. 20]

But between the unit and *A* two mean proportional numbers have fallen;
therefore two mean proportional numbers will also fall between *A*, *B*. [VIII. 8]

But, if two mean proportional numbers fall between two numbers, and the first be cube, the second will also be cube. [VIII. 23]

And *A* is cube;
therefore *B* is also cube.

Q. E. D.

The product of a^3 into itself, or $a^3 . a^3$, is a cube.

For $1 : a = a : a^2 = a^2 : a^3$.

Therefore between 1 and a^3 there are two mean proportionals.

Also $1 : a^3 = a^3 : a^3 . a^3$.

Therefore two mean proportionals fall between a^3 and $a^3 . a^3$. [VIII. 8]
(It is true that VIII. 8 is only enunciated of two pairs of numbers, but the proof is equally valid if one number of one pair is unity.)

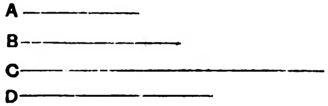
And a^3 is a cube number:
therefore $a^3 . a^3$ is also cube. [VIII. 23]

PROPOSITION 4.

If a cube number by multiplying a cube number make some number, the product will be cube.

For let the cube number A by multiplying the cube number B make C;

I say that C is cube.



For let A by multiplying itself make D;

therefore D is cube. [IX. 3]

And, since A by multiplying itself has made D, and by multiplying B has made C therefore, as A is to B, so is D to C. [VII. 17]

And, since A, B are cube numbers, A, B are similar solid numbers.

Therefore two mean proportional numbers fall between A, B; [VIII. 19]

so that two mean proportional numbers will fall between D, C also. [VIII. 8]

And D is cube;

therefore C is also cube [VIII. 23]

Q. E. D.

The product of two cubes, say $a^3 . b^3$, is a cube.

For $a^3 : b^3 = a^3 . a^3 : a^3 . b^3$. [VII. 17]

And two mean proportionals fall between a^3, b^3 which are similar solid numbers. [VIII. 19]

Therefore two mean proportionals fall between $a^3 . a^3$, $a^3 . b^3$ [VIII. 8]

But $a^3 . a^3$ is a cube: [IX. 3]

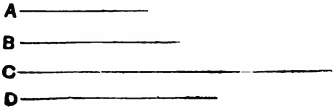
therefore $a^3 . b^3$ is a cube. [VIII. 23]

PROPOSITION 5.

If a cube number by multiplying any number make a cube number, the multiplied number will also be cube.

For let the cube number A by multiplying any number B make the cube number C;

I say that B is cube.



For let A by multiplying itself make D;

therefore D is cube. [IX. 3]

Now, since A by multiplying itself has made D, and by multiplying B has made C,
therefore, as A is to B, so is D to C. [VII. 17]

And since D, C are cube,
they are similar solid numbers.

Therefore two mean proportional numbers fall between D, C. [VIII. 19]

And, as D is to C, so is A to B;
therefore two mean proportional numbers fall between A, B also. [VIII. 8]

And A is cube;
therefore B is also cube. [VIII. 23]

If the product a^3b is a cube number, b is cube.

By IX. 3, the product $a^3 . a^3$ is a cube.

And $a^3 . a^3 : a^3b = a^3 : b$. [VII. 17]

The first two terms are cubes, and therefore "similar solids"; therefore there are two mean proportionals between them. [VIII. 19]

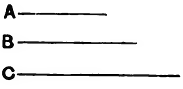
Therefore there are two mean proportionals between a^3, b. [VIII. 8]

And a^3 is a cube:
therefore b is a cube number. [VIII. 23]

PROPOSITION 6.

If a number by multiplying itself make a cube number, it will itself also be cube.

For let the number A by multiplying itself make the cube number B;
I say that A is also cube.



For let A by multiplying B make C.

Since, then, A by multiplying itself has made B, and by multiplying B has made C,
therefore C is cube.

And, since A by multiplying itself has made B,
therefore A measures B according to the units in itself.

But the unit also measures A according to the units in it.

Therefore, as the unit is to A, so is A to B. [VII. Def. 20]

And, since A by multiplying B has made C,
therefore B measures C according to the units in A.

But the unit also measures A according to the units in it.

Therefore, as the unit is to *A*, so is *B* to *C*. [VII. Def. 20]

But, as the unit is to *A*, so is *A* to *B*;

therefore also, as *A* is to *B*, so is *B* to *C*.

And, since *B*, *C* are cube,

they are similar solid numbers.

Therefore there are two mean proportional numbers between *B*, *C*. [VIII. 19]

And, as *B* is to *C*, so is *A* to *B*.

Therefore there are two mean proportional numbers between *A*, *B* also. [VIII. 8]

And *B* is cube;

therefore *A* is also cube. [cf. VIII. 23]

Q. E. D.

If a^2 is a cube number, a is also a cube.

For $$1 : a = a : a^2 = a^2 : a^3.$$

Now a^2, a^3 are both cubes, and therefore "similar solids"; therefore there are two mean proportionals between them. [VIII. 19]

Therefore there are two mean proportionals between a, a^2. [VIII. 8]

And a^2 is a cube;

therefore a is also a cube number. [VIII. 23]

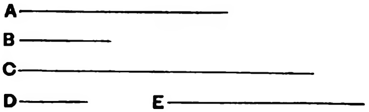
It will be noticed that the last step is not an exact quotation of the result of VIII. 23, because it is there the *first* of four terms which is known to be a cube, and the *last* which is proved to be a cube; here the case is reversed. But there is no difficulty. Without inverting the proportions, we have only to refer to VIII. 21 which proves that a, a^2, having two mean proportionals between them, are two similar solid numbers; whence, since a^2 is a cube, a is also a cube.

PROPOSITION 7.

If a composite number by multiplying any number make some number, the product will be solid.

For let the composite number *A* by multiplying any number *B* make *C*;

I say that *C* is solid.



For, since *A* is composite, it will be measured by some number. [VII. Def. 13]

Let it be measured by *D*;

and, as many times as *D* measures *A*, so many units let there be in *E*.

Since then D measures A according to the units in E, therefore E by multiplying D has made A. [VII. Def. 15]

And, since A by multiplying B has made C, and A is the product of D, E, therefore the product of D, E by multiplying B has made C.

Therefore C is solid, and D, E, B are its sides.

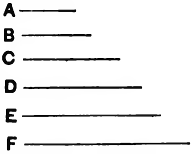
Q. E. D.

Since a composite number is the product of two factors, the result of multiplying it by another number is to produce a number which is the product of three factors, i.e. a "solid number."

Proposition 8.

If as many numbers as we please beginning from an unit be in continued proportion, the third from the unit will be square, as will also those which successively leave out one; the fourth will be cube, as will also all those which leave out two; and the seventh will be at once cube and square, as will also those which leave out five.

Let there be as many numbers as we please, A, B, C, D, E, F, beginning from an unit and in continued proportion;



I say that B, the third from the unit, is square, as are also all those which leave out one; C, the fourth, is cube, as are also all those which leave out two; and F, the seventh, is at once cube and square, as are also all those which leave out five.

For since, as the unit is to A, so is A to B, therefore the unit measures the number A the same number of times that A measures B. [VII. Def. 20]

But the unit measures the number A according to the units in it; therefore A also measures B according to the units in A.

Therefore A by multiplying itself has made B; therefore B is square.

And, since B, C, D are in continued proportion, and B is square, therefore D is also square. [VIII. 22]

For the same reason

F is also square.

Similarly we can prove that all those which leave out one are square.

I say next that *C*, the fourth from the unit, is cube, as are also all those which leave out two.

For since, as the unit is to *A*, so is *B* to *C*,

therefore the unit measures the number *A* the same number of times that *B* measures *C*.

But the unit measures the number *A* according to the units in *A*;

therefore *B* also measures *C* according to the units in *A*.

Therefore *A* by multiplying *B* has made *C*.

Since then *A* by multiplying itself has made *B*, and by multiplying *B* has made *C*,

therefore *C* is cube.

And, since *C*, *D*, *E*, *F* are in continued proportion, and *C* is cube,

therefore *F* is also cube. [VIII. 23]

But it was also proved square;

therefore the seventh from the unit is both cube and square.

Similarly we can prove that all the numbers which leave out five are also both cube and square.

Q. E. D.

If 1, a, a_2, a_3, ... be a geometrical progression, then a_2, a_4, a_6, ... are squares;

a_3, a_6, a_9, ... are cubes;

a_6, a_{12}, ... are both squares and cubes.

Since $$1 : a = a : a_2,$$ $$a_2 = a^2.$$

And, since a_2, a_3, a_4 are in geometrical progression and a_2 $(= a^2)$ is a square,

a_4 is a square. [VIII. 22]

Similarly a_6, a_8, ... are squares.

Next, $$1 : a = a_2 : a_3$$ $$= a^2 : a_3,$$

whence $a_3 = a^3$, a cube number.

And, since a_3, a_4, a_5, a_6 are in geometrical progression, and a_3 is a cube,

a_6 is a cube. [VIII. 23]

Similarly a_9, a_{12}, ... are cubes.

Clearly then a_6, a_{12}, a_{18}, ... are both squares and cubes.

The whole result is of course obvious if the geometrical progression is written, with our notation, as

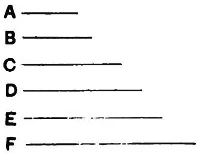
$$1, a, a^2, a^3, a^4, \dots a^n.$$

PROPOSITION 9.

If as many numbers as we please beginning from an unit be in continued proportion, and the number after the unit be square, all the rest will also be square. And, if the number after the unit be cube, all the rest will also be cube.

Let there be as many numbers as we please, *A*, *B*, *C*, *D*, *E*, *F*, beginning from an unit and in continued proportion, and let *A*, the number after the unit, be square;

I say that all the rest will also be square.



Now it has been proved that *B*, the third from the unit, is square, as are also all those which leave out one; [IX. 8]

I say that all the rest are also square.

For, since *A*, *B*, *C* are in continued proportion, and *A* is square,

therefore *C* is also square. [VIII. 22]

Again, since *B*, *C*, *D* are in continued proportion, and *B* is square,

D is also square. [VIII. 22]

Similarly we can prove that all the rest are also square.

Next, let *A* be cube;

I say that all the rest are also cube.

Now it has been proved that *C*, the fourth from the unit, is cube, as also are all those which leave out two; [IX. 8]

I say that all the rest are also cube.

For, since, as the unit is to *A*, so is *A* to *B*, therefore the unit measures *A* the same number of times as *A* measures *B*.

But the unit measures *A* according to the units in it; therefore *A* also measures *B* according to the units in itself; therefore *A* by multiplying itself has made *B*.

And A is cube.

But, if a cube number by multiplying itself make some number, the product is cube. [IX. 3]

Therefore B is also cube.

And, since the four numbers A, B, C, D are in continued proportion,
and A is cube,
D also is cube. [VIII. 23]

For the same reason
E is also cube, and similarly all the rest are cube.

Q. E. D.

If 1, a^2, a_2, a_3, a_4, ... are in geometrical progression, a_2, a_3, a_4, ... are all squares;
and, if 1, a^3, a_2, a_3, a_4, ... are in geometrical progression, a_2, a_3, ... are all cubes.

(1) By IX. 8, a_2, a_4, a_6, ... are all squares.

And, a^2, a_2, a_3 being in geometrical progression, and a^2 being a square,

a_3 is a square. [VIII. 22]

For the same reason a_5, a_7, ... are all squares.

(2) By IX. 8, a_3, a_6, a_9, ... are all cubes.

Now $1 : a^3 = a^3 : a_2$.

Therefore $a_2 = a^3 \,.\, a^3$, which is a cube, by IX. 3.

And, a^3, a_2, a_3, a_4 being in geometrical progression, and a^3 being cube,

a_4 is cube. [VIII. 23]

Similarly we prove that a_5 is cube, and so on.

The results are of course obvious in our notation, the series being

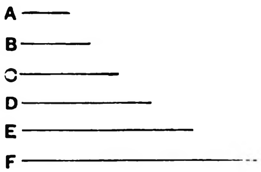
(1) $1, a^2, a^4, a^6, \ldots a^{2n}$,
(2) $1, a^3, a^9, a^{12}, \ldots a^{3n}$.

PROPOSITION 10.

If as many numbers as we please beginning from an unit be in continued proportion, and the number after the unit be not square, neither will any other be square except the third from the unit and all those which leave out one. And, if the number after the unit be not cube, neither will any other be cube except the fourth from the unit and all those which leave out two.

Let there be as many numbers as we please, A, B, C, D, E, F, beginning from an unit and in continued proportion, and let A, the number after the unit, not be square;

I say that neither will any other be square except the third from the unit <and those which leave out one>.



For, if possible, let *C* be square.

But *B* is also square; [IX. 8]

[therefore *B*, *C* have to one another the ratio which a square number has to a square number].

And, as *B* is to *C*, so is *A* to *B*;

therefore *A*, *B* have to one another the ratio which a square number has to a square number;

[so that *A*, *B* are similar plane numbers]. [VIII. 26, converse]

And *B* is square;

therefore *A* is also square:

which is contrary to the hypothesis.

Therefore *C* is not square.

Similarly we can prove that neither is any other of the numbers square except the third from the unit and those which leave out one.

Next, let *A* not be cube.

I say that neither will any other be cube except the fourth from the unit and those which leave out two.

For, if possible, let *D* be cube.

Now *C* is also cube; for it is fourth from the unit. [IX. 8]

And, as *C* is to *D*, so is *B* to *C*;

therefore *B* also has to *C* the ratio which a cube has to a cube.

And *C* is cube;

therefore *B* is also cube. [VIII. 25]

And since, as the unit is to *A*, so is *A* to *B*,

and the unit measures *A* according to the units in it,

therefore *A* also measures *B* according to the units in itself;

therefore *A* by multiplying itself has made the cube number *B*.

But, if a number by multiplying itself make a cube number, it is also itself cube. [IX. 6]

Therefore *A* is also cube:

which is contrary to the hypothesis.

Therefore *D* is not cube.

Similarly we can prove that neither is any other of the numbers cube except the fourth from the unit and those which leave out two.

Q. E. D.

If 1, a, a_2, a_3, a_4, ... be a geometrical progression, then (1), if a is not a square, none of the terms will be square except a_2, a_4, a_6, ...;
and (2), if a is not a cube, none of the terms will be cube except a_3, a_6, a_9, ...

With reference to the first part of the proof, viz. that which proves that, if a_3 is a square, a must be a square, Heiberg remarks that the words which I have bracketed are perhaps spurious; for it is easier to use VIII. 24 than the *converse* of VIII. 26, and a use of VIII. 24 would correspond better to the use of VIII. 25 in the second part relating to cubes. I agree in this view and have bracketed the words accordingly. (See however note, p. 383, on converses of VIII. 26, 27 given by Heron.) If this change be made, the proof runs as follows.

(1) If possible, let a_3 be square.

Now $$a_2 : a_3 = a : a_2.$$

But a_2 is a square. [IX. 8]

Therefore a is to a_2 in the ratio of a square to a square.

And a_2 is square;

therefore a is square [VIII. 24]: which is impossible.

(2) If possible, let a_4 be a cube.

Now $$a_3 : a_4 = a_2 : a_3.$$

And a_3 is a cube. [IX. 8]

Therefore a_2 is to a_3 in the ratio of a cube to a cube.

And a_3 is a cube:

therefore a_2 is a cube. [VIII. 25]

But, since $$1 : a = a : a_2,$$
$$a_2 = a^2.$$

And, since a^2 is a cube,

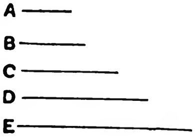
a must be a cube [IX. 6]: which is impossible.

The propositions VIII. 24, 25 are here not quoted in their exact form in that the *first* and *second* squares, or cubes, change places. But there is no difficulty, since the method by which the theorems are proved shows that either inference is equally correct.

PROPOSITION 11.

If as many numbers as we please beginning from an unit be in continued proportion, the less measures the greater according to some one of the numbers which have place among the proportional numbers.

Let there be as many numbers as we please, *B*, *C*, *D*, *E*, beginning from the unit *A* and in continued proportion;



I say that *B*, the least of the numbers *B*, *C*, *D*, *E*, measures *E* according to some one of the numbers *C*, *D*.

For since, as the unit *A* is to *B*, so is *D* to *E*,

therefore the unit *A* measures the number *B* the same number of times as *D* measures *E*;

therefore, alternately, the unit *A* measures *D* the same number of times as *B* measures *E*. [VII. 15]

But the unit *A* measures *D* according to the units in it;

therefore *B* also measures *E* according to the units in *D*;

so that *B* the less measures *E* the greater according to some number of those which have place among the proportional numbers.—

Porism. And it is manifest that, whatever place the measuring number has, reckoned from the unit, the same place also has the number according to which it measures, reckoned from the number measured, in the direction of the number before it.—

Q. E. D.

The proposition and the porism together assert that, if $1, a, a_2, \dots a_n$ be a geometrical progression, a_r measures a_n and gives the quotient a_{n-r} $(r < n)$.

Euclid only proves that $a_n = a \cdot a_{n-1}$, as follows.

$$1 : a = a_{n-1} : a_n.$$

Therefore 1 measures a the same number of times as a_{n-1} measures a_n.

Hence 1 measures a_{n-1} the same number of times as a measures a_n; [VII. 15]

that is, $$a_n = a \cdot a_{n-1}.$$

We can supply the proof of the porism as follows.

$$1 : a = a_r : a_{r+1},$$

$$a : a_2 = a_{r+1} : a_{r+2},$$

$$\dots\dots\dots\dots\dots\dots$$

$$a_{n-r-1} : a_{n-r} = a_{n-1} : a_n,$$

whence, *ex aequali*,

$$1 : a_{n-r} = a_r : a_n. \qquad \text{[VII. 14]}$$

It follows, by the same argument as before, that

$$a_n = a_r \cdot a_{n-r}.$$

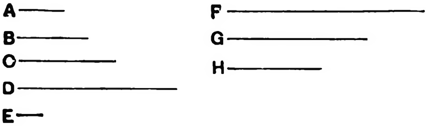
With our notation, we have the theorem of indices that

$$a^{m+n} = a^m \cdot a^n.$$

PROPOSITION 12.

If as many numbers as we please beginning from an unit be in continued proportion, by however many prime numbers the last is measured, the next to the unit will also be measured by the same.

Let there be as many numbers as we please, *A*, *B*, *C*, *D*, beginning from an unit, and in continued proportion;
I say that, by however many prime numbers *D* is measured, *A* will also be measured by the same.



For let *D* be measured by any prime number *E*;
I say that *E* measures *A*.

For suppose it does not;
now *E* is prime, and any prime number is prime to any which it does not measure; [VII. 29]
therefore *E*, *A* are prime to one another.

And, since *E* measures *D*, let it measure it according to *F*,
therefore *E* by multiplying *F* has made *D*.

Again, since *A* measures *D* according to the units in *C*, [IX. 11 and Por.]
therefore *A* by multiplying *C* has made *D*.

But, further, *E* has also by multiplying *F* made *D*;
therefore the product of *A*, *C* is equal to the product of *E*, *F*.

Therefore, as *A* is to *E*, so is *F* to *C*. [VII. 19]

But *A*, *E* are prime,
primes are also least, [VII. 21]
and the least measure those which have the same ratio the same number of times, the antecedent the antecedent and the consequent the consequent; [VII. 20]
therefore *E* measures *C*.

Let it measure it according to *G*;
therefore *E* by multiplying *G* has made *C*.

But, further, by the theorem before this,
A has also by multiplying *B* made *C*. [IX. 11 and Por.]

Therefore the product of A, B is equal to the product of E, G.

Therefore, as A is to E, so is G to B. [VII. 19]

But A, E are prime,
primes are also least, [VII. 21]
and the least numbers measure those which have the same ratio with them the same number of times, the antecedent the antecedent and the consequent the consequent: [VII. 20]
therefore E measures B.

Let it measure it according to H;
therefore E by multiplying H has made B.

But further A has also by multiplying itself made B; [IX. 8]
therefore the product of E, H is equal to the square on A.

Therefore, as E is to A, so is A to H. [VII. 19]

But A, E are prime,
primes are also least, [VII. 21]
and the least measure those which have the same ratio the same number of times, the antecedent the antecedent and the consequent the consequent; [VII. 20]
therefore E measures A, as antecedent antecedent.

But, again, it also does not measure it:
which is impossible.

Therefore E, A are not prime to one another.

Therefore they are composite to one another.

But numbers composite to one another are measured by some number. [VII. Def. 14]

And, since E is by hypothesis prime,
and the prime is not measured by any number other than itself,
therefore E measures A, E,
so that E measures A.

[But it also measures D;
therefore E measures A, D.]

Similarly we can prove that, by however many prime numbers D is measured, A will also be measured by the same.

Q. E. D.

If $1, a, a_2, \ldots a_n$ be a geometrical progression, and a_n be measured by any prime number p, a will also be measured by p.

For, if possible, suppose that p does not measure a; then, p being prime, p, a are prime to one another. [VII. 29]

Suppose $a_n = m \cdot p$.

Now $a_n = a \cdot a_{n-1}$. [IX. 11]

Therefore $a \cdot a_{n-1} = m \cdot p$,

and $a : p = m : a_{n-1}$. [VII. 19]

Hence, a, p being prime to one another,

p measures a_{n-1}. [VII. 20, 21]

By a repetition of the same process, we can prove that p measures a_{n-2} and therefore a_{n-3}, and so on, and finally that p measures a.

But, by hypothesis, p does not measure a: which is impossible.

Hence p, a are not prime to one another:

therefore they have some common factor. [VII. Def. 14]

But p is the only number which measures p;

therefore p measures a.

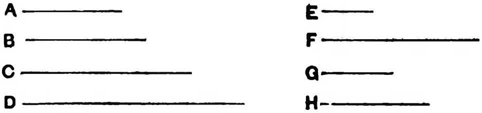
Heiberg remarks that, as, in the ἔκθεσις, Euclid sets himself to prove that E measures A, the words bracketed above are unnecessary and therefore perhaps interpolated.

PROPOSITION 13.

If as many numbers as we please beginning from an unit be in continued proportion, and the number after the unit be prime, the greatest will not be measured by any except those which have a place among the proportional numbers.

Let there be as many numbers as we please, A, B, C, D, beginning from an unit and in continued proportion, and let A, the number after the unit, be prime;

I say that D, the greatest of them, will not be measured by any other number except A, B, C.



For, if possible, let it be measured by E, and let E not be the same with any of the numbers A, B, C.

It is then manifest that E is not prime.

For, if E is prime and measures D,

it will also measure A [IX. 12], which is prime, though it is not the same with it:

which is impossible.

Therefore E is not prime.

Therefore it is composite.

But any composite number is measured by some prime number; [VII. 31]

therefore E is measured by some prime number.

I say next that it will not be measured by any other prime except A.

For, if E is measured by another,

and E measures D,

that other will also measure D;

so that it will also measure A [IX. 12], which is prime, though it is not the same with it:

which is impossible.

Therefore A measures E.

And, since E measures D, let it measure it according to F.

I say that F is not the same with any of the numbers A, B, C.

For, if F is the same with one of the numbers A, B, C, and measures D according to E,

therefore one of the numbers A, B, C also measures D according to E.

But one of the numbers A, B, C measures D according to some one of the numbers A, B, C; [IX. 11]

therefore E is also the same with one of the numbers A, B, C: which is contrary to the hypothesis.

Therefore F is not the same as any one of the numbers A. B, C.

Similarly we can prove that F is measured by A, by proving again that F is not prime.

For, if it is, and measures D,

it will also measure A [IX. 12], which is prime, though it is not the same with it:

which is impossible;

therefore F is not prime.

Therefore it is composite.

But any composite number is measured by some prime number; [VII. 31]

therefore F is measured by some prime number.

I say next that it will not be measured by any other prime except *A*.

For, if any other prime number measures *F*,
and *F* measures *D*,
that other will also measure *D*;
so that it will also measure *A* [IX. 12], which is prime, though it is not the same with it:
which is impossible.

Therefore *A* measures *F*.

And, since *E* measures *D* according to *F*,
therefore *E* by multiplying *F* has made *D*.

But, further, *A* has also by multiplying *C* made *D*; [IX. 11]
therefore the product of *A*, *C* is equal to the product of *E*, *F*.

Therefore, proportionally, as *A* is to *E*, so is *F* to *C*. [VII. 19]

But *A* measures *E*;
therefore *F* also measures *C*.

Let it measure it according to *G*.

Similarly, then, we can prove that *G* is not the same with any of the numbers *A*, *B*, and that it is measured by *A*.

And, since *F* measures *C* according to *G*
therefore *F* by multiplying *G* has made *C*.

But, further, *A* has also by multiplying *B* made *C*; [IX. 11]
therefore the product of *A*, *B* is equal to the product of *F*, *G*.

Therefore, proportionally, as *A* is to *F*, so is *G* to *B*. [VII. 19]

But *A* measures *F*;
therefore *G* also measures *B*.

Let it measure it according to *H*.

Similarly then we can prove that *H* is not the same with *A*.

And, since *G* measures *B* according to *H*,
therefore *G* by multiplying *H* has made *B*.

But further *A* has also by multiplying itself made *B*; [IX. 8]
therefore the product of *H*, *G* is equal to the square on *A*.

Therefore, as *H* is to *A*, so is *A* to *G*. [VII. 19]

But A measures G;
therefore H also measures A, which is prime, though it is not the same with it:
which is absurd.

Therefore D the greatest will not be measured by any other number except A, B, C.

Q. E. D.

If $1, a, a_2, \ldots a_n$ be a geometrical progression, and if a is prime, a_n will not be measured by any numbers except the preceding terms of the series.

If possible, let a_n be measured by b, a number different from all the preceding terms.

Now b cannot be prime, for, if it were, it would measure a. [IX. 12]

Therefore b is composite, and hence will be measured by *some* prime number [VII. 31], say p.

Thus p must measure a_n and therefore a [IX. 12]; so that p cannot be different from a, and b is not measured by any prime number except a.

Suppose that $$a_n = b \,.\, c.$$

Now c cannot be identical with any of the terms $a, a_2, \ldots a_{n-1}$; for, if it were, b would be identical with another of them: [IX. 11]
which is contrary to the hypothesis.

We can now prove (just as for b) that c cannot be prime and cannot be measured by any prime number except a.

Since $$b \,.\, c = a_n = a \,.\, a_{n-1}, \qquad \text{[IX. 11]}$$
$$a : b = c : a_{n-1},$$
whence, since a measures b,
$$c \text{ measures } a_{n-1}.$$

Let $$a_{n-1} = c \,.\, d.$$
We now prove in the same way that d is not identical with any of the terms $a, a_2, \ldots a_{n-2}$, is not prime, and is not measured by any prime except a, and also that
$$d \text{ measures } a_{n-2}.$$

Proceeding in this way, we get a last factor, say k, which measures a though different from it:
which is absurd, since a is prime.

Thus the original supposition that a_n can be measured by a number b different from all the terms $a, a_2, \ldots a_{n-1}$ must be incorrect.

Therefore etc.

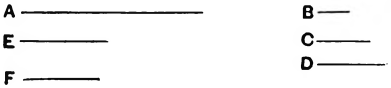
PROPOSITION 14.

If a number be the least that is measured by prime numbers, it will not be measured by any other prime number except those originally measuring it.

For let the number A be the least that is measured by the prime numbers B, C, D;

I say that A will not be measured by any other prime number except B, C, D.

For, if possible, let it be measured by the prime number E, and let E not be the same with any one of the numbers B, C, D.



Now, since E measures A, let it measure it according to F;

therefore E by multiplying F has made A.

And A is measured by the prime numbers B, C, D.

But, if two numbers by multiplying one another make some number, and any prime number measure the product, it will also measure one of the original numbers; [VII. 30]

therefore B, C, D will measure one of the numbers E, F.

Now they will not measure E;

for E is prime and not the same with any one of the numbers B, C, D.

Therefore they will measure F, which is less than A: which is impossible, for A is by hypothesis the least number measured by B, C, D.

Therefore no prime number will measure A except B, C, D.

Q. E. D.

In other words, a number can be resolved into prime factors in only one way.

Let a be the least number measured by each of the prime numbers $b, c, d, \dots k$.

If possible, suppose that a has a prime factor p different from $b, c, d, \dots k$.

Let $$a = p \,.\, m.$$

Now $b, c, d, \dots k$, measuring a, must measure one of the two factors p, m. [VII. 30]

They do not, by hypothesis, measure p;

therefore they must measure m, a number less than a:

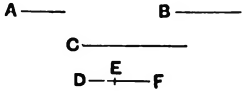
which is contrary to the hypothesis.

Therefore a has no prime factors except $b, c, d, \dots k$.

PROPOSITION 15.

If three numbers in continued proportion be the least of those which have the same ratio with them, any two whatever added together will be prime to the remaining number.

Let A, B, C, three numbers in continued proportion, be the least of those which have the same ratio with them;



I say that any two of the numbers A, B, C whatever added together are prime to the remaining number, namely A, B to C; B, C to A; and further A, C to B.

For let two numbers DE, EF, the least of those which have the same ratio with A, B, C, be taken. [VIII. 2]

It is then manifest that DE by multiplying itself has made A, and by multiplying EF has made B, and, further, EF by multiplying itself has made C. [VIII. 2]

Now, since DE, EF are least,
they are prime to one another. [VII. 22]

But, if two numbers be prime to one another,
their sum is also prime to each; [VII. 28]
therefore DF is also prime to each of the numbers DE, EF.

But further DE is also prime to EF;
therefore DF, DE are prime to EF.

But, if two numbers be prime to any number,
their product is also prime to the other; [VII. 24]
so that the product of FD, DE is prime to EF;
hence the product of FD, DE is also prime to the square on EF. [VII. 25]

But the product of FD, DE is the square on DE together with the product of DE, EF; [II. 3]
therefore the square on DE together with the product of DE, EF is prime to the square on EF.

And the square on DE is A,
the product of DE, EF is B,
and the square on EF is C;
therefore A, B added together are prime to C.

Similarly we can prove that *B*, *C* added together are prime to *A*.

I say next that *A*, *C* added together are also prime to *B*.

For, since *DF* is prime to each of the numbers *DE*, *EF*, the square on *DF* is also prime to the product of *DE*, *EF*. [VII. 24, 25]

But the squares on *DE*, *EF* together with twice the product of *DE*, *EF* are equal to the square on *DF*; [II. 4]
therefore the squares on *DE*, *EF* together with twice the product of *DE*, *EF* are prime to the product of *DE*, *EF*.

Separando, the squares on *DE*, *EF* together with once the product of *DE*, *EF* are prime to the product of *DE*, *EF*.

Therefore, *separando* again, the squares on *DE*, *EF* are prime to the product of *DE*, *EF*.

And the square on *DE* is *A*,
the product of *DE*, *EF* is *B*,
and the square on *EF* is *C*.

Therefore *A*, *C* added together are prime to *B*.

Q. E. D.

If a, b, c be a geometrical progression in the least terms which have a given common ratio, $(b+c)$, $(c+a)$, $(a+b)$ are respectively prime to a, b, c.

Let $\alpha : \beta$ be the common ratio in its lowest terms, so that the geometrical progression is

$$\alpha^2, \quad \alpha\beta, \quad \beta^2. \qquad [\text{VIII. } 2]$$

Now, α, β being prime to one another,
$\alpha+\beta$ is prime to both α and β. [VII. 28]

Therefore $(\alpha+\beta)$, α are both prime to β.
Hence $(\alpha+\beta)\,\alpha$ is prime to β, [VII. 24]
and therefore to β^2; [VII. 25]
i.e. $\alpha^2+\alpha\beta$ is prime to β^2,
or $a+b$ is prime to c.
Similarly, $\alpha\beta+\beta^2$ is prime to α^2,
or $b+c$ is prime to a.
Lastly, $\alpha+\beta$ being prime to both α and β,
$(\alpha+\beta)^2$ is prime to $\alpha\beta$, [VII. 24, 25]
or $\alpha^2+\beta^2+2\alpha\beta$ is prime to $\alpha\beta$:
whence $\alpha^2+\beta^2$ is prime to $\alpha\beta$.

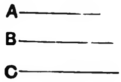
The latter inference, made in two steps, may be proved by *reductio ad absurdum* as Commandinus proves it.

If $\alpha^2+\beta^2$ is not prime to $\alpha\beta$, let x measure them;
therefore x measures $\alpha^2+\beta^2+2\alpha\beta$ as well as $\alpha\beta$;
hence $\alpha^2+\beta^2+2\alpha\beta$ and $\alpha\beta$ are not prime to one another, which is contrary to the hypothesis.

PROPOSITION 16.

If two numbers be prime to one another, the second will not be to any other number as the first is to the second.

For let the two numbers A, B be prime to one another; I say that B is not to any other number as A is to B.



For, if possible, as A is to B, so let B be to C.

Now A, B are prime,
primes are also least, [VII. 21]
and the least numbers measure those which have the same ratio the same number of times, the antecedent the antecedent and the consequent the consequent; [VII. 20]
therefore A measures B as antecedent antecedent.

But it also measures itself;
therefore A measures A, B which are prime to one another: which is absurd.

Therefore B will not be to C, as A is to B.

Q. E. D.

If a, b are prime to one another, they can have no integral third proportional.

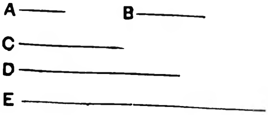
If possible, let $a : b = b : x$.

Therefore [VII. 20, 21] a measures b; and a, b have the common measure a, which is contrary to the hypothesis.

PROPOSITION 17.

If there be as many numbers as we please in continued proportion, and the extremes of them be prime to one another, the last will not be to any other number as the first to the second.

For let there be as many numbers as we please, A, B, C, D, in continued proportion,
and let the extremes of them, A, D, be prime to one another;
I say that D is not to any other number as A is to B.



For, if possible, as A is to B, so let D be to E;
therefore, alternately, as A is to D, so is B to E. [VII. 13]

But A, D are prime,
primes are also least, [VII. 21]
and the least numbers measure those which have the same ratio the same number of times, the antecedent the antecedent and the consequent the consequent. [VII. 20]

Therefore A measures B.

And, as A is to B, so is B to C.

Therefore B also measures C;
so that A also measures C.

And since, as B is to C, so is C to D,
and B measures C,
therefore C also measures D.

But A measured C;
so that A also measures D.

But it also measures itself;
therefore A measures A, D which are prime to one another: which is impossible.

Therefore D will not be to any other number as A is to B.

Q. E. D.

If a, a_2, a_3, ... a_n be a geometrical progression, and a, a_n are prime to one another, then a, a_2, a_n can have no integral fourth proportional.

For, if possible, let $a : a_2 = a_n : x$.

Therefore $a : a_n = a_2 : x$,
and hence [VII. 20, 21] a measures a_2.

Therefore a_2 measures a_3, [VII. Def. 20]
and hence a measures a_3, and therefore also ultimately a_n.

Thus a, a_n are both measured by a: which is contrary to the hypothesis.

PROPOSITION 18.

Given two numbers, to investigate whether it is possible to find a third proportional to them.

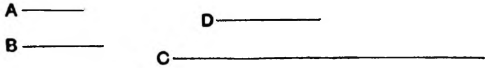
Let A, B be the given two numbers, and let it be required to investigate whether it is possible to find a third proportional to them.

Now A, B are either prime to one another or not.

And, if they are prime to one another, it has been proved that it is impossible to find a third proportional to them. [IX. 16]

Next, let A, B not be prime to one another, and let B by multiplying itself make C.

Then A either measures C or does not measure it.



First, let it measure it according to D; therefore A by multiplying D has made C.

But, further, B has also by multiplying itself made C; therefore the product of A, D is equal to the square on B.

Therefore, as A is to B, so is B to D; [VII. 19]
therefore a third proportional number D has been found to A, B.

Next, let A not measure C;
I say that it is impossible to find a third proportional number to A, B.

For, if possible, let D, such third proportional, have been found.

Therefore the product of A, D is equal to the square on B.

But the square on B is C;
therefore the product of A, D is equal to C.

Hence A by multiplying D has made C;
therefore A measures C according to D.

But, by hypothesis, it also does not measure it:
which is absurd.

Therefore it is not possible to find a third proportional number to A, B when A does not measure C. Q. E. D.

Given two numbers a, b, to find the condition that they may have an integral third proportional.

(1) a, b must not be prime to one another. [IX. 16]

(2) a must measure b^2.

For, if a, b, c be in continued proportion,

$$ac = b^2.$$

Therefore a measures b^2.

Condition (1) is included in condition (2) since, if $b^2 = ma$, a and b cannot be prime to one another.

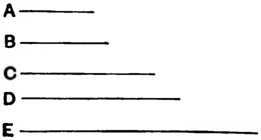
The result is of course easily seen if the three terms in continued proportion be written

$$a, \quad a\frac{b}{a}, \quad a\left(\frac{b}{a}\right)^2.$$

Proposition 19.

Given three numbers, to investigate when it is possible to find a fourth proportional to them.

Let A, B, C be the given three numbers, and let it be required to investigate when it is possible to find a fourth proportional to them.



Now either they are not in continued proportion, and the extremes of them are prime to one another;

or they are in continued proportion, and the extremes of them are not prime to one another;

or they are not in continued proportion, nor are the extremes of them prime to one another;

or they are in continued proportion, and the extremes of them are prime to one another.

If then A, B, C are in continued proportion, and the extremes of them A, C are prime to one another,

it has been proved that it is impossible to find a fourth proportional number to them. [IX. 17]

†Next, let A, B, C not be in continued proportion, the extremes being again prime to one another;

I say that in this case also it is impossible to find a fourth proportional to them.

For, if possible, let D have been found, so that,

as A is to B, so is C to D,

and let it be contrived that, as B is to C, so is D to E.

Now, since, as A is to B, so is C to D,

and, as B is to C, so is D to E,

therefore, *ex aequali*, as A is to C, so is C to E. [VII. 14]

But A, C are prime,

primes are also least, [VII. 21]

and the least numbers measure those which have the same ratio, the antecedent the antecedent and the consequent the consequent. [VII. 20]

Therefore A measures C as antecedent antecedent.

But it also measures itself;
therefore *A* measures *A*, *C* which are prime to one another:
which is impossible.

Therefore it is not possible to find a fourth proportional to *A*, *B*, *C*.†

Next, let *A*, *B*, *C* be again in continued proportion,
but let *A*, *C* not be prime to one another.

I say that it is possible to find a fourth proportional to them.

For let *B* by multiplying *C* make *D*;
therefore *A* either measures *D* or does not measure it.

First, let it measure it according to *E*;
therefore *A* by multiplying *E* has made *D*.

But, further, *B* has also by multiplying *C* made *D*;
therefore the product of *A*, *E* is equal to the product of *B*, *C*;
therefore, proportionally, as *A* is to *B*, so is *C* to *E*; [VII. 19]
therefore *E* has been found a fourth proportional to *A*, *B*, *C*.

Next, let *A* not measure *D*;
I say that it is impossible to find a fourth proportional number to *A*, *B*, *C*.

For, if possible, let *E* have been found;
therefore the product of *A*, *E* is equal to the product of *B*, *C*. [VII. 19]

But the product of *B*, *C* is *D*;
therefore the product of *A*, *E* is also equal to *D*.

Therefore *A* by multiplying *E* has made *D*;
therefore *A* measures *D* according to *E*,
so that *A* measures *D*.

But it also does not measure it:
which is absurd.

Therefore it is not possible to find a fourth proportional number to *A*, *B*, *C* when *A* does not measure *D*.

Next, let *A*, *B*, *C* not be in continued proportion, nor the extremes prime to one another.

And let *B* by multiplying *C* make *D*.

Similarly then it can be proved that, if *A* measures *D*, it is possible to find a fourth proportional to them, but, if it does not measure it, impossible. Q. E. D.

Given three numbers a, b, c, to find the condition that they may have an integral fourth proportional.

The Greek text of part of this proposition is hopelessly corrupt. According to it Euclid takes four cases.

(1) a, b, c not in continued proportion, and a, c prime to one another.

(2) a, b, c in continued proportion, and a, c not prime to one another.

(3) a, b, c not in continued proportion, and a, c not prime to one another.

(4) a, b, c in continued proportion, and a, c prime to one another.

(4) is the case dealt with in IX. 17, where it is shown that on hypothesis (4) a fourth proportional cannot be found.

The text now takes case (1) and asserts that a fourth proportional cannot be found in this case either. We have only to think of 4, 6, 9 in order to see that there is something wrong here. The supposed proof is also wrong. If possible, says the text, let d be a fourth proportional to a, b, c, *and let* e *be taken such that*

$$b : c = d : e.$$

Then, *ex aequali*, $$a : c = c : e,$$

whence a measures c: [VII. 20, 21]

which is impossible, since a, c are prime to one another.

But this does not prove that a fourth proportional d cannot be found; it only proves that, if d is a fourth proportional, no integer e can be found to satisfy the equation

$$b : c = d : e.$$

Indeed it is obvious from IX. 16 that in the equation

$$a : c = c : e$$

e cannot be integral.

The cases (2) and (3) are correctly given, the first in full, and the other as a case to be proved "similarly" to it.

These two cases really give all that is necessary.

Let the product bc be taken.

Then, if a measures bc, suppose $bc = ad$;

therefore $$a : b = c : d,$$

and d is a fourth proportional.

But, if a does *not* measure bc, no fourth proportional can be found. For, if x were a fourth proportional, ax would be equal to bc, and a would measure bc.

The sufficient condition in any case for the possibility of finding a fourth proportional to a, b, c is that a should measure bc.

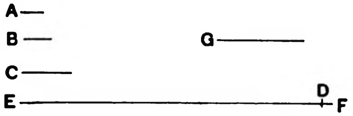
Theon appears to have corrected the proof by leaving out the incorrect portion which I have included between daggers and the last case (3) dealt with in the last lines. Also, in accordance with this arrangement, he does not distinguish four cases at the beginning but only two. "Either A, B, C are in continued proportion and the extremes of them A, C are prime to one another; or not." Then, instead of introducing case (2) by the words "Next let A, B, C...to find a fourth proportional to them," immediately following the second dagger above, Theon merely says "*But, if not,*" [i.e. if it is not the case that a, b, c are in G.P. and a, c prime to one another] "let B by multiplying C make D," and so on.

August adopts Theon's form of the proof. Heiberg does not feel able to do this, in view of the superiority of the authority for the text as given above (P); he therefore retains the latter without any attempt to emend it.

PROPOSITION 20.

Prime numbers are more than any assigned multitude of prime numbers.

Let A, B, C be the assigned prime numbers;



I say that there are more prime numbers than A, B, C.

For let the least number measured by A, B, C be taken, and let it be DE;

let the unit DF be added to DE.

Then EF is either prime or not.

First, let it be prime;

then the prime numbers A, B, C, EF have been found which are more than A, B, C.

Next, let EF not be prime;

therefore it is measured by some prime number. [VII. 31]

Let it be measured by the prime number G.

I say that G is not the same with any of the numbers A, B, C.

For, if possible, let it be so.

Now A, B, C measure DE;

therefore G also will measure DE.

But it also measures EF.

Therefore G, being a number, will measure the remainder, the unit DF:

which is absurd.

Therefore G is not the same with any one of the numbers A, B, C.

And by hypothesis it is prime.

Therefore the prime numbers A, B, C, G have been found which are more than the assigned multitude of A, B, C.

Q. E. D.

We have here the important proposition that *the number of prime numbers is infinite.*

The proof will be seen to be the same as that given in our algebraical text-books. Let $a, b, c, \ldots k$ be any prime numbers.

Take the product $abc \ldots k$ and add unity.

Then $(abc \ldots k + 1)$ is either a prime number or not a prime number.

(1) If it *is*, we have added another prime number to those given.

(2) If it is *not*, it must be measured by some prime number [VII. 31], say p.

Now p cannot be identical with any of the prime numbers $a, b, c, \ldots k$.

For, if it is, it will divide $abc \ldots k$.

Therefore, since it divides $(abc \ldots k + 1)$ also, it will measure the difference, or unity:

which is impossible.

Therefore in any case we have obtained one fresh prime number.

And the process can be carried on to any extent.

PROPOSITION 21.

If as many even numbers as we please be added together, the whole is even.

For let as many even numbers as we please, *AB*, *BC*, *CD*, *DE*, be added together;

I say that the whole *AE* is even.

For, since each of the numbers *AB*, *BC*, *CD*, *DE* is even, it has a half part; [VII. Def. 6]

so that the whole *AE* also has a half part.

But an even number is that which is divisible into two equal parts; [*id.*]

therefore *AE* is even.

Q. E. D.

In this and the following propositions up to IX. 34 inclusive we have a number of theorems about odd, even, "even-times even" and "even-times odd" numbers respectively. They are all simple and require no explanation in order to enable them to be followed easily.

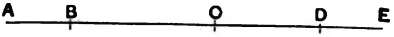
PROPOSITION 22.

If as many odd numbers as we please be added together, and their multitude be even, the whole will be even.

For let as many odd numbers as we please, *AB*, *BC*, *CD*, *DE*, even in multitude, be added together;

I say that the whole *AE* is even.

For, since each of the numbers *AB*, *BC*, *CD*, *DE* is odd, if an unit be subtracted from each, each of the remainders will be even; [VII. Def. 7]
so that the sum of them will be even. [IX. 21]



But the multitude of the units is also even.

Therefore the whole *AE* is also even. [IX. 21]

Q. E. D.

PROPOSITION 23.

If as many odd numbers as we please be added together, and their multitude be odd, the whole will also be odd.

For let as many odd numbers as we please, *AB*, *BC*, *CD*, the multitude of which is odd, be added together;
I say that the whole *AD* is also odd.

Let the unit *DE* be subtracted from *CD*;
therefore the remainder *CE* is even. [VII. Def. 7]

But *CA* is also even; [IX. 22]
therefore the whole *AE* is also even. [IX. 21]

And *DE* is an unit.

Therefore *AD* is odd. [VII. Def. 7]

Q. E. D.

PROPOSITION 24.

If from an even number an even number be subtracted, the remainder will be even.

For from the even number *AB* let the even number *BC* be subtracted:
I say that the remainder *CA* is even.

For, since *AB* is even, it has a half part. [VII. Def. 6]

---

3. Literally "let there be as many numbers as we please, of which *let* the multitude *be* odd." This form, natural in Greek, is awkward in English.

For the same reason BC also has a half part; so that the remainder [CA also has a half part, and] AC is therefore even.

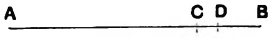
Q. E. D.

PROPOSITION 25.

If from an even number an odd number be subtracted, the remainder will be odd.

For from the even number AB let the odd number BC be subtracted;

I say that the remainder CA is odd.



For let the unit CD be subtracted from BC;

therefore DB is even. [VII. Def. 7]

But AB is also even;

therefore the remainder AD is also even. [IX. 24]

And CD is an unit;

therefore CA is odd. [VII. Def. 7]

Q. E. D.

PROPOSITION 26.

If from an odd number an odd number be subtracted, the remainder will be even.

For from the odd number AB let the odd number BC be subtracted;

I say that the remainder CA is even.

For, since AB is odd, let the unit BD be subtracted;

therefore the remainder AD is even. [VII. Def. 7]

For the same reason CD is also even; [VII. Def. 7]

so that the remainder CA is also even. [IX. 24]

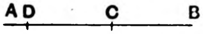
Q. E. D.

PROPOSITION 27.

If from an odd number an even number be subtracted, the remainder will be odd.

For from the odd number AB let the even number BC be subtracted;

I say that the remainder CA is odd.



Let the unit AD be subtracted;

therefore DB is even. [VII. Def. 7]

But BC is also even;

therefore the remainder CD is even. [IX. 24]

Therefore CA is odd. [VII. Def. 7]

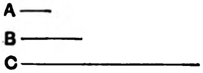
Q. E. D.

PROPOSITION 28.

If an odd number by multiplying an even number make some number, the product will be even.

For let the odd number A by multiplying the even number B make C;

I say that C is even.



For, since A by multiplying B has made C,

therefore C is made up of as many numbers equal to B as there are units in A. [VII. Def. 15]

And B is even;

therefore C is made up of even numbers.

But, if as many even numbers as we please be added together, the whole is even. [IX. 21]

Therefore C is even.

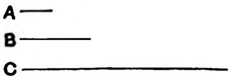
Q. E. D.

PROPOSITION 29.

If an odd number by multiplying an odd number make some number, the product will be odd.

For let the odd number A by multiplying the odd number B make C;

I say that C is odd.



For, since A by multiplying B has made C, therefore C is made up of as many numbers equal to B as there are units in A. [VII. Def. 15]

And each of the numbers A, B is odd;

therefore C is made up of odd numbers the multitude of which is odd.

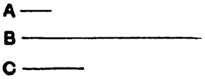
Thus C is odd. [IX. 23]

Q. E. D.

PROPOSITION 30.

If an odd number measure an even number, it will also measure the half of it.

For let the odd number A measure the even number B; I say that it will also measure the half of it.



For, since A measures B, let it measure it according to C; I say that C is not odd.

For, if possible, let it be so.

Then, since A measures B according to C, therefore A by multiplying C has made B.

Therefore B is made up of odd numbers the multitude of which is odd.

Therefore B is odd: [IX. 23]

which is absurd, for by hypothesis it is even.

Therefore C is not odd;

therefore C is even.

Thus A measures B an even number of times.

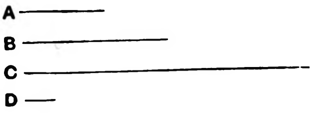
For this reason then it also measures the half of it.

Q. E. D.

PROPOSITION 31.

If an odd number be prime to any number, it will also be prime to the double of it.

For let the odd number A be prime to any number B, and let C be double of B; I say that A is prime to C.



For, if they are not prime to one another, some number will measure them.

Let a number measure them, and let it be *D*.

Now *A* is odd;
therefore *D* is also odd.

And since *D* which is odd measures *C*,
and *C* is even,
therefore [*D*] will measure the half of *C* also. [IX. 30]

But *B* is half of *C*;
therefore *D* measures *B*.

But it also measures *A*;
therefore *D* measures *A*, *B* which are prime to one another: which is impossible.

Therefore *A* cannot but be prime to *C*.

Therefore *A*, *C* are prime to one another.

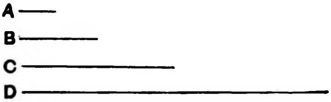
Q. E. D.

PROPOSITION 32.

Each of the numbers which are continually doubled beginning from a dyad is even-times even only.

For let as many numbers as we please, *B*, *C*, *D*, have been continually doubled beginning from the dyad *A*;

I say that *B*, *C*, *D* are even-times even only.



Now that each of the numbers *B*, *C*, *D* is even-times even is manifest; for it is doubled from a dyad.

I say that it is also even-times even only.

For let an unit be set out.

Since then as many numbers as we please beginning from an unit are in continued proportion,
and the number *A* after the unit is prime,
therefore *D*, the greatest of the numbers *A*, *B*, *C*, *D*, will not be measured by any other number except *A*, *B*, *C*. [IX. 13]

And each of the numbers *A*, *B*, *C* is even;
therefore *D* is even-times even only. [VII. Def. 8]

Similarly we can prove that each of the numbers *B*, *C* is even-times even only.

Q. E. D.

See the notes on VII. Deff. 8 to 11 for a discussion of the difficulties shown by Iamblichus to be involved by the Euclidean definitions of "even-times even," "even-times odd" and "odd-times even."

PROPOSITION 33.

If a number have its half odd, it is even-times odd only.

For let the number A have its half odd;

I say that A is even-times odd only.

Now that it is even-times odd is manifest; for the half of it, being odd, measures it an even number of times. [VII. Def. 9]

I say next that it is also even-times odd only.

For, if A is even-times even also,
it will be measured by an even number according to an even number; [VII. Def. 8]
so that the half of it will also be measured by an even number though it is odd:
which is absurd.

Therefore A is even-times odd only. Q. E. D.

PROPOSITION 34.

If a number neither be one of those which are continually doubled from a dyad, nor have its half odd, it is both even-times even and even-times odd.

For let the number A neither be one of those doubled from a dyad, nor have its half odd;

I say that A is both even-times even and even-times odd.

Now that A is even-times even is manifest;
for it has not its half odd. [VII. Def. 8]

I say next that it is also even-times odd.

For, if we bisect A, then bisect its half, and do this continually, we shall come upon some odd number which will measure A according to an even number.

For, if not, we shall come upon a dyad,
and A will be among those which are doubled from a dyad:
which is contrary to the hypothesis.

Thus A is even-times odd.

But it was also proved even-times even.

Therefore A is both even-times even and even-times odd.

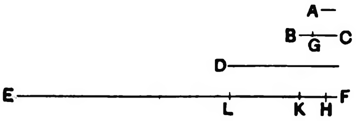
Q. E. D.

PROPOSITION 35.

If as many numbers as we please be in continued proportion, and there be subtracted from the second and the last numbers equal to the first, then, as the excess of the second is to the first, so will the excess of the last be to all those before it.

Let there be as many numbers as we please in continued proportion, A, BC, D, EF, beginning from A as least, and let there be subtracted from BC and EF the numbers BG, FH, each equal to A;

I say that, as GC is to A, so is EH to A, BC, D.



For let FK be made equal to BC, and FL equal to D.

Then, since FK is equal to BC,
and of these the part FH is equal to the part BG,
therefore the remainder HK is equal to the remainder GC.

And since, as EF is to D, so is D to BC, and BC to A,
while D is equal to FL, BC to FK, and A to FH,
therefore, as EF is to FL, so is LF to FK, and FK to FH.

Separando, as EL is to LF, so is LK to FK, and KH to FH. [VII. 11, 13]

Therefore also, as one of the antecedents is to one of the consequents, so are all the antecedents to all the consequents; [VII. 12]

therefore, as KH is to FH, so are EL, LK, KH to LF, FK, HF.

But KH is equal to CG, FH to A, and LF, FK, HF to D, BC, A;

therefore, as CG is to A, so is EH to D, BC, A.

Therefore, as the excess of the second is to the first, so is the excess of the last to all those before it.

Q. E. D.

This proposition is perhaps the most interesting in the arithmetical Books, since it gives a method, and a very elegant one, of *summing any series of terms in geometrical progression.*

Let $a_1, a_2, a_3, \ldots a_n, a_{n+1}$ be a series of terms in geometrical progression. Then Euclid's proposition proves that

$$(a_{n+1} - a_1) : (a_1 + a_2 + \ldots + a_n) = (a_2 - a_1) : a_1.$$

For clearness' sake we will on this occasion use the fractional notation of algebra to represent proportions.

Euclid's method then comes to this.

Since $$\frac{a_{n+1}}{a_n} = \frac{a_n}{a_{n-1}} = \ldots = \frac{a_2}{a_1},$$

we have, *separando,*

$$\frac{a_{n+1} - a_n}{a_n} = \frac{a_n - a_{n-1}}{a_{n-1}} = \ldots = \frac{a_3 - a_2}{a_2} = \frac{a_2 - a_1}{a_1},$$

whence, since, as one of the antecedents is to one of the consequents, so is the sum of all the antecedents to the sum of all the consequents, [VII. 12]

$$\frac{a_{n+1} - a_1}{a_n + a_{n-1} + \ldots + a_1} = \frac{a_2 - a_1}{a_1},$$

which gives $a_1 + a_2 + \ldots + a_n$, or S_n.

If, to compare the result with that arrived at in algebraical text-books, we write the series in the form

$$a, \quad ar, \quad ar^2, \ldots ar^{n-1} \quad (n \text{ terms}),$$

we have $$\frac{ar^n - a}{S_n} = \frac{ar - a}{a},$$

or $$S_n = \frac{a(r^n - 1)}{r - 1}.$$

PROPOSITION 36.

If as many numbers as we please beginning from an unit be set out continuously in double proportion, until the sum of all becomes prime, and if the sum multiplied into the last make some number, the product will be perfect.

For let as many numbers as we please, A, B, C, D, beginning from an unit be set out in double proportion, until the sum of all becomes prime,

let E be equal to the sum, and let E by multiplying D make FG;

I say that FG is perfect.

For, however many A, B, C, D are in multitude, let so many E, HK, L, M be taken in double proportion beginning from E;

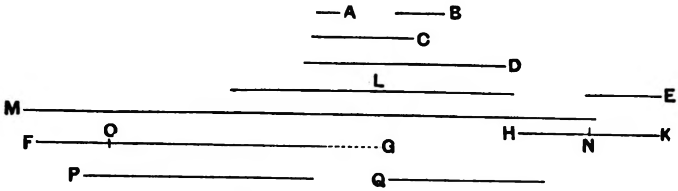
therefore, *ex aequali*, as A is to D, so is E to M. [VII. 14]

Therefore the product of *E*, *D* is equal to the product of *A*, *M*. [VII. 19]

And the product of *E*, *D* is *FG*;
therefore the product of *A*, *M* is also *FG*.

Therefore *A* by multiplying *M* has made *FG*;
therefore *M* measures *FG* according to the units in *A*.

And *A* is a dyad;
therefore *FG* is double of *M*.



But *M*, *L*, *HK*, *E* are continuously double of each other;
therefore *E*, *HK*, *L*, *M*, *FG* are continuously proportional in double proportion.

Now let there be subtracted from the second *HK* and the last *FG* the numbers *HN*, *FO*, each equal to the first *E*;
therefore, as the excess of the second is to the first, so is the excess of the last to all those before it. [IX. 35]

Therefore, as *NK* is to *E*, so is *OG* to *M*, *L*, *KH*, *E*.

And *NK* is equal to *E*;
therefore *OG* is also equal to *M*, *L*, *HK*, *E*.

But *FO* is also equal to *E*,
and *E* is equal to *A*, *B*, *C*, *D* and the unit.

Therefore the whole *FG* is equal to *E*, *HK*, *L*, *M* and *A*, *B*, *C*, *D* and the unit;
and it is measured by them.

I say also that *FG* will not be measured by any other number except *A*, *B*, *C*, *D*, *E*, *HK*, *L*, *M* and the unit.

For, if possible, let some number *P* measure *FG*,
and let *P* not be the same with any of the numbers *A*, *B*, *C*, *D*, *E*, *HK*, *L*, *M*.

And, as many times as *P* measures *FG*, so many units let there be in *Q*;
therefore *Q* by multiplying *P* has made *FG*.

But, further, E has also by multiplying D made FG;
therefore, as E is to Q, so is P to D. [VII. 19]

And, since A, B, C, D are continuously proportional beginning from an unit,
therefore D will not be measured by any other number except A, B, C. [IX. 13]

And, by hypothesis, P is not the same with any of the numbers A, B, C;
therefore P will not measure D.

But, as P is to D, so is E to Q;
therefore neither does E measure Q. [VII. Def. 20]

And E is prime;
and any prime number is prime to any number which it does not measure. [VII. 29]

Therefore E, Q are prime to one another.

But primes are also least, [VII. 21]
and the least numbers measure those which have the same ratio the same number of times, the antecedent the antecedent and the consequent the consequent; [VII. 20]
and, as E is to Q, so is P to D;
therefore E measures P the same number of times that Q measures D.

But D is not measured by any other number except A, B, C;
therefore Q is the same with one of the numbers A, B, C.

Let it be the same with B.

And, however many B, C, D are in multitude, let so many E, HK, L be taken beginning from E.

Now E, HK, L are in the same ratio with B, C, D;
therefore, *ex aequali*, as B is to D, so is E to L. [VII. 14]

Therefore the product of B, L is equal to the product of D, E. [VII. 19]

But the product of D, E is equal to the product of Q, P;
therefore the product of Q, P is also equal to the product of B, L.

Therefore, as Q is to B, so is L to P. [VII. 19]

And Q is the same with B;
therefore L is also the same with P:

which is impossible, for by hypothesis *P* is not the same with any of the numbers set out.

Therefore no number will measure *FG* except *A*, *B*, *C*, *D*, *E*, *HK*, *L*, *M* and the unit.

And *FG* was proved equal to *A*, *B*, *C*, *D*, *E*, *HK*, *L*, *M* and the unit;
and a perfect number is that which is equal to its own parts; [VII. Def. 22]
therefore *FG* is perfect.

Q. E. D.

If the sum of any number of terms of the series

$$1, 2, 2^2, \dots 2^{n-1}$$

be prime, and the said sum be multiplied by the last term, the product will be a "perfect" number, i.e. equal to the sum of all its factors.

Let $1 + 2 + 2^2 + \dots + 2^{n-1}$ $(= S_n)$ be prime;
then shall $S_n \cdot 2^{n-1}$ be "perfect."

Take $(n - 1)$ terms of the series

$$S_n, 2S_n, 2^2S_n, \dots 2^{n-2}S_n.$$

These are then terms proportional to the terms

$$2, 2^2, 2^3, \dots 2^{n-1}.$$

Therefore, *ex aequali*,

$$2 : 2^{n-1} = S_n : 2^{n-2} S_n, \qquad \text{[VII. 14]}$$

or

$$2 \cdot 2^{n-2}S_n = 2^{n-1} \cdot S_n. \qquad \text{[VII. 19]}$$

(This is of course obvious algebraically, but Euclid's notation requires him to prove it.)

Now, by IX. 35, we can sum the series $S_n + 2S_n + \dots + 2^{n-2}S_n$,
and $(2S_n - S_n) : S_n = (2^{n-1} S_n - S_n) : (S_n + 2S_n + \dots + 2^{n-2}S_n)$.

Therefore $S_n + 2S_n + 2^2S_n + \dots + 2^{n-2}S_n = 2^{n-1}S_n - S_n$,
or

$$\begin{aligned} 2^{n-1}S_n &= S_n + 2S_n + 2^2S_n + \dots + 2^{n-2}S_n + S_n \\ &= S_n + 2S_n + \dots + 2^{n-2}S_n + (1 + 2 + 2^2 + \dots + 2^{n-1}), \end{aligned}$$

and $2^{n-1} S_n$ is measured by every term of the right hand expression.

It is now necessary to prove that $2^{n-1}S_n$ cannot have any factor except those terms.

Suppose, if possible, that it has a factor x different from all of them, and let

$$2^{n-1}S_n = x \cdot m.$$

Therefore

$$S_n : m = x : 2^{n-1}. \qquad \text{[VII. 19]}$$

Now 2^{n-1} can only be measured by the preceding terms of the series $1, 2, 2^2, \dots 2^{n-1}$, [IX. 13]
and x is different from all of these;
therefore x does not measure 2^{n-1},
so that S_n does not measure m. [VII. Def. 20]

And S_n is prime; therefore it is prime to m. [VII. 29]

It follows [VII. 20, 21] that

$$m \text{ measures } 2^{n-1}.$$

Suppose that $m = 2^r$.

Now, *ex aequali*, $2^r : 2^{n-1} = S_n : 2^{n-r-1} S_n$.

Therefore $2^r \,.\, 2^{n-r-1} S_n = 2^{n-1} S_n$ [VII. 19]

$= x \,.\, m$, from above.

And $m = 2^r$;

therefore $x = 2^{n-r-1} S_n$, one of the terms of the series $S_n, 2S_n, 2^2 S_n, \dots 2^{n-2} S_n$: which contradicts the hypothesis.

There $2^{n-1} S_n$ has no factors except

$$S_n, 2S_n, 2^2 S_n, \dots 2^{n-2} S_n, 1, 2, 2^2, \dots 2^{n-1}.$$

Theon of Smyrna and Nicomachus both define a "perfect" number and give the law of its formation. Nicomachus gives four perfect numbers and no more, namely 6, 28, 496, 8128. He says they are formed in "ordered" fashion, there being one among the units (i.e. less than 10), one among the tens (less than 100), one among the hundreds (less than 1000) and one among the thousands (less than 10000); he adds that they terminate in 6 or 8 alternately. They do all terminate in 6 or 8, as can easily be proved by means of the formula $(2^n - 1) 2^{n-1}$ (cf. Loria, *Le scienze esatte nell' antica Grecia*, pp. 840—1), but not alternately, for the fifth and sixth perfect numbers both end in 6, and the seventh and eighth both end in 8. Iamblichus adds a tentative suggestion that perhaps there may be, in like manner, one perfect number among the "first myriads" (less than 10000^2), one among the "second myriads" (less than 10000^3), and so on. This is, as we shall see, incorrect.

It is natural that the subject of perfect numbers should, ever since Euclid's time, have had a fascination for mathematicians. Fermat (1601—1655), in a letter to Mersenne (*Œuvres de Fermat*, ed. Tannery and Henry, Vol. II., 1894, pp. 197—9), enunciated three propositions which much facilitate the investigation whether a given number of the form $2^n - 1$ is prime or not. If we write in one line the exponents 1, 2, 3, 4, etc. of the successive powers of 2 and underneath them respectively the numbers representing the corresponding powers of 2 diminished by 1, thus,

1	2	3	4	5	6	7	8	9	10	11 ...n
1	3	7	15	31	63	127	255	511	1023	2047...$2^n - 1$,

the following relations are found to subsist between the numbers in the first line and those directly below them in the second line.

1. If the exponent is not a prime number, the corresponding number is not a prime number either (since $a^{pq} - 1$ is always divisible by $a^p - 1$ as well as by $a^q - 1$).

2. If the exponent is a prime number, the corresponding number diminished by 1 is divisible by twice the exponent. [$(2^n - 2)/2n = (2^{n-1} - 1)/n$; so that this is a special case of "Fermat's theorem" that, if p is a prime number and a is prime to p, then a^{p-1} is divisible by p.]

3. If the exponent n is a prime number, the corresponding number is only divisible by numbers of the form $(2mn + 1)$. If therefore the corresponding number in the second line has no factors of this form, it has no integral factor.

The first and third of these propositions are those which are specially useful for the purpose in question. As usual, Fermat does not give his proofs but merely adds: "Voilà trois fort belles propositions que j'ay trouvées et prouvées non sans peine. Je les puis appeller les fondements de l'invention des nombres parfaits."

I append a few details of discoveries of further perfect numbers after the first four. The next are as follows:

fifth, $2^{12}(2^{13}-1) = 33\ 550\ 336$
sixth, $2^{16}(2^{17}-1) = 8\ 589\ 869\ 056$
seventh, $2^{18}(2^{19}-1) = 137\ 438\ 691\ 328$
eighth, $2^{30}(2^{31}-1) = 2\ 305\ 843\ 008\ 139\ 952\ 128$
ninth, $2^{60}(2^{61}-1) = 2\ 658\ 455\ 991\ 569\ 831\ 744\ 654\ 692\ 615\ 953\ 842\ 176$
tenth, $2^{88}(2^{89}-1)$.

It has further been proved that $2^{107}-1$ is prime, and so is $2^{127}-1$. Hence $2^{106}(2^{107}-1)$ and $2^{126}(2^{127}-1)$ are two more perfect numbers.

The fifth perfect number may have been known to Iamblichus, though he does not give it; it was however known, with all its factors, in the fifteenth century, as appears from a tract written in German which was discovered by Curtze (Cod. lat. Monac. 14908). The first eight perfect numbers were calculated by Jean Prestet (d. 1670). Fermat had stated, and Euler proved, that $2^{31}-1$ is prime. The ninth perfect number was found by P. Seelhoff (*Zeitschrift für Math. u. Physik*, XXXI., 1886, pp. 174—8) and verified by E. Lucas (*Mathésis*, VII., 1887, pp. 45—6). The tenth was discovered by R. E. Powers (see *Bulletin of the American Mathematical Society*, XVIII., 1912, p. 162). $2^{107}-1$ was proved to be prime by E. Fauquembergue and R. E. Powers (1914), while Fauquembergue proved that $2^{127}-1$ is prime.

There have been attempts, so far unsuccessful, to solve the question whether there exist other "perfect numbers" than those of Euclid, and, in particular, perfect numbers which are *odd*. (Cf. several notes by Sylvester in *Comptes rendus*, CVI., 1888; Catalan, "Mélanges mathématiques" in *Mém. de la Soc. de Liége*, 2^e^ Série, XV., 1888, pp. 205—7; C. Servais in *Mathésis*, VII., pp. 228—30 and VIII., pp. 92—93, 135; E. Cesàro in *Mathésis*, VII., pp. 245—6; E. Lucas in *Mathésis*, X., pp. 74—6).

For the detailed history of the whole subject see L. E. Dickson, *History of the Theory of Numbers*, Vol. I., 1919, pp. iii—iv, 3—33.

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THE THIRTEEN BOOKS OF EUCLID'S ELEMENTS

TRANSLATED FROM THE TEXT OF HEIBERG

WITH INTRODUCTION AND COMMENTARY

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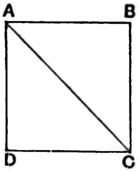
BOOK X.

INTRODUCTORY NOTE.

We have seen (Vol. I., p. 351 etc.) that the discovery of the irrational is due to the Pythagoreans. The first scholium on Book x. of the *Elements* states that the Pythagoreans were the first to address themselves to the investigation of commensurability, having discovered it by means of their observation of numbers. They discovered, the scholium continues, that not all magnitudes have a common measure. "They called all magnitudes measureable by the same measure commensurable, but those which are not subject to the same measure incommensurable, and again such of these as are measured by some other common measure commensurable with one another, and such as are not, incommensurable with the others. And thus by *assuming* their measures they referred everything to *different* commensurabilities, but, though they were different, even so (they proved that) not all magnitudes are commensurable with any. (They showed that) all magnitudes can be rational (ῥητά) and all irrational (ἄλογα) in a relative sense (ὡς πρός τι); hence the commensurable and the incommensurable would be for them *natural* (kinds) (φύσει), while the rational and irrational would rest on *assumption* or *convention* (θέσει)." The scholium quotes further the legend according to which "the first of the Pythagoreans who made public the investigation of these matters perished in a shipwreck," conjecturing that the authors of this story "perhaps spoke allegorically, hinting that everything irrational and formless is properly concealed, and, if any soul should rashly invade this region of life and lay it open, it would be carried away into the sea of becoming and be overwhelmed by its unresting currents." There would be a reason also for keeping the discovery of irrationals secret for the time in the fact that it rendered unstable so much of the groundwork of geometry as the Pythagoreans had based upon the imperfect theory of proportions which applied only to numbers. We have already, after Tannery, referred to the probability that the discovery of incommensurability must have necessitated a great recasting of the whole fabric of elementary geometry, pending the discovery of the general theory of proportion applicable to incommensurable as well as to commensurable magnitudes.

It seems certain that it was with reference to the length of the diagonal of a square or the hypotenuse of an isosceles right-angled triangle that the irrational was discovered. Plato (*Theaetetus*, 147 D) tells us that Theodorus of Cyrene wrote about square roots (δυνάμεις), proving that the square roots of three square feet and five square feet are not commensurable with that of one square foot, and so on, selecting each such square root up to that of 17 square feet, at which for some reason he stopped. No mention is here made of $\sqrt{2}$, doubtless for the reason that its incommensurability had been proved before. Now we are told that Pythagoras invented a formula for finding right-angled triangles in rational numbers, and in connexion with this it was inevitable that the Pythagoreans should investigate the relations between sides and hypotenuse in other right-angled triangles. They would naturally give special attention to the isosceles right-angled triangle; they would try to measure the diagonal, would arrive at successive approximations, in rational fractions, to the value of $\sqrt{2}$, and would find that successive efforts to obtain an exact expression for it failed. It was however an enormous step to conclude that such exact expression was *impossible*, and it was this step which the Pythagoreans made. We now know that the formation of the *side-* and *diagonal-* numbers explained by Theon of Smyrna and others was Pythagorean, and also that the theorems of Eucl. II. 9, 10 were used by the Pythagoreans in direct connexion with this method of approximating to the value of $\sqrt{2}$. The very method by which Euclid proves these propositions is itself an indication of their connexion with the investigation of $\sqrt{2}$, since he uses a figure made up of two isosceles right-angled triangles.

The actual method by which the Pythagoreans proved the incommensurability of $\sqrt{2}$ with unity was no doubt that referred to by Aristotle (*Anal. prior.* I. 23, 41 a 26—7), a *reductio ad absurdum* by which it is proved that, if the diagonal is commensurable with the side, it will follow that the same number is both odd and even. The proof formerly appeared in the texts of Euclid as X. 117, but it is undoubtedly an interpolation, and August and Heiberg accordingly relegate it to an Appendix. It is in substance as follows.



Suppose AC, the diagonal of a square, to be commensurable with AB, its side. Let $\alpha : \beta$ be their ratio expressed in the smallest numbers.

Then $\alpha > \beta$ and therefore necessarily > 1.

Now $AC^2 : AB^2 = \alpha^2 : \beta^2$,

and, since $AC^2 = 2AB^2$, [Eucl. I. 47]

$$\alpha^2 = 2\beta^2.$$

Therefore α^2 is even, and therefore α is even.

Since $\alpha : \beta$ is in its lowest terms, it follows that β must be *odd*.

Put $\alpha = 2\gamma$;

therefore $4\gamma^2 = 2\beta^2$,

or $\beta^2 = 2\gamma^2$,

so that β^2, and therefore β, must be *even*.

But β was also odd:

which is impossible.

This proof only enables us to prove the incommensurability of the diagonal of a square with its side, or of $\sqrt{2}$ with unity. In order to prove the incommensurability of the sides of squares, one of which has *three* times the area of another, an entirely different procedure is necessary; and we find in fact that, even a century after Pythagoras' time, it was still necessary to use *separate* proofs (as the passage of the *Theaetetus* shows that Theodorus did) to establish the incommensurability with unity of $\sqrt{3}$, $\sqrt{5}$, ... up to $\sqrt{17}$.

This fact indicates clearly that the general theorem in Eucl. x. 9 that *squares which have not to one another the ratio of a square number to a square number have their sides incommensurable in length* was not arrived at all at once, but was, in the manner of the time, developed out of the separate consideration of special cases (Hankel, p. 103).

The proposition x. 9 of Euclid is definitely ascribed by the scholiast to Theaetetus. Theaetetus was a pupil of Theodorus, and it would seem clear that the theorem was not known to Theodorus. Moreover the Platonic passage itself (*Theaet.* 147 D sqq.) represents the young Theaetetus as striving after a general conception of what we call a *surd*. "The idea occurred to me, seeing that *square roots* (δυνάμεις) appeared to be unlimited in multitude, to try to arrive at one collective term by which we could designate all these square roots.... I divided number in general into two classes. The number which can be expressed as equal multiplied by equal (ἴσον ἰσάκις) I likened to a square in form, and I called it square and equilateral....The intermediate number, such as three, five, and any number which cannot be expressed as equal multiplied by equal, but is either less times more or more times less, so that it is always contained by a greater and less side, I likened to an oblong figure and called an oblong number....Such straight lines then as square the equilateral and plane number I defined as length (μῆκος), and such as square the oblong *square roots* (δυνάμεις), as not being commensurable with the others in length but only in the plane areas to which their squares are equal."

There is further evidence of the contributions of Theaetetus to the theory of incommensurables in a commentary on Eucl. x. discovered, in an Arabic translation, by Woepcke (*Mémoires présentés à l'Académie des Sciences*, XIV., 1856, pp. 658—720). It is certain that this commentary is of Greek origin. Woepcke conjectures that it was by Vettius Valens, an astronomer, apparently of Antioch, and a contemporary of Claudius Ptolemy (2nd cent. A.D.). Heiberg, with greater probability, thinks that we have here a fragment of the commentary of Pappus (*Euklid-studien*, pp. 169—71), and this is rendered practically certain by Suter (*Die Mathematiker und Astronomen der Araber und ihre Werke*, pp. 49 and 211). This commentary states that the theory of irrational magnitudes "had its origin in the school of Pythagoras. It was considerably developed by Theaetetus the Athenian, who gave proof, in this part of mathematics, as in others, of ability which has been justly admired. He was one of the most happily endowed of men, and gave himself up, with a fine enthusiasm, to the investigation of the truths contained in these sciences, as Plato bears witness for him in the work which he called after his name. As for the exact distinctions of the above-named magnitudes and the rigorous demonstrations of the propositions to which this theory gives rise, I believe that they were chiefly established by this mathematician; and, later, the great Apollonius, whose genius touched the highest point of excellence in mathematics, added to these discoveries a number of remarkable theories after many efforts and much labour.

"For Theaetetus had distinguished square roots [*puissances* must be the δυνάμεις of the Platonic passage] commensurable in length from those which are incommensurable, and had divided the well-known species of irrational lines after the different means, assigning the *medial* to geometry, the *binomial* to arithmetic, and the *apotome* to harmony, as is stated by Eudemus the Peripatetic.

"As for Euclid, he set himself to give rigorous rules, which he established, relative to commensurability and incommensurability in general; he made precise the definitions and the distinctions between rational and irrational magnitudes, he set out a great number of orders of irrational magnitudes, and finally he clearly showed their whole extent."

The allusion in the last words must apparently be to x. 115, where it is proved that from the *medial* straight line an unlimited number of other irrationals can be derived, all different from it and from one another.

The connexion between the *medial* straight line and the geometric mean is obvious, because it is in fact the mean proportional between two rational straight lines "commensurable in square only." Since $\frac{1}{2}(x+y)$ is the *arithmetic* mean between x, y, the reference to it of the binomial can be understood. The connexion between the apotome and the harmonic mean is explained by some propositions in the second book of the Arabic commentary. The harmonic mean between x, y is $\frac{2xy}{x+y}$, and propositions of which Woepcke quotes the enunciations prove that, if a rational or a medial area has for one of its sides a *binomial* straight line, the other side will be an *apotome* of corresponding order (these propositions are generalised from Eucl. x. 111—4); the fact is that $\frac{2xy}{x+y} = \frac{2xy}{x^2-y^2} \cdot (x-y)$.

One other predecessor of Euclid appears to have written on irrationals, though we know no more of the work than its title as handed down by Diogenes Laertius[1]. According to this tradition, Democritus wrote *περὶ ἀλόγων γραμμῶν καὶ ναστῶν β′*, *two Books on irrational straight lines and solids* (or *atoms*). Hultsch (*Neue Jahrbücher für Philologie und Pädagogik*, 1881, pp. 578—9) conjectures that the true reading may be *περὶ ἀλόγων γραμμῶν κλαστῶν*, "on irrational broken lines." Hultsch seems to have in mind *straight* lines divided into two parts one of which is rational and the other irrational ("Aus einer Art von Umkehr des Pythagoreischen Lehrsatzes über das rechtwinklige Dreieck gieng zunächst mit Leichtigkeit hervor, dass man eine Linie construiren könne, welche als irrational zu bezeichnen ist, aber durch Brechung sich darstellen lässt als die Summe einer rationalen und einer irrationalen Linie"). But I doubt the use of *κλαστός* in the sense of breaking one straight line into parts; it should properly mean a bent line, i.e. two straight lines forming an angle or *broken short off* at their point of meeting. It is also to be observed that *ναστόν* is quoted as a Democritean word (opposite to *κενόν*) in a fragment of Aristotle (202). I see therefore no reason for questioning the correctness of the title of Democritus' book as above quoted[2].

I will here quote a valuable remark of Zeuthen's relating to the classification of irrationals. He says (*Geschichte der Mathematik im Altertum und Mittelalter*, p. 56) "Since such roots of equations of the second degree as are incommensurable with the given magnitudes cannot be expressed by means of the latter and of numbers, it is conceivable that the Greeks, in exact investigations, introduced no approximate values but worked on with the magnitudes they had found, which were represented by straight lines obtained by the construction corresponding to the solution of the equation. That is exactly the same thing which happens when we do not evaluate roots but content ourselves with expressing them by radical signs and other algebraical symbols. But, inasmuch as one straight line looks like another, the Greeks did not get the same clear view of what they denoted (i.e. by simple inspection) as our system of symbols assures to us. For this reason it was necessary to undertake a classification of the irrational magnitudes which had been arrived at by successive solution of equations of the second degree." To much the same effect Tannery wrote in 1882 (*De la solution géométrique des problèmes du second degré avant Euclide* in *Mémoires de la Société des sciences physiques et naturelles de Bordeaux*, 2ᵉ Série, IV. pp. 395—416). Accordingly Book X. formed a repository of results to which could be referred problems which depended on the solution of certain types of equations, quadratic and biquadratic but reducible to quadratics.

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[1] Diog. Laert. IX. 47, p. 239 (ed. Cobet).

[2] Cf. *ante*, Vol. I., p. 413.

Consider the quadratic equations

$$x^2 \pm 2\alpha x \,.\, \rho \pm \beta \,.\, \rho^2 = 0,$$

where ρ is a rational straight line, and α, β are coefficients. Our quadratic equations in algebra leave out the ρ; but I put it in, because it has always to be remembered that Euclid's x is a *straight line*, not an algebraical quantity, and is therefore to be found in terms of, or in relation to, a certain assumed *rational straight line*, and also because with Euclid ρ may be not only of the form a, where a represents a units of length, but also of the form $\sqrt{\frac{m}{n}} \,.\, a$, which represents a length "commensurable in square only" with the unit of length, or $\sqrt{A}$ where A represents a number (not square) of units of *area*. The use therefore of ρ in our equations makes it unnecessary to multiply different *cases* according to the relation of ρ to the unit of length, and has the further advantage that, e.g., the expression $\rho \pm \sqrt{k} \,.\, \rho$ is just as general as the expression $\sqrt{k} \,.\, \rho \pm \sqrt{\lambda} \,.\, \rho$, since ρ covers the form $\sqrt{k} \,.\, \rho$, both expressions covering a length either commensurable in length, or "commensurable in square only," with the unit of length.

Now the *positive* roots of the quadratic equations

$$x^2 \pm 2\alpha x \,.\, \rho \pm \beta \,.\, \rho^2 = 0$$

can only have the following forms

$$\left.\begin{array}{l} x_1 = \rho\,(\alpha + \sqrt{\alpha^2 - \beta}),\ x_1' = \rho\,(\alpha - \sqrt{\alpha^2 - \beta}) \\ x_2 = \rho\,(\sqrt{\alpha^2 + \beta} + \alpha),\ x_2' = \rho\,(\sqrt{\alpha^2 + \beta} - \alpha) \end{array}\right\}.$$

The negative roots do not come in, since x must be a *straight line*. The omission however to bring in negative roots constitutes no loss of generality, since the Greeks would write the equation leading to negative roots in another form so as to make them positive, i.e. they would change the sign of x in the equation.

Now the positive roots x_1, x_1', x_2, x_2' may be classified according to the character of the coefficents α, β and their relation to one another.

I. Suppose that α, β do not contain any surds, i.e. are either integers or of the form m/n, where m, n are integers.

Now in the expressions for x_1, x_1' it may be that

(1) β is of the form $\frac{m^2}{n^2}\alpha^2$.

Euclid expresses this by saying that the square on $\alpha\rho$ exceeds the square on $\rho\sqrt{\alpha^2 - \beta}$ by the square on a straight line commensurable in length with $\alpha\rho$.

In this case x_1 is, in Euclid's terminology, a *first binomial* straight line, and x_1' a *first apotome*.

(2) In general, β not being of the form $\frac{m^2}{n^2}\alpha^2$,

x_1 is a *fourth binomial*,
x_1' a *fourth apotome*.

Next, in the expressions for x_2, x_2' it may be that

(1) β is equal to $\frac{m^2}{n^2}(\alpha^2+\beta)$, where m, n are integers, i.e. β is of the form

$$\frac{m^2}{n^2-m^2}\alpha^2.$$

Euclid expresses this by saying that the square on $\rho\sqrt{\alpha^2+\beta}$ exceeds the square on $\alpha\rho$ by the square on a straight line commensurable in length with $\rho\sqrt{\alpha^2+\beta}$.

In this case x_2 is, in Euclid's terminology, a *second binomial*,
x_2' a *second apotome*.

(2) In general, β not being of the form $\frac{m^2}{n^2-m^2}\alpha^2$,

x_2 is a *fifth binomial*,
x_2' a *fifth apotome*.

II. Now suppose that α is of the form $\sqrt{\frac{m}{n}}$, where m, n are integers, and let us denote it by $\sqrt{\lambda}$.

Then in this case

$$x_1=\rho(\sqrt{\lambda}+\sqrt{\lambda-\beta}),\quad x_1'=\rho(\sqrt{\lambda}-\sqrt{\lambda-\beta}),$$
$$x_2=\rho(\sqrt{\lambda+\beta}+\sqrt{\lambda}),\quad x_2'=\rho(\sqrt{\lambda+\beta}-\sqrt{\lambda}).$$

Thus x_1, x_1' are of the same form as x_2, x_2'.

If $\sqrt{\lambda-\beta}$ in x_1, x_1' is not surd but of the form m/n, and if $\sqrt{\lambda+\beta}$ in x_2, x_2' is not surd but of the form m/n, the roots are comprised among the forms already shown, the first, second, fourth and fifth binomials and apotomes.

If $\sqrt{\lambda-\beta}$ in x_1, x_1' is surd, then

(1) we may have β of the form $\frac{m^2}{n^2}\lambda$, and in this case

x_1 is a *third binomial* straight line,
x_1' a *third apotome*;

(2) in general, β not being of the form $\frac{m^2}{n^2}\lambda$,

x_1 is a *sixth binomial* straight line,
x_1' a *sixth apotome*.

With the expressions for x_2, x_2' the distinction between the third and sixth binomials and apotomes is of course the distinction between the cases

(1) in which $\beta=\frac{m^2}{n^2}(\lambda+\beta)$, or β is of the form $\frac{m^2}{n^2-m^2}\lambda$,

and (2) in which β is not of this form.

If we take the square root of the product of ρ and each of the six binomials and six apotomes just classified, i.e.

$$\rho^2(\alpha\pm\sqrt{\alpha^2-\beta}),\quad \rho^2(\sqrt{\alpha^2+\beta}\pm\alpha),$$

in the six different forms that each may take, we find six new irrationals with a positive sign separating the two terms, and six corresponding irrationals with a negative sign. These are of course roots of the equations

$$x^4 \pm 2\alpha x^2 . \rho^2 \pm \beta . \rho^4 = 0.$$

These irrationals really come before the others in Euclid's order (x. 36—41 for the positive sign and x. 73—78 for the negative sign). As we shall see in due course, the straight lines actually found by Euclid are

1. $\rho \pm \sqrt{k} . \rho$, the *binomial* (ἡ ἐκ δύο ὀνομάτων)

and the *apotome* (ἀποτομή),

which are the positive roots of the biquadratic (reducible to a quadratic)

$$x^4 - 2(1+k)\rho^2 . x^2 + (1-k)^2 \rho^4 = 0.$$

2. $k^{\frac{1}{4}}\rho \pm k^{\frac{3}{4}}\rho$, the *first bimedial* (ἐκ δύο μέσων πρώτη)

and the *first apotome of a medial* (μέσης ἀποτομὴ πρώτη),

which are the positive roots of

$$x^4 - 2\sqrt{k}(1+k)\rho^2 . x^2 + k(1-k)^2 \rho^4 = 0.$$

3. $k^{\frac{1}{4}}\rho \pm \dfrac{\sqrt{\lambda}}{k^{\frac{1}{4}}}\rho$, the *second bimedial* (ἐκ δύο μέσων δευτέρα)

and the *second apotome of a medial* (μέσης ἀποτομὴ δευτέρα),

which are the positive roots of the equation

$$x^4 - 2\frac{k+\lambda}{\sqrt{k}}\rho^2 . x^2 + \frac{(k-\lambda)^2}{k}\rho^4 = 0.$$

4. $\dfrac{\rho}{\sqrt{2}}\sqrt{1 + \dfrac{k}{\sqrt{1+k^2}}} \pm \dfrac{\rho}{\sqrt{2}}\sqrt{1 - \dfrac{k}{\sqrt{1+k^2}}}$,

the *major* (irrational straight line) (μείζων)

and the *minor* (irrational straight line) (ἐλάσσων),

which are the positive roots of the equation

$$x^4 - 2\rho^2 . x^2 + \frac{k^2}{1+k^2}\rho^4 = 0.$$

5. $\dfrac{\rho}{\sqrt{2(1+k^2)}}\sqrt{\sqrt{1+k^2}+k} \pm \dfrac{\rho}{\sqrt{2(1+k^2)}}\sqrt{\sqrt{1+k^2}-k}$,

the "*side*" *of a rational plus a medial* (area) (ῥητὸν καὶ μέσον δυναμένη) and the "*side*" *of a medial minus a rational area* (in the Greek ἡ μετὰ ῥητοῦ μέσον τὸ ὅλον ποιοῦσα),

which are the positive roots of the equation

$$x^4 - \frac{2}{\sqrt{1+k^2}}\rho^2 . x^2 + \frac{k^2}{(1+k^2)^2}\rho^4 = 0,$$

6. $\dfrac{\lambda^{\frac{1}{4}}\rho}{\sqrt{2}}\sqrt{1 + \dfrac{k}{\sqrt{1+k^2}}} \pm \dfrac{\lambda^{\frac{1}{4}}\rho}{\sqrt{2}}\sqrt{1 - \dfrac{k}{\sqrt{1+k^2}}}$,

the "*side*" *of the sum of two medial areas* (ἡ δύο μέσα δυναμένη) and the "*side*" *of a medial minus a medial area* (in the Greek ἡ μετὰ μέσου μέσον τὸ ὅλον ποιοῦσα),

which are the positive roots of the equation

$$x^4 - 2\sqrt{\lambda} . x^2\rho^2 + \lambda\frac{k^2}{1+k^2}\rho^4 = 0.$$

The above facts and formulae admit of being stated in a great variety of ways according to the notation and the particular letters used. Consequently the summaries which have been given of Eucl. x. by various writers differ much in appearance while expressing the same thing in substance. The first summary in algebraical form (and a very elaborate one) seems to have been that of Cossali (*Origine, trasporto in Italia, primi progressi in essa dell' Algebra*, Vol. II., pp. 242—65) who takes credit accordingly (p. 265). In 1794 Meier Hirsch published at Berlin an *Algebraischer Commentar über das zehente Buch der Elemente des Euklides* which gives the *contents* in algebraical form but fails to give any indication of Euclid's methods, using modern forms of proof only. In 1834 Poselger wrote a paper, *Ueber das zehnte Buch der Elemente des Euklides*, in which he pointed out the defects of Hirsch's reproduction and gave a summary of his own, which however, though nearer to Euclid's form, is difficult to follow in consequence of an elaborate system of abbreviations, and is open to the objection that it is not algebraical enough to enable the character of Euclid's irrationals to be seen at a glance. Other summaries will be found (1) in Nesselmann, *Die Algebra der Griechen*, pp. 165—84; (2) in Loria, *Le scienze esatte nell' antica Grecia*, 1914, pp. 221—34; (3) in Christensen's article "Ueber Gleichungen vierten Grades im zehnten Buch der Elemente Euklids" in the *Zeitschrift für Math. u. Physik* (*Historisch-litterarische Abtheilung*), XXXIV. (1889), pp. 201—17. The only summary in English that I know is that in the *Penny Cyclopaedia*, under "Irrational quantity," by De Morgan, who yielded to none in his admiration of Book x. "Euclid investigates," says De Morgan, "every possible variety of lines which can be represented by $\sqrt{(\sqrt{a} \pm \sqrt{b})}$, a and b representing two commensurable lines....This book has a completeness which none of the others (not even the fifth) can boast of: and we could almost suspect that Euclid, having arranged his materials in his own mind, and having completely elaborated the 10th Book, wrote the preceding books after it and did not live to revise them thoroughly."

Much attention was given to Book x. by the early algebraists. Thus Leonardo of Pisa (fl. about 1200 A.D.) wrote in the 14th section of his *Liber Abaci* on the theory of irrationalities (*de tractatu binomiorum et recisorum*), without however (except in treating of irrational trinomials and cubic irrationalities) adding much to the substance of Book x.; and, in investigating the equation

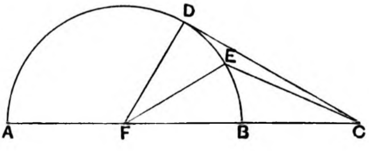
$$x^3 + 2x^2 + 10x = 20,$$

propounded by Johannes of Palermo, he proved that none of the irrationals in Eucl. x. would satisfy it (Hankel, pp. 344—6, Cantor, II_1, p. 43). Luca Paciuolo (about 1445—1514 A.D.) in his algebra based himself largely, as he himself expressly says, on Euclid x. (Cantor, II_1, p. 293). Michael Stifel (1486 or 1487 to 1567) wrote on irrational numbers in the second Book of his *Arithmetica integra*, which Book may be regarded, says Cantor (II_1, p. 402), as an elucidation of Eucl. x. The works of Cardano (1501—76) abound in speculations regarding the irrationals of Euclid, as may be seen by reference to Cossali (Vol. II., especially pp. 268—78 and 382—99); the character of the various odd and even powers of the binomials and apotomes is therein investigated, and Cardano considers in detail of what particular forms of equations, quadratic, cubic, and biquadratic, each class of Euclidean irrationals can be roots. Simon Stevin (1548—1620) gave an *Appendice des incommensurables grandeurs en laquelle est sommairement déclaré le contenu du Dixiesme Livre d'Euclide* (*Oeuvres mathématiques*, Leyde, 1634, pp. 218-22); he speaks thus of the book: "La difficulté du dixiesme Livre d'Euclide est à plusieurs devenue en horreur, voire jusque à l'appeler la croix des mathématiciens, matière trop dure à digérer, et en la quelle n'aperçoivent aucune utilité," a passage quoted by Loria (*op. cit.*, p. 222).

It will naturally be asked, what use did the Greek geometers actually make of the theory of irrationals developed at such length in Book x.? The answer is that Euclid himself, in Book XIII., makes considerable use of the second portion of Book x. dealing with the irrationals affected with a negative sign, the *apotomes* etc. One object of Book XIII. is to investigate the relation of the sides of a pentagon inscribed in a circle and of an icosahedron and dodecahedron inscribed in a sphere to the diameter of the circle or sphere respectively, supposed rational. The connexion with the regular pentagon of a straight line cut in extreme and mean ratio is well known, and Euclid first proves (XIII. 6) that, if a *rational* straight line is so divided, the parts are the irrationals called *apotomes*, the lesser part being a *first apotome*. Then, on the assumption that the diameters of a circle and sphere respectively are rational, he proves (XIII. 11) that the side of the inscribed regular pentagon is the irrational straight line called *minor*, as is also the side of the inscribed icosahedron (XIII. 16), while the side of the inscribed dodecahedron is the irrational called an *apotome* (XIII. 17).

Of course the investigation in Book x. would not have been complete if it had dealt only with the irrationals affected with a *negative* sign. Those affected with the positive sign, the *binomials* etc., had also to be discussed, and we find both portions of Book x., with its nomenclature, made use of by Pappus in two propositions, of which it may be of interest to give the enunciations here.

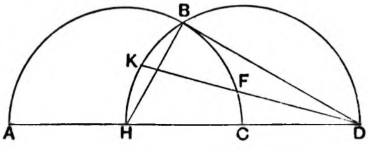
If, says Pappus (IV. p. 178), AB be the rational diameter of a semicircle, and if AB be produced to C so that BC is equal to the radius, if CD be a tangent,



if E be the middle point of the arc BD, and if CE be joined, then CE is the irrational straight line called *minor*. As a matter of fact, if ρ is the radius,

$$CE^2 = \rho^2(5 - 2\sqrt{3}) \text{ and } CE = \sqrt{\frac{5+\sqrt{13}}{2}} - \sqrt{\frac{5-\sqrt{13}}{2}}.$$

If, again (p. 182), CD be equal to the radius of a semicircle supposed



rational, and if the tangent DB be drawn and the angle ADB be bisected by DF meeting the circumference in F, then DF is the excess by which the *binomial* exceeds the *straight line which produces with a rational area a medial* *whole* (see Eucl. x. 77). (In the figure *DK* is the *binomial* and *KF* the other irrational straight line.) As a matter of fact, if ρ be the radius,

$$KD=\rho\,.\,\frac{\sqrt{3}+1}{\sqrt{2}},\ \text{and}\ KF=\rho\,.\,\sqrt{\sqrt{3}-1}=\rho\,\left(\sqrt{\frac{\sqrt{3}+\sqrt{2}}{2}}-\sqrt{\frac{\sqrt{3}-\sqrt{2}}{2}}\right).$$

Proclus tells us that Euclid left out, as alien to a selection of *elements*, the discussion of the more complicated irrationals, "the unordered irrationals which Apollonius worked out more fully" (Proclus, p. 74, 23), while the scholiast to Book x. remarks that Euclid does not deal with all rationals and irrationals but only the simplest kinds by the combination of which an infinite number of irrationals are obtained, of which Apollonius also gave some. The author of the commentary on Book x. found by Woepcke in an Arabic translation, and above alluded to, also says that "it was Apollonius who, beside the *ordered* irrational magnitudes, showed the existence of the *unordered* and by accurate methods set forth a great number of them." It can only be vaguely gathered, from such hints as the commentator proceeds to give, what the character of the extension of the subject given by Apollonius may have been. See note at end of Book.

DEFINITIONS.

1. Those magnitudes are said to be **commensurable** which are measured by the same measure, and those **incommensurable** which cannot have any common measure.

2. Straight lines are **commensurable in square** when the squares on them are measured by the same area, and **incommensurable in square** when the squares on them cannot possibly have any area as a common measure.

3. With these hypotheses, it is proved that there exist straight lines infinite in multitude which are commensurable and incommensurable respectively, some in length only, and others in square also, with an assigned straight line. Let then the assigned straight line be called **rational**, and those straight lines which are commensurable with it, whether in length and in square or in square only, **rational**, but those which are incommensurable with it **irrational**.

4. And let the square on the assigned straight line be called **rational** and those areas which are commensurable with it **rational**, but those which are incommensurable with it **irrational**, and the straight lines which produce them **irrational**, that is, in case the areas are squares, the sides themselves, but in case they are any other rectilineal figures, the straight lines on which are described squares equal to them.

DEFINITION 1.

Σύμμετρα μεγέθη λέγεται τὰ τῷ αὐτῷ μέτρῳ μετρούμενα, ἀσύμμετρα δέ, ὧν μηδὲν ἐνδέχεται κοινὸν μέτρον γενέσθαι

DEFINITION 2.

Εὐθεῖαι δυνάμει σύμμετροί εἰσιν, ὅταν τὰ ἀπ᾽ αὐτῶν τετράγωνα τῷ αὐτῷ χωρίῳ μετρῆται, ἀσύμμετροι δέ, ὅταν τοῖς ἀπ᾽ αὐτῶν τετραγώνοις μηδὲν ἐνδέχηται χωρίον κοινὸν μέτρον γενέσθαι.

Commensurable in square is in the Greek δυνάμει σύμμετρος. In earlier translations (e.g. Williamson's) δυνάμει has been translated "in power," but, as the particular *power* represented by δύναμις in Greek geometry is *square*, I have thought it best to use the latter word throughout. It will be observed that Euclid's expression *commensurable in square only* (used in Def. 3 and constantly) corresponds to what Plato makes Theaetetus call a *square root* (δύναμις) in the sense of a *surd*. If a is any straight line, a and $a\sqrt{m}$, or $a\sqrt{m}$ and $a\sqrt{n}$ (where m, n are integers or arithmetical fractions in their lowest terms, proper or improper, but not square) are *commensurable in square only*. Of course (as explained in the Porism to X. 10) all straight lines *commensurable in length* (μήκει), in Euclid's phrase, are commensurable *in square* also; but not all straight lines which are commensurable *in square* are commensurable *in length* as well. On the other hand, straight lines *incommensurable in square* are necessarily incommensurable *in length* also; but not all straight lines which are incommensurable *in length* are incommensurable *in square*. In fact, straight lines which are *commensurable in square only* are incommensurable *in length*, but obviously not incommensurable in square.

DEFINITION 3.

Τούτων ὑποκειμένων δείκνυται, ὅτι τῇ προτεθείσῃ εὐθείᾳ ὑπάρχουσιν εὐθεῖαι πλήθει ἄπειροι σύμμετροί τε καὶ ἀσύμμετροι αἳ μὲν μήκει μόνον, αἳ δὲ καὶ δυνάμει. καλείσθω οὖν ἡ μὲν προτεθεῖσα εὐθεῖα ῥητή, καὶ αἱ ταύτῃ σύμμετροι εἴτε μήκει καὶ δυνάμει εἴτε δυνάμει μόνον ῥηταί, αἱ δὲ ταύτῃ ἀσύμμετροι ἄλογοι καλείσθωσαν.

The first sentence of the definition is decidedly elliptical. It should, strictly speaking, assert that "with a given straight line there are an infinite number of straight lines which are (1) commensurable either (a) in square only or (b) in square and in length also, and (2) incommensurable, either (a) in length only or (b) in length and in square also."

The relativity of the terms *rational* and *irrational* is well brought out in this definition. We may set out *any straight line* and call it rational, and it is then with reference to this assumed rational straight line that others are called *rational* or *irrational*.

We should carefully note that the signification of *rational* in Euclid is wider than in our terminology. With him, not only is a straight line commensurable *in length* with a rational straight line rational, but a straight line is rational which is commensurable with a rational straight line *in square only*. That is, if ρ is a rational straight line, not only is $\frac{m}{n}\rho$ rational, where m, n are integers and

m/n in its lowest terms is not square, but $\sqrt{\frac{m}{n}}\,.\,\rho$ is *rational* also. We should in this case call $\sqrt{\frac{m}{n}}\,.\,\rho$ irrational. It would appear that Euclid's terminology here differed as much from that of his predecessors as it does from ours. We are familiar with the phrase ἄρρητος διάμετρος τῆς πεμπάδος by which Plato (evidently after the Pythagoreans) describes the diagonal of a square on a straight line containing 5 units of length. This "inexpressible diameter of five (squared)" means $\sqrt{50}$, in contrast to the ῥητὴ διάμετρος, the "expressible diameter" of the same square, by which is meant the approximation $\sqrt{50 - 1}$, or 7. Thus for Euclid's predecessors $\sqrt{\frac{m}{n}}\,.\,\rho$ would apparently not have been rational but ἄρρητος, "inexpressible," i.e. irrational.

I shall throughout my notes on this Book denote a *rational* straight line in Euclid's sense by ρ, and by ρ and σ when two different rational straight lines are required. Wherever then I use ρ or σ, it must be remembered that ρ, σ may have either of the forms a, $\sqrt{k}\,.\,a$, where a represents a units of length, a being either an integer or of the form m/n, where m, n are both integers, and k is an integer or of the form m/n (where both m, n are integers) but not square. In other words, ρ, σ may have either of the forms a or $\sqrt{A}$, where A represents A units of *area* and A is integral or of the form m/n, where m, n are both integers. It has been the habit of writers to give a and $\sqrt{a}$ as the alternative forms of ρ, but I shall always use $\sqrt{A}$ for the second in order to keep the dimensions right, because it must be borne in mind throughout that ρ is an irrational *straight line*.

As Euclid extends the signification of *rational* (ῥητός, literally *expressible*), so he limits the scope of the term ἄλογος (literally *having no ratio*) as applied to straight lines. That this limitation was started by himself may perhaps be inferred from the form of words "*let* straight lines incommensurable with it *be called* irrational." Irrational straight lines then are with Euclid straight lines commensurable *neither in length nor in square* with the assumed rational straight line. $\sqrt{k}\,.\,a$ where k is not square is not irrational; $\sqrt[4]{k}\,.\,a$ is irrational, and so (as we shall see later on) is $(\sqrt{k} \pm \sqrt{\lambda})\,a$.

DEFINITION 4.

Καὶ τὸ μὲν ἀπὸ τῆς προτεθείσης εὐθείας τετράγωνον ῥητόν, καὶ τὰ τούτῳ σύμμετρα ῥητά, τὰ δὲ τούτῳ ἀσύμμετρα ἄλογα καλείσθω, καὶ αἱ δυνάμεναι αὐτὰ ἄλογοι, εἰ μὲν τετράγωνα εἴη, αὐταὶ αἱ πλευραί, εἰ δὲ ἕτερά τινα εὐθύγραμμα, αἱ ἴσα αὐτοῖς τετράγωνα ἀναγράφουσαι.

As applied to *areas*, the terms *rational* and *irrational* have, on the other hand, the same sense with Euclid as we should attach to them. According to Euclid, if ρ is a rational straight line in *his* sense, ρ^2 is *rational* and any area commensurable with it, i.e. of the form $k\rho^2$ (where k is an integer, or of the form m/n, where m, n are integers), is rational; but any area of the form $\sqrt{k}\,.\,\rho^2$ is *irrational*. Euclid's *rational area* thus contains A *units of area*, where A is an integer or of the form m/n, where m, n are integers; and his *irrational area* is of the form $\sqrt{k}\,.\,A$. His irrational *area* is then connected with his irrational *straight line* by making the latter the square root of the former. This would give us for the irrational *straight line* $\sqrt[4]{k} \,.\, \sqrt{A}$, which of course includes $\sqrt[4]{k} \,.\, a$.

αἱ δυνάμεναι αὐτά are the straight lines the squares on which are equal to the areas, in accordance with the regular meaning of δύνασθαι. It is scarcely possible, in a book written in geometrical language, to translate δυναμένη as the *square root* (of an *area*) and δύνασθαι as *to be the square root* (of an *area*), although I can use the term "square root" when in my notes I am using an algebraical expression to represent an area; I shall therefore hereafter use the word "side" for δυναμένη and "to be the side of" for δύνασθαι, so that "side" will in such expressions be a short way of expressing the "side of a *square equal to* (*an area*)." In this particular passage it is not quite practicable to use the words "side of" or "straight line the square on which is equal to," for these expressions occur just afterwards for two alternatives which the word δυναμένη covers. I have therefore exceptionally translated "the straight lines which produce them" (i.e. if squares are described upon them as sides).

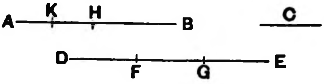
αἱ ἴσα αὐτοῖς τετράγωνα ἀναγράφουσαι, literally "the (straight lines) which *describe* squares equal to them": a peculiar use of the active of ἀναγράφειν, the meaning being of course "the straight lines on which *are described* the squares" which are equal to the rectilineal figures.

BOOK X. PROPOSITIONS.

PROPOSITION 1.

Two unequal magnitudes being set out, if from the greater there be subtracted a magnitude greater than its half, and from that which is left a magnitude greater than its half, and if this process be repeated continually, there will be left some magnitude which will be less than the lesser magnitude set out.

Let *AB*, *C* be two unequal magnitudes of which *AB* is the greater:



I say that, if from *AB* there be subtracted a magnitude greater than its half, and from that which is left a magnitude greater than its half, and if this process be repeated continually, there will be left some magnitude which will be less than the magnitude *C*.

For *C* if multiplied will sometime be greater than *AB*.
[cf. v. Def. 4]

Let it be multiplied, and let *DE* be a multiple of *C*, and greater than *AB*;

let *DE* be divided into the parts *DF*, *FG*, *GE* equal to *C*,

from *AB* let there be subtracted *BH* greater than its half,

and, from *AH*, *HK* greater than its half,

and let this process be repeated continually until the divisions in *AB* are equal in multitude with the divisions in *DE*.

Let, then, *AK*, *KH*, *HB* be divisions which are equal in multitude with *DF*, *FG*, *GE*.

Now, since *DE* is greater than *AB*,

and from *DE* there has been subtracted *EG* less than its half,

and, from *AB*, *BH* greater than its half,

therefore the remainder *GD* is greater than the remainder *HA*.

And, since *GD* is greater than *HA*,
and there has been subtracted, from *GD*, the half *GF*,
and, from *HA*, *HK* greater than its half,
therefore the remainder *DF* is greater than the remainder *AK*.

But *DF* is equal to *C*;
therefore *C* is also greater than *AK*.

Therefore *AK* is less than *C*.

Therefore there is left of the magnitude *AB* the magnitude *AK* which is less than the lesser magnitude set out, namely *C*.

Q. E. D.

And the theorem can be similarly proved even if the parts subtracted be halves.

This proposition will be remembered because it is the lemma required in Euclid's proof of XII. 2 to the effect that circles are to one another as the squares on their diameters. Some writers appear to be under the impression that XII. 2 and the other propositions in Book XII. in which the method of exhaustion is used are the only places where Euclid makes use of X. 1; and it is commonly remarked that X. 1 might just as well have been deferred till the beginning of Book XII. Even Cantor (*Gesch. d. Math.* I_3, p. 269) remarks that "Euclid draws no inference from it [X. 1], not even that which we should more than anything else expect, namely that, if two magnitudes are incommensurable, we can always form a magnitude commensurable with the first which shall differ from the second magnitude by as little as we please." But, so far from making no use of X. 1 before XII. 2, Euclid actually uses it in the very next proposition, X. 2. This being so, as the next note will show, it follows that, since X. 2 gives the criterion for the incommensurability of two magnitudes (a very necessary preliminary to the study of incommensurables), X. 1 comes exactly where it should be.

Euclid uses X. 1 to prove not only XII. 2 but XII. 5 (that pyramids with the same height and triangular bases are to one another as their bases), by means of which he proves (XII. 7 and Por.) that any pyramid is a third part of the prism which has the same base and equal height, and XII. 10 (that any cone is a third part of the cylinder which has the same base and equal height), besides other similar propositions. Now XII. 7 Por. and XII. 10 are theorems specifically attributed to Eudoxus by Archimedes (*On the Sphere and Cylinder*, Preface), who says in another place (*Quadrature of the Parabola*, Preface) that the first of the two, and the theorem that circles are to one another as the squares on their diameters, were proved by means of a certain lemma which he states as follows: "Of unequal lines, unequal surfaces, or unequal solids, the greater exceeds the less by such a magnitude as is capable, if added [continually] to itself, of exceeding any magnitude of those which are comparable with one another," i.e. of magnitudes of the same kind as the original magnitudes. Archimedes also says (*loc. cit.*) that the second of the two theorems which he attributes to Eudoxus (Eucl. XII. 10) was proved by means of "a lemma similar to the aforesaid." The lemma stated thus by Archimedes is decidedly different from X. 1, which, however, Archimedes himself uses several times, while he refers to the use of it in xii. 2 (*On the Sphere and Cylinder*, i. 6). As I have before suggested (*The Works of Archimedes*, p. xlviii), the apparent difficulty caused by the mention of *two* lemmas in connexion with the theorem of Eucl. xii. 2 may be explained by reference to the proof of x. 1. Euclid there takes the lesser magnitude and says that it is possible, by multiplying it, to make it some time exceed the greater, and this statement he clearly bases on the 4th definition of Book v., to the effect that "magnitudes are said to bear a ratio to one another which can, if multiplied, exceed one another." Since then the smaller magnitude in x. 1 may be regarded as the difference between some two unequal magnitudes, it is clear that the lemma stated by Archimedes is in substance used to prove the lemma in x. 1, which appears to play so much larger a part in the investigations of quadrature and cubature which have come down to us.

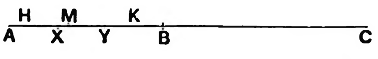
Besides being employed in Eucl. x. 1, the "Axiom of Archimedes" appears in Aristotle, who also practically quotes the result of x. 1 itself. Thus he says, *Physics* viii. 10, 266 b 2, "By continually adding to a finite (magnitude) I shall exceed any definite (magnitude), and similarly by continually subtracting from it I shall arrive at something less than it," and *ibid.* iii. 7, 207 b 10 "For bisections of a magnitude are endless." It is thus somewhat misleading to use the term "Archimedes' Axiom" for the "lemma" quoted by him, since he makes no claim to be the discoverer of it, and it was obviously much earlier.

Stolz (see G. Vitali in *Questioni riguardanti le matematiche elementari*, i., pp. 129—30) showed how to prove the so-called Axiom or Postulate of Archimedes by means of the Postulate of Dedekind, thus. Suppose the two magnitudes to be straight lines. It is required to prove that, *given two straight lines, there always exists a multiple of the smaller which is greater than the other.*

Let the straight lines be so placed that they have a common extremity and the smaller lies along the other on the same side of the common extremity.

If AC be the greater and AB the smaller, we have to prove that there exists an integral number n such that $n \,.\, AB > AC$.

Suppose that this is not true but that there are some points, like B, not coincident with the extremity A, and such that, n being any integer however great, $n \,.\, AB < AC$; and we have to prove that this assumption leads to an absurdity.



The points of AC may be regarded as distributed into two "parts," namely

(1) points H for which there exists no integer n such that $n \,.\, AH > AC$,

(2) points K for which an integer n does exist such that $n \,.\, AK > AC$.

This division into parts satisfies the conditions for the application of Dedekind's Postulate, and therefore there exists a point M such that the points of AM belong to the first part and those of MC to the second part.

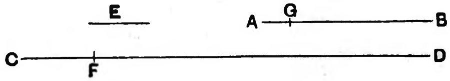
Take now a point Y on MC such that $MY < AM$. The middle point (X) of AY will fall between A and M and will therefore belong to the first part; but, since there exists an integer n such that $n \,.\, AY > AC$, it follows that $2n \,.\, AX > AC$: which is contrary to the hypothesis.

PROPOSITION 2.

If, when the less of two unequal magnitudes is continually subtracted in turn from the greater, that which is left never measures the one before it, the magnitudes will be incommensurable.

For, there being two unequal magnitudes *AB*, *CD*, and *AB* being the less, when the less is continually subtracted in turn from the greater, let that which is left over never measure the one before it;

I say that the magnitudes *AB*, *CD* are incommensurable.



For, if they are commensurable, some magnitude will measure them.

Let a magnitude measure them, if possible, and let it be *E*;
let *AB*, measuring *FD*, leave *CF* less than itself,
let *CF* measuring *BG*, leave *AG* less than itself,
and let this process be repeated continually, until there is left some magnitude which is less than *E*.

Suppose this done, and let there be left *AG* less than *E*.

Then, since *E* measures *AB*,
while *AB* measures *DF*,
therefore *E* will also measure *FD*.

But it measures the whole *CD* also;
therefore it will also measure the remainder *CF*.

But *CF* measures *BG*;
therefore *E* also measures *BG*.

But it measures the whole *AB* also;
therefore it will also measure the remainder *AG*, the greater the less:
which is impossible.

Therefore no magnitude will measure the magnitudes *AB*, *CD*;
therefore the magnitudes *AB*, *CD* are incommensurable.
[x. Def. 1]

Therefore etc.

This proposition states the test for incommensurable magnitudes, founded on the usual operation for finding the greatest common measure. The sign of the incommensurability of two magnitudes is that this operation never comes to an end, while the successive remainders become smaller and smaller until they are less than any assigned magnitude.

Observe that Euclid says "let this process be repeated continually until there is left some magnitude which is less than E." Here he evidently assumes that the process *will* some time produce a remainder less than any assigned magnitude E. Now this is by no means self-evident, and yet Heiberg (though so careful to supply references) and Lorenz do not refer to the basis of the assumption, which is in reality x. 1, as Billingsley and Williamson were shrewd enough to see. The fact is that, if we set off a smaller magnitude once or oftener along a greater which it does not exactly measure, until the remainder is less than the smaller magnitude, we take away from the greater *more than its half.* Thus, in the figure, FD is more than the half of CD, and BG more than the half of AB. If we continued the process, AG marked off along CF as many times as possible would cut off more than its half; next, more than half AG would be cut off, and so on. Hence along CD, AB *alternately* the process would cut off more than half, then more than half the remainder and so on, so that on *both* lines we should ultimately arrive at a remainder less than any assigned length.

The method of finding the greatest common measure exhibited in this proposition and the next is of course again the same as that which we use and which may be shown thus:

$$\begin{array}{l} b)\,a\,(p \\ \underline{pb} \\ c)\,b\,(q \\ \underline{qc} \\ d)\,c\,(r \\ \underline{rd} \\ \;e \end{array}$$

The proof too is the same as ours, taking just the same form, as shown in the notes to the similar propositions vii. 1, 2 above. In the present case the hypothesis is that the process never stops, and it is required to prove that a, b cannot in that case have any common measure, as f. For suppose that f is a common measure, and suppose the process to be continued until the remainder e, say, is less than f.

Then, since f measures a, b, it measures $a - pb$, or c.

Since f measures b, c, it measures $b - qc$, or d; and, since f measures c, d, it measures $c - rd$, or e: which is impossible, since $e < f$.

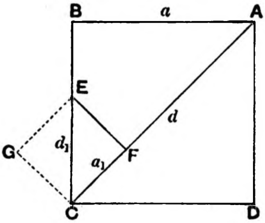
Euclid assumes as axiomatic that, if f measures a, b, it measures $ma \pm nb$.

In practice, of course, it is often unnecessary to carry the process far in order to see that it will never stop, and consequently that the magnitudes are incommensurable. A good instance is pointed out by Allman (*Greek Geometry from Thales to Euclid*, pp. 42, 137—8). Euclid proves in xiii. 5 that, if AB be cut in extreme and mean ratio at C, and if DA equal to AC be added, then DB is also cut in extreme and mean ratio at A. This is indeed obvious from the proof of ii. 11. It follows conversely that, if BD is cut into extreme and mean ratio at A, and AC, equal to the lesser segment AD, be subtracted from the greater AB, AB is similarly divided at C. We can then

mark off from AC a portion equal to CB, and AC will then be similarly divided, and so on. Now the greater segment in a line thus divided is greater than half the line, but it follows from XIII. 3 that it is less than twice the lesser segment, i.e. the lesser segment can never be marked off more than *once* from the greater. Our process of marking off the lesser segment from the greater continually is thus exactly that of finding the greatest common measure. If, therefore, the segments were commensurable, the process would stop. But it clearly does not; therefore the segments are incommensurable.

Allman expresses the opinion that it was rather in connexion with the line cut in extreme and mean ratio than with reference to the diagonal and side of a square that the Pythagoreans discovered the incommensurable. But the evidence seems to put it beyond doubt that the Pythagoreans did discover the incommensurability of $\sqrt{2}$ and devoted much attention to this particular case. The view of Allman does not therefore commend itself to me, though it is likely enough that the Pythagoreans were aware of the incommensurability of the segments of a line cut in extreme and mean ratio. At all events the Pythagoreans could hardly have carried their investigations into the incommensurability of the segments of this line very far, since Theaetetus is said to have made the first classification of irrationals, and to him is also, with reasonable probability, attributed the substance of the first part of Eucl. XIII., in the sixth proposition of which occurs the proof that the segments of a rational straight line cut in extreme and mean ratio are *apotomes*.

Again, the incommensurability of $\sqrt{2}$ can be proved by a method practically equivalent to that of x. 2, and without carrying the process very far. This method is given in Chrystal's *Text-book of Algebra* (I. p. 270). Let d, a be the diagonal and side respectively of a square $ABCD$. Mark off AF along AC equal to a. Draw FE at right angles to AC meeting BC in E.



It is easily proved that

$$BE = EF = FC,$$

$$CF = AC - AB = d - a \text{ ....(1).}$$

$$CE = CB - CF = a - (d - a)$$

$$= 2a - d \text{.........(2).}$$

Suppose, if possible, that d, a are commensurable. If d, a are both commensurably expressible in terms of any finite unit, each must be an integral multiple of a certain finite unit.

But from (1) it follows that CF, and from (2) it follows that CE, is an integral multiple of the same unit.

And CF, CE are the side and diagonal of a square $CFEG$, the side of which is *less than half the side of the original square*. If a_1, d_1 are the side and diagonal of this square,

$$\left.\begin{aligned} a_1 &= d - a \\ d_1 &= 2a - d \end{aligned}\right\}.$$

Similarly we can form a square with side a_2 and diagonal d_2 which are less than half a_1, d_1 respectively, and a_2, d_2 must be integral multiples of the same unit, where

$$a_2 = d_1 - a_1,$$

$$d_2 = 2a_1 - d_1;$$

and this process may be continued indefinitely until (x. 1) we have a square as small as we please, the side and diagonal of which are integral multiples of a finite unit: which is absurd.

Therefore a, d are incommensurable.

It will be observed that this method is the opposite of that shown in the Pythagorean series of side- and diagonal-numbers, the squares being successively smaller instead of larger.

PROPOSITION 3.

Given two commensurable magnitudes, to find their greatest common measure.

Let the two given commensurable magnitudes be *AB*, *CD* of which *AB* is the less;

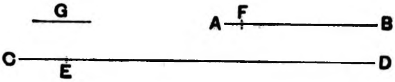
thus it is required to find the greatest common measure of *AB*, *CD*.

Now the magnitude *AB* either measures *CD* or it does not.

If then it measures it—and it measures itself also—*AB* is a common measure of *AB*, *CD*.

And it is manifest that it is also the greatest;

for a greater magnitude than the magnitude *AB* will not measure *AB*.



Next, let *AB* not measure *CD*.

Then, if the less be continually subtracted in turn from the greater, that which is left over will sometime measure the one before it, because *AB*, *CD* are not incommensurable; [cf. x. 2]

let *AB*, measuring *ED*, leave *EC* less than itself,

let *EC*, measuring *FB*, leave *AF* less than itself,

and let *AF* measure *CE*.

Since, then, *AF* measures *CE*,

while *CE* measures *FB*,

therefore *AF* will also measure *FB*.

But it measures itself also;

therefore *AF* will also measure the whole *AB*.

But *AB* measures *DE*;
therefore *AF* will also measure *ED*.

But it measures *CE* also;
therefore it also measures the whole *CD*.

Therefore *AF* is a common measure of *AB*, *CD*.

I say next that it is also the greatest.

For, if not, there will be some magnitude greater than *AF* which will measure *AB*, *CD*.

Let it be *G*.

Since then *G* measures *AB*,
while *AB* measures *ED*,
therefore *G* will also measure *ED*.

But it measures the whole *CD* also;
therefore *G* will also measure the remainder *CE*.

But *CE* measures *FB*;
therefore *G* will also measure *FB*.

But it measures the whole *AB* also,
and it will therefore measure the remainder *AF*, the greater the less:
which is impossible.

Therefore no magnitude greater than *AF* will measure *AB*, *CD*;
therefore *AF* is the greatest common measure of *AB*, *CD*.

Therefore the greatest common measure of the two given commensurable magnitudes *AB*, *CD* has been found.

Q. E. D.

PORISM. From this it is manifest that, if a magnitude measure two magnitudes, it will also measure their greatest common measure.

This proposition for two commensurable *magnitudes* is, *mutatis mutandis*, exactly the same as VII. 2 for numbers. We have the process

$$\begin{array}{l} b)\,a\,(p \\ \underline{pb} \\ \quad c)\,b\,(q \\ \quad \underline{qc} \\ \qquad d)\,c\,(r \\ \qquad \underline{rd} \end{array}$$

where c is equal to rd and therefore there is no remainder, It is then proved that d is a common measure of a, b; and next, by a *reductio ad absurdum*, that it is the *greatest* common measure, since any common measure must measure d, and no magnitude greater than d can measure d. The *reductio ad absurdum* is of course one of form only.

The Porism corresponds exactly to the Porism to VII. 2.

The process of finding the greatest common measure is probably given in this Book, not only for the sake of completeness, but because in x. 5 a common measure of two magnitudes A, B is assumed and used, and therefore it is important to show that such a measure can be *found* if not already known.

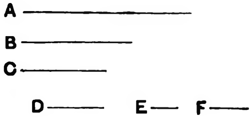
PROPOSITION 4.

Given three commensurable magnitudes, to find their greatest common measure.

Let A, B, C be the three given commensurable magnitudes; thus it is required to find the greatest common measure of A, B, C.

Let the greatest common measure of the two magnitudes A, B be taken, and let it be D; [x. 3]

then D either measures C, or does not measure it.



First, let it measure it.

Since then D measures C,

while it also measures A, B,

therefore D is a common measure of A, B, C.

And it is manifest that it is also the greatest;

for a greater magnitude than the magnitude D does not measure A, B.

Next, let D not measure C.

I say first that C, D are commensurable.

For, since A, B, C are commensurable,

some magnitude will measure them,

and this will of course measure A, B also;

so that it will also measure the greatest common measure of A, B, namely D. [x. 3, Por.]

But it also measures C;

so that the said magnitude will measure C, D;

therefore C, D are commensurable.

Now let their greatest common measure be taken, and let it be *E*. [x. 3]

Since then *E* measures *D*,
while *D* measures *A*, *B*,
therefore *E* will also measure *A*, *B*.

But it measures *C* also;
therefore *E* measures *A*, *B*, *C*;
therefore *E* is a common measure of *A*, *B*, *C*.

I say next that it is also the greatest.

For, if possible, let there be some magnitude *F* greater than *E*, and let it measure *A*, *B*, *C*.

Now, since *F* measures *A*, *B*, *C*,
it will also measure *A*, *B*,
and will measure the greatest common measure of *A*, *B*. [x. 3, Por.]

But the greatest common measure of *A*, *B* is *D*;
therefore *F* measures *D*.

But it measures *C* also;
therefore *F* measures *C*, *D*;
therefore *F* will also measure the greatest common measure of *C*, *D*. [x. 3, Por.]

But that is *E*;
therefore *F* will measure *E*, the greater the less:
which is impossible.

Therefore no magnitude greater than the magnitude *E* will measure *A*, *B*, *C*;
therefore *E* is the greatest common measure of *A*, *B*, *C* if *D* do not measure *C*,
and, if it measure it, *D* is itself the greatest common measure.

Therefore the greatest common measure of the three given commensurable magnitudes has been found.

PORISM. From this it is manifest that, if a magnitude measure three magnitudes, it will also measure their greatest common measure.

Similarly too, with more magnitudes, the greatest common measure can be found, and the porism can be extended.

Q. E. D.

This proposition again corresponds exactly to VII. 3 for numbers. As there Euclid thinks it necessary to prove that, *a*, *b*, *c* not being prime to one another, *d* and *c* are also not prime to one another, so here he thinks it necessary to prove that *d*, *c* are commensurable, as they must be since any common measure of *a*, *b* must be a measure of their greatest common measure *d* (X. 3, Por.).

The argument in the proof that *e*, the greatest common measure of *d*, *c*, is the greatest common measure of *a*, *b*, *c*, is the same as that in VII. 3 and X. 3.

The Porism contains the extension of the process to the case of four or more magnitudes, corresponding to Heron's remark with regard to the similar extension of VII. 3 to the case of four or more *numbers*.

PROPOSITION 5.

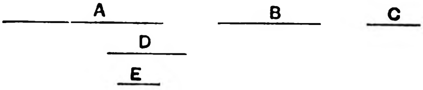
Commensurable magnitudes have to one another the ratio which a number has to a number.

Let *A*, *B* be commensurable magnitudes;

I say that *A* has to *B* the ratio which a number has to a number.

For, since *A*, *B* are commensurable, some magnitude will measure them.

Let it measure them, and let it be *C*.



And, as many times as *C* measures *A*, so many units let there be in *D*;

and, as many times as *C* measures *B*, so many units let there be in *E*.

Since then *C* measures *A* according to the units in *D*,

while the unit also measures *D* according to the units in it,

therefore the unit measures the number *D* the same number of times as the magnitude *C* measures *A*;

therefore, as *C* is to *A*, so is the unit to *D*; [VII. Def. 20]

therefore, inversely, as *A* is to *C*, so is *D* to the unit. [cf. V. 7, Por.]

Again, since *C* measures *B* according to the units in *E*,

while the unit also measures *E* according to the units in it, therefore the unit measures E the same number of times as C measures B;

therefore, as C is to B, so is the unit to E.

But it was also proved that,

as A is to C, so is D to the unit;

therefore, *ex aequali*,

as A is to B, so is the number D to E. [v. 22]

Therefore the commensurable magnitudes A, B have to one another the ratio which the number D has to the number E.

Q. E. D.

The argument is as follows. If a, b be commensurable magnitudes, they have some common measure c, and

$$a = mc,$$
$$b = nc,$$

where m, n are integers.

It follows that $c : a = 1 : m \quad \dots\dots\dots\dots (1),$

or, inversely, $a : c = m : 1;$

and also that $c : b = 1 : n,$

so that, *ex aequali*, $a : b = m : n.$

It will be observed that, in stating the proportion (1), Euclid is merely expressing the fact that a is the same multiple of c that m is of 1. In other words, he rests the statement on the definition of proportion in VII. Def. 20. This, however, is applicable only to four *numbers*, and c, a are not numbers but magnitudes. Hence the statement of the proportion is not legitimate unless it is proved that it is true in the sense of V. Def. 5 with regard to magnitudes in general, the numbers 1, m being *magnitudes*. Similarly with regard to the other proportions in the proposition.

There is, therefore, a hiatus. Euclid ought to have proved that magnitudes which are proportional in the sense of VII. Def. 20 are also proportional in the sense of V. Def. 5, or that the proportion of numbers is included in the proportion of magnitudes as a particular case. Simson has proved this in his Proposition C inserted in Book V. (see Vol. II. pp. 126—8). The portion of that proposition which is required here is the proof that,

if
$$\left.\begin{aligned} a &= mb \\ c &= md \end{aligned}\right\},$$

then $a : b = c : d$, in the sense of V. Def. 5.

Take any equimultiples pa, pc of a, c and any equimultiples qb, qd of b, d.

Now
$$\left.\begin{aligned} pa &= pmb \\ pc &= pmd \end{aligned}\right\}.$$

But, according as $pmb >=< qb$, $pmd >=< qd$.

Therefore, according as $pa >=< qb$, $pc >=< qd$.

And pa, pc are *any* equimultiples of a, c, and qb, qd *any* equimultiples of b, d.

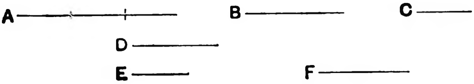
Therefore $a : b = c : d.$ [V. Def. 5.]

PROPOSITION 6.

If two magnitudes have to one another the ratio which a number has to a number, the magnitudes will be commensurable.

For let the two magnitudes A, B have to one another the ratio which the number D has to the number E;

I say that the magnitudes A, B are commensurable.



For let A be divided into as many equal parts as there are units in D,

and let C be equal to one of them;

and let F be made up of as many magnitudes equal to C as there are units in E.

Since then there are in A as many magnitudes equal to C as there are units in D,

whatever part the unit is of D, the same part is C of A also;

therefore, as C is to A, so is the unit to D. [VII. Def. 20]

But the unit measures the number D;

therefore C also measures A.

And since, as C is to A, so is the unit to D,

therefore, inversely, as A is to C, so is the number D to the unit. [cf. V. 7, Por.]

Again, since there are in F as many magnitudes equal to C as there are units in E,

therefore, as C is to F, so is the unit to E. [VII. Def. 20]

But it was also proved that,

as A is to C, so is D to the unit;

therefore, *ex aequali*, as A is to F, so is D to E. [V. 22]

But, as D is to E, so is A to B;

therefore also, as A is to B, so is it to F also. [V. 11]

Therefore A has the same ratio to each of the magnitudes B, F;

therefore B is equal to F. [V. 9]

But C measures F;

therefore it measures B also.

Further it measures A also;

therefore C measures A, B.

Therefore *A* is commensurable with *B*.

Therefore etc.

PORISM. From this it is manifest that, if there be two numbers, as *D*, *E*, and a straight line, as *A*, it is possible to make a straight line [*F*] such that the given straight line is to it as the number *D* is to the number *E*.

And, if a mean proportional be also taken between *A*, *F*, as *B*,

as *A* is to *F*, so will the square on *A* be to the square on *B*, that is, as the first is to the third, so is the figure on the first to that which is similar and similarly described on the second. [VI. 19, Por.]

But, as *A* is to *F*, so is the number *D* to the number *E*; therefore it has been contrived that, as the number *D* is to the number *E*, so also is the figure on the straight line *A* to the figure on the straight line *B*. Q. E. D.

15. **But the unit measures the number D; therefore C also measures A.** These words are redundant, though they are apparently found in all the MSS.

The same link to connect the proportion of numbers with the proportion of magnitudes as was necessary in the last proposition is necessary here. This being premised, the argument is as follows.

Suppose $a : b = m : n$,

where m, n are (integral) numbers.

Divide a into m parts, each equal to c, say,

so that $a = mc$.

Now take d such that $d = nc$.

Therefore we have $a : c = m : 1$,

and $c : d = 1 : n$,

so that, *ex aequali*, $a : d = m : n$

$= a : b$, by hypothesis.

Therefore $b = d = nc$,

so that c measures b n times, and a, b are commensurable.

The Porism is often used in the later propositions. It follows (1) that, if a be a given straight line, and m, n any numbers, a straight line x can be found such that

$$a : x = m : n.$$

(2) We can find a straight line y such that

$$a^2 : y^2 = m : n.$$

For we have only to take y, a mean proportional between a and x, as previously found, in which case a, y, x are in continued proportion and [v. Def. 9]

$$a^2 : y^2 = a : x$$
$$= m : n.$$

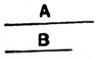
PROPOSITION 7.

Incommensurable magnitudes have not to one another the ratio which a number has to a number.

Let A, B be incommensurable magnitudes;
I say that A has not to B the ratio which a number has to a number.

For, if A has to B the ratio which a number has to a number, A will be commensurable with B. [x. 6]

But it is not;
therefore A has not to B the ratio which a number has to a number.

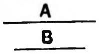


Therefore etc.

PROPOSITION 8.

If two magnitudes have not to one another the ratio which a number has to a number, the magnitudes will be incommensurable.

For let the two magnitudes A, B not have to one another the ratio which a number has to a number;
I say that the magnitudes A, B are incommensurable.



For, if they are commensurable, A will have to B the ratio which a number has to a number. [x. 5]

But it has not;
therefore the magnitudes A, B are incommensurable.

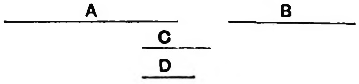
Therefore etc.

PROPOSITION 9.

The squares on straight lines commensurable in length have to one another the ratio which a square number has to a square number; and squares which have to one another the ratio which a square number has to a square number will also have their sides commensurable in length. But the squares on straight lines incommensurable in length have not to one another the ratio which a square number has to a square number; and squares which have not to one another the ratio which a square number has to a square number will not have their sides commensurable in length either.

For let A, B be commensurable in length;

I say that the square on A has to the square on B the ratio which a square number has to a square number.



For, since A is commensurable in length with B, therefore A has to B the ratio which a number has to a number. [x. 5]

Let it have to it the ratio which C has to D.

Since then, as A is to B, so is C to D,
while the ratio of the square on A to the square on B is duplicate of the ratio of A to B,
for similar figures are in the duplicate ratio of their corresponding sides; [vi. 20, Por.]
and the ratio of the square on C to the square on D is duplicate of the ratio of C to D,
for between two square numbers there is one mean proportional number, and the square number has to the square number the ratio duplicate of that which the side has to the side; [viii. 11]
therefore also, as the square on A is to the square on B, so is the square on C to the square on D.

Next, as the square on A is to the square on B, so let the square on C be to the square on D;
I say that A is commensurable in length with B.

For since, as the square on A is to the square on B, so is the square on C to the square on D,
while the ratio of the square on A to the square on B is duplicate of the ratio of A to B,
and the ratio of the square on C to the square on D is duplicate of the ratio of C to D,
therefore also, as A is to B, so is C to D.

Therefore A has to B the ratio which the number C has to the number D;
therefore A is commensurable in length with B. [x. 6]

Next, let A be incommensurable in length with B;
I say that the square on A has not to the square on B the ratio which a square number has to a square number.

For, if the square on A has to the square on B the ratio which a square number has to a square number, *A* will be commensurable with *B*.

But it is not;

therefore the square on *A* has not to the square on *B* the ratio which a square number has to a square number.

Again, let the square on *A* not have to the square on *B* the ratio which a square number has to a square number;

I say that *A* is incommensurable in length with *B*.

For, if *A* is commensurable with *B*, the square on *A* will have to the square on *B* the ratio which a square number has to a square number.

But it has not;

therefore *A* is not commensurable in length with *B*.

Therefore etc.

PORISM. And it is manifest from what has been proved that straight lines commensurable in length are always commensurable in square also, but those commensurable in square are not always commensurable in length also.

[LEMMA. It has been proved in the arithmetical books that similar plane numbers have to one another the ratio which a square number has to a square number, [VIII. 26]

and that, if two numbers have to one another the ratio which a square number has to a square number, they are similar plane numbers. [Converse of VIII. 26]

And it is manifest from these propositions that numbers which are not similar plane numbers, that is, those which have not their sides proportional, have not to one another the ratio which a square number has to a square number.

For, if they have, they will be similar plane numbers: which is contrary to the hypothesis.

Therefore numbers which are not similar plane numbers have not to one another the ratio which a square number has to a square number.]

A scholium to this proposition (Schol. x. No. 62) says categorically that the theorem proved in it was the discovery of Theaetetus.

If a, b be straight lines, and

$$a : b = m : n,$$

where m, n are numbers,

then

$$a^2 : b^2 = m^2 : n^2;$$

and conversely.

This inference, which looks so easy when thus symbolically expressed, was by no means so easy for Euclid owing to the fact that a, b are straight lines, and m, n numbers. He has to pass from $a : b$ to $a^2 : b^2$ by means of VI. 20, Por. through the duplicate ratio; the square on a is to the square on b in the duplicate ratio of the corresponding sides a, b. On the other hand, m, n being *numbers*, it is VIII. 11 which has to be used to show that $m^2 : n^2$ is the ratio duplicate of $m : n$.

Then, in order to establish his result, Euclid *assumes* that, *if two ratios are equal, the ratios which are their duplicates are also equal.* This is nowhere proved in Euclid, but it is an easy inference from V. 22, as shown in my note on VI. 22.

The converse has to be established in the same careful way, and Euclid *assumes* that *ratios the duplicates of which are equal are themselves equal.* This is much more troublesome to prove than the converse; for proofs I refer to the same note on VI. 22.

The second part of the theorem, deduced by *reductio ad absurdum* from the first, requires no remark.

In the Greek text there is an addition to the Porism which Heiberg brackets as superfluous and not in Euclid's manner. It consists (1) of a sort of proof, or rather explanation, of the Porism and (2) of a statement and explanation to the effect that straight lines incommensurable in length are not necessarily incommensurable in square also, and that straight lines incommensurable in square are, on the other hand, always incommensurable in length also.

The Lemma gives expressions for two numbers which have to one another the ratio of a square number to a square number. *Similar plane numbers* are of the form $pm \,.\, pn$ and $qm \,.\, qn$, or mnp^2 and mnq^2, the ratio of which is of course the ratio of p^2 to q^2.

The converse theorem that, if two numbers have to one another the ratio of a square number to a square number, the numbers are similar plane numbers is not, as a matter of fact, proved in the arithmetical Books. It is the converse of VIII. 26 and is used in IX. 10. Heron gave it (see note on VIII. 27 above).

Heiberg however gives strong reason for supposing the Lemma to be an interpolation. It has reference to the next proposition, X. 10, and, as we shall see, there are so many objections to X. 10 that it can hardly be accepted as genuine. Moreover there is no reason why, in the Lemma itself, numbers which are *not* similar plane numbers should be brought in as they are.

[PROPOSITION 10.

To find two straight lines incommensurable, the one in length only, and the other in square also, with an assigned straight line.

Let A be the assigned straight line;

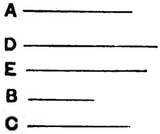
thus it is required to find two straight lines incommensurable, the one in length only, and the other in square also, with A.

Let two numbers B, C be set out which have not to one another the ratio which a square number has to a square number, that is, which are not similar plane numbers;

and let it be contrived that,

as B is to C, so is the square on A to the square on D

—for we have learnt how to do this— [x. 6, Por.]



therefore the square on A is commensurable with the square on D. [x. 6]

And, since B has not to C the ratio which a square number has to a square number,

therefore neither has the square on A to the square on D the ratio which a square number has to a square number;

therefore A is incommensurable in length with D. [x. 9]

Let E be taken a mean proportional between A, D;

therefore, as A is to D, so is the square on A to the square on E. [v. Def. 9]

But A is incommensurable in length with D;

therefore the square on A is also incommensurable with the square on E; [x. 11]

therefore A is incommensurable in square with E.

Therefore two straight lines D, E have been found incommensurable, D in length only, and E in square and of course in length also, with the assigned straight line A.]

It would appear as though this proposition was intended to supply a justification for the statement in x. Def. 3 that *it is proved* that there are an infinite number of straight lines (*a*) incommensurable in length only, or commensurable in square only, and (*b*) incommensurable in square, with any given straight line.

But in truth the proposition could well be dispensed with; and the positive objections to its genuineness are considerable.

In the first place, it depends on the following proposition, x. 11; for the last step concludes that, since

$$a^2 : y^2 = a : x,$$

and a, x are incommensurable in length, therefore a^2, y^2 are incommensurable. But Euclid never commits the irregularity of proving a theorem by means of a later one. Gregory sought to get over the difficulty by putting x. 10 after x. 11; but of course, if the order were so inverted, the Lemma would still be in the wrong place.

Further, the expression ἐμάθομεν γάρ, "for we have learnt (how to do this)," is not in Euclid's manner and betrays the hand of a learner (though the same expression is found in the *Sectio Canonis* of Euclid, where the reference is to the *Elements*).

Lastly the manuscript P has the number 10, in the first hand, at the top of x. 11, from which it may perhaps be concluded that x. 10 had at first no number.

It seems best therefore to reject as spurious both the Lemma and x. 10.

The argument of x. 10 is simple. If a be a given straight line and m, n numbers which have not to one another the ratio of square to square, take x such that

$$a^2 : x^2 = m : n, \qquad \text{[x. 6, Por.]}$$

whence a, x are incommensurable in length. [x. 9]

Then take y a mean proportional between a, x, whence

$$a^2 : y^2 = a : x \qquad \text{[v. Def. 9]}$$

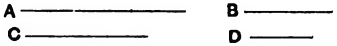
$$[= \sqrt{m} : \sqrt{n}],$$

and x is incommensurable in length only, while y is incommensurable in square as well as in length, with a.

PROPOSITION 11.

If four magnitudes be proportional, and the first be commensurable with the second, the third will also be commensurable with the fourth; and, if the first be incommensurable with the second, the third will also be incommensurable with the fourth.

Let A, B, C, D be four magnitudes in proportion, so that, as A is to B, so is C to D,



and let A be commensurable with B;

I say that C will also be commensurable with D.

For, since A is commensurable with B,

therefore A has to B the ratio which a number has to a number. [x. 5]

And, as A is to B, so is C to D;

therefore C also has to D the ratio which a number has to a number;

therefore C is commensurable with D. [x. 6]

Next, let A be incommensurable with B;

I say that C will also be incommensurable with D.

For, since A is incommensurable with B,

therefore A has not to B the ratio which a number has to a number. [x. 7]

And, as A is to B, so is C to D;
therefore neither has C to D the ratio which a number has to a number;
therefore C is incommensurable with D. [x. 8]

Therefore etc.

I shall henceforth, for the sake of brevity, use symbols for the terms "commensurable (with)" and "incommensurable (with)" according to the varieties described in x. Deff. 1—4. The symbols are taken from Lorenz and seem convenient.

Commensurable and *commensurable with*, in relation to areas, and *commensurable in length* and *commensurable in length with*, in relation to straight lines, will be denoted by $\frown$.

Commensurable in square only or *commensurable in square only with* (terms applicable only to straight lines) will be denoted by $\frown\!\!-$.

Incommensurable (*with*), of areas, and *incommensurable* (*with*), of straight lines will be denoted by $\smile$.

Incommensurable in square (*with*) (a term applicable to straight lines only) will be denoted by $\smile\!\!-$.

Suppose a, b, c, d to be four magnitudes such that

$$a : b = c : d.$$

Then (1), if $a \frown b$, $a : b = m : n$, where m, n are integers, [x. 5]
whence $c : d = m : n$,
and therefore $c \frown d$. [x. 6]

(2) If $a \smile b$, $a : b \neq m : n$, [x. 7]
so that $c : d \neq m : n$,
whence $c \smile d$. [x. 8]

PROPOSITION 12.

Magnitudes commensurable with the same magnitude are commensurable with one another also.

For let each of the magnitudes A, B be commensurable with C;
I say that A is also commensurable with B.

A
C
B
D
E
F
G
H
K
L

For, since A is commensurable with C,
therefore A has to C the ratio which a number has to a number. [x. 5]

Let it have the ratio which D has to E.

Again, since C is commensurable with B,
therefore C has to B the ratio which a number has to a number. [X. 5]

Let it have the ratio which F has to G.

And, given any number of ratios we please, namely the ratio which D has to E and that which F has to G,
let the numbers H, K, L be taken continuously in the given ratios; [cf. VIII. 4]
so that, as D is to E, so is H to K,
and, as F is to G, so is K to L.

Since, then, as A is to C, so is D to E,
while, as D is to E, so is H to K,
therefore also, as A is to C, so is H to K. [V. 11]

Again, since, as C is to B, so is F to G,
while, as F is to G, so is K to L,
therefore also, as C is to B, so is K to L. [V. 11]

But also, as A is to C, so is H to K;
therefore, *ex aequali*, as A is to B, so is H to L. [V. 22]

Therefore A has to B the ratio which a number has to a number;
therefore A is commensurable with B. [X. 6]

Therefore etc.

Q. E. D.

We have merely to go through the process of compounding two ratios in numbers.

Suppose a, b each $\frown c$.
Therefore $a : c = m : n$, say, [X. 5]
$c : b = p : q$, say.

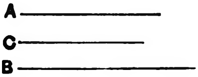
Now $m : n = mp : np$,
and $p : q = np : nq$.
Therefore $a : c = mp : np$,
$c : b = np : nq$,
whence, *ex aequali*, $a : b = mp : nq$,
so that $a \frown b$. [X. 6]

PROPOSITION 13.

If two magnitudes be commensurable, and the one of them be incommensurable with any magnitude, the remaining one will also be incommensurable with the same.

Let A, B be two commensurable magnitudes, and let one of them, A, be incommensurable with any other magnitude C;

I say that the remaining one, B, will also be incommensurable with C.



For, if B is commensurable with C, while A is also commensurable with B, A is also commensurable with C. [x. 12]

But it is also incommensurable with it: which is impossible.

Therefore B is not commensurable with C; therefore it is incommensurable with it.

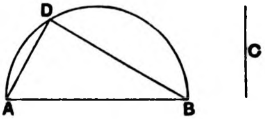
Therefore etc.

LEMMA.

Given two unequal straight lines, to find by what square the square on the greater is greater than the square on the less.

Let AB, C be the given two unequal straight lines, and let AB be the greater of them;

thus it is required to find by what square the square on AB is greater than the square on C.



Let the semicircle ADB be described on AB, and let AD be fitted into it equal to C; [iv. 1]

let DB be joined.

It is then manifest that the angle ADB is right, [iii. 31]

and that the square on AB is greater than the square on AD, that is, C, by the square on DB. [i. 47]

Similarly also, if two straight lines be given, the straight line the square on which is equal to the sum of the squares on them is found in this manner.

Let *AD*, *DB* be the given two straight lines, and let it be required to find the straight line the square on which is equal to the sum of the squares on them.

Let them be placed so as to contain a right angle, that formed by *AD*, *DB*;

and let *AB* be joined.

It is again manifest that the straight line the square on which is equal to the sum of the squares on *AD*, *DB* is *AB*. [I. 47]

Q. E. D.

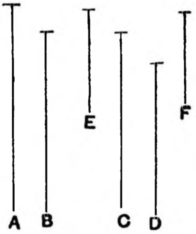
The lemma gives an obvious method of finding a straight line (c) equal to $\sqrt{a^2 \mp b^2}$, where a, b are given straight lines of which a is the greater.

PROPOSITION 14.

If four straight lines be proportional, and the square on the first be greater than the square on the second by the square on a straight line commensurable with the first, the square on the third will also be greater than the square on the fourth by the square on a straight line commensurable with the third.

And, if the square on the first be greater than the square on the second by the square on a straight line incommensurable with the first, the square on the third will also be greater than the square on the fourth by the square on a straight line incommensurable with the third.

Let *A*, *B*, *C*, *D* be four straight lines in proportion, so that, as *A* is to *B*, so is *C* to *D*;



and let the square on *A* be greater than the square on *B* by the square on *E*, and let the square on *C* be greater than the square on *D* by the square on *F*;

I say that, if *A* is commensurable with *E*, *C* is also commensurable with *F*,

and, if *A* is incommensurable with *E*, *C* is also incommensurable with *F*.

For since, as *A* is to *B*, so is *C* to *D*,

therefore also, as the square on *A* is to the square on *B*, so is the square on *C* to the square on *D*. [VI. 22]

But the squares on *E*, *B* are equal to the square on *A*, and the squares on *D*, *F* are equal to the square on *C*.

Therefore, as the squares on *E*, *B* are to the square on *B*, so are the squares on *D*, *F* to the square on *D*;

therefore, *separando*, as the square on *E* is to the square on *B*, so is the square on *F* to the square on *D*; [V. 17]

therefore also, as *E* is to *B*, so is *F* to *D*; [VI. 22]

therefore, inversely, as *B* is to *E*, so is *D* to *F*.

But, as *A* is to *B*, so also is *C* to *D*;

therefore, *ex aequali*, as *A* is to *E*, so is *C* to *F*. [V. 22]

Therefore, if *A* is commensurable with *E*, *C* is also commensurable with *F*,

and, if *A* is incommensurable with *E*, *C* is also incommensurable with *F*. [X. 11]

Therefore etc.

3, 5, 8, 10. Euclid speaks of the square on the first (third) being greater than the square on the second (fourth) by the square on a straight line commensurable (incommensurable) "with *itself* (ἑαυτῇ)," and similarly in all like phrases throughout the Book. For clearness' sake I substitute "the first," "the third," or whatever it may be, for "itself" in these cases.

Suppose a, b, c, d to be straight lines such that

$$a : b = c : d \quad \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

It follows [VI. 22] that

$$a^2 : b^2 = c^2 : d^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

In order to prove that, *convertendo*,

$$a^2 : (a^2 - b^2) = c^2 : (c^2 - d^2)$$

Euclid has to use a somewhat roundabout method owing to the absence of a *convertendo* proposition in his Book V. (which omission Simson supplied by his Prop. E).

It follows from (2) that

$$\{(a^2 - b^2) + b^2\} : b^2 = \{(c^2 - d^2) + d^2\} : d^2,$$

whence, *separando*,

$$(a^2 - b^2) : b^2 = (c^2 - d^2) : d^2, \qquad [\text{V. } 17]$$

and, inversely,

$$b^2 : (a^2 - b^2) = d^2 : (c^2 - d^2).$$

From this and (2), *ex aequali*,

$$a^2 : (a^2 - b^2) = c^2 : (c^2 - d^2). \qquad [\text{V. } 22]$$

Hence

$$a : \sqrt{a^2 - b^2} = c : \sqrt{c^2 - d^2}. \qquad [\text{VI. } 22]$$

According therefore as $a \frown$ or $\smile \sqrt{a^2 - b^2}$,

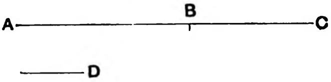
$$c \frown \text{or} \smile \sqrt{c^2 - d^2}. \qquad [\text{X. } 11]$$

If $a \frown \sqrt{a^2 - b^2}$, we may put $\sqrt{a^2 - b^2} = ka$, where k is of the form m/n and m, n are integers. And if $\sqrt{a^2 - b^2} = ka$, it follows in this case that $\sqrt{c^2 - d^2} = kc$.

Proposition 15.

If two commensurable magnitudes be added together, the whole will also be commensurable with each of them; and, if the whole be commensurable with one of them, the original magnitudes will also be commensurable.

For let the two commensurable magnitudes AB, BC be added together;



I say that the whole AC is also commensurable with each of the magnitudes AB, BC.

For, since AB, BC are commensurable, some magnitude will measure them.

Let it measure them, and let it be D.

Since then D measures AB, BC, it will also measure the whole AC.

But it measures AB, BC also;

therefore D measures AB, BC, AC;

therefore AC is commensurable with each of the magnitudes AB, BC. [x. Def. 1]

Next, let AC be commensurable with AB;

I say that AB, BC are also commensurable.

For, since AC, AB are commensurable, some magnitude will measure them.

Let it measure them, and let it be D.

Since then D measures CA, AB, it will also measure the remainder BC.

But it measures AB also;

therefore D will measure AB, BC;

therefore AB, BC are commensurable. [x. Def. 1]

Therefore etc.

(1) If a, b be any two commensurable magnitudes, they are of the form mc, nc, where c is a common measure of a, b and m, n some integers.

It follows that $$a + b = (m + n)\,c;$$
therefore $(a + b)$, being measured by c, is commensurable with both a and b.

(2) If $a + b$ is commensurable with either a or b, say a, we may put $a + b = mc$, $a = nc$, where c is a common measure of $(a + b)$, a, and m, n are integers.

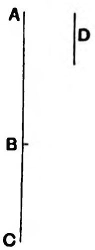
Subtracting, we have $$b = (m - n)\,c,$$
whence $b \frown a$.

PROPOSITION 16.

If two incommensurable magnitudes be added together, the whole will also be incommensurable with each of them; and, if the whole be incommensurable with one of them, the original magnitudes will also be incommensurable.

For let the two incommensurable magnitudes *AB*, *BC* be added together;

I say that the whole *AC* is also incommensurable with each of the magnitudes *AB*, *BC*.



For, if *CA*, *AB* are not incommensurable, some magnitude will measure them.

Let it measure them, if possible, and let it be *D*.

Since then *D* measures *CA*, *AB*,
therefore it will also measure the remainder *BC*.

But it measures *AB* also;
therefore *D* measures *AB*, *BC*.

Therefore *AB*, *BC* are commensurable;
but they were also, by hypothesis, incommensurable:
which is impossible.

Therefore no magnitude will measure *CA*, *AB*;
therefore *CA*, *AB* are incommensurable. [X. Def. 1]

Similarly we can prove that *AC*, *CB* are also incommensurable.

Therefore *AC* is incommensurable with each of the magnitudes *AB*, *BC*.

Next, let *AC* be incommensurable with one of the magnitudes *AB*, *BC*.

First, let it be incommensurable with *AB*;
I say that *AB*, *BC* are also incommensurable.

For, if they are commensurable, some magnitude will measure them.

Let it measure them, and let it be *D*.

Since then *D* measures *AB*, *BC*,
therefore it will also measure the whole *AC*.

But it measures *AB* also;
therefore *D* measures *CA*, *AB*.

Therefore CA, AB are commensurable;
but they were also, by hypothesis, incommensurable:
which is impossible.

Therefore no magnitude will measure AB, BC;
therefore AB, BC are incommensurable. [x. Def. 1]

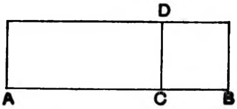
Therefore etc.

LEMMA.

If to any straight line there be applied a parallelogram deficient by a square figure, the applied parallelogram is equal to the rectangle contained by the segments of the straight line resulting from the application.

For let there be applied to the straight line AB the parallelogram AD deficient by the square figure DB;

I say that AD is equal to the rectangle contained by AC, CB.



This is indeed at once manifest;
for, since DB is a square,
DC is equal to CB;
and AD is the rectangle AC, CD, that is, the rectangle AC, CB.

Therefore etc.

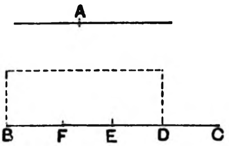
If a be the given straight line, and x the side of the square by which the applied rectangle is to be deficient, the rectangle is equal to $ax - x^2$, which is of course equal to $x(a-x)$. The rectangle may be written xy, where $x+y=a$. Given the area $x(a-x)$, or xy (where $x+y=a$), two different *applications* will give rectangles equal to this area, the sides of the defect being x or $a-x$ (x or y) respectively; but the second mode of expression shows that the rectangles do not differ in form but only in position.

PROPOSITION 17.

If there be two unequal straight lines, and to the greater there be applied a parallelogram equal to the fourth part of the square on the less and deficient by a square figure, and if it divide it into parts which are commensurable in length, then the square on the greater will be greater than the square on the less by the square on a straight line commensurable with the greater.

And, if the square on the greater be greater than the square on the less by the square on a straight line commensurable with *the greater, and if there be applied to the greater a parallelogram equal to the fourth part of the square on the less and deficient by a square figure, it will divide it into parts which are commensurable in length.*

Let A, BC be two unequal straight lines, of which BC is the greater,



and let there be applied to BC a parallelogram equal to the fourth part of the square on the less, A, that is, equal to the square on the half of A, and deficient by a square figure. Let this be the rectangle BD, DC, [cf. Lemma]

and let BD be commensurable in length with DC;

I say that the square on BC is greater than the square on A by the square on a straight line commensurable with BC.

For let BC be bisected at the point E,

and let EF be made equal to DE.

Therefore the remainder DC is equal to BF.

And, since the straight line BC has been cut into equal parts at E, and into unequal parts at D,

therefore the rectangle contained by BD, DC, together with the square on ED, is equal to the square on EC; [II. 5]

And the same is true of their quadruples;

therefore four times the rectangle BD, DC, together with four times the square on DE, is equal to four times the square on EC.

But the square on A is equal to four times the rectangle BD, DC;

and the square on DF is equal to four times the square on DE, for DF is double of DE.

And the square on BC is equal to four times the square on EC, for again BC is double of CE.

Therefore the squares on A, DF are equal to the square on BC,

so that the square on BC is greater than the square on A by the square on DF.

It is to be proved that BC is also commensurable with DF.

Since BD is commensurable in length with DC,

therefore BC is also commensurable in length with CD. [x. 15]

But *CD* is commensurable in length with *CD*, *BF*, for *CD* is equal to *BF*. [x. 6]

Therefore *BC* is also commensurable in length with *BF*, *CD*, [x. 12]

so that *BC* is also commensurable in length with the remainder *FD*; [x. 15]

therefore the square on *BC* is greater than the square on *A* by the square on a straight line commensurable with *BC*.

Next, let the square on *BC* be greater than the square on *A* by the square on a straight line commensurable with *BC*, let a parallelogram be applied to *BC* equal to the fourth part of the square on *A* and deficient by a square figure, and let it be the rectangle *BD*, *DC*.

It is to be proved that *BD* is commensurable in length with *DC*.

With the same construction, we can prove similarly that the square on *BC* is greater than the square on *A* by the square on *FD*.

But the square on *BC* is greater than the square on *A* by the square on a straight line commensurable with *BC*.

Therefore *BC* is commensurable in length with *FD*, so that *BC* is also commensurable in length with the remainder, the sum of *BF*, *DC*. [x. 15]

But the sum of *BF*, *DC* is commensurable with *DC*, [x. 6]

so that *BC* is also commensurable in length with *CD*; [x. 12]

and therefore, *separando*, *BD* is commensurable in length with *DC*. [x. 15]

Therefore etc.

45. After saying literally that "the square on *BC* is greater than the square on *A* by the square on *DF*," Euclid adds the equivalent expression with δύναται in its technical sense, ἡ ΒΓ ἄρα τῆς Α μεῖζον δύναται τῇ ΔΖ. As this is untranslatable in English except by a paraphrase in practically the same words as have preceded, I have not attempted to reproduce it.

This proposition gives the condition that the roots of the equation in x,

$$ax - x^2 = \beta\left(=\frac{b^2}{4}, \text{ say}\right),$$

are commensurable with a, or that x is expressible in terms of a and integral numbers, i.e. is of the form $\frac{m}{n}a$. No better proof can be found for the fact that Euclid and the Greeks used their solutions of quadratic equations for *numerical* problems. On no other assumption could an elaborate discussion of the conditions of incommensurability of the roots with given lengths or with a given number of units of length be explained. In a purely *geometrical* solution the distinction between commensurable and incommensurable roots has no point, because each can equally easily be represented by straight lines. On the other hand, on the assumption that the *numerical* solution of quadratic equations was an important part of the system of the Greek geometers, the distinction between the cases where the roots are commensurable and incommensurable respectively with a given length or unit becomes of great importance. Since the Greeks had no means of *expressing* what we call an irrational number, the case of an equation with incommensurable roots could *only* be represented by them geometrically; and the geometrical representations had to serve instead of what we can express by formulae involving surds.

Euclid proves in this proposition and the next that, x being determined from the equation

$$x(a-x)=\frac{b^2}{4} \quad \text{..................................} (1),$$

x, $(a-x)$ are commensurable in length when $\sqrt{a^2-b^2}$, a are so, and incommensurable in length when $\sqrt{a^2-b^2}$, a are incommensurable; and conversely.

Observe the similarity of his proof to our algebraical method of solving the equation. a being represented in the figure by BC, and x by CD,

$$EF=ED=\frac{a}{2}-x$$

and $$x(a-x)+\left(\frac{a}{2}-x\right)^2=\frac{a^2}{4}, \qquad \text{by Eucl. II. 5.}$$

If we multiply throughout by 4,

$$4x(a-x)+4\left(\frac{a}{2}-x\right)^2=a^2,$$

whence, by (1), $$b^2+(a-2x)^2=a^2,$$

or $$a^2-b^2=(a-2x)^2,$$

and $$\sqrt{a^2-b^2}=a-2x.$$

We have to prove in this proposition

(1) that, if x, $(a-x)$ are commensurable in length, so are a, $\sqrt{a^2-b^2}$,

(2) that, if a, $\sqrt{a^2-b^2}$ are commensurable in length, so are x, $(a-x)$.

(1) To prove that a, $a-2x$ are commensurable in length Euclid employs several successive steps, thus.

Since $(a-x) \frown x$, $\quad a \frown x$. [x. 15]
But $\quad x \frown 2x$. [x. 6]
Therefore $\quad a \frown 2x$ [x. 12]
$\quad \frown (a-2x)$. [x. 15]
That is, $\quad a \frown \sqrt{a^2-b^2}$.

(2) Since $a \frown \sqrt{a^2-b^2}$, $\quad a \frown a-2x$,
whence $\quad a \frown 2x$. [x. 15]
But $\quad 2x \frown x$; [x. 6]
therefore $\quad a \frown x$, [x. 12]
and hence $\quad (a-x) \frown x$. [x. 15]

It is often more convenient to use the symmetrical form of equation in this and similar cases, viz.

$$\left.\begin{aligned} xy &= \frac{b^2}{4} \\ x + y &= a \end{aligned}\right\}.$$

The result with this mode of expression is that

(1) if $x \frown y$, then $a \frown \sqrt{a^2 - b^2}$; and

(2) if $a \frown \sqrt{a^2 - b^2}$, then $x \frown y$.

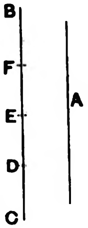
The truth of the proposition is even easier to see in this case, since $(x - y)^2 = (a^2 - b^2)$.

PROPOSITION 18.

If there be two unequal straight lines, and to the greater there be applied a parallelogram equal to the fourth part of the square on the less and deficient by a square figure, and if it divide it into parts which are incommensurable, the square on the greater will be greater than the square on the less by the square on a straight line incommensurable with the greater.

And, if the square on the greater be greater than the square on the less by the square on a straight line incommensurable with the greater, and if there be applied to the greater a parallelogram equal to the fourth part of the square on the less and deficient by a square figure, it divides it into parts which are incommensurable.

Let *A*, *BC* be two unequal straight lines, of which *BC* is the greater,



and to *BC* let there be applied a parallelogram equal to the fourth part of the square on the less, *A*, and deficient by a square figure. Let this be the rectangle *BD*, *DC*, [cf. Lemma before x. 17]

and let *BD* be incommensurable in length with *DC*;

I say that the square on *BC* is greater than the square on *A* by the square on a straight line incommensurable with *BC*.

For, with the same construction as before, we can prove similarly that the square on *BC* is greater than the square on *A* by the square on *FD*.

It is to be proved that *BC* is incommensurable in length with *DF*.

Since *BD* is incommensurable in length with *DC*, therefore *BC* is also incommensurable in length with *CD*. [x. 16]

But *DC* is commensurable with the sum of *BF*, *DC*; [x. 6] therefore *BC* is also incommensurable with the sum of *BF*, *DC*; [x. 13] so that *BC* is also incommensurable in length with the remainder *FD*. [x. 16]

And the square on *BC* is greater than the square on *A* by the square on *FD*;
therefore the square on *BC* is greater than the square on *A* by the square on a straight line incommensurable with *BC*.

Again, let the square on *BC* be greater than the square on *A* by the square on a straight line incommensurable with *BC*, and let there be applied to *BC* a parallelogram equal to the fourth part of the square on *A* and deficient by a square figure. Let this be the rectangle *BD*, *DC*.

It is to be proved that *BD* is incommensurable in length with *DC*.

For, with the same construction, we can prove similarly that the square on *BC* is greater than the square on *A* by the square on *FD*.

But the square on *BC* is greater than the square on *A* by the square on a straight line incommensurable with *BC*;
therefore *BC* is incommensurable in length with *FD*,
so that *BC* is also incommensurable with the remainder, the sum of *BF*, *DC*. [x. 16]

But the sum of *BF*, *DC* is commensurable in length with *DC*; [x. 6]
therefore *BC* is also incommensurable in length with *DC*, [x. 13]
so that, *separando*, *BD* is also incommensurable in length with *DC*. [x. 16]

Therefore etc.

With the same notation as before, we have to prove in this proposition that

(1) if $(a - x)$, x are incommensurable in length, so are a, $\sqrt{a^2 - b^2}$, and
(2) if a, $\sqrt{a^2 - b^2}$ are incommensurable in length, so are $(a - x)$, x.

Or, with the equations

$$\left.\begin{aligned} xy &= \frac{b^2}{4} \\ x + y &= a \end{aligned}\right\},$$

(1) if $x \smile y$, then $a \smile \sqrt{a^2 - b^2}$, and

(2) if $a \smile \sqrt{a^2 - b^2}$, then $x \smile y$.

The steps are exactly the same as shown under (1) and (2) of the last note, with $\smile$ instead of $\frown$, except only in the lines "$x \frown 2x$" and "$2x \frown x$" which are unaltered, while, in the references, x. 13, 16 take the place of x. 12, 15 respectively.

[LEMMA.

Since it has been proved that straight lines commensurable in length are always commensurable in square also, while those commensurable in square are not always commensurable in length also, but can of course be either commensurable or incommensurable in length, it is manifest that, if any straight line be commensurable in length with a given rational straight line, it is called rational and commensurable with the other not only in length but in square also, since straight lines commensurable in length are always commensurable in square also.

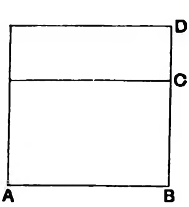
But, if any straight line be commensurable in square with a given rational straight line, then, if it is also commensurable in length with it, it is called in this case also rational and commensurable with it both in length and in square; but, if again any straight line, being commensurable in square with a given rational straight line, be incommensurable in length with it, it is called in this case also rational but commensurable in square only.]

PROPOSITION 19.

The rectangle contained by rational straight lines commensurable in length is rational.

For let the rectangle *AC* be contained by the rational straight lines *AB*, *BC* commensurable in length;

I say that *AC* is rational.



For on *AB* let the square *AD* be described;

therefore *AD* is rational. [x. Def. 4]

And, since *AB* is commensurable in length with *BC*,

while *AB* is equal to *BD*,

therefore *BD* is commensurable in length with *BC*.

And, as BD is to BC, so is DA to AC. [vi. 1]

Therefore DA is commensurable with AC. [x. 11]

But DA is rational;

therefore AC is also rational. [x. Def. 4]

Therefore etc.

There is a difficulty in the text of the enunciation of this proposition. The Greek runs *τὸ ὑπὸ ῥητῶν μήκει συμμέτρων κατά τινα τῶν προειρημένων τρόπων εὐθειῶν περιεχόμενον ὀρθογώνιον ῥητόν ἐστιν*, where the rectangle is said to be contained by "rational straight lines commensurable in length *in any of the aforesaid ways.*" Now straight lines can only be commensurable *in length* in *one* way, the degrees of commensurability being commensurability in length and commensurability in square only. But a straight line may be *rational* in two ways in relation to a *given* rational straight line, since it may be either commensurable *in length*, or commensurable *in square only*, with the latter. Hence Billingsley takes *κατά τινα τῶν προειρημένων τρόπων* with *ῥητῶν*, translating "straight lines commensurable in length and rational in any of the aforesaid ways," and this agrees with the expression in the next proposition "a straight line once more rational in any of the aforesaid ways"; but the order of words in the Greek seems to be fatal to this way of translating the passage.

The best solution of the difficulty seems to be to reject the words "in any of the aforesaid ways" altogether. They have reference to the Lemma which immediately precedes and which is itself open to the gravest suspicion. It is very prolix, and cannot be called necessary; it appears moreover in connexion with an addition clearly spurious and therefore relegated by Heiberg to the Appendix. The addition does not even pretend to be Euclid's, for it begins with the words "for *he calls* rational straight lines those...." Hence we should no doubt relegate the Lemma itself to the Appendix. August does so and leaves out the suspected words in the enunciation, as I have done.

Exactly the same arguments apply to the Lemma added (without the heading "Lemma") to x. 23 and the same words "in any of the aforesaid ways" used with "medial straight lines commensurable in length" in the enunciation of x. 24. The said Lemma must stand or fall with that now in question, since it refers to it in terms: "And in the same way as was explained in the case of rationals...."

Hence I have bracketed the Lemma added to x. 23 and left out the objectionable words in the enunciation of x. 24.

If ρ be one of the given rational straight lines (rational of course in the sense of x. Def. 3), the other can be denoted by $k\rho$, where k is, as usual, of the form m/n (where m, n are integers). Thus the rectangle is $k\rho^2$, which is obviously rational since it is commensurable with ρ^2. [x. Def. 4.]

A rational rectangle may have any of the forms ab, ka^2, kA or A, where a, b are commensurable with the unit of length, and A with the unit of area.

Since Euclid is not able to use $k\rho$ as a symbol for a straight line commensurable in length with ρ, he has to put his proof in a form corresponding to

$$\rho^2 : k\rho^2 = \rho \cdot k\rho,$$

whence, ρ, $k\rho$ being commensurable, ρ^2, $k\rho^2$ are so also. [x. 11]

PROPOSITION 20.

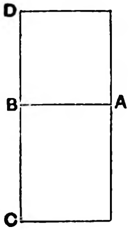
If a rational area be applied to a rational straight line, it produces as breadth a straight line rational and commensurable in length with the straight line to which it is applied.

For let the rational area *AC* be applied to *AB*, a straight line once more rational in any of the aforesaid ways, producing *BC* as breadth;

I say that *BC* is rational and commensurable in length with *BA*.

For on *AB* let the square *AD* be described;

therefore *AD* is rational. [x. Def. 4]



But *AC* is also rational;

therefore *DA* is commensurable with *AC*.

And, as *DA* is to *AC*, so is *DB* to *BC*. [vi. 1]

Therefore *DB* is also commensurable with *BC*; [x. 11]

and *DB* is equal to *BA*;

therefore *AB* is also commensurable with *BC*.

But *AB* is rational;

therefore *BC* is also rational and commensurable in length with *AB*.

Therefore etc.

The converse of the last. If ρ is a rational straight line, any rational area is of the form $k\rho^2$. If this be "applied" to ρ, the breadth is $k\rho$ commensurable in length with ρ and therefore rational. We should reach the same result if we applied the area to *another* rational straight line σ. The breadth is then $\frac{k\rho^2}{\sigma} = \frac{k\rho^2}{\sigma^2} \cdot \sigma = \frac{m}{n} k \cdot \sigma$ or $k'\sigma$, say.

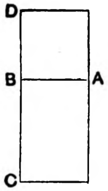
PROPOSITION 21.

The rectangle contained by rational straight lines commensurable in square only is irrational, and the side of the square equal to it is irrational. Let the latter be called medial.

For let the rectangle *AC* be contained by the rational straight lines *AB*, *BC* commensurable in square only;

I say that AC is irrational, and the side of the square equal to it is irrational;

and let the latter be called **medial**.



For on AB let the square AD be described;
therefore AD is rational. [x. Def. 4]

And, since AB is incommensurable in length with BC,
for by hypothesis they are commensurable in square only,
while AB is equal to BD,
therefore DB is also incommensurable in length with BC.

And, as DB is to BC, so is AD to AC; [vi. 1]
therefore DA is incommensurable with AC. [x. 11]

But DA is rational;
therefore AC is irrational,
so that the side of the square equal to AC is also irrational. [x. Def. 4]

And let the latter be called **medial**.

Q. E. D.

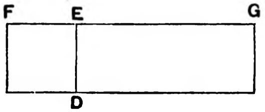
A *medial* straight line, now defined for the first time, is so called because it is a mean proportional between two rational straight lines commensurable in square only. Such straight lines can be denoted by ρ, $\rho\sqrt{k}$. A medial straight line is therefore of the form $\sqrt{\rho^2\sqrt{k}}$ or $k^{\frac{1}{4}}\rho$. Euclid's proof that this is irrational is equivalent to the following. Take ρ, $\rho\sqrt{k}$ commensurable in square only, so that they are incommensurable in length.

Now $$\rho : \rho\sqrt{k} = \rho^2 : \rho^2\sqrt{k},$$
whence [x. 11] $\rho^2\sqrt{k}$ is incommensurable with ρ^2 and therefore irrational [x. Def. 4], so that $\sqrt{\rho^2\sqrt{k}}$ is also irrational [*ibid.*].

A medial straight line may evidently take either of the forms $\sqrt{a\sqrt{B}}$ or $\sqrt[4]{AB}$, where of course B is not of the form k^2A.

LEMMA.

If there be two straight lines, then, as the first is to the second, so is the square on the first to the rectangle contained by the two straight lines.



Let FE, EG be two straight lines.

I say that, as FE is to EG, so is the square on FE to the rectangle FE, EG.

For on *FE* let the square *DF* be described, and let *GD* be completed.

Since then, as *FE* is to *EG*, so is *FD* to *DG*, [VI. 1]
and *FD* is the square on *FE*,
and *DG* the rectangle *DE*, *EG*, that is, the rectangle *FE*, *EG*,
therefore, as *FE* is to *EG*, so is the square on *FE* to the rectangle *FE*, *EG*.

Similarly also, as the rectangle *GE*, *EF* is to the square on *EF*, that is, as *GD* is to *FD*, so is *GE* to *EF*.

Q. E. D.

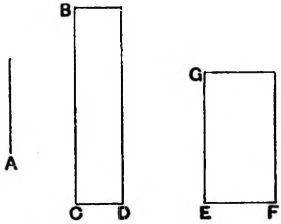
If a, b be two straight lines,

$$a : b = a^2 : ab.$$

PROPOSITION 22.

The square on a medial straight line, if applied to a rational straight line, produces as breadth a straight line rational and incommensurable in length with that to which it is applied.

Let *A* be medial and *CB* rational,
and let a rectangular area *BD* equal to the square on *A* be applied to *BC*, producing *CD* as breadth;



I say that *CD* is rational and incommensurable in length with *CB*.

For, since *A* is medial, the square on it is equal to a rectangular area contained by rational straight lines commensurable in square only. [X. 21]

Let the square on it be equal to *GF*.

But the square on it is also equal to *BD*;
therefore *BD* is equal to *GF*.

But it is also equiangular with it;
and in equal and equiangular parallelograms the sides about the equal angles are reciprocally proportional; [VI. 14]
therefore, proportionally, as *BC* is to *EG*, so is *EF* to *CD*.

Therefore also, as the square on *BC* is to the square on *EG*, so is the square on *EF* to the square on *CD*. [VI. 22]

But the square on *CB* is commensurable with the square on *EG*, for each of these straight lines is rational;
therefore the square on *EF* is also commensurable with the square on *CD*. [X. 11]

But the square on *EF* is rational;
therefore the square on *CD* is also rational; [X. Def. 4]
therefore *CD* is rational.

And, since *EF* is incommensurable in length with *EG*, for they are commensurable in square only,
and, as *EF* is to *EG*, so is the square on *EF* to the rectangle *FE*, *EG*, [Lemma]
therefore the square on *EF* is incommensurable with the rectangle *FE*, *EG*. [X. 11]

But the square on *CD* is commensurable with the square on *EF*, for the straight lines are rational in square;
and the rectangle *DC*, *CB* is commensurable with the rectangle *FE*, *EG*, for they are equal to the square on *A*;
therefore the square on *CD* is also incommensurable with the rectangle *DC*, *CB*. [X. 13]

But, as the square on *CD* is to the rectangle *DC*, *CB*, so is *DC* to *CB*; [Lemma]
therefore *DC* is incommensurable in length with *CB*. [X. 11]

Therefore *CD* is rational and incommensurable in length with *CB*.

Q. E. D.

Our algebraical notation makes the result of this proposition almost self-evident. We have seen that the square of a medial straight line is of the form $\sqrt{k}\,.\,\rho^2$. If we "apply" this area to another rational straight line σ, the breadth is $\dfrac{\sqrt{k}\,.\,\rho^2}{\sigma}$.

This is equal to $\dfrac{\sqrt{k}\,.\,\rho^2}{\sigma^2}\,.\,\sigma = \sqrt{k}\,.\,\dfrac{m}{n}\,\sigma$, where m, n are integers. The latter straight line, which we may express, if we please, in the form $\sqrt{k'}\,.\,\sigma$, is clearly commensurable with σ in square only, and therefore rational but incommensurable in length with σ.

Euclid's proof, necessarily longer, is in two parts.

Suppose that the rectangle $\sqrt{k}\,.\,\rho^2 = \sigma\,.\,x$.

Then (1) $\sigma : \rho = \sqrt{k}\,.\,\rho : x$, [VI. 14]
whence $\sigma^2 : \rho^2 = k\rho^2 : x^2$. [VI. 22]

But $\sigma^2 \frown \rho^2$, and therefore $k\rho^2 \frown x^2$. [X. 11]

And $k\rho^2$ is rational,
therefore x^2, and therefore x, is rational. [X. Def. 4]

(2) Since $\sqrt{k}\,.\,\rho \frown\!\!- \rho$, $\sqrt{k}\,.\,\rho \smile \rho$.

But [Lemma] $\sqrt{k}\,.\,\rho : \rho = k\rho^2 : \sqrt{k}\,.\,\rho^2$,

whence $k\rho^2 \smile \sqrt{k}\,.\,\rho^2$. [X. 11]

But $\sqrt{k}\,.\,\rho^2 = \sigma x$, and $k\rho^2 \frown x^2$ (from above);

therefore $x^2 \smile \sigma x$; [X. 13]

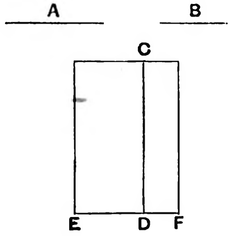
and, since $x^2 : \sigma x = x : \sigma$, [Lemma]

$$x \smile \sigma.$$

Proposition 23.

A straight line commensurable with a medial straight line is medial.

Let *A* be medial, and let *B* be commensurable with *A*; I say that *B* is also medial.



For let a rational straight line *CD* be set out, and to *CD* let the rectangular area *CE* equal to the square on *A* be applied, producing *ED* as breadth;

therefore *ED* is rational and incommensurable in length with *CD*. [X. 22]

And let the rectangular area *CF* equal to the square on *B* be applied to *CD*, producing *DF* as breadth.

Since then *A* is commensurable with *B*, the square on *A* is also commensurable with the square on *B*.

But *EC* is equal to the square on *A*, and *CF* is equal to the square on *B*;

therefore *EC* is commensurable with *CF*.

And, as *EC* is to *CF*, so is *ED* to *DF*; [VI. 1]

therefore *ED* is commensurable in length with *DF*. [X. 11]

But *ED* is rational and incommensurable in length with *DC*;

therefore *DF* is also rational [X. Def. 3] and incommensurable in length with *DC*. [X. 13]

Therefore *CD*, *DF* are rational and commensurable in square only.

But the straight line the square on which is equal to the rectangle contained by rational straight lines commensurable in square only is medial; [x. 21]
therefore the side of the square equal to the rectangle *CD*, *DF* is medial.

And *B* is the side of the square equal to the rectangle *CD*, *DF*;
therefore *B* is medial.

PORISM. From this it is manifest that an area commensurable with a medial area is medial.

[And in the same way as was explained in the case of rationals [Lemma following x. 18] it follows, as regards medials, that a straight line commensurable in length with a medial straight line is called **medial** and commensurable with it not only in length but in square also, since, in general, straight lines commensurable in length are always commensurable in square also.

But, if any straight line be commensurable in square with a medial straight line, then, if it is also commensurable in length with it, the straight lines are called, in this case too, medial and commensurable in length and in square, but, if in square only, they are called medial straight lines commensurable in square only.]

As explained in the bracketed passage following this proposition, a straight line commensurable with a medial straight line *in square only*, as well as a straight line commensurable with it in length, is medial.

Algebraical notation shows this easily.

If $k^{\frac{1}{4}}\rho$ be the given straight line, $\lambda k^{\frac{1}{4}}\rho$ is a straight line commensurable in length with it and $\sqrt{\lambda} . k^{\frac{1}{4}}\rho$ a straight line commensurable with it in square only.

But $\lambda\rho$ and $\sqrt{\lambda} . \rho$ are both rational [x. Def. 3] and therefore can be expressed by ρ', and we thus arrive at $k^{\frac{1}{4}}\rho'$, which is clearly medial.

Euclid's proof amounts to the following.

Apply both the areas $\sqrt{k} . \rho^2$ and $\lambda^2\sqrt{k} . \rho^2$ (or $\lambda\sqrt{k} . \rho^2$) to a rational straight line σ.

The breadths $\sqrt{k} . \frac{\rho^2}{\sigma}$ and $\lambda^2\sqrt{k} . \frac{\rho^2}{\sigma}\left(\text{or } \lambda\sqrt{k} . \frac{\rho^2}{\sigma}\right)$ are in the ratio of the areas $\sqrt{k} . \rho^2$ and $\lambda^2\sqrt{k} . \rho^2$ (or $\lambda\sqrt{k} . \rho^2$) themselves and are therefore commensurable.

Now [x. 22] $\sqrt{k} . \frac{\rho^2}{\sigma}$ is rational but incommensurable with σ.

Therefore $\lambda^2\sqrt{k} . \frac{\rho^2}{\sigma}\left(\text{or } \lambda\sqrt{k} . \frac{\rho^2}{\sigma}\right)$ is so also;

whence the area $\lambda^2\sqrt{k} \,.\, \rho^2$ (or $\lambda\sqrt{k} \,.\, \rho^2$) is contained by two rational straight lines commensurable in square only, so that $\lambda k^{\frac{1}{4}}\rho$ (or $\sqrt{\lambda} \,.\, k^{\frac{1}{4}}\rho$) is a medial straight line.

It is in the Porism that we have the first mention of a medial *area*. It is the area which is equal to the square on a medial straight line, an area, therefore, of the form $k^{\frac{1}{2}}\rho^2$, which is, as a matter of fact, arrived at, though not named, before the medial *straight line* itself (x. 21).

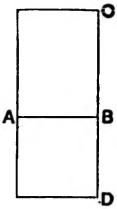
The Porism states that $\lambda k^{\frac{1}{2}}\rho^2$ is a medial area, which is indeed obvious.

PROPOSITION 24.

The rectangle contained by medial straight lines commensurable in length is medial.

For let the rectangle AC be contained by the medial straight lines AB, BC which are commensurable in length;

I say that AC is medial.



For on AB let the square AD be described;

therefore AD is medial.

And, since AB is commensurable in length with BC,

while AB is equal to BD,

therefore DB is also commensurable in length with BC;

so that DA is also commensurable with AC. [vi. 1, x. 11]

But DA is medial;

therefore AC is also medial. [x. 23, Por.]

Q. E. D.

There is the same difficulty in the text of this enunciation as in that of x. 19. The Greek says "medial straight lines commensurable in length in any of the aforesaid ways"; but straight lines can only be *commensurable in length* in one way, though they can be medial in two ways, as explained in the addition to the preceding proposition, i.e. they can be either commensurable in length or commensurable in square only with a *given* medial straight line. For the same reason as that explained in the note on x. 19 I have omitted "in any of the aforesaid ways" in the enunciation and bracketed the addition to x. 23 to which it refers.

$k^{\frac{1}{4}}\rho$ and $\lambda k^{\frac{1}{4}}\rho$ are medial straight lines commensurable in length. The rectangle contained by them is $\lambda k^{\frac{1}{2}}\rho^2$, which may be written $k^{\frac{1}{2}}\rho'^2$ and is therefore clearly medial.

Euclid's proof proceeds thus. Let x, λx be the two medial straight lines commensurable in length.

Therefore $$x^2 : x \,.\, \lambda x = x : \lambda x.$$

But $x \frown \lambda x$, so that $x^2 \frown x \,.\, \lambda x$. [X. 11]

Now x^2 is medial [X. 21];

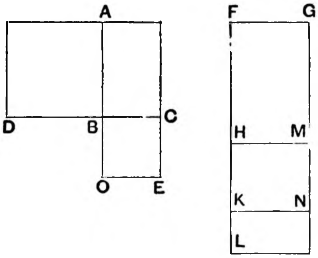
therefore $x \cdot \lambda x$ is also medial. [X. 23, Por.]

We may of course write two medial straight lines commensurable in length in the forms $mk^{\frac{1}{4}}\rho$, $nk^{\frac{1}{4}}\rho$; and these may either be $m\sqrt{a}\sqrt{B}$, $n\sqrt{a}\sqrt{B}$, or $m\sqrt[4]{AB}$, $n\sqrt[4]{AB}$.

Proposition 25.

The rectangle contained by medial straight lines commensurable in square only is either rational or medial.

For let the rectangle AC be contained by the medial straight lines AB, BC which are commensurable in square only; I say that AC is either rational or medial.



For on AB, BC let the squares AD, BE be described; therefore each of the squares AD, BE is medial.

Let a rational straight line FG be set out,

to FG let there be applied the rectangular parallelogram GH equal to AD, producing FH as breadth,

to HM let there be applied the rectangular parallelogram MK equal to AC, producing HK as breadth,

and further to KN let there be similarly applied NL equal to BE, producing KL as breadth;

therefore FH, HK, KL are in a straight line.

Since then each of the squares AD, BE is medial,

and AD is equal to GH, and BE to NL,

therefore each of the rectangles GH, NL is also medial.

And they are applied to the rational straight line FG;

therefore each of the straight lines FH, KL is rational and incommensurable in length with FG. [X. 22]

And, since AD is commensurable with BE,

therefore GH is also commensurable with NL.

And, as GH is to NL, so is FH to KL; [VI. 1]

therefore FH is commensurable in length with KL. [X. 11]

Therefore *FH*, *KL* are rational straight lines commensurable in length;
therefore the rectangle *FH*, *KL* is rational. [x. 19]

And, since *DB* is equal to *BA*, and *OB* to *BC*,
therefore, as *DB* is to *BC*, so is *AB* to *BO*.

But, as *DB* is to *BC*, so is *DA* to *AC*, [vi. 1]
and, as *AB* is to *BO*, so is *AC* to *CO*; [*id.*]
therefore, as *DA* is to *AC*, so is *AC* to *CO*.

But *AD* is equal to *GH*, *AC* to *MK* and *CO* to *NL*;
therefore, as *GH* is to *MK*, so is *MK* to *NL*;
therefore also, as *FH* is to *HK*, so is *HK* to *KL*; [vi. 1, v. 11]
therefore the rectangle *FH*, *KL* is equal to the square on *HK*. [vi. 17]

But the rectangle *FH*, *KL* is rational;
therefore the square on *HK* is also rational.

Therefore *HK* is rational.

And, if it is commensurable in length with *FG*,
HN is rational; [x. 19]
but, if it is incommensurable in length with *FG*,
KH, *HM* are rational straight lines commensurable in square only, and therefore *HN* is medial. [x. 21]

Therefore *HN* is either rational or medial.

But *HN* is equal to *AC*;
therefore *AC* is either rational or medial.

Therefore etc.

Two medial straight lines commensurable in square only are of the form $k^{\frac{1}{4}}\rho$, $\sqrt{\lambda} . k^{\frac{1}{4}}\rho$

The rectangle contained by them is $\sqrt{\lambda} . k^{\frac{1}{2}}\rho^2$. Now this is in general *medial*; but, if $\sqrt{\lambda} = k' \sqrt{k}$, the rectangle is $kk'\rho^2$, which is *rational*.

Euclid's argument is as follows. Let us, for convenience, put x for $k^{\frac{1}{4}}\rho$, so that the medial straight lines are x, $\sqrt{\lambda} . x$.

Form the areas x^2, $x . \sqrt{\lambda} . x$, λx^2,
and let these be respectively equal to σu, σv, σw, where σ is a rational straight line.

Since x^2, λx^2 are medial *areas*,
so are σu, σw,
whence u, w are respectively rational and $\frown \sigma$.

But $x^2 \frown \lambda x^2$,

so that $\sigma u \frown \sigma w$,

or $u \frown w$ ....(1).

Therefore, u, w being both rational, uw is rational .. ..(2).

Now $x^2 : \sqrt{\lambda} \,.\, x^2 = \sqrt{\lambda} \,.\, x^2 : \lambda x^2$

or $\sigma u : \sigma v = \sigma v : \sigma w$,

so that $u : v = v : w$,

and $uw = v^2$.

Hence, by (2), v^2, and therefore v, is rational ....(3).

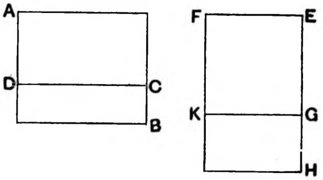
Now (α) if $v \frown \sigma$, σv or $\sqrt{\lambda} \,.\, x^2$ is *rational*;

(β) if $v \smile \sigma$, so that $v \frown\!\!- \sigma$, σv or $\sqrt{\lambda} \,.\, x^2$ is *medial*.

PROPOSITION 26.

A medial area does not exceed a medial area by a rational area.

For, if possible, let the medial area *AB* exceed the medial area *AC* by the rational area *DB*,



and let a rational straight line *EF* be set out;

to *EF* let there be applied the rectangular parallelogram *FH* equal to *AB*, producing *EH* as breadth,

and let the rectangle *FG* equal to *AC* be subtracted;

therefore the remainder *BD* is equal to the remainder *KH*.

But *DB* is rational;

therefore *KH* is also rational.

Since, then, each of the rectangles *AB*, *AC* is medial, and *AB* is equal to *FH*, and *AC* to *FG*,

therefore each of the rectangles *FH*, *FG* is also medial.

And they are applied to the rational straight line *EF*;

therefore each of the straight lines *HE*, *EG* is rational and incommensurable in length with *EF*. [x. 22]

And, since [*DB* is rational and is equal to *KH*, therefore] *KH* is [also] rational;

and it is applied to the rational straight line *EF*;

therefore *GH* is rational and commensurable in length with *EF*. [X. 20]

But *EG* is also rational, and is incommensurable in length with *EF*;

therefore *EG* is incommensurable in length with *GH*. [X. 13]

And, as *EG* is to *GH*, so is the square on *EG* to the rectangle *EG*, *GH*;

therefore the square on *EG* is incommensurable with the rectangle *EG*, *GH*. [X. 11]

But the squares on *EG*, *GH* are commensurable with the square on *EG*, for both are rational;

and twice the rectangle *EG*, *GH* is commensurable with the rectangle *EG*, *GH*, for it is double of it; [X. 6]

therefore the squares on *EG*, *GH* are incommensurable with twice the rectangle *EG*, *GH*; [X. 13]

therefore also the sum of the squares on *EG*, *GH* and twice the rectangle *EG*, *GH*, that is, the square on *EH* [II. 4], is incommensurable with the squares on *EG*, *GH*. [X. 16]

But the squares on *EG*, *GH* are rational;

therefore the square on *EH* is irrational. [X. Def. 4]

Therefore *EH* is irrational.

But it is also rational:

which is impossible.

Therefore etc.

Q. E. D.

"Apply" the two given medial areas to one and the same rational straight line ρ. They can then be written in the form $\rho \,.\, k^{\frac{1}{2}}\rho$, $\rho \,.\, \lambda^{\frac{1}{2}}\rho$.

The difference is then $(\sqrt{k} - \sqrt{\lambda})\,\rho^2$; and the proposition asserts that this cannot be rational, i.e. $(\sqrt{k} - \sqrt{\lambda})$ cannot be equal to k'. Cf. the proposition corresponding to this in algebraical text-books.

To make Euclid's proof clear we will put x for $k^{\frac{1}{2}}\rho$ and y for $\lambda^{\frac{1}{2}}\rho$.

Suppose $$\rho\,(x - y) = \rho z,$$

and, if possible, let ρz be rational, so that z must be rational and $\frown \rho$ ...(1).

Since ρx, ρy are medial,

x and y are respectively rational and $\smile \rho$ ....(2).

From (1) and (2), $$y \smile z.$$

Now $$y : z = y^2 : yz,$$

so that $$y^2 \smile yz.$$

But $y^2 + z^2 \frown y^2$,
and $2yz \frown yz$.
Therefore $y^2 + z^2 \smile 2yz$,
whence $(y + z)^2 \smile (y^2 + z^2)$,
or $x^2 \smile (y^2 + z^2)$.

And $(y^2 + z^2)$ is rational;
therefore x^2, and consequently x, is irrational.

But, by (2), x is rational:
which is impossible.

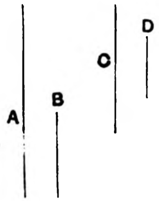
Therefore ρz is *not* rational.

PROPOSITION 27.

To find medial straight lines commensurable in square only which contain a rational rectangle.

Let two rational straight lines A, B commensurable in square only be set out;
let C be taken a mean proportional between A, B, [VI. 13]
and let it be contrived that,

as A is to B, so is C to D. [VI. 12]



Then, since A, B are rational and commensurable in square only,
the rectangle A, B, that is, the square on C [VI 17], is medial. [X. 21]

Therefore C is medial. [X. 21]

And since, as A is to B, so is C to D,
and A, B are commensurable in square only,
therefore C, D are also commensurable in square only. [X. 11]

And C is medial;
therefore D is also medial. [X. 23, addition]

Therefore C, D are medial and commensurable in square only.

I say that they also contain a rational rectangle.

For since, as A is to B, so is C to D,
therefore, alternately, as A is to C, so is B to D. [V. 16]

But, as A is to C, so is C to B;
therefore also, as C is to B, so is B to D;
therefore the rectangle C, D is equal to the square on B.

But the square on B is rational;
therefore the rectangle C, D is also rational.

Therefore medial straight lines commensurable in square only have been found which contain a rational rectangle.

Q. E. D.

Euclid takes two rational straight lines commensurable in square only, say ρ, $k^{\frac{1}{2}}\rho$.

Find the mean proportional, i.e. $k^{\frac{1}{4}}\rho$.

Take x such that $\rho : k^{\frac{1}{2}}\rho = k^{\frac{1}{4}}\rho : x$ ....(1).

This gives $x = k^{\frac{3}{4}}\rho$,
and the lines required are $k^{\frac{1}{4}}\rho$, $k^{\frac{3}{4}}\rho$.

For (α) $k^{\frac{1}{4}}\rho$ is medial.

And (β), by (1), since $\rho \frown k^{\frac{1}{2}}\rho$,

$$k^{\frac{1}{4}}\rho \frown k^{\frac{3}{4}}\rho,$$

whence [addition to X. 23], since $k^{\frac{1}{4}}\rho$ is medial,

$$k^{\frac{3}{4}}\rho \text{ is also medial.}$$

The medial straight lines thus found may take either of the forms

$$(1)\quad \sqrt{a\sqrt{B}},\quad \sqrt{\frac{B\sqrt{B}}{a}}\quad \text{or}\quad (2)\quad \sqrt[4]{AB},\quad \sqrt{B\frac{\sqrt{B}}{\sqrt{A}}}.$$

PROPOSITION 28.

To find medial straight lines commensurable in square only which contain a medial rectangle.

Let the rational straight lines A, B, C commensurable in square only be set out;
let D be taken a mean proportional between A, B, [VI. 13]
and let it be contrived that,

as B is to C, so is D to E. [VI. 12]

Since A, B are rational straight lines commensurable in square only,
therefore the rectangle A, B, that is, the square on D [VI. 17], is medial. [X. 21]

Therefore *D* is medial. [x. 21]

And since *B*, *C* are commensurable in square only,
and, as *B* is to *C*, so is *D* to *E*,
therefore *D*, *E* are also commensurable in square only. [x. 11]

But *D* is medial;
therefore *E* is also medial. [x. 23, addition]

Therefore *D*, *E* are medial straight lines commensurable in square only.

I say next that they also contain a medial rectangle.

For since, as *B* is to *C*, so is *D* to *E*,
therefore, alternately, as *B* is to *D*, so is *C* to *E*. [v. 16]

But, as *B* is to *D*, so is *D* to *A*;
therefore also, as *D* is to *A*, so is *C* to *E*;
therefore the rectangle *A*, *C* is equal to the rectangle *D*, *E*. [vi. 16]

But the rectangle *A*, *C* is medial; [x. 21]
therefore the rectangle *D*, *E* is also medial.

Therefore medial straight lines commensurable in square only have been found which contain a medial rectangle.

Q. E. D.

Euclid takes three straight lines commensurable in square only, i.e. of the form ρ, $k^{\frac{1}{2}}\rho$, $\lambda^{\frac{1}{2}}\rho$, and proceeds as follows.

Take the mean proportional to ρ, $k^{\frac{1}{2}}\rho$, i.e. $k^{\frac{1}{4}}\rho$.

Then take x such that

$$k^{\frac{1}{2}}\rho : \lambda^{\frac{1}{2}}\rho = k^{\frac{1}{4}}\rho : x \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

so that $x = \lambda^{\frac{1}{2}}\rho / k^{\frac{1}{4}}$.

$k^{\frac{1}{4}}\rho$, $\lambda^{\frac{1}{2}}\rho / k^{\frac{1}{4}}$ are the required medial straight lines.

For $k^{\frac{1}{4}}\rho$ is medial.

Now, by (1), since $k^{\frac{1}{2}}\rho \frown \lambda^{\frac{1}{2}}\rho$,

$$k^{\frac{1}{4}}\rho \frown x,$$

whence x is also medial [x. 23, addition], while $\frown k^{\frac{1}{4}}\rho$.

Next, by (1),
$$\lambda^{\frac{1}{2}}\rho : x = k^{\frac{1}{2}}\rho : k^{\frac{1}{4}}\rho$$
$$= k^{\frac{1}{4}}\rho : \rho,$$
whence
$$x \,.\, k^{\frac{1}{4}}\rho = \lambda^{\frac{1}{2}}\rho^2, \text{ which is medial.}$$

The straight lines $k^{\frac{1}{4}}\rho$, $\lambda^{\frac{1}{2}}\rho / k^{\frac{1}{4}}$ of course take different forms according as the original straight lines are of the forms (1) a, $\sqrt{B}$, $\sqrt{C}$, (2) $\sqrt{A}$, $\sqrt{B}$, $\sqrt{C}$, (3) $\sqrt{A}$, b, $\sqrt{C}$, and (4) $\sqrt{A}$, $\sqrt{B}$, c.

E.g. in case (1) they are $\sqrt{a\sqrt{B}}$, $\sqrt{\dfrac{aC}{\sqrt{B}}}$,

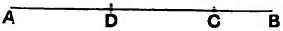
in case (2) they are $\sqrt[4]{AB}$, $\sqrt{\dfrac{C\sqrt{A}}{\sqrt{B}}}$,

and so on.

LEMMA 1.

To find two square numbers such that their sum is also square.

Let two numbers AB, BC be set out, and let them be either both even or both odd.



Then since, whether an even number is subtracted from an even number, or an odd number from an odd number, the remainder is even, [IX. 24, 26]

therefore the remainder AC is even.

Let AC be bisected at D.

Let AB, BC also be either similar plane numbers, or square numbers, which are themselves also similar plane numbers.

Now the product of AB, BC together with the square on CD is equal to the square on BD. [II. 6]

And the product of AB, BC is square, inasmuch as it was proved that, if two similar plane numbers by multiplying one another make some number the product is square. [IX. 1]

Therefore two square numbers, the product of AB, BC, and the square on CD, have been found which, when added together, make the square on BD.

And it is manifest that two square numbers, the square on BD and the square on CD, have again been found such that their difference, the product of AB, BC, is a square, whenever AB, BC are similar plane numbers.

But when they are not similar plane numbers, two square numbers, the square on BD and the square on DC, have been found such that their difference, the product of AB, BC, is not square.

Q. E. D.

Euclid's method of forming right-angled triangles in integral numbers, already alluded to in the note on I. 47, is as follows.

Take two similar plane numbers, e.g. mnp^2, mnq^2, *which are either both even or both odd*, so that their difference is divisible by 2.

Now the product of the two numbers, or $m^2n^2p^2q^2$, is square, [IX. 1]
and, by II. 6,

$$mnp^2 \cdot mnq^2 + \left(\frac{mnp^2 - mnq^2}{2}\right)^2 = \left(\frac{mnp^2 + mnq^2}{2}\right)^2,$$

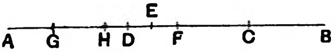
so that the numbers $mnpq$, $\frac{1}{2}(mnp^2 - mnq^2)$ satisfy the condition that the sum of their squares is also a square number.

It is also clear that $\frac{1}{2}(mnp^2 + mnq^2)$, $mnpq$ are numbers such that the *difference* of their squares is also square.

LEMMA 2.

To find two square numbers such that their sum is not square.

For let the product of *AB*, *BC*, as we said, be square,
and *CA* even,
and let *CA* be bisected by *D*.



It is then manifest that the square product of *AB*, *BC* together with the square on *CD* is equal to the square on *BD*. [See Lemma 1]

Let the unit *DE* be subtracted;
therefore the product of *AB*, *BC* together with the square on *CE* is less than the square on *BD*.

I say then that the square product of *AB*, *BC* together with the square on *CE* will not be square.

For, if it is square, it is either equal to the square on *BE*, or less than the square on *BE*, but cannot any more be greater, lest the unit be divided.

First, if possible, let the product of *AB*, *BC* together with the square on *CE* be equal to the square on *BE*,
and let *GA* be double of the unit *DE*.

Since then the whole *AC* is double of the whole *CD*,
and in them *AG* is double of *DE*,
therefore the remainder *GC* is also double of the remainder *EC*;
therefore *GC* is bisected by *E*.

Therefore the product of *GB*, *BC* together with the square on *CE* is equal to the square on *BE*. [II. 6]

But the product of *AB*, *BC* together with the square on *CE* is also, by hypothesis, equal to the square on *BE*;

therefore the product of *GB*, *BC* together with the square on *CE* is equal to the product of *AB*, *BC* together with the square on *CE*.

And, if the common square on *CE* be subtracted,
it follows that *AB* is equal to *GB*:
which is absurd.

Therefore the product of *AB*, *BC* together with the square on *CE* is not equal to the square on *BE*.

I say next that neither is it less than the square on *BE*.

For, if possible, let it be equal to the square on *BF*,
and let *HA* be double of *DF*.

Now it will again follow that *HC* is double of *CF*;
so that *CH* has also been bisected at *F*,
and for this reason the product of *HB*, *BC* together with the square on *FC* is equal to the square on *BF*. [II. 6]

But, by hypothesis, the product of *AB*, *BC* together with the square on *CE* is also equal to the square on *BF*.

Thus the product of *HB*, *BC* together with the square on *CF* will also be equal to the product of *AB*, *BC* together with the square on *CE*:
which is absurd.

Therefore the product of *AB*, *BC* together with the square on *CE* is not less than the square on *BE*.

And it was proved that neither is it equal to the square on *BE*.

Therefore the product of *AB*, *BC* together with the square on *CE* is not square.

Q. E. D.

We can, of course, write the identity in the note on Lemma 1 above (p. 64) in the simpler form

$$mp^2 \cdot mq^2 + \left(\frac{mp^2 - mq^2}{2}\right)^2 = \left(\frac{mp^2 + mq^2}{2}\right)^2,$$

where, as before, mp^2, mq^2 are both odd or both even.

Now, says Euclid,

$$mp^2 \cdot mq^2 + \left(\frac{mp^2 - mq^2}{2} - 1\right)^2 \text{ is not a square number.}$$

This is proved by *reductio ad absurdum*.

The number is clearly less than $mp^2 \,.\, mq^2 + \left(\frac{mp^2 - mq^2}{2}\right)^2$, i.e. less than $\left(\frac{mp^2 + mq^2}{2}\right)^2$.

If then the number is square, its side must be greater than, equal to, or less than $\left(\frac{mp^2 + mq^2}{2} - 1\right)$, the number next less than $\frac{mp^2 + mq^2}{2}$.

But (1) the side cannot be $> \left(\frac{mp^2 + mq^2}{2} - 1\right)$ without being equal to $\frac{mp^2 + mq^2}{2}$, since they are consecutive numbers.

(2) $\quad (mp^2 - 2)\, mq^2 + \left(\frac{mp^2 - mq^2}{2} - 1\right)^2 = \left(\frac{mp^2 + mq^2}{2} - 1\right)^2$. [II. 6]

If then $mp^2 \,.\, mq^2 + \left(\frac{mp^2 - mq^2}{2} - 1\right)^2$ is *also* equal to $\left(\frac{mp^2 + mq^2}{2} - 1\right)^2$,

we must have $\qquad (mp^2 - 2)\, mq^2 = mp^2 \,.\, mq^2$,

or $\qquad mp^2 - 2 = mp^2$:

which is impossible.

(3) If $\quad mp^2 \,.\, mq^2 + \left(\frac{mp^2 - mq^2}{2} - 1\right)^2 < \left(\frac{mp^2 + mq^2}{2} - 1\right)^2$,

suppose it equal to $\left(\frac{mp^2 + mq^2}{2} - r\right)^2$.

But [II. 6] $(mp^2 - 2r)\, mq^2 + \left(\frac{mp^2 - mq^2}{2} - r\right)^2 = \left(\frac{mp^2 + mq^2}{2} - r\right)^2$.

Therefore

$$(mp^2 - 2r)\, mq^2 + \left(\frac{mp^2 - mq^2}{2} - r\right)^2 = mp^2 \,.\, mq^2 + \left(\frac{mp^2 - mq^2}{2} - 1\right)^2:$$

which is impossible.

Hence all three hypotheses are false, and the sum of the squares $mp^2 \,.\, mq^2$ and $\left(\frac{mp^2 - mq^2}{2} - 1\right)^2$ is *not* square.

PROPOSITION 29.

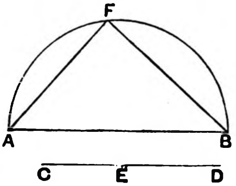
To find two rational straight lines commensurable in square only and such that the square on the greater is greater than the square on the less by the square on a straight line commensurable in length with the greater.

For let there be set out any rational straight line *AB*, and two square numbers *CD*, *DE* such that their difference *CE* is not square; [Lemma 1]

let there be described on *AB* the semicircle *AFB*, and let it be contrived that,

as DC is to CE, so is the square on BA to the square on AF. [x. 6, Por.]

Let FB be joined.



Since, as the square on BA is to the square on AF, so is DC to CE,

therefore the square on BA has to the square on AF the ratio which the number DC has to the number CE;

therefore the square on BA is commensurable with the square on AF. [x. 6]

But the square on AB is rational; [x. Def. 4]

therefore the square on AF is also rational; [*id.*]

therefore AF is also rational.

And, since DC has not to CE the ratio which a square number has to a square number,

neither has the square on BA to the square on AF the ratio which a square number has to a square number;

therefore AB is incommensurable in length with AF. [x. 9]

Therefore BA, AF are rational straight lines commensurable in square only.

And since, as DC is to CE, so is the square on BA to the square on AF,

therefore, *convertendo*, as CD is to DE, so is the square on AB to the square on BF. [v. 19, Por., III. 31, I. 47]

But CD has to DE the ratio which a square number has to a square number;

therefore also the square on AB has to the square on BF the ratio which a square number has to a square number;

therefore AB is commensurable in length with BF. [x. 9]

And the square on AB is equal to the squares on AF, FB;

therefore the square on AB is greater than the square on AF by the square on BF commensurable with AB.

Therefore there have been found two rational straight lines BA, AF commensurable in square only and such that the square on the greater AB is greater than the square on the less AF by the square on BF commensurable in length with AB.

Take a rational straight line ρ and two numbers m^2, n^2 such that $(m^2 - n^2)$ is not a square.

Take a straight line x such that

$$m^2 : m^2 - n^2 = \rho^2 : x^2 \quad\text{.............................}(1),$$

whence
$$x^2 = \frac{m^2 - n^2}{m^2}\rho^2,$$

and
$$x = \rho\sqrt{1 - k^2}, \qquad \text{where } k = \frac{n}{m}.$$

Then ρ, $\rho\sqrt{1 - k^2}$ are the straight lines required.

It follows from (1) that $x^2 \frown \rho^2$,

and x is rational, but $x \smile \rho$.

By (1), *convertendo*, $m^2 : n^2 = \rho^2 : \rho^2 - x^2$,

so that $\sqrt{\rho^2 - x^2} \frown \rho$, and in fact $= k\rho$.

According as ρ is of the form a or $\sqrt{A}$, the straight lines are (1) a, $\sqrt{a^2 - b^2}$ or (2) $\sqrt{A}$, $\sqrt{A - k^2 A}$.

PROPOSITION 30.

To find two rational straight lines commensurable in square only and such that the square on the greater is greater than the square on the less by the square on a straight line incommensurable in length with the greater.

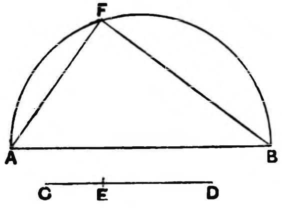
Let there be set out a rational straight line *AB*, and two square numbers *CE*, *ED* such that their sum *CD* is not square; [Lemma 2]

let there be described on *AB* the semicircle *AFB*,

let it be contrived that,

as *DC* is to *CE*, so is the square on *BA* to the square on *AF*, [x. 6, Por.]

and let *FB* be joined.



Then, in a similar manner to the preceding, we can prove that *BA*, *AF* are rational straight lines commensurable in square only.

And since, as *DC* is to *CE*, so is the square on *BA* to the square on *AF*,

therefore, *convertendo*, as *CD* is to *DE*, so is the square on *AB* to the square on *BF*. [v. 19, Por., III. 31, I. 47]

But *CD* has not to *DE* the ratio which a square number has to a square number;

therefore neither has the square on *AB* to the square on *BF* the ratio which a square number has to a square number; therefore *AB* is incommensurable in length with *BF*. [x. 9]

And the square on *AB* is greater than the square on *AF* by the square on *FB* incommensurable with *AB*.

Therefore *AB*, *AF* are rational straight lines commensurable in square only, and the square on *AB* is greater than the square on *AF* by the square on *FB* incommensurable in length with *AB*.

Q. E. D.

In this case we take m^2, n^2 such that $m^2 + n^2$ is not square.

Find x such that $m^2 + n^2 : m^2 = \rho^2 : x^2$,

whence
$$x^2 = \frac{m^2}{m^2 + n^2} \rho^2,$$

or
$$x = \frac{\rho}{\sqrt{1 + k^2}}, \qquad \text{where } k = \frac{n}{m}.$$

Then ρ, $\dfrac{\rho}{\sqrt{1 + k^2}}$ satisfy the condition.

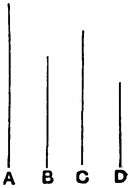
The proof is after the manner of the proof of the preceding proposition and need not be repeated.

According as ρ is of the form a or $\sqrt{A}$, the straight lines take the form (1) a, $\sqrt{a^2 - \dfrac{k^2 a^2}{1 + k^2}}$, that is, a, $\sqrt{a^2 - B}$, or (2) $\sqrt{A}$, $\sqrt{A - B}$ and $\sqrt{A}$, $\sqrt{A - b^2}$.

Proposition 31.

To find two medial straight lines commensurable in square only, containing a rational rectangle, and such that the square on the greater is greater than the square on the less by the square on a straight line commensurable in length with the greater.

Let there be set out two rational straight lines *A*, *B* commensurable in square only and such that the square on *A*, being the greater, is greater than the square on *B* the less by the square on a straight line commensurable in length with *A*. [x. 29]



And let the square on *C* be equal to the rectangle *A*, *B*.

Now the rectangle *A*, *B* is medial; [x. 21]
therefore the square on *C* is also medial;
therefore *C* is also medial. [x. 21]

Let the rectangle *C*, *D* be equal to the square on *B*.

Now the square on *B* is rational;
therefore the rectangle *C*, *D* is also rational.

And since, as *A* is to *B*, so is the rectangle *A*, *B* to the square on *B*,
while the square on *C* is equal to the rectangle *A*, *B*,
and the rectangle *C*, *D* is equal to the square on *B*,
therefore, as *A* is to *B*, so is the square on *C* to the rectangle *C*, *D*.

But, as the square on *C* is to the rectangle *C*, *D*, so is *C* to *D*;
therefore also, as *A* is to *B*, so is *C* to *D*.

But *A* is commensurable with *B* in square only;
therefore *C* is also commensurable with *D* in square only. [x. 11]

And *C* is medial;
therefore *D* is also medial. [x. 23, addition]

And since, as *A* is to *B*, so is *C* to *D*,
and the square on *A* is greater than the square on *B* by the square on a straight line commensurable with *A*,
therefore also the square on *C* is greater than the square on *D* by the square on a straight line commensurable with *C*. [x. 14]

Therefore two medial straight lines *C*, *D*, commensurable in square only and containing a rational rectangle, have been found, and the square on *C* is greater than the square on *D* by the square on a straight line commensurable in length with *C*.

Similarly also it can be proved that the square on *C* exceeds the square on *D* by the square on a straight line incommensurable with *C*, when the square on *A* is greater than the square on *B* by the square on a straight line incommensurable with *A*. [x. 30]

I. Take the rational straight lines commensurable in square only found in x. 29, i.e. ρ, $\rho\sqrt{1-k^2}$.

Take the mean proportional $\rho(1-k^2)^{\frac{1}{4}}$ and x such that

$$\rho(1-k^2)^{\frac{1}{4}} : \rho\sqrt{1-k^2} = \rho\sqrt{1-k^2} : x.$$

Then $\rho(1-k^2)^{\frac{1}{4}}$, x, or $\rho(1-k^2)^{\frac{1}{4}}$, $\rho(1-k^2)^{\frac{3}{4}}$ are straight lines satisfying the given conditions.

For (α) $\rho^2\sqrt{1-k^2}$ is a medial area, and therefore $\rho(1-k^2)^{\frac{1}{4}}$ is a medial straight line ....(1); and $x \cdot \rho(1-k^2)^{\frac{1}{4}} = \rho^2(1-k^2)$ and is therefore a rational area.

(β) ρ, $\rho(1-k^2)^{\frac{1}{4}}$, $\rho\sqrt{1-k^2}$, x are straight lines in continued proportion, by construction.

Therefore $\rho : \rho\sqrt{1-k^2} = \rho(1-k^2)^{\frac{1}{4}} : x$ ....(2).

(This Euclid has to prove in a somewhat roundabout way by means of the lemma after X. 21 to the effect that $a : b = ab : b^2$.)

From (2) it follows [X. 11] that $x \frown \rho(1-k^2)^{\frac{1}{4}}$; whence, since $\rho(1-k^2)^{\frac{1}{4}}$ is medial, x or $\rho(1-k^2)^{\frac{3}{4}}$ is medial also.

(γ) From (2), since ρ, $\rho\sqrt{1-k^2}$ satisfy the remaining condition of the problem, $\rho(1-k^2)^{\frac{1}{4}}$, $\rho(1-k^2)^{\frac{3}{4}}$ do so also [X. 14].

According as ρ is of the form a or $\sqrt{A}$, the straight lines take the forms

$$(1)\qquad \sqrt{a\sqrt{a^2-b^2}},\quad \frac{a^2-b^2}{\sqrt{a\sqrt{a^2-b^2}}},$$

$$\text{or}\qquad (2)\qquad \sqrt[4]{A(A-k^2A)},\quad \frac{A-k^2A}{\sqrt[4]{A(A-k^2A)}}.$$

II. To find medial straight lines commensurable in square only containing a rational rectangle, and such that the square on one exceeds the square on the other by the square on a straight line *incommensurable* with the former, we simply begin with the rational straight lines having the corresponding property [X. 30], viz. ρ, $\frac{\rho}{\sqrt{1+k^2}}$, and we arrive at the straight lines

$$\frac{\rho}{(1+k^2)^{\frac{1}{4}}},\quad \frac{\rho}{(1+k^2)^{\frac{3}{4}}}.$$

According as ρ is of the form a or $\sqrt{A}$, these (if we use the same transformation as at the end of the note on X. 30) may take any of the forms

$$(1)\qquad \sqrt{a\sqrt{a^2-B}},\quad \frac{a^2-B}{\sqrt{a\sqrt{a^2-B}}},$$

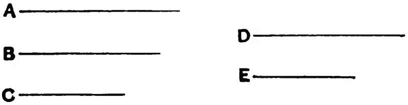
$$\text{or}\qquad (2)\qquad \sqrt[4]{A(A-B)},\quad \frac{A-B}{\sqrt[4]{A(A-B)}},$$

$$\text{or}\qquad \sqrt[4]{A(A-b^2)},\quad \frac{A-b^2}{\sqrt[4]{A(A-b^2)}}.$$

PROPOSITION 32.

To find two medial straight lines commensurable in square only, containing a medial rectangle, and such that the square on the greater is greater than the square on the less by the square on a straight line commensurable with the greater.

Let there be set out three rational straight lines *A*, *B*, *C* commensurable in square only, and such that the square on *A* is greater than the square on *C* by the square on a straight line commensurable with *A*, [X. 29]
and let the square on *D* be equal to the rectangle *A*, *B*.



Therefore the square on *D* is medial;
therefore *D* is also medial. [X. 21]

Let the rectangle *D*, *E* be equal to the rectangle *B*, *C*.

Then since, as the rectangle *A*, *B* is to the rectangle *B*, *C*, so is *A* to *C*;
while the square on *D* is equal to the rectangle *A*, *B*,
and the rectangle *D*, *E* is equal to the rectangle *B*, *C*,
therefore, as *A* is to *C*, so is the square on *D* to the rectangle *D*, *E*.

But, as the square on *D* is to the rectangle *D*, *E*, so is *D* to *E*;
therefore also, as *A* is to *C*, so is *D* to *E*.

But *A* is commensurable with *C* in square only;
therefore *D* is also commensurable with *E* in square only. [X. 11]

But *D* is medial;
therefore *E* is also medial. [X. 23, addition]

And, since, as *A* is to *C*, so is *D* to *E*,
while the square on *A* is greater than the square on *C* by the square on a straight line commensurable with *A*,
therefore also the square on *D* will be greater than the square on *E* by the square on a straight line commensurable with *D*. [X. 14]

I say next that the rectangle *D*, *E* is also medial.

For, since the rectangle *B*, *C* is equal to the rectangle *D*, *E*,
while the rectangle *B*, *C* is medial, [X. 21]
therefore the rectangle *D*, *E* is also medial.

Therefore two medial straight lines *D*, *E*, commensurable in square only, and containing a medial rectangle, have been found such that the square on the greater is greater than the square on the less by the square on a straight line commensurable with the greater.

Similarly again it can be proved that the square on D is greater than the square on E by the square on a straight line incommensurable with D, when the square on A is greater than the square on C by the square on a straight line incommensurable with A. [x. 30]

I. Euclid takes three straight lines of the form ρ, $\rho\sqrt{\lambda}$, $\rho\sqrt{1-k^2}$, takes the mean proportional $\rho\lambda^{\frac{1}{4}}$ between the first two ....(1), and then finds x such that

$$\rho\lambda^{\frac{1}{4}} : \rho\lambda^{\frac{1}{2}} = \rho\sqrt{1-k^2} : x \quad \text{.........................(2),}$$

whence $x = \rho\lambda^{\frac{1}{4}}\sqrt{1-k^2}$,

and the straight lines $\rho\lambda^{\frac{1}{4}}$, $\rho\lambda^{\frac{1}{4}}\sqrt{1-k^2}$ satisfy the given conditions.

Now (α) $\rho\lambda^{\frac{1}{4}}$ is medial.

(β) We have, from (1) and (2),

$$\rho : \rho\sqrt{1-k^2} = \rho\lambda^{\frac{1}{4}} : x \quad \text{.........................(3),}$$

whence $x \smallfrown \rho\lambda^{\frac{1}{4}}$; and x is therefore medial and $\smallfrown \rho\lambda^{\frac{1}{4}}$.

(γ) $x \cdot \rho\lambda^{\frac{1}{4}} = \rho\sqrt{\lambda} \cdot \rho\sqrt{1-k^2}$.

But the latter is medial; [x. 21]

therefore $x \cdot \rho\lambda^{\frac{1}{4}}$, or $\rho\lambda^{\frac{1}{4}} \cdot \rho\lambda^{\frac{1}{4}}\sqrt{1-k^2}$, is medial.

Lastly (δ) ρ, $\rho\sqrt{1-k^2}$ have the remaining property in the enunciation; therefore $\rho\lambda^{\frac{1}{4}}$, $\rho\lambda^{\frac{1}{4}}\sqrt{1-k^2}$ have it also. [x. 14]

(Euclid has not the assistance of symbols to prove the proportion (3) above. He therefore uses the lemmas $ab : bc = a : c$ and $d^2 : de = d : e$ to deduce from the relations

$$\left.\begin{aligned} ab &= d^2 \\ d : b &= c : e \end{aligned}\right\}$$

and

that

$$a : c = d : e.)$$

The straight lines $\rho\lambda^{\frac{1}{4}}$, $\rho\lambda^{\frac{1}{4}}\sqrt{1-k^2}$ may take any of the following forms according as the straight lines first taken are

(1) a, $\sqrt{B}$, $\sqrt{a^2-c^2}$, (2) $\sqrt{A}$, $\sqrt{B}$, $\sqrt{A-k^2A}$, (3) $\sqrt{A}$, b, $\sqrt{A-k^2A}$.

$$(1)\quad \sqrt{a\sqrt{B}},\quad \frac{\sqrt{B(a^2-c^2)}}{\sqrt{a\sqrt{B}}};$$

$$(2)\quad \sqrt[4]{AB},\quad \frac{\sqrt{B(A-k^2A)}}{\sqrt[4]{AB}};$$

$$(3)\quad \sqrt{b\sqrt{A}},\quad \frac{b\sqrt{A-k^2A}}{\sqrt{b\sqrt{A}}}.$$

II. If the other conditions are the same, but the square on the first medial straight line is to exceed the square on the second by the square on a straight line *incommensurable* with the first, we begin with the three straight lines ρ, $\rho\sqrt{\lambda}$, $\frac{\rho}{\sqrt{1+k^2}}$, and the medial straight lines are

$$\rho\lambda^{\frac{1}{4}},\quad \frac{\rho\lambda^{\frac{1}{4}}}{\sqrt{1+k^2}}.$$

The possible forms are even more various in this case owing to the more various forms that the original lines may take, e.g.

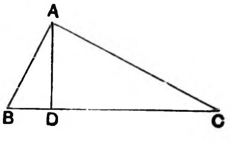
$$\begin{array}{llll} (1) & a, & \sqrt{B}, & \sqrt{a^2-C}; \\ (2) & \sqrt{A}, & b, & \sqrt{A-c^2}; \\ (3) & \sqrt{A}, & b, & \sqrt{A-C}; \\ (4) & \sqrt{A}, & \sqrt{B}, & \sqrt{A-c^2}; \\ (5) & \sqrt{A}, & \sqrt{B}, & \sqrt{A-C}; \end{array}$$

the medial straight lines corresponding to these being

$$\begin{array}{lll} (1) & \sqrt{a\sqrt{B}}, & \dfrac{\sqrt{B(a^2-C)}}{\sqrt{a\sqrt{B}}}; \\ (2) & \sqrt{b\sqrt{A}}, & \dfrac{b\sqrt{A-c^2}}{\sqrt{b\sqrt{A}}}; \\ (3) & \sqrt{b\sqrt{A}}, & \dfrac{b\sqrt{A-C}}{\sqrt{b\sqrt{A}}}; \\ (4) & \sqrt[4]{AB}, & \dfrac{\sqrt{B(A-c^2)}}{\sqrt[4]{AB}}; \\ (5) & \sqrt[4]{AB}, & \dfrac{\sqrt{B(A-C)}}{\sqrt[4]{AB}}. \end{array}$$

LEMMA.

Let ABC be a right-angled triangle having the angle A right, and let the perpendicular AD be drawn;



I say that the rectangle CB, BD is equal to the square on BA,

the rectangle BC, CD equal to the square on CA,

the rectangle BD, DC equal to the square on AD,

and, further, the rectangle BC, AD equal to the rectangle BA, AC.

And first that the rectangle *CB*, *BD* is equal to the square on *BA*.

For, since in a right-angled triangle *AD* has been drawn from the right angle perpendicular to the base,

therefore the triangles *ABD*, *ADC* are similar both to the whole *ABC* and to one another. [VI. 8]

And since the triangle *ABC* is similar to the triangle *ABD*, therefore, as *CB* is to *BA*, so is *BA* to *BD*; [VI. 4]

therefore the rectangle *CB*, *BD* is equal to the square on *AB*. [VI. 17]

For the same reason the rectangle *BC*, *CD* is also equal to the square on *AC*.

And since, if in a right-angled triangle a perpendicular be drawn from the right angle to the base, the perpendicular so drawn is a mean proportional between the segments of the base, [VI. 8, Por.]

therefore, as *BD* is to *DA*, so is *AD* to *DC*;

therefore the rectangle *BD*, *DC* is equal to the square on *AD*. [VI. 17]

I say that the rectangle *BC*, *AD* is also equal to the rectangle *BA*, *AC*.

For since, as we said, *ABC* is similar to *ABD*,

therefore, as *BC* is to *CA*, so is *BA* to *AD*. [VI. 4]

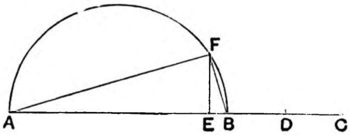
Therefore the rectangle *BC*, *AD* is equal to the rectangle *BA*, *AC*. [VI. 16]

Q. E. D.

PROPOSITION 33.

To find two straight lines incommensurable in square which make the sum of the squares on them rational but the rectangle contained by them medial.

Let there be set out two rational straight lines *AB*, *BC* commensurable in square only and such that the square on the greater *AB* is greater than the square on the less *BC* by the square on a straight line incommensurable with *AB*, [x. 30]



let *BC* be bisected at *D*,

let there be applied to *AB* a parallelogram equal to the square on either of the straight lines *BD*, *DC* and deficient by a square figure, and let it be the rectangle *AE*, *EB*; [vi. 28]

let the semicircle *AFB* be described on *AB*,

let *EF* be drawn at right angles to *AB*,

and let *AF*, *FB* be joined.

Then, since *AB*, *BC* are unequal straight lines,

and the square on *AB* is greater than the square on *BC* by the square on a straight line incommensurable with *AB*,

while there has been applied to *AB* a parallelogram equal to the fourth part of the square on *BC*, that is, to the square on half of it, and deficient by a square figure, making the rectangle *AE*, *EB*,

therefore *AE* is incommensurable with *EB*. [x. 18]

And, as *AE* is to *EB*, so is the rectangle *BA*, *AE* to the rectangle *AB*, *BE*,

while the rectangle *BA*, *AE* is equal to the square on *AF*,

and the rectangle *AB*, *BE* to the square on *BF*;

therefore the square on *AF* is incommensurable with the square on *FB*;

therefore *AF*, *FB* are incommensurable in square.

And, since *AB* is rational,

therefore the square on *AB* is also rational;

so that the sum of the squares on *AF*, *FB* is also rational. [i. 47]

And since, again, the rectangle *AE*, *EB* is equal to the square on *EF*,

and, by hypothesis, the rectangle *AE*, *EB* is also equal to the square on *BD*,

therefore *FE* is equal to *BD*;

therefore *BC* is double of *FE*,

so that the rectangle *AB*, *BC* is also commensurable with the rectangle *AB*, *EF*.

But the rectangle *AB*, *BC* is medial; [x. 21]

therefore the rectangle *AB*, *EF* is also medial. [x. 23, Por.]

But the rectangle *AB*, *EF* is equal to the rectangle *AF*, *FB*; [Lemma]

therefore the rectangle *AF*, *FB* is also medial.

But it was also proved that the sum of the squares on these straight lines is rational.

Therefore two straight lines *AF*, *FB* incommensurable in square have been found which make the sum of the squares on them rational, but the rectangle contained by them medial.

Q. E. D.

Euclid takes the straight lines found in x. 30, viz. ρ, $\dfrac{\rho}{\sqrt{1+k^2}}$.

He then solves *geometrically* the equations

$$\left.\begin{aligned} x+y&=\rho \\ xy&=\frac{\rho^2}{4(1+k^2)} \end{aligned}\right\} \qquad \text{.........................(1).}$$

If x, y are the values found, he takes u, v such that

$$\left.\begin{aligned} u^2&=\rho x \\ v^2&=\rho y \end{aligned}\right\} \qquad \text{...................................(2),}$$

and u, v are straight lines satisfying the conditions of the problem.

Solving algebraically, we get (if $x>y$)

$$x=\frac{\rho}{2}\left(1+\frac{k}{\sqrt{1+k^2}}\right),\quad y=\frac{\rho}{2}\left(1-\frac{k}{\sqrt{1+k^2}}\right),$$

whence

$$\left.\begin{aligned} u&=\frac{\rho}{\sqrt{2}}\sqrt{1+\frac{k}{\sqrt{1+k^2}}} \\ v&=\frac{\rho}{\sqrt{2}}\sqrt{1-\frac{k}{\sqrt{1+k^2}}} \end{aligned}\right\} \qquad \text{........................(3).}$$

Euclid's proof that these straight lines fulfil the requirements is as follows.

(α) The constants in the equations (1) satisfy the conditions of x. 18; therefore $x \smile y$.

But $x:y=u^2:v^2$.

Therefore $u^2 \smile v^2$,

and u, v are thus *incommensurable in square*.

(β) $u^2+v^2=\rho^2$, which is *rational*.

(γ) By (1),

$$\sqrt{xy}=\frac{\rho}{2\sqrt{1+k^2}}.$$

By (2),

$$\begin{aligned} uv&=\rho\,.\,\sqrt{xy} \\ &=\frac{\rho^2}{2\sqrt{1+k^2}}. \end{aligned}$$

But $\dfrac{\rho^2}{\sqrt{1+k^2}}$ is a medial area,

therefore uv is medial.

Since ρ, $\dfrac{\rho}{\sqrt{1+k^2}}$ may have any of the three forms

$$(1)\ a,\ \sqrt{a^2-B},\quad (2)\ \sqrt{A},\ \sqrt{A-B},\quad (3)\ \sqrt{A},\ \sqrt{A-b^2},$$

u, v may have any of the forms

$$(1)\quad \sqrt{\frac{a^2+a\sqrt{B}}{2}},\quad \sqrt{\frac{a^2-a\sqrt{B}}{2}};$$

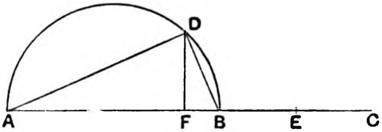
$$(2)\quad \sqrt{\frac{A+\sqrt{AB}}{2}},\quad \sqrt{\frac{A-\sqrt{AB}}{2}};$$

$$(3)\quad \sqrt{\frac{A+b\sqrt{A}}{2}},\quad \sqrt{\frac{A-b\sqrt{A}}{2}}.$$

PROPOSITION 34.

To find two straight lines incommensurable in square which make the sum of the squares on them medial but the rectangle contained by them rational.

Let there be set out two medial straight lines *AB*, *BC*, commensurable in square only, such that the rectangle which they contain is rational, and the square on *AB* is greater than the square on *BC* by the square on a straight line incommensurable with *AB*; [x. 31, *ad fin.*]



let the semicircle *ADB* be described on *AB*,

let *BC* be bisected at *E*,

let there be applied to *AB* a parallelogram equal to the square on *BE* and deficient by a square figure, namely the rectangle *AF*, *FB*; [vi. 28]

therefore *AF* is incommensurable in length with *FB*. [x. 18]

Let *FD* be drawn from *F* at right angles to *AB*, and let *AD*, *DB* be joined.

Since *AF* is incommensurable in length with *FB*, therefore the rectangle *BA*, *AF* is also incommensurable with the rectangle *AB*, *BF*. [x. 11]

But the rectangle *BA*, *AF* is equal to the square on *AD*, and the rectangle *AB*, *BF* to the square on *DB*; therefore the square on *AD* is also incommensurable with the square on *DB*.

And, since the square on *AB* is medial, therefore the sum of the squares on *AD*, *DB* is also medial. [III. 31, I. 47]

And, since *BC* is double of *DF*, therefore the rectangle *AB*, *BC* is also double of the rectangle *AB*, *FD*.

But the rectangle *AB*, *BC* is rational; therefore the rectangle *AB*, *FD* is also rational. [x. 6]

But the rectangle *AB*, *FD* is equal to the rectangle *AD*, *DB*; [Lemma] so that the rectangle *AD*, *DB* is also rational.

Therefore two straight lines *AD*, *DB* incommensurable in square have been found which make the sum of the squares on them medial, but the rectangle contained by them rational.

Q. E. D.

In this case we take [x. 31, 2nd part] the medial straight lines

$$\frac{\rho}{(1+k^2)^{\frac{1}{4}}},\ \frac{\rho}{(1+k^2)^{\frac{3}{4}}}.$$

Solve the equations

$$\left.\begin{aligned} x+y &= \frac{\rho}{(1+k^2)^{\frac{1}{4}}} \\ xy &= \frac{\rho^2}{4(1+k^2)^{\frac{3}{2}}} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Take u, v such that, if x, y be the result of the solution,

$$\left.\begin{aligned} u^2 &= \frac{\rho}{(1+k^2)^{\frac{1}{4}}}\cdot x \\ v^2 &= \frac{\rho}{(1+k^2)^{\frac{1}{4}}}\cdot y \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

and u, v are straight lines satisfying the given conditions.

Euclid's proof is similar to the preceding.

(α) From (1) it follows [x. 18] that

$$x \smile y,$$

whence

$$u^2 \smile v^2,$$

and u, v are thus incommensurable in square.

(β) $u^2 + v^2 = \dfrac{\rho^2}{\sqrt{1+k^2}}$, which is a medial area.

(γ)
$$uv = \frac{\rho}{(1+k^2)^{\frac{1}{4}}} \cdot \sqrt{xy}$$
$$= \frac{1}{2} \cdot \frac{\rho^2}{1+k^2}, \text{ which is a rational area.}$$

Therefore uv is rational.

To find the actual form of u, v, we have, by solving the equations (1) (if $x > y$),

$$x = \frac{\rho}{2(1+k^2)^{\frac{3}{4}}}(\sqrt{1+k^2} + k),$$
$$y = \frac{\rho}{2(1+k^2)^{\frac{3}{4}}}(\sqrt{1+k^2} - k);$$

and hence
$$u = \frac{\rho}{\sqrt{2(1+k^2)}}\sqrt{\sqrt{1+k^2}+k},$$
$$v = \frac{\rho}{\sqrt{2(1+k^2)}}\sqrt{\sqrt{1+k^2}-k}.$$

Bearing in mind the forms which $\dfrac{\rho}{(1+k^2)^{\frac{1}{4}}}$, $\dfrac{\rho}{(1+k^2)^{\frac{3}{4}}}$ may take (see note on x. 31), we shall find that u, v may have any of the forms

(1) $\sqrt{\dfrac{(a+\sqrt{B})\sqrt{a^2-B}}{2}}$, $\sqrt{\dfrac{(a-\sqrt{B})\sqrt{a^2-B}}{2}}$;

(2) $\sqrt{\dfrac{(\sqrt{A}+\sqrt{B})\sqrt{A-B}}{2}}$, $\sqrt{\dfrac{(\sqrt{A}-\sqrt{B})\sqrt{A-B}}{2}}$;

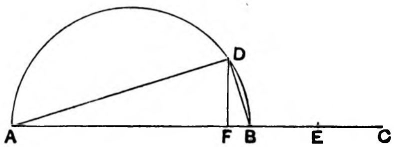
(3) $\sqrt{\dfrac{(\sqrt{A}+b)\sqrt{A-b^2}}{2}}$, $\sqrt{\dfrac{(\sqrt{A}-b)\sqrt{A-b^2}}{2}}$.

PROPOSITION 35.

To find two straight lines incommensurable in square which make the sum of the squares on them medial and the rectangle contained by them medial and moreover incommensurable with the sum of the squares on them.

Let there be set out two medial straight lines *AB*, *BC* commensurable in square only, containing a medial rectangle, and such that the square on *AB* is greater than the square on *BC* by the square on a straight line incommensurable with *AB*; [x. 32, *ad fin.*]

let the semicircle *ADB* be described on *AB*,
and let the rest of the construction be as above.



Then, since *AF* is incommensurable in length with *FB*, [x. 18]
AD is also incommensurable in square with *DB*. [x. 11]

And, since the square on *AB* is medial,
therefore the sum of the squares on *AD*, *DB* is also medial. [III. 31, I. 47]

And, since the rectangle *AF*, *FB* is equal to the square on each of the straight lines *BE*, *DF*,
therefore *BE* is equal to *DF*;
therefore *BC* is double of *FD*,
so that the rectangle *AB*, *BC* is also double of the rectangle *AB*, *FD*.

But the rectangle *AB*, *BC* is medial;
therefore the rectangle *AB*, *FD* is also medial. [x. 32, Por.]

And it is equal to the rectangle *AD*, *DB*; [Lemma after x. 32]
therefore the rectangle *AD*, *DB* is also medial.

And, since *AB* is incommensurable in length with *BC*,
while *CB* is commensurable with *BE*,
therefore *AB* is also incommensurable in length with *BE*, [x. 13]
so that the square on *AB* is also incommensurable with the rectangle *AB*, *BE*. [x. 11]

But the squares on *AD*, *DB* are equal to the square on *AB*, [I. 47]
and the rectangle *AB*, *FD*, that is, the rectangle *AD*, *DB*, is equal to the rectangle *AB*, *BE*;
therefore the sum of the squares on *AD*, *DB* is incommensurable with the rectangle *AD*, *DB*.

Therefore two straight lines *AD*, *DB* incommensurable in square have been found which make the sum of the squares on them medial and the rectangle contained by them medial and moreover incommensurable with the sum of the squares on them.

Q. E. D.

Take the medial straight lines found in x. 32 (2nd part), viz.

$$\rho\lambda^{\frac{1}{4}},\quad \rho\lambda^{\frac{1}{4}}/\sqrt{1+k^2}.$$

Solve the equations

$$\left.\begin{aligned} x+y &= \rho\lambda^{\frac{1}{4}} \\ xy &= \frac{\rho^2\sqrt{\lambda}}{4(1+k^2)} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

and then put

$$\left.\begin{aligned} u^2 &= \rho\lambda^{\frac{1}{4}}\,.\,x \\ v^2 &= \rho\lambda^{\frac{1}{4}}\,.\,y \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

where x, y are the ascertained values of x, y.

Then u, v are straight lines satisfying the given conditions.

Euclid proves this as follows.

(α) From (1) it follows [x. 18] that $x \smile y$.

Therefore $$u^2 \smile v^2,$$

and $$u \smile\!\!\!\!- \; v.$$

(β) $$u^2+v^2 = \rho^2\sqrt{\lambda}, \text{ which is a medial area} \ldots\ldots\ldots\ldots(3).$$

(γ) $$\begin{aligned} uv &= \rho\lambda^{\frac{1}{4}}\,.\,\sqrt{xy} \\ &= \frac{1}{2}\frac{\rho^2\sqrt{\lambda}}{\sqrt{1+k^2}}, \text{ which is a medial area} \ldots\ldots\ldots(4); \end{aligned}$$

therefore uv is medial.

(δ) $$\rho\lambda^{\frac{1}{4}} \smile \frac{1}{2}\frac{\rho\lambda^{\frac{1}{4}}}{\sqrt{1+k^2}},$$

whence $$\rho^2\sqrt{\lambda} \smile \frac{1}{2}\frac{\rho^2\sqrt{\lambda}}{\sqrt{1+k^2}}.$$

That is, by (3) and (4),

$$(u^2+v^2) \smile uv.$$

The actual values are found thus. Solving the equations (1), we have

$$x = \frac{\rho\lambda^{\frac{1}{4}}}{2}\left(1+\frac{k}{\sqrt{1+k^2}}\right),$$

$$y = \frac{\rho\lambda^{\frac{1}{4}}}{2}\left(1-\frac{k}{\sqrt{1+k^2}}\right),$$

whence $$u = \frac{\rho\lambda^{\frac{1}{4}}}{\sqrt{2}}\sqrt{1+\frac{k}{\sqrt{1+k^2}}},$$

$$v = \frac{\rho\lambda^{\frac{1}{4}}}{\sqrt{2}}\sqrt{1-\frac{k}{\sqrt{1+k^2}}}.$$

According as ρ is of the form a or $\sqrt{A}$, we have a variety of forms for u, v, arrived at by using the same transformations as in the notes on x. 30 and x. 32 (second part), e.g.

$$(1)\quad \sqrt{\frac{(a+\sqrt{C})\sqrt{B}}{2}},\quad \sqrt{\frac{(a-\sqrt{C})\sqrt{B}}{2}};$$

$$(2)\quad \sqrt{\frac{(\sqrt{A}+\sqrt{C})\sqrt{B}}{2}},\quad \sqrt{\frac{(\sqrt{A}-\sqrt{C})\sqrt{B}}{2}};$$

$$(3)\quad \sqrt{\frac{(\sqrt{A}+c)\sqrt{B}}{2}},\quad \sqrt{\frac{(\sqrt{A}-c)\sqrt{B}}{2}};$$

and the expressions in (2), (3) with b in place of $\sqrt{B}$.

PROPOSITION 36.

If two rational straight lines commensurable in square only be added together, the whole is irrational; and let it be called **binomial.**

For let two rational straight lines AB, BC commensurable in square only be added together;

I say that the whole AC is irrational.

For, since AB is incommensurable in length with BC—for they are commensurable in square only—

and, as AB is to BC, so is the rectangle AB, BC to the square on BC,

therefore the rectangle AB, BC is incommensurable with the square on BC. [x. 11]

But twice the rectangle AB, BC is commensurable with the rectangle AB, BC [x. 6], and the squares on AB, BC are commensurable with the square on BC—for AB, BC are rational straight lines commensurable in square only— [x. 15]

therefore twice the rectangle AB, BC is incommensurable with the squares on AB, BC. [x. 13]

And, *componendo*, twice the rectangle AB, BC together with the squares on AB, BC, that is, the square on AC [II. 4], is incommensurable with the sum of the squares on AB, BC. [x. 16]

But the sum of the squares on AB, BC is rational;

therefore the square on AC is irrational,

so that AC is also irrational. [x. Def. 4]

And let it be called **binomial.**

Here begins the first hexad of propositions relating to compound irrational straight lines. The six compound irrational straight lines are formed by *adding* two parts, as the corresponding six in Props. 73—78 are formed by *subtraction*. The relation between the six irrational straight lines in this and the next five propositions with those described in Definitions II. and the Props. 48—53 following thereon (the *first*, *second*, *third*, *fourth*, *fifth* and *sixth binomials*) will be seen when we come to Props. 54—59; but it may be stated here that the six compound irrationals in Props. 36—41 can be found by means of the equivalent of *extracting the square root* of the compound irrationals in x. 48—53 (the process being, strictly speaking, the finding of the sides of the squares equal to the rectangles contained by the latter irrationals respectively and a rational straight line as the other side), and it is therefore the further removed compound irrational, so to speak, which is treated first.

In reproducing the proofs of the propositions, I shall for the sake of simplicity call the two parts of the compound irrational straight line x, y, explaining at the outset the forms which x, y really have in each case; x will always be supposed to be the greater segment.

In this proposition x, y are of the form ρ, $\sqrt{k} \cdot \rho$, and $(x+y)$ is proved to be irrational thus.

$x \frown\!\!-\, y$, so that $x \smile y$.

Now $$x : y = x^2 : xy,$$

so that $$x^2 \smile xy.$$

But $x^2 \frown (x^2+y^2)$, and $xy \frown 2xy$;

therefore $$(x^2+y^2) \smile 2xy,$$

and hence $$(x^2+y^2+2xy) \smile (x^2+y^2).$$

But (x^2+y^2) is rational;

therefore $(x+y)^2$, and therefore $(x+y)$, is *irrational*.

This irrational straight line, $\rho + \sqrt{k} \cdot \rho$, is called a *binomial* straight line.

This and the corresponding *apotome* $(\rho - \sqrt{k} \cdot \rho)$ found in x. 73 are the positive roots of the equation

$$x^4 - 2(1+k)\rho^2 \cdot x^2 + (1-k)^2 \rho^4 = 0.$$

PROPOSITION 37.

If two medial straight lines commensurable in square only and containing a rational rectangle be added together, the whole is irrational; and let it be called a **first bimedial** *straight line.*

For let two medial straight lines *AB*, *BC* commensurable in square only and containing a rational rectangle be added together;

I say that the whole *AC* is irrational.

For, since *AB* is incommensurable in length with *BC*, therefore the squares on *AB*, *BC* are also incommensurable with twice the rectangle *AB*, *BC*; [cf. x. 36, ll. 9—20] and, *componendo*, the squares on AB, BC together with twice the rectangle AB, BC, that is, the square on AC [II. 4], is incommensurable with the rectangle AB, BC. [x. 16]

But the rectangle AB, BC is rational, for, by hypothesis, AB, BC are straight lines containing a rational rectangle; therefore the square on AC is irrational; therefore AC is irrational. [x. Def. 4]

And let it be called a **first bimedial** straight line.

Q. E. D.

Here x, y have the forms $k^{\frac{1}{4}}\rho$, $k^{\frac{3}{4}}\rho$ respectively, as found in x. 27.

Exactly as in the last case we prove that

$$x^2 + y^2 \smile 2xy,$$

whence $$(x+y)^2 \smile 2xy.$$

But xy is rational;

therefore $(x+y)^2$, and consequently $(x+y)$, is *irrational.*

The irrational straight line $k^{\frac{1}{4}}\rho + k^{\frac{3}{4}}\rho$ is called a *first bimedial* straight line.

This and the corresponding first *apotome of a medial* $(k^{\frac{1}{4}}\rho - k^{\frac{3}{4}}\rho)$ found in x. 74 are the positive roots of the equation

$$x^4 - 2\sqrt{k}\,(1+k)\,\rho^2 . x^2 + k\,(1-k)^2\rho^4 = 0.$$

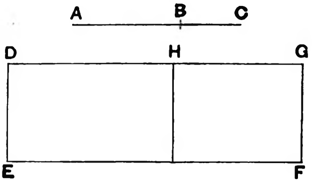
PROPOSITION 38.

If two medial straight lines commensurable in square only and containing a medial rectangle be added together, the whole is irrational; and let it be called a **second bimedial** *straight line.*

For let two medial straight lines AB, BC commensurable in square only and containing a medial rectangle be added together;

I say that AC is irrational.

For let a rational straight line DE be set out, and let the parallelogram DF equal to the square on AC be applied to DE, producing DG as breadth. [I. 44]



Then, since the square on AC is equal to the squares on AB, BC and twice the rectangle AB, BC, [II. 4] let EH, equal to the squares on AB, BC, be applied to DE;

therefore the remainder HF is equal to twice the rectangle AB, BC.

And, since each of the straight lines AB, BC is medial, therefore the squares on AB, BC are also medial.

But, by hypothesis, twice the rectangle AB, BC is also medial.

And EH is equal to the squares on AB, BC, while FH is equal to twice the rectangle AB, BC; therefore each of the rectangles EH, HF is medial.

And they are applied to the rational straight line DE; therefore each of the straight lines DH, HG is rational and incommensurable in length with DE. [x. 22]

Since then AB is incommensurable in length with BC, and, as AB is to BC, so is the square on AB to the rectangle AB, BC, therefore the square on AB is incommensurable with the rectangle AB, BC. [x. 11]

But the sum of the squares on AB, BC is commensurable with the square on AB, [x. 15] and twice the rectangle AB, BC is commensurable with the rectangle AB, BC. [x. 6]

Therefore the sum of the squares on AB, BC is incommensurable with twice the rectangle AB, BC. [x. 13]

But EH is equal to the squares on AB, BC, and HF is equal to twice the rectangle AB, BC.

Therefore EH is incommensurable with HF, so that DH is also incommensurable in length with HG. [vi. 1, x. 11]

Therefore DH, HG are rational straight lines commensurable in square only; so that DG is irrational. [x. 36]

But DE is rational; and the rectangle contained by an irrational and a rational straight line is irrational; [cf. x. 20] therefore the area DF is irrational, and the side of the square equal to it is irrational. [x. Def. 4]

But AC is the side of the square equal to DF; therefore AC is irrational.

And let it be called a **second bimedial** straight line.

Q. E. D.

After proving (l. 21) that *each* of the squares on AB, BC is medial, Euclid states (ll. 24, 26) that EH, which is equal to the *sum* of the squares, is a medial area, but does not explain why. It is because, by hypothesis, the squares on AB, BC are commensurable, so that the sum of the squares is commensurable with either [x. 15] and is therefore a medial area [x. 23, Por.].

In this case [x. 28, note] x, y are of the forms $k^{\frac{1}{4}}\rho$, $\lambda^{\frac{1}{2}}\rho/k^{\frac{1}{4}}$ respectively.

Apply each of the areas (x^2+y^2) and $2xy$ to a rational straight line σ, i.e. suppose

$$x^2+y^2=\sigma u,$$

$$2xy=\sigma v.$$

Now it follows from the hypothesis, x. 15 and x. 23, Por. that (x^2+y^2) is a medial area; and so is $2xy$, by hypothesis; therefore σu, σv are medial areas.

Therefore each of the straight lines u, v is rational and $\smile \sigma$ .... (1).

Again $$x \smile y;$$
therefore $$x^2 \smile xy.$$
But $$x^2 \frown x^2+y^2 \text{ and } xy \frown 2xy;$$
therefore $$x^2+y^2 \smile 2xy,$$
or $$\sigma u \smile \sigma v,$$
whence $$u \smile v \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Therefore, by (1), (2), u, v are rational and $\frown\!\!-$.

It follows, by x. 36, that $(u+v)$ is irrational.

Therefore $(u+v)\,\sigma$ is an irrational area [this can be deduced from x. 20 by *reductio ad absurdum*], whence $(x+y)^2$, and consequently $(x+y)$, is irrational.

The irrational straight line $k^{\frac{1}{4}}\rho+\dfrac{\lambda^{\frac{1}{2}}\rho}{k^{\frac{1}{4}}}$ is called a *second bimedial* straight line.

This and the corresponding *second apotome of a medial* $\left(k^{\frac{1}{4}}\rho-\dfrac{\sqrt{\lambda}}{k^{\frac{1}{4}}}\rho\right)$ found in x. 75 are the positive roots of the equation

$$x^4-2\,\frac{k+\lambda}{\sqrt{k}}\,\rho^2\,.\,x^2+\frac{(k-\lambda)^2}{k}\,\rho^4=0.$$

PROPOSITION 39.

If two straight lines incommensurable in square which make the sum of the squares on them rational, but the rectangle contained by them medial, be added together, the whole straight line is irrational: and let it be called **major**.

For let two straight lines *AB*, *BC* incommensurable in square, and fulfilling the given conditions [x. 33], be added together;

I say that *AC* is irrational.

For, since the rectangle *AB*, *BC* is medial,
twice the rectangle *AB*, *BC* is also medial. [x. 6 and 23, Por.]

But the sum of the squares on *AB*, *BC* is rational;
therefore twice the rectangle *AB*, *BC* is incommensurable with the sum of the squares on *AB*, *BC*,
so that the squares on *AB*, *BC* together with twice the rectangle *AB*, *BC*, that is, the square on *AC*, is also incommensurable with the sum of the squares on *AB*, *BC*; [x. 16]
therefore the square on *AC* is irrational,
so that *AC* is also irrational. [x. Def. 4]

And let it be called **major**.

Q. E. D.

Here x, y are of the form found in x. 33, viz.

$$\frac{\rho}{\sqrt{2}}\sqrt{1+\frac{k}{\sqrt{1+k^2}}},\quad \frac{\rho}{\sqrt{2}}\sqrt{1-\frac{k}{\sqrt{1+k^2}}}.$$

By hypothesis, the rectangle xy is medial;
therefore $2xy$ is medial.

Also (x^2+y^2) is a rational area.

Therefore $$x^2+y^2 \smile 2xy,$$
whence $$(x+y)^2 \smile (x^2+y^2),$$
so that $(x+y)^2$, and therefore $(x+y)$, is irrational.

The irrational straight line $\frac{\rho}{\sqrt{2}}\sqrt{1+\frac{k}{\sqrt{1+k^2}}}+\frac{\rho}{\sqrt{2}}\sqrt{1-\frac{k}{\sqrt{1+k^2}}}$ is called a *major* (irrational) straight line.

This and the corresponding *minor* irrational found in x. 76 are the positive roots of the equation

$$x^4-2\rho^2 . x^2+\frac{k^2}{1+k^2}\rho^4=0.$$

PROPOSITION 40.

If two straight lines incommensurable in square which make the sum of the squares on them medial, but the rectangle contained by them rational, be added together, the whole straight line is irrational; and let it be called the **side of a rational plus a medial area.**

For let two straight lines *AB*, *BC* incommensurable in square, and fulfilling the given conditions [x. 34], be added together;

I say that *AC* is irrational.

For, since the sum of the squares on *AB*, *BC* is medial, while twice the rectangle *AB*, *BC* is rational,

therefore the sum of the squares on *AB*, *BC* is incommensurable with twice the rectangle *AB*, *BC*;

so that the square on *AC* is also incommensurable with twice the rectangle *AB*, *BC*. [x. 16]

But twice the rectangle *AB*, *BC* is rational;

therefore the square on *AC* is irrational.

Therefore *AC* is irrational. [x. Def. 4]

And let it be called the **side of a rational plus a medial area.**

Q. E. D.

Here x, y have [x. 34] the forms

$$\frac{\rho}{\sqrt{2(1+k^2)}}\sqrt{\sqrt{1+k^2}+k},\quad \frac{\rho}{\sqrt{2(1+k^2)}}\sqrt{\sqrt{1+k^2}-k}.$$

In this case (x^2+y^2) is a medial, and $2xy$ a rational, area; thus

$$x^2+y^2 \smile 2xy.$$

Therefore $$(x+y)^2 \smile 2xy,$$

whence, since $2xy$ is rational,

$(x+y)^2$, and consequently $(x+y)$, is irrational.

The irrational straight line

$$\frac{\rho}{\sqrt{2(1+k^2)}}\sqrt{\sqrt{1+k^2}+k}+\frac{\rho}{\sqrt{2(1+k^2)}}\sqrt{\sqrt{1+k^2}-k}$$

is called (for an obvious reason) the "*side*" *of a rational plus a medial* (area).

This and the corresponding irrational with a minus sign found in x. 77 are the positive roots of the equation

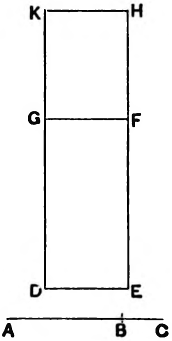
$$x^4-\frac{2}{\sqrt{1+k^2}}\rho^2 . x^2+\frac{k^2}{(1+k^2)^2}\rho^4=0.$$

PROPOSITION 41.

If two straight lines incommensurable in square which make the sum of the squares on them medial, and the rectangle contained by them medial and also incommensurable with the sum of the squares on them, be added together, the whole straight line is irrational; and let it be called the **side of the sum of two medial areas.**

For let two straight lines *AB*, *BC* incommensurable in square and satisfying the given conditions [x. 35] be added together;

I say that *AC* is irrational.



Let a rational straight line *DE* be set out, and let there be applied to *DE* the rectangle *DF* equal to the squares on *AB*, *BC*, and the rectangle *GH* equal to twice the rectangle *AB*, *BC*;

therefore the whole *DH* is equal to the square on *AC*. [II. 4]

Now, since the sum of the squares on *AB*, *BC* is medial,

and is equal to *DF*,

therefore *DF* is also medial.

And it is applied to the rational straight line *DE*;

therefore *DG* is rational and incommensurable in length with *DE*. [x. 22]

For the same reason *GK* is also rational and incommensurable in length with *GF*, that is, *DE*.

And, since the squares on *AB*, *BC* are incommensurable with twice the rectangle *AB*, *BC*,

DF is incommensurable with *GH*;

so that *DG* is also incommensurable with *GK*. [VI. 1, x. 11]

And they are rational;

therefore *DG*, *GK* are rational straight lines commensurable in square only;

therefore *DK* is irrational and what is called binomial. [x. 36]

But *DE* is rational;

therefore *DH* is irrational, and the side of the square which is equal to it is irrational. [x. Def. 4]

But *AC* is the side of the square equal to *HD*;

therefore *AC* is irrational.

And let it be called the **side of the sum of two medial areas.**

Q. E. D.

In this case x, y are of the form

$$\frac{\rho\lambda^{\frac{1}{4}}}{\sqrt{2}}\sqrt{1+\frac{k}{\sqrt{1+k^2}}},\quad \frac{\rho\lambda^{\frac{1}{4}}}{\sqrt{2}}\sqrt{1-\frac{k}{\sqrt{1+k^2}}}.$$

By hypothesis, (x^2+y^2) and $2xy$ are medial areas, and

$$x^2+y^2 \smile 2xy \quad \text{..............................}(1).$$

'Apply' these areas respectively to a rational straight line σ, and suppose

$$\left.\begin{aligned} x^2+y^2 &= \sigma u \\ 2xy &= \sigma v \end{aligned}\right\} \quad \text{..............................}(2).$$

Since then σu and σv are both medial areas, u, v are rational and both are $\smile \sigma$ ....(3).

Now, by (1) and (2),

$$\sigma u \smile \sigma v,$$

so that

$$u \smile v.$$

By this and (3), u, v are rational and $\frown$.

Therefore [x. 36] $(u+v)$ is irrational.

Hence $\sigma(u+v)$ is irrational [deduction from x. 20].

Thus $(x+y)^2$, and therefore $(x+y)$, is irrational.

The irrational straight line

$$\frac{\rho\lambda^{\frac{1}{4}}}{\sqrt{2}}\sqrt{1+\frac{k}{\sqrt{1+k^2}}}+\frac{\rho\lambda^{\frac{1}{4}}}{\sqrt{2}}\sqrt{1-\frac{k}{\sqrt{1+k^2}}}$$

is called (again for an obvious reason) the "*side*" *of the sum of two medials* (medial areas).

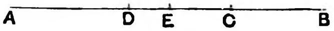
This and the corresponding irrational with a minus sign found in x. 78 are the positive roots of the equation

$$x^4-2\sqrt{\lambda}\,.\,x^2\rho^2+\lambda\frac{k^2}{1+k^2}\rho^4=0.$$

LEMMA.

And that the aforesaid irrational straight lines are divided only in one way into the straight lines of which they are the sum and which produce the types in question, we will now prove after premising the following lemma.

Let the straight line *AB* be set out, let the whole be cut into unequal parts at each of the points *C*, *D*, and let *AC* be supposed greater than *DB*;



I say that the squares on *AC*, *CB* are greater than the squares on *AD*, *DB*.

For let *AB* be bisected at *E*.

Then, since *AC* is greater than *DB*, let *DC* be subtracted from each;

therefore the remainder *AD* is greater than the remainder *CB*.

But *AE* is equal to *EB*;

therefore *DE* is less than *EC*;

therefore the points *C*, *D* are not equidistant from the point of bisection.

And, since the rectangle *AC*, *CB* together with the square on *EC* is equal to the square on *EB*, [II. 5]
and, further, the rectangle *AD*, *DB* together with the square on *DE* is equal to the square on *EB*, [*id.*]
therefore the rectangle *AC*, *CB* together with the square on *EC* is equal to the rectangle *AD*, *DB* together with the square on *DE*.

And of these the square on *DE* is less than the square on *EC*;
therefore the remainder, the rectangle *AC*, *CB*, is also less than the rectangle *AD*, *DB*,
so that twice the rectangle *AC*, *CB* is also less than twice the rectangle *AD*, *DB*.

Therefore also the remainder, the sum of the squares on *AC*, *CB*, is greater than the sum of the squares on *AD*, *DB*.

Q. E. D.

3. **and which produce the types in question.** The Greek is ποιουσῶν τὰ προκείμενα εἴδη, and I have taken εἴδη to mean "types (of irrational straight lines)," though the expression might perhaps mean "satisfying the *conditions* in question."

This proves that, if $x + y = u + v$, and if u, v are more nearly equal than x, y (i.e. if the straight line is divided in the second case nearer to the point of bisection), then

$$(x^2 + y^2) > (u^2 + v^2).$$

It is first proved by means of II. 5 that

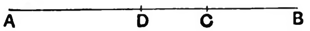
$$2xy < 2uv,$$

whence, since $(x + y)^2 = (u + v)^2$, the required result follows.

PROPOSITION 42.

A binomial straight line is divided into its terms at one point only.

Let *AB* be a binomial straight line divided into its terms at *C*;



therefore *AC*, *CB* are rational straight lines commensurable in square only. [x. 36]

I say that *AB* is not divided at another point into two rational straight lines commensurable in square only.

For, if possible, let it be divided at *D* also, so that *AD*, *DB* are also rational straight lines commensurable in square only.

It is then manifest that *AC* is not the same with *DB*.

For, if possible, let it be so.

Then *AD* will also be the same as *CB*,

and, as *AC* is to *CB*, so will *BD* be to *DA*;

thus *AB* will be divided at *D* also in the same way as by the division at *C*:

which is contrary to the hypothesis.

Therefore *AC* is not the same with *DB*.

For this reason also the points *C*, *D* are not equidistant from the point of bisection.

Therefore that by which the squares on *AC*, *CB* differ from the squares on *AD*, *DB* is also that by which twice the rectangle *AD*, *DB* differs from twice the rectangle *AC*, *CB*,

because both the squares on *AC*, *CB* together with twice the rectangle *AC*, *CB*, and the squares on *AD*, *DB* together with twice the rectangle *AD*, *DB*, are equal to the square on *AB*. [II. 4]

But the squares on *AC*, *CB* differ from the squares on *AD*, *DB* by a rational area,

for both are rational;

therefore twice the rectangle *AD*, *DB* also differs from twice the rectangle *AC*, *CB* by a rational area, though they are medial [x. 21]:

which is absurd, for a medial area does not exceed a medial by a rational area. [x. 26]

Therefore a binomial straight line is not divided at different points;

therefore it is divided at one point only.

Q. E. D.

This proposition proves the equivalent of the well-known theorem in surds that,

if $a + \sqrt{b} = x + \sqrt{y}$,

then $a = x, \quad b = y$,

and if $\sqrt{a} + \sqrt{b} = \sqrt{x} + \sqrt{y}$,

then $a = x, \quad b = y$ (or $a = y, \quad b = x$).

The proposition states that a *binomial* straight line cannot be split up into *terms* (ὀνόματα) in two ways. For, if possible, let

$$x + y = x' + y',$$

where x, y, and also x', y', are the *terms* of a binomial straight line, x', y' being different from x, y (or y, x).

One pair is necessarily more nearly equal than the other. Let x', y' be more nearly equal than x, y.

Then $$(x^2 + y^2) - (x'^2 + y'^2) = 2x'y' - 2xy.$$

Now by hypothesis $(x^2 + y^2)$, $(x'^2 + y'^2)$ are *rational* areas, being of the form $\rho^2 + k\rho^2$;

but $2x'y'$, $2xy$ are *medial* areas, being of the form $\sqrt{k} \cdot \rho^2$;

therefore the difference of two medial areas is rational:

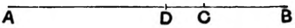
which is impossible. [x. 26]

Therefore x', y' cannot be different from x, y (or y, x).

PROPOSITION 43.

A first bimedial straight line is divided at one point only.

Let AB be a first bimedial straight line divided at C, so that AC, CB are medial straight lines commensurable in square only and containing a rational rectangle; [x. 37]



I say that AB is not so divided at another point.

For, if possible, let it be divided at D also, so that AD, DB are also medial straight lines commensurable in square only and containing a rational rectangle.

Since, then, that by which twice the rectangle AD, DB differs from twice the rectangle AC, CB is that by which the squares on AC, CB differ from the squares on AD, DB, while twice the rectangle AD, DB differs from twice the rectangle AC, CB by a rational area—for both are rational—therefore the squares on AC, CB also differ from the squares on AD, DB by a rational area, though they are medial: which is absurd. [x. 26]

Therefore a first bimedial straight line is not divided into its terms at different points;

therefore it is so divided at one point only.

In this case, with the same hypothesis, viz. that

$$x + y = x' + y',$$

and x', y' are more nearly equal than x, y,

we have as before $(x^2 + y^2) - (x'^2 + y'^2) = 2x'y' - 2xy$.

But, from the given properties of x, y, and x', y', it follows that $2xy$, $2x'y'$ are *rational*, and $(x^2 + y^2)$, $(x'^2 + y'^2)$ *medial*, areas.

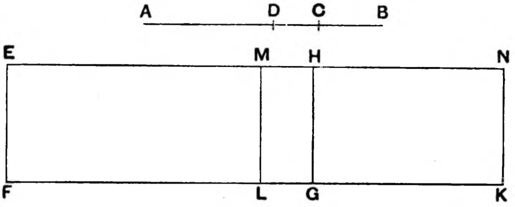
Therefore the difference between two medial areas is rational: which is impossible. [X. 26]

PROPOSITION 44.

A second bimedial straight line is divided at one point only.

Let *AB* be a second bimedial straight line divided at *C*, so that *AC*, *CB* are medial straight lines commensurable in square only and containing a medial rectangle; [X. 38] it is then manifest that *C* is not at the point of bisection, because the segments are not commensurable in length.

I say that *AB* is not so divided at another point.



For, if possible, let it be divided at *D* also, so that *AC* is not the same with *DB*, but *AC* is supposed greater;

it is then clear that the squares on *AD*, *DB* are also, as we proved above [Lemma], less than the squares on *AC*, *CB*;

and suppose that *AD*, *DB* are medial straight lines commensurable in square only and containing a medial rectangle.

Now let a rational straight line *EF* be set out,

let there be applied to *EF* the rectangular parallelogram *EK* equal to the square on *AB*,

and let *EG* equal to the squares on *AC*, *CB* be subtracted; therefore the remainder *HK* is equal to twice the rectangle *AC*, *CB*. [II. 4]

Again, let there be subtracted *EL*, equal to the squares on *AD*, *DB*, which were proved less than the squares on *AC*, *CB* [Lemma];

therefore the remainder *MK* is also equal to twice the rectangle *AD*, *DB*.

Now, since the squares on *AC*, *CB* are medial, therefore *EG* is medial.

And it is applied to the rational straight line *EF*; therefore *EH* is rational and incommensurable in length with *EF*. [x. 22]

For the same reason *HN* is also rational and incommensurable in length with *EF*.

And, since *AC*, *CB* are medial straight lines commensurable in square only, therefore *AC* is incommensurable in length with *CB*.

But, as *AC* is to *CB*, so is the square on *AC* to the rectangle *AC*, *CB*; therefore the square on *AC* is incommensurable with the rectangle *AC*, *CB*. [x. 11]

But the squares on *AC*, *CB* are commensurable with the square on *AC*; for *AC*, *CB* are commensurable in square. [x. 15]

And twice the rectangle *AC*, *CB* is commensurable with the rectangle *AC*, *CB*. [x. 6]

Therefore the squares on *AC*, *CB* are also incommensurable with twice the rectangle *AC*, *CB*. [x. 13]

But *EG* is equal to the squares on *AC*, *CB*, and *HK* is equal to twice the rectangle *AC*, *CB*; therefore *EG* is incommensurable with *HK*, so that *EH* is also incommensurable in length with *HN*. [vi. 1, x. 11]

And they are rational; therefore *EH*, *HN* are rational straight lines commensurable in square only.

But, if two rational straight lines commensurable in square only be added together, the whole is the irrational which is called binomial. [x. 36]

Therefore *EN* is a binomial straight line divided at *H*.

In the same way *EM*, *MN* will also be proved to be rational straight lines commensurable in square only; and *EN* will be a binomial straight line divided at different points, *H* and *M*.

And *EH* is not the same with *MN*.

For the squares on *AC*, *CB* are greater than the squares on *AD*, *DB*.

But the squares on *AD*, *DB* are greater than twice the rectangle *AD*, *DB*;

therefore also the squares on *AC*, *CB*, that is, *EG*, are much greater than twice the rectangle *AD*, *DB*, that is, *MK*,

so that *EH* is also greater than *MN*.

Therefore *EH* is not the same with *MN*.

Q. E. D.

As the irrationality of the *second bimedial* straight line [x. 38] is proved by means of the irrationality of the binomial straight line [x. 36], so the present theorem is reduced to that of x. 42.

Suppose, if possible, that the second bimedial straight line can be divided into its terms as such in two ways, i.e. that

$$x+y=x'+y',$$

where x', y' are nearer equality than x, y.

Apply x^2+y^2, $2xy$ to a rational straight line σ, i.e. let

$$x^2+y^2=\sigma u,$$
$$2xy=\sigma v.$$

Then, as in x. 38, the areas x^2+y^2, $2xy$ are medial, so that σu, σv are medial;

therefore u, v are both rational and $\smile \sigma$ ....(1).

Again, by hypothesis, x, y are medial straight lines commensurable in square only;

therefore $x \smile y$.

Hence $x^2 \smile xy$.

And $x^2 \frown (x^2+y^2)$, while $xy \frown 2xy$;

therefore $(x^2+y^2) \smile 2xy$,

or $\sigma u \smile \sigma v$,

and hence $u \smile v$ ....(2).

Therefore, by (1) and (2), u, v are rational straight lines commensurable in square only;

therefore $u+v$ is a *binomial* straight line.

Similarly, if $x'^2+y'^2=\sigma u'$ and $2x'y'=\sigma v'$,

$u'+v'$ will be proved to be a binomial straight line.

And, since $(x+y)^2=(x'+y')^2$, and therefore $(u+v)=(u'+v')$, it follows that a binomial straight line is divided as such in two ways:

which is impossible. [x. 42]

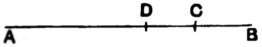
Therefore $x+y$, the given second bimedial straight line, can only be so divided in one way.

In order to prove that $u+v$, $u'+v'$ represent a *different* division of the same straight line, Euclid assumes that $x^2+y^2>2xy$. This is of course an easy inference from II. 7; but the assumption of it here renders it probable that the Lemma after x. 59 is interpolated.

PROPOSITION 45.

A major straight line is divided at one and the same point only.

Let AB be a major straight line divided at C, so that AC, CB are incommensurable in square and make the sum of the squares on AC, CB rational, but the rectangle AC, CB medial; [x. 39]



I say that AB is not so divided at another point.

For, if possible, let it be divided at D also, so that AD, DB are also incommensurable in square and make the sum of the squares on AD, DB rational, but the rectangle contained by them medial.

Then, since that by which the squares on AC, CB differ from the squares on AD, DB is also that by which twice the rectangle AD, DB differs from twice the rectangle AC, CB,

while the squares on AC, CB exceed the squares on AD, DB by a rational area—for both are rational—

therefore twice the rectangle AD, DB also exceeds twice the rectangle AC, CB by a rational area, though they are medial:

which is impossible. [x. 26]

Therefore a major straight line is not divided at different points;

therefore it is only divided at one and the same point.

Q. E. D.

If possible, let the *major* irrational straight line be divided into terms in two ways, viz. as $(x+y)$ and $(x'+y')$, where x', y' are supposed to be nearer equality than x, y.

We have then, as in x. 42, 43,

$$(x^2+y^2)-(x'^2+y'^2)=2x'y'-2xy.$$

But, by hypothesis, (x^2+y^2), $(x'^2+y'^2)$ are both *rational*, so that their difference is rational.

Also, by hypothesis, $2x'y'$, $2xy$ are both *medial* areas;

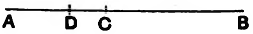
therefore the difference of two medial areas is a rational area:

which is impossible. [x. 26]

Therefore etc.

PROPOSITION 46.

The side of a rational plus a medial area is divided at one point only.



Let AB be the side of a rational plus a medial area divided at C, so that AC, CB are incommensurable in square and make the sum of the squares on AC, CB medial, but twice the rectangle AC, CB rational; [x. 40]
I say that AB is not so divided at another point.

For, if possible, let it be divided at D also, so that AD, DB are also incommensurable in square and make the sum of the squares on AD, DB medial, but twice the rectangle AD, DB rational.

Since then that by which twice the rectangle AC, CB differs from twice the rectangle AD, DB is also that by which the squares on AD, DB differ from the squares on AC, CB,
while twice the rectangle AC, CB exceeds twice the rectangle AD, DB by a rational area,
therefore the squares on AD, DB also exceed the squares on AC, CB by a rational area, though they are medial:
which is impossible. [x. 26]

Therefore the side of a rational plus a medial area is not divided at different points;
therefore it is divided at one point only.

Q. E. D.

Here, as before, if we use the same notation,

$$(x^2+y^2)-(x'^2+y'^2)=2x'y'-2xy,$$

and the areas on the left side are, by hypothesis, both medial, while the areas on the right side are both rational.

Thus the result of x. 26 is contradicted, as before.

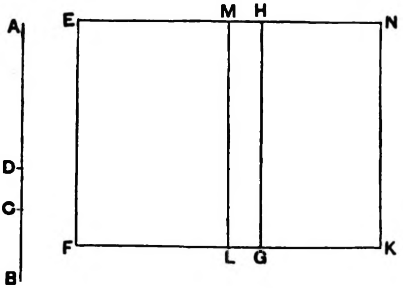
Therefore etc.

PROPOSITION 47.

The side of the sum of two medial areas is divided at one point only.

Let AB be divided at C, so that AC, CB are incommensurable in square and make the sum of the squares on AC, *CB* medial, and the rectangle *AC*, *CB* medial and also incommensurable with the sum of the squares on them;

I say that *AB* is not divided at another point so as to fulfil the given conditions.



For, if possible, let it be divided at *D*, so that again *AC* is of course not the same as *BD*, but *AC* is supposed greater;

let a rational straight line *EF* be set out,

and let there be applied to *EF* the rectangle *EG* equal to the squares on *AC*, *CB*,

and the rectangle *HK* equal to twice the rectangle *AC*, *CB*;

therefore the whole *EK* is equal to the square on *AB*. [II. 4]

Again, let *EL*, equal to the squares on *AD*, *DB*, be applied to *EF*;

therefore the remainder, twice the rectangle *AD*, *DB*, is equal to the remainder *MK*.

And since, by hypothesis, the sum of the squares on *AC*, *CB* is medial,

therefore *EG* is also medial.

And it is applied to the rational straight line *EF*;

therefore *HE* is rational and incommensurable in length with *EF*. [X. 22]

For the same reason

HN is also rational and incommensurable in length with *EF*.

And, since the sum of the squares on *AC*, *CB* is incommensurable with twice the rectangle *AC*, *CB*,

therefore *EG* is also incommensurable with *GN*,

so that *EH* is also incommensurable with *HN*. [VI. 1, X. 11]

And they are rational;

therefore *EH*, *HN* are rational straight lines commensurable in square only;

therefore *EN* is a binomial straight line divided at *H*. [x. 36]

Similarly we can prove that it is also divided at *M*.

And *EH* is not the same with *MN*;

therefore a binomial has been divided at different points: which is absurd. [x. 42]

Therefore a side of the sum of two medial areas is not divided at different points;

therefore it is divided at one point only.

Using the same notation as in the note on x. 44, we suppose that, if possible,

$$x + y = x' + y',$$

and we put

$$\left.\begin{aligned} x^2 + y^2 &= \sigma u \\ 2xy &= \sigma v \end{aligned}\right\} \text{ and } \left.\begin{aligned} x'^2 + y'^2 &= \sigma u' \\ 2x'y' &= \sigma v' \end{aligned}\right\}.$$

Then, since $x^2 + y^2$, $2xy$ are medial areas, and σ rational,

u, v are both rational and $\smile \sigma$ ....(1).

Also, by hypothesis, $x^2 + y^2 \smile 2xy$,

whence $u \smile v$ ....(2).

Therefore, by (1) and (2), u, v are rational and $\frown$.

Hence $u + v$ is a *binomial* straight line. [x. 36]

Similarly $u' + v'$ is a binomial straight line.

But $u + v = u' + v'$;

therefore a binomial straight line is divided into terms in two ways: which is impossible. [x. 42]

Therefore etc.

DEFINITIONS II.

1. Given a rational straight line and a binomial, divided into its terms, such that the square on the greater term is greater than the square on the lesser by the square on a straight line commensurable in length with the greater, then, if the greater term be commensurable in length with the rational straight line set out, let the whole be called a **first binomial** straight line;

2. but if the lesser term be commensurable in length with the rational straight line set out, let the whole be called a **second binomial**;

3. and if neither of the terms be commensurable in length with the rational straight line set out, let the whole be called a **third binomial.**

4. Again, if the square on the greater term be greater than the square on the lesser by the square on a straight line incommensurable in length with the greater, then, if the greater term be commensurable in length with the rational straight line set out, let the whole be called a **fourth binomial**;

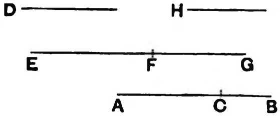
5. if the lesser, a **fifth binomial**;

6. and if neither, a **sixth binomial.**

PROPOSITION 48.

To find the first binomial straight line.

Let two numbers AC, CB be set out such that the sum of them AB has to BC the ratio which a square number has to a square number, but has not to CA the ratio which a square number has to a square number; [Lemma 1 after X. 28]



let any rational straight line D be set out, and let EF be commensurable in length with D.

Therefore EF is also rational.

Let it be contrived that,

as the number BA is to AC, so is the square on EF to the square on FG. [X. 6, Por.]

But AB has to AC the ratio which a number has to a number;

therefore the square on EF also has to the square on FG the ratio which a number has to a number,

so that the square on EF is commensurable with the square on FG. [X. 6]

And EF is rational;

therefore FG is also rational.

And, since BA has not to AC the ratio which a square number has to a square number,

neither, therefore, has the square on *EF* to the square on *FG* the ratio which a square number has to a square number;
therefore *EF* is incommensurable in length with *FG*. [x. 9]

Therefore *EF*, *FG* are rational straight lines commensurable in square only;
therefore *EG* is binomial. [x. 36]

I say that it is also a first binomial straight line.

For since, as the number *BA* is to *AC*, so is the square on *EF* to the square on *FG*,
while *BA* is greater than *AC*,
therefore the square on *EF* is also greater than the square on *FG*.

Let then the squares on *FG*, *H* be equal to the square on *EF*.

Now since, as *BA* is to *AC*, so is the square on *EF* to the square on *FG*,
therefore, *convertendo*,
as *AB* is to *BC*, so is the square on *EF* to the square on *H*. [v. 19, Por.]

But *AB* has to *BC* the ratio which a square number has to a square number;
therefore the square on *EF* also has to the square on *H* the ratio which a square number has to a square number.

Therefore *EF* is commensurable in length with *H*; [x. 9]
therefore the square on *EF* is greater than the square on *FG* by the square on a straight line commensurable with *EF*.

And *EF*, *FG* are rational, and *EF* is commensurable in length with *D*.

Therefore *EF* is a first binomial straight line.

Q. E. D.

Let $k\rho$ be a straight line commensurable in length with ρ, a given rational straight line.

The two numbers taken may be written $p(m^2 - n^2)$, pn^2, where $(m^2 - n^2)$ is not a square.

Take x such that

$$pm^2 : p(m^2 - n^2) = k^2\rho^2 : x^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

whence

$$x = k\rho\frac{\sqrt{m^2 - n^2}}{m}.$$

Then $k\rho + x$, or $k\rho + k\rho\dfrac{\sqrt{m^2 - n^2}}{m}$, is a *first binomial* straight line ......(2).

To prove this we have, from (1),

$$x^2 \frown k^2\rho^2,$$

and x is rational, but $x \smile k\rho$;
that is, x is rational and $\frown\!\!-\ k\rho$,
so that $k\rho + x$ is a binomial straight line.

Also, $k^2\rho^2$ being greater than x^2, suppose $k^2\rho^2 - x^2 = y^2$.

Then, from (1), $\quad pm^2 : pn^2 = k^2\rho^2 : y^2,$
whence y is rational and $\frown k\rho$.

Therefore $k\rho + x$ is a *first binomial* straight line [x. Deff. II. 1].

This binomial straight line may be written thus,

$$k\rho + k\rho\sqrt{1 - \lambda^2}.$$

When we come to x. 85, we shall find that the corresponding straight line with a negative sign is the *first apotome*,

$$k\rho - k\rho\sqrt{1 - \lambda^2}.$$

Consider now the equation of which these two expressions are the roots. The equation is

$$x^2 - 2k\rho \,.\, x + \lambda^2 k^2\rho^2 = 0.$$

In other words, the first binomial and the first apotome correspond to the roots of the equation

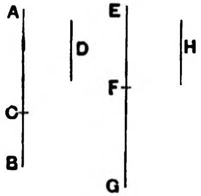
$$x^2 - 2\alpha x + \lambda^2\alpha^2 = 0,$$

where $\alpha = k\rho$.

PROPOSITION 49.

To find the second binomial straight line.

Let two numbers AC, CB be set out such that the sum of them AB has to BC the ratio which a square number has to a square number, but has not to AC the ratio which a square number has to a square number;
let a rational straight line D be set out, and let EF be commensurable in length with D;
therefore EF is rational.



Let it be contrived then that,
as the number CA is to AB, so also is the square on EF to the square on FG; [x. 6, Por.]
therefore the square on EF is commensurable with the square on FG. [x. 6]

Therefore FG is also rational.

Now, since the number CA has not to AB the ratio which a square number has to a square number, neither has the square on *EF* to the square on *FG* the ratio which a square number has to a square number.

Therefore *EF* is incommensurable in length with *FG*; [x. 9]

therefore *EF*, *FG* are rational straight lines commensurable in square only;

therefore *EG* is binomial. [x. 36]

It is next to be proved that it is also a second binomial straight line.

For since, inversely, as the number *BA* is to *AC*, so is the square on *GF* to the square on *FE*,

while *BA* is greater than *AC*,

therefore the square on *GF* is greater than the square on *FE*.

Let the squares on *EF*, *H* be equal to the square on *GF*;

therefore, *convertendo*, as *AB* is to *BC*, so is the square on *FG* to the square on *H*. [v. 19, Por.]

But *AB* has to *BC* the ratio which a square number has to a square number;

therefore the square on *FG* also has to the square on *H* the ratio which a square number has to a square number.

Therefore *FG* is commensurable in length with *H*; [x. 9]

so that the square on *FG* is greater than the square on *FE* by the square on a straight line commensurable with *FG*.

And *FG*, *FE* are rational straight lines commensurable in square only, and *EF*, the lesser term, is commensurable in length with the rational straight line *D* set out.

Therefore *EG* is a second binomial straight line.

Q. E. D.

Taking a rational straight line $k\rho$ commensurable in length with ρ, and selecting numbers of the same form as before, viz. $p(m^2 - n^2)$, pn^2, we put

$$p(m^2 - n^2) : pm^2 = k^2\rho^2 : x^2 \quad \text{.......................(1)},$$

so that

$$x = k\rho \frac{m}{\sqrt{m^2 - n^2}}$$

$$= k\rho \frac{1}{\sqrt{1 - \lambda^2}}, \text{ say} \quad \text{..........................(2)}.$$

Just as before, x is rational and $\frown k\rho$,

whence $k\rho + x$ is a *binomial* straight line.

By (1), $x^2 > k^2\rho^2$.

Let $$x^2 - k^2\rho^2 = y^2,$$
whence, from (1), $$pm^2 : pn^2 = x^2 : y^2,$$
and y is therefore rational and $\frown x$.

The greater term of the binomial straight line is x and the lesser $k\rho$, and

$$\frac{k\rho}{\sqrt{1 - \lambda^2}} + k\rho$$

satisfies the definition of the *second binomial* straight line.

The corresponding *second apotome* [x. 86] is

$$\frac{k\rho}{\sqrt{1 - \lambda^2}} - k\rho.$$

The equation of which the two expressions are the roots is

$$x^2 - \frac{2k\rho}{\sqrt{1 - \lambda^2}} \cdot x + \frac{\lambda^2}{1 - \lambda^2} k^2\rho^2 = 0,$$

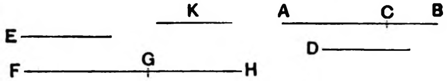
or $$x^2 - 2ax + \lambda^2 a^2 = 0,$$

where $$a = \frac{k\rho}{\sqrt{1 - \lambda^2}}.$$

PROPOSITION 50.

To find the third binomial straight line.

Let two numbers AC, CB be set out such that the sum of them AB has to BC the ratio which a square number has to a square number, but has not to AC the ratio which a square number has to a square number.



Let any other number D, not square, be set out also, and let it not have to either of the numbers BA. AC the ratio which a square number has to a square number.

Let any rational straight line E be set out,

and let it be contrived that, as D is to AB, so is the square on E to the square on FG; [x. 6, Por.]

therefore the square on E is commensurable with the square on FG. [x. 6]

And E is rational;

therefore FG is also rational.

And, since D has not to AB the ratio which a square number has to a square number,
neither has the square on E to the square on FG the ratio which a square number has to a square number;
therefore E is incommensurable in length with FG. [x. 9]

Next let it be contrived that, as the number BA is to AC, so is the square on FG to the square on GH; [x. 6, Por.]
therefore the square on FG is commensurable with the square on GH. [x. 6]

But FG is rational;
therefore GH is also rational.

And, since BA has not to AC the ratio which a square number has to a square number,
neither has the square on FG to the square on HG the ratio which a square number has to a square number;
therefore FG is incommensurable in length with GH. [x. 9]

Therefore FG, GH are rational straight lines commensurable in square only;
therefore FH is binomial. [x. 36]

I say next that it is also a third binomial straight line.

For since, as D is to AB, so is the square on E to the square on FG,
and, as BA is to AC, so is the square on FG to the square on GH,
therefore, *ex aequali*, as D is to AC, so is the square on E to the square on GH. [v. 22]

But D has not to AC the ratio which a square number has to a square number;
therefore neither has the square on E to the square on GH the ratio which a square number has to a square number;
therefore E is incommensurable in length with GH. [x. 9]

And since, as BA is to AC, so is the square on FG to the square on GH,
therefore the square on FG is greater than the square on GH.

Let then the squares on GH, K be equal to the square on FG;

therefore, *convertendo*, as *AB* is to *BC*, so is the square on *FG* to the square on *K*. [v. 19, Por.]

But *AB* has to *BC* the ratio which a square number has to a square number;
therefore the square on *FG* also has to the square on *K* the ratio which a square number has to a square number;
therefore *FG* is commensurable in length with *K*. [x. 9]

Therefore the square on *FG* is greater than the square on *GH* by the square on a straight line commensurable with *FG*.

And *FG*, *GH* are rational straight lines commensurable in square only, and neither of them is commensurable in length with *E*.

Therefore *FH* is a third binomial straight line.

Q. E. D.

Let ρ be a rational straight line.

Take the numbers $q(m^2 - n^2)$, qn^2,
and let p be a third number which is not a square and which has not to qm^2 or $q(m^2 - n^2)$ the ratio of square to square.

Take x such that $p : qm^2 = \rho^2 : x^2$ ....(1).

Thus x is rational and $\smile \rho$ ....(2).

Next suppose that $qm^2 : q(m^2 - n^2) = x^2 : y^2$ ....(3).

It follows that y is rational and $\frown\!\!\!-\, x$ ....(4).

Thus $(x + y)$ is a *binomial* straight line.

Again, from (1) and (3), *ex aequali*,

$p : q(m^2 - n^2) = \rho^2 : y^2$ ....(5),

whence $y \smile \rho$ ....(6).

Suppose that $x^2 - y^2 = z^2$.

Then, from (3), *convertendo*,

$$qm^2 : qn^2 = x^2 : z^2,$$

whence $z \frown x$.

Thus $\sqrt{x^2 - y^2} \frown x$,

and x, y are both $\smile \rho$;
therefore $x + y$ is a *third binomial* straight line.

Now, from (1), $x = \rho \cdot \dfrac{m\sqrt{q}}{\sqrt{p}}$,

and, by (5), $y = \rho \cdot \dfrac{\sqrt{m^2 - n^2} \cdot \sqrt{q}}{\sqrt{p}}$.

Thus the *third binomial* is

$$\sqrt{\frac{q}{p}} \cdot \rho\left(m + \sqrt{m^2 - n^2}\right),$$

which we may write in the form

$$m\sqrt{k} \cdot \rho + m\sqrt{k} \cdot \rho\sqrt{1 - \lambda^2}.$$

The corresponding *third apotome* [x. 87] is

$$m\sqrt{k}\,.\,\rho - m\sqrt{k}\,.\,\rho\sqrt{1-\lambda^2}.$$

The two expressions are accordingly the roots of the equation

$$x^2 - 2m\sqrt{k}\,.\,\rho x + \lambda^2 m^2 k\rho^2 = 0,$$

or

$$x^2 - 2ax + \lambda^2 a^2 = 0,$$

where

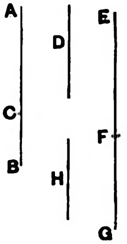
$$a = m\sqrt{k}\,.\,\rho.$$

See also note on x. 53 (*ad fin.*).

PROPOSITION 51.

To find the fourth binomial straight line.

Let two numbers AC, CB be set out such that AB neither has to BC, nor yet to AC, the ratio which a square number has to a square number.



Let a rational straight line D be set out, and let EF be commensurable in length with D; therefore EF is also rational.

Let it be contrived that, as the number BA is to AC, so is the square on EF to the square on FG; [x. 6, Por.]
therefore the square on EF is commensurable with the square on FG; [x. 6]
therefore FG is also rational.

Now, since BA has not to AC the ratio which a square number has to a square number,
neither has the square on EF to the square on FG the ratio which a square number has to a square number;
therefore EF is incommensurable in length with FG. [x. 9]

Therefore EF, FG are rational straight lines commensurable in square only;
so that EG is binomial.

I say next that it is also a fourth binomial straight line.

For since, as BA is to AC, so is the square on EF to the square on FG,
therefore the square on EF is greater than the square on FG.

Let then the squares on FG, H be equal to the square on EF;

therefore, *convertendo*, as the number AB is to BC, so is the square on EF to the square on H. [v. 19, Por.]

But AB has not to BC the ratio which a square number has to a square number;

therefore neither has the square on EF to the square on H the ratio which a square number has to a square number.

Therefore EF is incommensurable in length with H; [x. 9]

therefore the square on EF is greater than the square on GF by the square on a straight line incommensurable with EF.

And EF, FG are rational straight lines commensurable in square only, and EF is commensurable in length with D.

Therefore EG is a fourth binomial straight line.

Q. E. D.

Take numbers m, n such that $(m+n)$ has not to either m or n the ratio of square to square.

Take x such that $$(m+n) : m = k^2\rho^2 : x^2,$$

whence $$x = k\rho\sqrt{\frac{m}{m+n}}$$
$$= \frac{k\rho}{\sqrt{1+\lambda}}, \text{ say.}$$

Then $k\rho + x$, or $k\rho + \dfrac{k\rho}{\sqrt{1+\lambda}}$, is a *fourth binomial* straight line.

For $\sqrt{k^2\rho^2 - x^2}$ is incommensurable in length with $k\rho$, and $k\rho$ is commensurable in length with ρ.

The corresponding *fourth apotome* [x. 88] is
$$k\rho - \frac{k\rho}{\sqrt{1+\lambda}}.$$

The equation of which the two expressions are the roots is
$$x^2 - 2k\rho \,.\, x + \frac{\lambda}{1+\lambda} k^2\rho^2 = 0,$$
or $$x^2 - 2ax + \frac{\lambda}{1+\lambda} a^2 = 0,$$
where $$a = k\rho.$$

PROPOSITION 52.

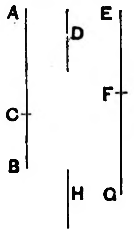
To find the fifth binomial straight line.

Let two numbers AC, CB be set out such that AB has not to either of them the ratio which a square number has to a square number;

let any rational straight line D be set out,

and let *EF* be commensurable with *D*;
therefore *EF* is rational.

Let it be contrived that, as *CA* is to *AB*, so is the square on *EF* to the square on *FG*. [x. 6, Por.]



But *CA* has not to *AB* the ratio which a square number has to a square number;
therefore neither has the square on *EF* to the square on *FG* the ratio which a square number has to a square number.

Therefore *EF*, *FG* are rational straight lines commensurable in square only; [x. 9]
therefore *EG* is binomial. [x. 36]

I say next that it is also a fifth binomial straight line.

For since, as *CA* is to *AB*, so is the square on *EF* to the square on *FG*,
inversely, as *BA* is to *AC*, so is the square on *FG* to the square on *FE*;
therefore the square on *GF* is greater than the square on *FE*.

Let then the squares on *EF*, *H* be equal to the square on *GF*;
therefore, *convertendo*, as the number *AB* is to *BC*, so is the square on *GF* to the square on *H*. [v. 19, Por.]

But *AB* has not to *BC* the ratio which a square number has to a square number;
therefore neither has the square on *FG* to the square on *H* the ratio which a square number has to a square number.

Therefore *FG* is incommensurable in length with *H*; [x. 9]
so that the square on *FG* is greater than the square on *FE* by the square on a straight line incommensurable with *FG*.

And *GF*, *FE* are rational straight lines commensurable in square only, and the lesser term *EF* is commensurable in length with the rational straight line *D* set out.

Therefore *EG* is a fifth binomial straight line.

Q. E. D.

If m, n be numbers of the kind taken in the last proposition, take x such that

$$m : (m + n) = k^2\rho^2 : x^2.$$

In this case
$$x = k\rho\sqrt{\frac{m+n}{m}}$$
$$= k\rho\sqrt{1+\lambda}, \text{ say,}$$
and $x > k\rho$.

Then $k\rho\sqrt{1+\lambda} + k\rho$ is a *fifth binomial* straight line.

For $\sqrt{x^2 - k^2\rho^2}$, or $\sqrt{\lambda} \,.\, k\rho$, is incommensurable in length with $k\rho\sqrt{1+\lambda}$, or x;

and $k\rho$, but not $k\rho\sqrt{1+\lambda}$, is commensurable in length with ρ.

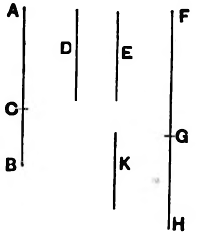
The corresponding *fifth apotome* [x. 89] is
$$k\rho\sqrt{1+\lambda} - k\rho.$$

The equation of which the fifth binomial and the fifth apotome are the roots is
$$x^2 - 2k\rho\sqrt{1+\lambda} \,.\, x + \lambda k^2\rho^2 = 0,$$
or
$$x^2 - 2ax + \frac{\lambda}{1+\lambda}a^2 = 0,$$
where
$$a = k\rho\sqrt{1+\lambda}.$$

PROPOSITION 53.

To find the sixth binomial straight line.

Let two numbers AC, CB be set out such that AB has not to either of them the ratio which a square number has to a square number;



and let there also be another number D which is not square and which has not to either of the numbers BA, AC the ratio which a square number has to a square number.

Let any rational straight line E be set out,

and let it be contrived that, as D is to AB, so is the square on E to the square on FG; [x. 6, Por.]

therefore the square on E is commensurable with the square on FG. [x. 6]

And E is rational;

therefore FG is also rational.

Now, since D has not to AB the ratio which a square number has to a square number,

neither has the square on E to the square on FG the ratio which a square number has to a square number;
therefore E is incommensurable in length with FG. [x. 9]

Again, let it be contrived that, as BA is to AC, so is the square on FG to the square on GH. [x. 6, Por.]

Therefore the square on FG is commensurable with the square on HG. [x. 6]

Therefore the square on HG is rational;
therefore HG is rational.

And, since BA has not to AC the ratio which a square number has to a square number,
neither has the square on FG to the square on GH the ratio which a square number has to a square number;
therefore FG is incommensurable in length with GH. [x. 9]

Therefore FG, GH are rational straight lines commensurable in square only;
therefore FH is binomial. [x. 36]

It is next to be proved that it is also a sixth binomial straight line.

For since, as D is to AB, so is the square on E to the square on FG,
and also, as BA is to AC, so is the square on FG to the square on GH,
therefore, *ex aequali*, as D is to AC, so is the square on E to the square on GH. [v. 22]

But D has not to AC the ratio which a square number has to a square number;
therefore neither has the square on E to the square on GH the ratio which a square number has to a square number;
therefore E is incommensurable in length with GH. [x. 9]

But it was also proved incommensurable with FG;
therefore each of the straight lines FG, GH is incommensurable in length with E.

And, since, as BA is to AC, so is the square on FG to the square on GH,
therefore the square on FG is greater than the square on GH.

Let then the squares on GH, K be equal to the square on FG;

therefore, *convertendo*, as *AB* is to *BC*, so is the square on *FG* to the square on *K*. [v. 19, Por.]

But *AB* has not to *BC* the ratio which a square number has to a square number;
so that neither has the square on *FG* to the square on *K* the ratio which a square number has to a square number.

Therefore *FG* is incommensurable in length with *K*; [x. 9]
therefore the square on *FG* is greater than the square on *GH* by the square on a straight line incommensurable with *FG*.

And *FG*, *GH* are rational straight lines commensurable in square only, and neither of them is commensurable in length with the rational straight line *E* set out.

Therefore *FH* is a sixth binomial straight line.

Q. E. D.

Take numbers m, n such that $(m+n)$ has not to either of the numbers m, n the ratio of square to square; take also a third number p, which is not square, and which has not to either of the numbers $(m+n)$, m the ratio of square to square.

Let $$p : (m+n) = \rho^2 : x^2 \quad\text{.............................}(1)$$
and $$(m+n) : m = x^2 : y^2 \quad\text{.............................}(2).$$

Then shall $(x+y)$ be a *sixth binomial* straight line.

For, by (1), x is rational and $\smile \rho$.
By (2), since x is rational,
$$y \text{ is rational and } \smile x.$$

Hence x, y are rational and commensurable in square only, so that $(x+y)$ is a binomial straight line.

Again, *ex aequali*, from (1) and (2),
$$p : m = \rho^2 : y^2 \quad\text{.................................}(3),$$
whence $y \smile \rho$.

Thus x, y are both incommensurable in length with ρ.

Lastly, from (2), *convertendo*,
$$(m+n) : n = x^2 : (x^2 - y^2),$$
so that $\sqrt{x^2 - y^2} \smile x$.

Therefore $(x+y)$ is a *sixth binomial* straight line.

Now, from (1) and (3),
$$x = \rho \cdot \sqrt{\frac{m+n}{p}} = \rho\sqrt{k}, \text{ say,}$$
$$y = \rho \cdot \sqrt{\frac{m}{p}} = \rho\sqrt{\lambda}, \text{ say,}$$
and the *sixth binomial* straight line may be written
$$\sqrt{k} \cdot \rho + \sqrt{\lambda} \cdot \rho.$$
The corresponding *sixth apotome* is [x. 90]
$$\sqrt{k} \cdot \rho - \sqrt{\lambda} \cdot \rho;$$

and the equation of which the two expressions are the roots is

$$x^2 - 2\sqrt{k} \,.\, \rho x + (k - \lambda)\rho^2 = 0,$$

or

$$x^2 - 2ax + \frac{k-\lambda}{k} a^2 = 0,$$

where $a = \sqrt{k} \,.\, \rho$.

Tannery remarks ("De la solution géométrique des problèmes du second degré avant Euclide" in *Mémoires de la Société des sciences physiques et naturelles de Bordeaux*, 2e Série, T. IV.) that Euclid admits as binomials and apotomes the *third* and *sixth* binomials and apotomes which are the square roots of first binomials and apotomes respectively. Hence the third and sixth binomials and apotomes are the positive roots of *biquadratic* equations of the same form as the quadratics which give as roots the first and fourth binomials and apotomes. But this remark seems to be of no value because (as was pointed out a hundred years ago by Cossali, II. p. 260) the squares of *all the six* binomials and apotomes (including the first and fourth) give *first* binomials and apotomes respectively. Hence we may equally well regard them all as roots of biquadratics reducible to quadratics, or generally as roots of equations of the form

$$x^{2^n} \pm 2a \,.\, x^{2^{n-1}} \pm q = 0;$$

and nothing is gained by raising the degree of the equations in this way.

It is, of course, easy to see that the most general form of binomial and apotome, viz.

$$\rho \,.\, \sqrt{k} \pm \rho \,.\, \sqrt{\lambda},$$

give *first* binomials and apotomes when squared.

For the square is $\rho\,\{(k+\lambda)\rho \pm 2\sqrt{k\lambda} \,.\, \rho\}$; and the expression within the bracket is a first binomial or apotome, because

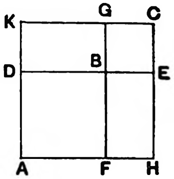
(1) $k + \lambda > 2\sqrt{k\lambda}$,

(2) $\sqrt{(k+\lambda)^2 - 4k\lambda} = k - \lambda$, which is $\frown (k+\lambda)$,

(3) $(k+\lambda)\rho \frown \rho$.

LEMMA.

Let there be two squares *AB*, *BC*, and let them be placed so that *DB* is in a straight line with *BE*; therefore *FB* is also in a straight line with *BG*.



Let the parallelogram *AC* be completed; I say that *AC* is a square, that *DG* is a mean proportional between *AB*, *BC*, and further that *DC* is a mean proportional between *AC*, *CB*.

For, since *DB* is equal to *BF*, and *BE* to *BG*, therefore the whole *DE* is equal to the whole *FG*.

But *DE* is equal to each of the straight lines *AH*, *KC*, and *FG* is equal to each of the straight lines *AK*, *HC*; [I. 34] therefore each of the straight lines AH, KC is also equal to each of the straight lines AK, HC.

Therefore the parallelogram AC is equilateral.

And it is also rectangular;
therefore AC is a square.

And since, as FB is to BG, so is DB to BE,
while, as FB is to BG, so is AB to DG,
and, as DB is to BE, so is DG to BC, [VI. 1]
therefore also, as AB is to DG, so is DG to BC. [V. 11]

Therefore DG is a mean proportional between AB, BC.

I say next that DC is also a mean proportional between AC, CB.

For since, as AD is to DK, so is KG to GC—
for they are equal respectively—
and, *componendo*, as AK is to KD, so is KC to CG, [V. 18]
while, as AK is to KD, so is AC to CD,
and, as KC is to CG, so is DC to CB, [VI. 1]
therefore also, as AC is to DC, so is DC to BC. [V. 11]

Therefore DC is a mean proportional between AC, CB.

Being what it was proposed to prove.

It is here proved that

$$x^2 : xy = xy : y^2,$$

and

$$(x+y)^2 : (x+y)y = (x+y)y : y^2.$$

The first of the two results is proved in the course of X. 25 (lines 6—8 on p. 57 above). This fact may, I think, suggest doubt as to the genuineness of this Lemma.

PROPOSITION 54.

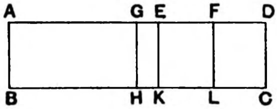
If an area be contained by a rational straight line and the first binomial, the "side" of the area is the irrational straight line which is called binomial.

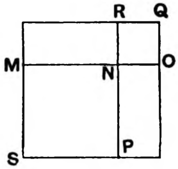
For let the area AC be contained by the rational straight line AB and the first binomial AD;
I say that the "side" of the area AC is the irrational straight line which is called binomial.

For, since AD is a first binomial straight line, let it be divided into its terms at E,
and let AE be the greater term.

It is then manifest that *AE*, *ED* are rational straight lines commensurable in square only,
the square on *AE* is greater than the square on *ED* by the square on a straight line commensurable with *AE*,
and *AE* is commensurable in length with the rational straight line *AB* set out. [x. Deff. II. 1]

Let *ED* be bisected at the point *F*.





Then, since the square on *AE* is greater than the square on *ED* by the square on a straight line commensurable with *AE*,
therefore, if there be applied to the greater *AE* a parallelogram equal to the fourth part of the square on the less, that is, to the square on *EF*, and deficient by a square figure, it divides it into commensurable parts. [x. 17]

Let then the rectangle *AG*, *GE* equal to the square on *EF* be applied to *AE*;
therefore *AG* is commensurable in length with *EG*.

Let *GH*, *EK*, *FL* be drawn from *G*, *E*, *F* parallel to either of the straight lines *AB*, *CD*;
let the square *SN* be constructed equal to the parallelogram *AH*, and the square *NQ* equal to *GK*, [II. 14]
and let them be placed so that *MN* is in a straight line with *NO*;
therefore *RN* is also in a straight line with *NP*.

And let the parallelogram *SQ* be completed;
therefore *SQ* is a square. [Lemma]

Now, since the rectangle *AG*, *GE* is equal to the square on *EF*,
therefore, as *AG* is to *EF*, so is *FE* to *EG*; [VI. 17]
therefore also, as *AH* is to *EL*, so is *EL* to *KG*; [VI. 1]
therefore *EL* is a mean proportional between *AH*, *GK*.

But *AH* is equal to *SN*, and *GK* to *NQ*;
therefore *EL* is a mean proportional between *SN*, *NQ*.

But *MR* is also a mean proportional between the same *SN*, *NQ*; [Lemma]
therefore *EL* is equal to *MR*,
so that it is also equal to *PO*.

But *AH*, *GK* are also equal to *SN*, *NQ*;
therefore the whole *AC* is equal to the whole *SQ*, that is, to the square on *MO*;
therefore *MO* is the "side" of *AC*.

I say next that *MO* is binomial.

For, since *AG* is commensurable with *GE*,
therefore *AE* is also commensurable with each of the straight lines *AG*, *GE*. [x. 15]

But *AE* is also, by hypothesis, commensurable with *AB*;
therefore *AG*, *GE* are also commensurable with *AB*. [x. 12]

And *AB* is rational;
therefore each of the straight lines *AG*, *GE* is also rational;
therefore each of the rectangles *AH*, *GK* is rational, [x. 19]
and *AH* is commensurable with *GK*.

But *AH* is equal to *SN*, and *GK* to *NQ*;
therefore *SN*, *NQ*, that is, the squares on *MN*, *NO*, are rational and commensurable.

And, since *AE* is incommensurable in length with *ED*,
while *AE* is commensurable with *AG*, and *DE* is commensurable with *EF*,
therefore *AG* is also incommensurable with *EF*, [x. 13]
so that *AH* is also incommensurable with *EL*. [vi. 1, x. 11]

But *AH* is equal to *SN*, and *EL* to *MR*;
therefore *SN* is also incommensurable with *MR*.

But, as *SN* is to *MR*, so is *PN* to *NR*; [vi. 1]
therefore *PN* is incommensurable with *NR*. [x. 11]

But *PN* is equal to *MN*, and *NR* to *NO*;
therefore *MN* is incommensurable with *NO*.

And the square on *MN* is commensurable with the square on *NO*,
and each is rational;
therefore *MN*, *NO* are rational straight lines commensurable in square only.

Therefore MO is binomial [x. 36] and the "side" of AC.

Q. E. D.

2. "side." I use the word "side" in the sense explained in the note on x. Def. 4 (p. 13 above), i.e. as short for "side of a square equal to." The Greek is ἡ τὸ χωρίον δυναμένη.

A *first binomial* straight line being, as we have seen in x. 48, of the form

$$k\rho + k\rho\sqrt{1-\lambda^2},$$

the problem solved in this proposition is the equivalent of *finding the square root of this expression multiplied by* ρ, or of

$$\rho\,(k\rho + k\rho\sqrt{1-\lambda^2}),$$

and of proving that the said square root represents a *binomial* straight line as defined in x. 36.

The geometrical method corresponds sufficiently closely to the algebraical one which we should use.

First solve the equations

$$\left.\begin{aligned} u + v &= k\rho \\ uv &= \tfrac{1}{4}k^2\rho^2(1-\lambda^2) \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Then, if u, v represent the straight lines so found, put

$$\left.\begin{aligned} x^2 &= \rho u \\ y^2 &= \rho v \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2);$$

and the straight line $(x+y)$ is the square root required.

The actual algebraical solution of (1) gives

$$u - v = k\rho\,.\,\lambda,$$

so that

$$u = \tfrac{1}{2}k\rho\,(1+\lambda),$$

$$v = \tfrac{1}{2}k\rho\,(1-\lambda),$$

and therefore

$$x = \rho\sqrt{\frac{k}{2}(1+\lambda)},$$

$$y = \rho\sqrt{\frac{k}{2}(1-\lambda)},$$

and

$$x + y = \rho\sqrt{\frac{k}{2}(1+\lambda)} + \rho\sqrt{\frac{k}{2}(1-\lambda)}.$$

This is clearly a *binomial* straight line as defined in x. 36.

Since Euclid has to express his results by straight lines in his figure, and has no symbols to make the result obvious by inspection, he is obliged to prove (1) that $(x+y)$ is the square root of $\rho\,(k\rho + k\rho\sqrt{1-\lambda^2})$, and (2) that $(x+y)$ is a binomial straight line, in the following manner.

First, he proves, by means of the preceding Lemma, that

$$xy = \frac{k}{2}\rho^2\sqrt{1-\lambda^2}\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3);$$

therefore

$$\begin{aligned}(x+y)^2 &= x^2 + y^2 + 2xy \\ &= \rho\,(u+v) + 2xy \\ &= k\rho^2 + k\rho^2\sqrt{1-\lambda^2}, \text{ by (1) and (3)},\end{aligned}$$

so that

$$x + y = \sqrt{\rho\,(k\rho + k\rho\sqrt{1-\lambda^2})}.$$

Secondly, it results from (1), [by x. 17], that

$$u \frown v,$$

so that u, v are both $\frown (u+v)$, and therefore $\frown \rho$ ....(4);
thus u, v are rational,
whence ρu, ρv are both rational, and

$$\rho u \frown \rho v.$$

Therefore x^2, y^2 are rational and commensurable ....(5).

Next, $k\rho \smile k\rho\sqrt{1-\lambda^2}$,
and $k\rho \frown u$, while $k\rho\sqrt{1-\lambda^2} \frown \frac{1}{2}k\rho\sqrt{1-\lambda^2}$;

therefore $u \smile \frac{1}{2}k\rho\sqrt{1-\lambda^2}$,

whence $\rho u \smile \frac{1}{2}k\rho^2\sqrt{1-\lambda^2}$,

or $x^2 \smile xy$,

so that $x \smile y$.

By this and (5), x, y are rational and $\frown\!\!-$, so that $(x+y)$ is a binomial straight line. [x. 36]

x. 91 will prove in like manner that a like theorem holds for *apotomes*, viz. that

$$\rho\sqrt{\frac{k}{2}(1+\lambda)} - \rho\sqrt{\frac{k}{2}(1-\lambda)} = \sqrt{\rho\left(k\rho - k\rho\sqrt{1-\lambda^2}\right)}.$$

Since the *first binomial* straight line and the *first apotome* are the roots of the equation

$$x^2 - 2k\rho\,.\,x + \lambda^2k^2\rho^2 = 0,$$

this proposition and x. 91 give us the solution of the biquadratic equation

$$x^4 - 2k\rho^2\,.\,x^2 + \lambda^2k^2\rho^4 = 0.$$

PROPOSITION 55.

If an area be contained by a rational straight line and the second binomial, the "side" of the area is the irrational straight line which is called a first bimedial.

For let the area *ABCD* be contained by the rational straight line *AB* and the second binomial *AD*;

I say that the "side" of the area *AC* is a first bimedial straight line.

For, since *AD* is a second binomial straight line, let it be divided into its terms at *E*, so that *AE* is the greater term;

therefore *AE*, *ED* are rational straight lines commensurable in square only,

the square on *AE* is greater than the square on *ED* by the square on a straight line commensurable with *AE*,

and the lesser term *ED* is commensurable in length with *AB*. [x. Deff. II. 2]

Let *ED* be bisected at *F*,

and let there be applied to *AE* the rectangle *AG*, *GE* equal to the square on *EF* and deficient by a square figure;

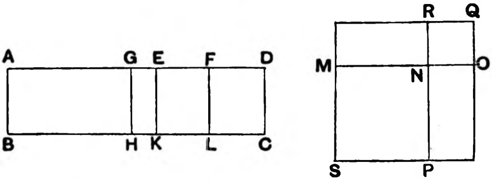
therefore *AG* is commensurable in length with *GE*. [x. 17]

Through *G*, *E*, *F* let *GH*, *EK*, *FL* be drawn parallel to *AB*, *CD*,

let the square *SN* be constructed equal to the parallelogram *AH*, and the square *NQ* equal to *GK*,

and let them be placed so that *MN* is in a straight line with *NO*;

therefore *RN* is also in a straight line with *NP*.



Let the square *SQ* be completed.

It is then manifest from what was proved before that *MR* is a mean proportional between *SN*, *NQ* and is equal to *EL*, and that *MO* is the "side" of the area *AC*.

It is now to be proved that *MO* is a first bimedial straight line.

Since *AE* is incommensurable in length with *ED*,

while *ED* is commensurable with *AB*,

therefore *AE* is incommensurable with *AB*. [x. 13]

And, since *AG* is commensurable with *EG*,

AE is also commensurable with each of the straight lines *AG*, *GE*. [x. 15]

But *AE* is incommensurable in length with *AB*;

therefore *AG*, *GE* are also incommensurable with *AB*. [x. 13]

Therefore *BA*, *AG* and *BA*, *GE* are pairs of rational straight lines commensurable in square only;

so that each of the rectangles *AH*, *GK* is medial. [x. 21]

Hence each of the squares *SN*, *NQ* is medial.

Therefore *MN*, *NO* are also medial.

And, since *AG* is commensurable in length with *GE*,

AH is also commensurable with *GK*, [vi. 1, x. 11]

that is, *SN* is commensurable with *NQ*,

that is, the square on *MN* with the square on *NO*.

And, since *AE* is incommensurable in length with *ED*, while *AE* is commensurable with *AG*, and *ED* is commensurable with *EF*, therefore *AG* is incommensurable with *EF*; [X. 13] so that *AH* is also incommensurable with *EL*, that is, *SN* is incommensurable with *MR*, that is, *PN* with *NR*, [VI. 1, X. 11] that is, *MN* is incommensurable in length with *NO*.

But *MN*, *NO* were proved to be both medial and commensurable in square; therefore *MN*, *NO* are medial straight lines commensurable in square only.

I say next that they also contain a rational rectangle.

For, since *DE* is, by hypothesis, commensurable with each of the straight lines *AB*, *EF*, therefore *EF* is also commensurable with *EK*. [X. 12]

And each of them is rational; therefore *EL*, that is, *MR* is rational, [X. 19] and *MR* is the rectangle *MN*, *NO*.

But, if two medial straight lines commensurable in square only and containing a rational rectangle be added together, the whole is irrational and is called a first bimedial straight line. [X. 37]

Therefore *MO* is a first bimedial straight line.

Q. E. D.

39. **Therefore BA, AG and BA, GE are pairs of rational straight lines commensurable in square only.** The text has "Therefore *BA*, *AG*, *GE* are rational straight lines commensurable in square only," which I have altered because it would naturally convey the impression that *any two* of the three straight lines are commensurable in square only, whereas *AG*, *GE* are commensurable in length (l. 18), and it is only the other two pairs which are commensurable in square only.

A *second binomial* straight line being [X. 49] of the form

$$\frac{k\rho}{\sqrt{1-\lambda^2}} + k\rho,$$

the present proposition is equivalent to finding the *square root of the expression*

$$\rho\left(\frac{k\rho}{\sqrt{1-\lambda^2}} + k\rho\right).$$

As in the last proposition, Euclid finds u, v from the equations

$$\left.\begin{aligned} u+v &= \frac{k\rho}{\sqrt{1-\lambda^2}} \\ uv &= \tfrac{1}{4}k^2\rho^2 \end{aligned}\right\} \text{..............................}(1),$$

then finds x, y from the equations

$$\left.\begin{aligned} x^2 &= \rho u \\ y^2 &= \rho v \end{aligned}\right\} \text{..................................}(2),$$

and then proves (α) that

$$x+y=\sqrt{\rho\left(\frac{k\rho}{\sqrt{1-\lambda^2}}+k\rho\right)},$$

and (β) that $(x+y)$ is a first bimedial straight line [X. 37].

The steps in the proof are as follows.

For (α) reference to the corresponding part of the previous proposition suffices.

(β) By (1) and X. 17,

$$u \frown v;$$

therefore u, v are both rational and $\frown (u+v)$, and therefore $\smile \rho$ [by (1)]...(3).

Hence ρu, ρv, or x^2, y^2, are *medial* areas,

so that x, y are also medial ....(4).

But, since $u \frown v$,

$$x^2 \frown y^2 \text{ ....}(5).$$

Again $(u+v)$, or $\dfrac{k\rho}{\sqrt{1-\lambda^2}}$, $\underset{\cdot}{\smile}\, k\rho$,

so that $u \smile \tfrac{1}{2}k\rho$,

whence $\rho u \smile \tfrac{1}{2}k\rho^2$,

or $x^2 \smile xy$,

and $x \smile y$ ....(6).

Thus [(4), (5), (6)] x, y are medial and $\frown\!\!\!-$.

Lastly, $xy=\tfrac{1}{2}k\rho^2$, which is rational.

Therefore $(x+y)$ is a *first bimedial* straight line.

The actual straight lines obtained from (1) are

$$\left.\begin{aligned} u &= \tfrac{1}{2}\frac{1+\lambda}{\sqrt{1-\lambda^2}}k\rho \\ v &= \tfrac{1}{2}\frac{1-\lambda}{\sqrt{1-\lambda^2}}k\rho \end{aligned}\right\},$$

so that $$x+y=\rho\sqrt{\frac{k}{2}\left(\frac{1+\lambda}{1-\lambda}\right)^{\frac{1}{2}}}+\rho\sqrt{\frac{k}{2}\left(\frac{1-\lambda}{1+\lambda}\right)^{\frac{1}{2}}}.$$

The corresponding *first apotome of a medial* straight line found in X. 92 being the same thing with a *minus* sign between the terms, the two expressions are the roots of the biquadratic

$$x^4-\frac{2k\rho^2}{\sqrt{1-\lambda^2}}x^2+\frac{\lambda^2}{1-\lambda^2}k^2\rho^4=0,$$

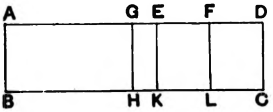
being the equation in x^2 corresponding to that in x in X. 49.

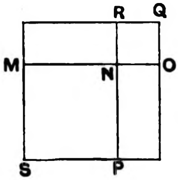
PROPOSITION 56.

If an area be contained by a rational straight line and the third binomial, the "side" of the area is the irrational straight line called a second bimedial.

For let the area $ABCD$ be contained by the rational straight line AB and the third binomial AD divided into its terms at E, of which terms AE is the greater;
I say that the "side" of the area AC is the irrational straight line called a second bimedial.

For let the same construction be made as before.





Now, since AD is a third binomial straight line,
therefore AE, ED are rational straight lines commensurable in square only,
the square on AE is greater than the square on ED by the square on a straight line commensurable with AE,
and neither of the terms AE, ED is commensurable in length with AB. [x. Deff. II. 3]

Then, in manner similar to the foregoing, we shall prove that MO is the "side" of the area AC,
and MN, NO are medial straight lines commensurable in square only;
so that MO is bimedial.

It is next to be proved that it is also a second bimedial straight line.

Since DE is incommensurable in length with AB, that is, with EK,
and DE is commensurable with EF,
therefore EF is incommensurable in length with EK. [x. 13]

And they are rational;

therefore *FE*, *EK* are rational straight lines commensurable in square only.

Therefore *EL*, that is, *MR*, is medial. [X. 21]

And it is contained by *MN*, *NO*;

therefore the rectangle *MN*, *NO* is medial.

Therefore *MO* is a second bimedial straight line. [X. 38]

Q. E. D.

This proposition in like manner is the equivalent of finding the square root of the product of ρ and the *third binomial* [X. 50], i.e. of the expression

$$\rho\,(\sqrt{k}\,.\,\rho + \sqrt{k}\,.\,\rho\sqrt{1-\lambda^2}).$$

As before, put

$$\left.\begin{aligned} u+v &= \sqrt{k}\,.\,\rho \\ uv &= \tfrac{1}{4}k\rho^2(1-\lambda^2) \end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Next, u, v being found, let

$$x^2 = \rho u,$$
$$y^2 = \rho v;$$

then $(x+y)$ is the square root required and is a *second bimedial* straight line. [X. 38]

For, as in the last proposition, it is proved that $(x+y)$ is the square root, and x, y are medial and $\frown -$.

Again, $xy = \frac{1}{2}\sqrt{k}\,.\,\rho^2\sqrt{1-\lambda^2}$, which is *medial*.

Hence $(x+y)$ is a *second bimedial* straight line.

By solving equations (1), we find

$$u = \tfrac{1}{2}(\sqrt{k}\,.\,\rho + \lambda\sqrt{k}\,.\,\rho),$$
$$v = \tfrac{1}{2}(\sqrt{k}\,.\,\rho - \lambda\sqrt{k}\,.\,\rho),$$

and
$$x+y = \rho\sqrt{\frac{\sqrt{k}}{2}(1+\lambda)} + \rho\sqrt{\frac{\sqrt{k}}{2}(1-\lambda)}.$$

The corresponding *second apotome of a medial* found in X. 93 is the same thing with a minus sign between the terms, and the two are the roots (cf. note on X. 50) of the biquadratic equation

$$x^4 - 2\sqrt{k}\,.\,\rho^2x^2 + \lambda^2k\rho^4 = 0.$$

PROPOSITION 57.

If an area be contained by a rational straight line and the fourth binomial, the "side" of the area is the irrational straight line called major.

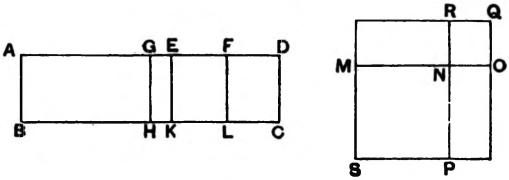
For let the area *AC* be contained by the rational straight line *AB* and the fourth binomial *AD* divided into its terms at *E*, of which terms let *AE* be the greater;

I say that the "side" of the area *AC* is the irrational straight line called major.

For, since AD is a fourth binomial straight line,
therefore AE, ED are rational straight lines commensurable in square only,
the square on AE is greater than the square on ED by the square on a straight line incommensurable with AE,
and AE is commensurable in length with AB. [x. Deff. II. 4]

Let DE be bisected at F,
and let there be applied to AE a parallelogram, the rectangle AG, GE, equal to the square on EF;
therefore AG is incommensurable in length with GE. [x. 18]

Let GH, EK, FL be drawn parallel to AB,
and let the rest of the construction be as before;
it is then manifest that MO is the "side" of the area AC.



It is next to be proved that MO is the irrational straight line called major.

Since AG is incommensurable with EG,
AH is also incommensurable with GK, that is, SN with NQ; [vi. 1, x. 11]
therefore MN, NO are incommensurable in square.

And, since AE is commensurable with AB,
AK is rational; [x. 19]
and it is equal to the squares on MN, NO;
therefore the sum of the squares on MN, NO is also rational.

And, since DE is incommensurable in length with AB, that is, with EK,
while DE is commensurable with EF,
therefore EF is incommensurable in length with EK. [x. 13]

Therefore EK, EF are rational straight lines commensurable in square only;
therefore LE, that is, MR, is medial. [x. 21]

And it is contained by MN, NO;
therefore the rectangle MN, NO is medial.

And the [sum] of the squares on *MN*, *NO* is rational, and *MN*, *NO* are incommensurable in square.

But, if two straight lines incommensurable in square and making the sum of the squares on them rational, but the rectangle contained by them medial, be added together, the whole is irrational and is called major. [X. 39]

Therefore *MO* is the irrational straight line called major and is the "side" of the area *AC*. Q. E. D.

The problem here is to find the square root of the expression [cf. X. 51]

$$\rho\left(k\rho + \frac{k\rho}{\sqrt{1+\lambda}}\right).$$

The procedure is the same.

Find u, v from the equations

$$\left.\begin{aligned} u + v &= k\rho \\ uv &= \tfrac{1}{4}\frac{k^2\rho^2}{1+\lambda} \end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

and, if

$$\left.\begin{aligned} x^2 &= \rho u \\ y^2 &= \rho v \end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

$(x+y)$ is the required square root.

To prove that $(x+y)$ is the *major* irrational straight line Euclid argues thus.

By X. 18, $u \smile v$,

therefore $\rho u \smile \rho v$,

or $x^2 \smile y^2$,

so that $x \smile\!\!- \, y$ ……………………………(3).

Now, since $(u+v) \frown \rho$,

$(u+v)\rho$, or (x^2+y^2), is a rational area………………(4).

Lastly, $xy = \frac{1}{2}\frac{k\rho^2}{\sqrt{1+\lambda}}$, which is a *medial* area ……………………(5).

Thus [(3), (4), (5)] $(x+y)$ is a *major* irrational straight line. [X. 39]

Actual solution gives

$$u = \tfrac{1}{2}k\rho\left(1 + \sqrt{\frac{\lambda}{1+\lambda}}\right),$$

$$v = \tfrac{1}{2}k\rho\left(1 - \sqrt{\frac{\lambda}{1+\lambda}}\right),$$

and

$$x + y = \rho\,.\sqrt{\frac{k}{2}\left(1 + \sqrt{\frac{\lambda}{1+\lambda}}\right)} + \rho\,.\sqrt{\frac{k}{2}\left(1 - \sqrt{\frac{\lambda}{1+\lambda}}\right)}$$

The corresponding square root found in X. 94 is the *minor* irrational straight line, the terms being separated by a *minus* sign, and the two straight lines are the roots (cf. note on X. 51) of the biquadratic equation

$$x^4 - 2k\rho^2\,.\,x^2 + \frac{\lambda}{1+\lambda}k^2\rho^4 = 0.$$

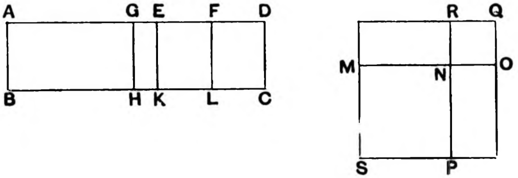
PROPOSITION 58.

If an area be contained by a rational straight line and the fifth binomial, the "side" of the area is the irrational straight line called the side of a rational plus a medial area.

For let the area AC be contained by the rational straight line AB and the fifth binomial AD divided into its terms at E, so that AE is the greater term;

I say that the "side" of the area AC is the irrational straight line called the side of a rational plus a medial area.

For let the same construction be made as before shown; it is then manifest that MO is the "side" of the area AC.



It is then to be proved that MO is the side of a rational plus a medial area.

For, since AG is incommensurable with GE, [x. 18]
therefore AH is also commensurable with HE, [vi. 1, x. 11]
that is, the square on MN with the square on NO;
therefore MN, NO are incommensurable in square.

And, since AD is a fifth binomial straight line, and ED the lesser segment,
therefore ED is commensurable in length with AB. [x. Deff. ii. 5]

But AE is incommensurable with ED;
therefore AB is also incommensurable in length with AE. [x. 13]

Therefore AK, that is, the sum of the squares on MN, NO, is medial. [x. 21]

And, since DE is commensurable in length with AB, that is, with EK,
while DE is commensurable with EF,
therefore EF is also commensurable with EK. [x. 12]

And *EK* is rational;
therefore *EL*, that is, *MR*, that is, the rectangle *MN*, *NO*, is also rational. [x. 19]

Therefore *MN*, *NO* are straight lines incommensurable in square which make the sum of the squares on them medial, but the rectangle contained by them rational.

Therefore *MO* is the side of a rational plus a medial area [x. 40] and is the "side" of the area *AC*.

Q. E. D.

We have here to find the square root of the expression [cf. x. 52]

$$\rho\,(k\rho\sqrt{1+\lambda}+k\rho).$$

As usual, we put

$$\left.\begin{aligned} u+v &= k\rho\sqrt{1+\lambda} \\ uv &= \tfrac{1}{4}k^2\rho^2 \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Then, u, v being found, we take

$$\left.\begin{aligned} x^2 &= \rho u \\ y^2 &= \rho v \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

and $(x+y)$, so found, is our required square root.

Euclid's proof of the *class* of $(x+y)$ is as follows:

By x. 18, $\quad u \smile v$;

therefore $\quad \rho u \smile \rho v$,

so that $\quad x^2 \smile y^2$,

and $\quad x \smile\!\!\!\!-\; y \quad\ldots\ldots\ldots\ldots\ldots\ldots(3)$.

Next $\quad u+v \smile k\rho$

$\quad \smile \rho$,

whence $\quad \rho\,(u+v)$, or (x^2+y^2), is a *medial* area $\ldots\ldots\ldots(4)$.

Lastly, $\quad xy = \frac{1}{2}k\rho^2$, which is a *rational* area $\ldots\ldots\ldots(5)$.

Hence [(3), (4), (5)] $(x+y)$ is the *side of a rational plus a medial area.* [x. 40]

If we solve algebraically, we obtain

$$u = \frac{k\rho}{2}\left(\sqrt{1+\lambda}+\sqrt{\lambda}\right),$$

$$v = \frac{k\rho}{2}\left(\sqrt{1+\lambda}-\sqrt{\lambda}\right),$$

and

$$x+y = \rho\sqrt{\frac{k}{2}\left(\sqrt{1+\lambda}+\sqrt{\lambda}\right)}+\rho\sqrt{\frac{k}{2}\left(\sqrt{1+\lambda}-\sqrt{\lambda}\right)}.$$

The corresponding "side" found in x. 95 is a *straight line which produces with a rational area a medial whole*, being of the form $(x-y)$, where x, y have the same values as above.

The two square roots are (cf. note on x. 52) the roots of the biquadratic equation

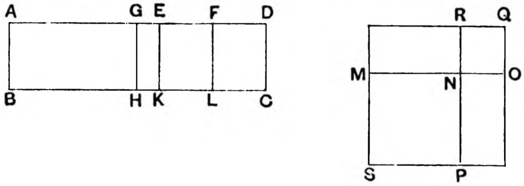
$$x^4 - 2k\rho^2\sqrt{1+\lambda}\,.\,x^2 + \lambda k^2\rho^4 = 0.$$

PROPOSITION 59.

If an area be contained by a rational straight line and the sixth binomial, the "side" of the area is the irrational straight line called the side of the sum of two medial areas.

For let the area *ABCD* be contained by the rational straight line *AB* and the sixth binomial *AD*, divided into its terms at *E*, so that *AE* is the greater term;
I say that the "side" of *AC* is the side of the sum of two medial areas.

Let the same construction be made as before shown.



It is then manifest that *MO* is the "side" of *AC*, and that *MN* is incommensurable in square with *NO*.

Now, since *EA* is incommensurable in length with *AB*, therefore *EA*, *AB* are rational straight lines commensurable in square only;
therefore *AK*, that is, the sum of the squares on *MN*, *NO*, is medial. [x. 21]

Again, since *ED* is incommensurable in length with *AB*, therefore *FE* is also incommensurable with *EK*; [x. 13]
therefore *FE*, *EK* are rational straight lines commensurable in square only;
therefore *EL*, that is, *MR*, that is, the rectangle *MN*, *NO*, is medial. [x. 21]

And, since *AE* is incommensurable with *EF*,
AK is also incommensurable with *EL*. [vi. 1, x. 11]

But *AK* is the sum of the squares on *MN*, *NO*,
and *EL* is the rectangle *MN*, *NO*;
therefore the sum of the squares on *MN*, *NO* is incommensurable with the rectangle *MN*, *NO*.

And each of them is medial, and *MN*, *NO* are incommensurable in square.

Therefore *MO* is the side of the sum of two medial areas [x. 41], and is the "side" of *AC*.

Q. E. D.

Euclid here finds the square root of the expression [cf. x. 53]

$$\rho\,(\sqrt{k}\cdot\rho+\sqrt{\lambda}\cdot\rho).$$

As usual, we solve the equations

$$\left.\begin{aligned}u+v&=\sqrt{k}\cdot\rho\\ uv&=\tfrac{1}{4}\lambda\rho^2\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1);$$

then, u, v being found, we put

$$\left.\begin{aligned}x^2&=\rho u\\ y^2&=\rho v\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

and $(x+y)$ is the square root required.

Euclid proves that $(x+y)$ is the *side of* (*the sum of*) *two medial areas*, as follows.

As in the last two propositions, x, y are proved to be incommensurable in square.

Now $\sqrt{k}\cdot\rho$, ρ are commensurable in square only;

therefore $\rho\,(u+v)$, or (x^2+y^2), is a *medial* area ....(3).

Next, $xy=\frac{1}{2}\sqrt{\lambda}\cdot\rho^2$, which is again a *medial area* ....(4).

Lastly, $\sqrt{k}\cdot\rho \smile \frac{1}{2}\sqrt{\lambda}\cdot\rho$,

so that $\sqrt{k}\cdot\rho^2 \smile \frac{1}{2}\sqrt{\lambda}\cdot\rho^2$;

that is, $(x^2+y^2)\smile xy$ ....(5).

Hence [(3), (4), (5)] $(x+y)$ is the *side of the sum of two medial areas.*

Solving the equations algebraically, we have

$$u=\frac{\rho}{2}\,(\sqrt{k}+\sqrt{k-\lambda}),$$

$$v=\frac{\rho}{2}\,(\sqrt{k}-\sqrt{k-\lambda}),$$

and $$x+y=\rho\sqrt{\tfrac{1}{2}\,(\sqrt{k}+\sqrt{k-\lambda})}+\rho\sqrt{\tfrac{1}{2}\,(\sqrt{k}-\sqrt{k-\lambda})}.$$

The corresponding square root found in x. 96 is $x-y$, where x, y are the same as here.

The two square roots are (cf. note on x. 53) the roots of the biquadratic equation

$$x^4-2\sqrt{k}\cdot\rho^2x^2+(k-\lambda)\,\rho^4=0.$$

[LEMMA.

If a straight line be cut into unequal parts, the squares on the unequal parts are greater than twice the rectangle contained by the unequal parts.

Let *AB* be a straight line, and let it be cut into unequal parts at *C*, and let *AC* be the greater;

I say that the squares on *AC*, *CB* are greater than twice the rectangle *AC*, *CB*.

For let AB be bisected at D.

Since then a straight line has been cut into equal parts at D, and into unequal parts at C,

therefore the rectangle AC, CB together with the square on CD is equal to the square on AD, [II. 5]

so that the rectangle AC, CB is less than the square on AD;

therefore twice the rectangle AC, CB is less than double of the square on AD.

But the squares on AC, CB are double of the squares on AD, DC; [II. 9]

therefore the squares on AC, CB are greater than twice the rectangle AC, CB.

Q. E. D.]

We have already remarked (note on x. 44) that the Lemma here proving that

$$x^2 + y^2 > 2xy$$

can hardly be genuine, since the result is used in x. 44.

PROPOSITION 60.

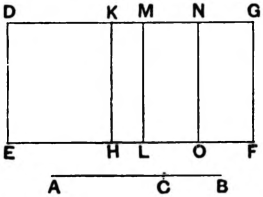
The square on the binomial straight line applied to a rational straight line produces as breadth the first binomial.

Let AB be a binomial straight line divided into its terms at C, so that AC is the greater term;

let a rational straight line DE be set out,

and let $DEFG$ equal to the square on AB be applied to DE producing DG as its breadth;

I say that DG is a first binomial straight line.



For let there be applied to DE the rectangle DH equal to the square on AC, and KL equal to the square on BC;

therefore the remainder, twice the rectangle AC, CB, is equal to MF.

Let MG be bisected at N, and let NO be drawn parallel [to ML or GF].

Therefore each of the rectangles MO, NF is equal to once the rectangle AC, CB.

Now, since AB is a binomial divided into its terms at C,

therefore *AC*, *CB* are rational straight lines commensurable in square only; [x. 36]
therefore the squares on *AC*, *CB* are rational and commensurable with one another,
so that the sum of the squares on *AC*, *CB* is also rational. [x. 15]

And it is equal to *DL*;
therefore *DL* is rational.

And it is applied to the rational straight line *DE*;
therefore *DM* is rational and commensurable in length with *DE*. [x. 20]

Again, since *AC*, *CB* are rational straight lines commensurable in square only,
therefore twice the rectangle *AC*, *CB*, that is *MF*, is medial. [x. 21]

And it is applied to the rational straight line *ML*;
therefore *MG* is also rational and incommensurable in length with *ML*, that is, *DE*. [x. 22]

But *MD* is also rational and is commensurable in length with *DE*;
therefore *DM* is incommensurable in length with *MG*. [x. 13]

And they are rational;
therefore *DM*, *MG* are rational straight lines commensurable in square only;
therefore *DG* is binomial. [x. 36]

It is next to be proved that it is also a first binomial straight line.

Since the rectangle *AC*, *CB* is a mean proportional between the squares on *AC*, *CB*, [cf. Lemma after x. 53]
therefore *MO* is also a mean proportional between *DH*, *KL*.

Therefore, as *DH* is to *MO*, so is *MO* to *KL*,
that is, as *DK* is to *MN*, so is *MN* to *MK*; [vi. 1]
therefore the rectangle *DK*, *KM* is equal to the square on *MN*. [vi. 17]

And, since the square on *AC* is commensurable with the square on *CB*,
DH is also commensurable with *KL*,
so that *DK* is also commensurable with *KM*. [vi. 1, x. 11]

And, since the squares on *AC*, *CB* are greater than twice the rectangle *AC*, *CB*, [Lemma]
therefore *DL* is also greater than *MF*,
so that *DM* is also greater than *MG*. [vi. 1]

And the rectangle *DK*, *KM* is equal to the square on *MN*, that is, to the fourth part of the square on *MG*,
and *DK* is commensurable with *KM*.

But, if there be two unequal straight lines, and to the greater there be applied a parallelogram equal to the fourth part of the square on the less and deficient by a square figure, and if it divide it into commensurable parts, the square on the greater is greater than the square on the less by the square on a straight line commensurable with the greater; [x. 17]
therefore the square on *DM* is greater than the square on *MG* by the square on a straight line commensurable with *DM*.

And *DM*, *MG* are rational,
and *DM*, which is the greater term, is commensurable in length with the rational straight line *DE* set out.

Therefore *DG* is a first binomial straight line. [x. Deff. II. 1]

Q. E. D.

In the hexad of propositions beginning with this we have the solution of the converse problem to that of x. 54—59. We find the *squares* of the irrational straight lines of x. 36—41 and prove that they are respectively equal to the rectangles contained by a rational straight line and the *first*, *second*, *third*, *fourth*, *fifth* and *sixth binomials*.

In x. 60 we prove that, $\rho + \sqrt{k} \cdot \rho$ being a *binomial* straight line [x. 36],

$$\frac{(\rho + \sqrt{k} \cdot \rho)^2}{\sigma}$$

is a *first binomial* straight line, and we find it geometrically.

The procedure may be represented thus.

Take x, y, z such that

$$\sigma x = \rho^2,$$
$$\sigma y = k\rho^2$$
$$\sigma \cdot 2z = 2\sqrt{k} \cdot \rho^2,$$

ρ^2, $k\rho^2$ being of course the squares on the terms of the original binomial, and $2\sqrt{k} \cdot \rho^2$ twice the rectangle contained by them.

Then $$(x+y) + 2z = \frac{(\rho + \sqrt{k} \cdot \rho)^2}{\sigma},$$

and we have to prove that $(x+y) + 2z$ is a *first binomial* straight line of which $(x+y)$, $2z$ are the terms and $(x+y)$ the greater.

Euclid divides the proof into two parts, showing first that $(x+y) + 2z$ is *some* binomial, and secondly that it is the *first* binomial.

(α) $\rho \sim \sqrt{k}\,.\,\rho$, so that ρ^2, $k\rho^2$ are rational and commensurable;
therefore $\rho^2 + k\rho^2$, or $\sigma(x+y)$, is a rational area,
whence $(x+y)$ is rational and $\frown \sigma$ ....(1).

Next, $2\rho\,.\,\sqrt{k}\,.\,\rho$ is a medial area,
so that $\sigma\,.\,2z$ is a medial area,
whence $2z$ is rational but $\smile \sigma$ ....(2).

Hence [(1), (2)] $(x+y)$, $2z$ are rational and commensurable in square only ....(3);
thus $(x+y) + 2z$ is a *binomial* straight line. [x. 36]

(β) $\rho^2 : \sqrt{k}\,.\,\rho^2 = \sqrt{k}\,.\,\rho^2 : k\rho^2$,
so that $\sigma x : \sigma z = \sigma z : \sigma y$,
and $x : z = z : y$,
or $xy = z^2 = \frac{1}{4}(2z)^2$ ....(4).

Now ρ^2, $k\rho^2$ are commensurable, so that σx, σy are commensurable, and therefore

$x \frown y$ ....(5).

And, since [Lemma] $\rho^2 + k\rho^2 > 2\sqrt{k}\,.\,\rho^2$,

$x + y > 2z$.

But $(x+y)$ is given, being equal to $\dfrac{\rho^2 + k\rho^2}{\sigma}$ ....(6).

Therefore [(4), (5), (6), and x. 17] $\sqrt{(x+y)^2 - (2z)^2} \frown (x+y)$.

And $(x+y)$, $2z$ are rational and $\sim$ [(3)],
while $(x+y) \frown \sigma$ [(1)].

Hence $(x+y) + 2z$ is a *first binomial.*

The actual value of $(x+y) + 2z$ is, of course,

$$\frac{\rho^2}{\sigma}(1 + k + 2\sqrt{k}).$$

PROPOSITION 61.

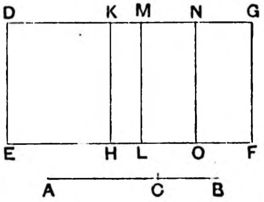
The square on the first bimedial straight line applied to a rational straight line produces as breadth the second binomial.

Let AB be a first bimedial straight line divided into its medials at C, of which medials AC is the greater;

let a rational straight line DE be set out,

and let there be applied to DE the parallelogram DF equal to the square on AB, producing DG as its breadth;

I say that DG is a second binomial straight line.



For let the same construction as before be made.

Then, since *AB* is a first bimedial divided at *C*, therefore *AC*, *CB* are medial straight lines commensurable in square only, and containing a rational rectangle, [x. 37] so that the squares on *AC*, *CB* are also medial. [x. 21]

Therefore *DL* is medial. [x. 15 and 23, Por.]

And it has been applied to the rational straight line *DE*; therefore *MD* is rational and incommensurable in length with *DE*. [x. 22]

Again, since twice the rectangle *AC*, *CB* is rational, *MF* is also rational.

And it is applied to the rational straight line *ML*; therefore *MG* is also rational and commensurable in length with *ML*, that is, *DE*; [x. 20] therefore *DM* is incommensurable in length with *MG*. [x. 13]

And they are rational; therefore *DM*, *MG* are rational straight lines commensurable in square only; therefore *DG* is binomial. [x. 36]

It is next to be proved that it is also a second binomial straight line.

For, since the squares on *AC*, *CB* are greater than twice the rectangle *AC*, *CB*, therefore *DL* is also greater than *MF*, so that *DM* is also greater than *MG*. [vi. 1]

And, since the square on *AC* is commensurable with the square on *CB*, *DH* is also commensurable with *KL*, so that *DK* is also commensurable with *KM*. [vi. 1, x. 11]

And the rectangle *DK*, *KM* is equal to the square on *MN*; therefore the square on *DM* is greater than the square on *MG* by the square on a straight line commensurable with *DM*. [x. 17]

And *MG* is commensurable in length with *DE*.

Therefore *DG* is a second binomial straight line. [x. Deff. ii. 2]

In this case we have to prove that, $(k^{\frac{1}{4}}\rho + k^{\frac{3}{4}}\rho)$ being a *first bimedial* straight line, as found in x. 37,

$$\frac{(k^{\frac{1}{4}}\rho + k^{\frac{3}{4}}\rho)^2}{\sigma}$$

is a *second binomial* straight line.

The form of the proposition and the figure being similar to those of X. 60, I can somewhat abbreviate the reproduction of the proof.

Take x, y, z such that

$$\sigma x = k^{\frac{1}{2}}\rho^2,$$
$$\sigma y = k^{\frac{3}{2}}\rho^3,$$
$$\sigma \,.\, 2z = 2k\rho^2.$$

Then shall $(x + y) + 2z$ be a second binomial.

(α) $k^{\frac{1}{4}}\rho$, $k^{\frac{3}{4}}\rho$ are medial straight lines commensurable in square only and containing a rational rectangle. [X. 37]

The squares $k^{\frac{1}{2}}\rho^2$, $k^{\frac{3}{2}}\rho^4$ are medial;
thus the sum, or $\sigma(x + y)$, is medial. [X. 23, Por.]

Therefore $(x + y)$ is rational and $\smile \sigma$.

And $\sigma \,.\, 2z$ is rational;
therefore $2z$ is rational and $\frown \sigma$ ....(1).

Therefore $(x + y)$, $2z$ are rational and $\frown\!\!\!\!-$ ....(2),
so that $(x + y) + 2z$ is a *binomial.*

(β) As before, $(x + y) > 2z$.

Now, $k^{\frac{1}{2}}\rho^2$, $k^{\frac{3}{2}}\rho^2$ being commensurable,

$$x \frown y.$$

And $$xy = z^2,$$

while $x + y = \dfrac{k^{\frac{1}{2}}\rho^2 + k^{\frac{3}{2}}\rho^2}{\sigma}$.

Hence [X. 17] $\sqrt{(x + y)^2 - (2z)^2} \frown (x + y)$ ....(3).

But $2z \frown \sigma$, by (1).

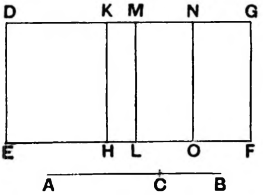
Therefore [(1), (2), (3)] $(x + y) + 2z$ is a *second binomial* straight line.

Of course $(x + y) + 2z = \dfrac{\rho^2}{\sigma}\{\sqrt{k}(1 + k) + 2k\}$.

PROPOSITION 62.

The square on the second bimedial straight line applied to a rational straight line produces as breadth the third binomial.

Let AB be a second bimedial straight line divided into its medials at C, so that AC is the greater segment;



let DE be any rational straight line, and to DE let there be applied the parallelogram DF equal to the square on AB and producing DG as its breadth;

I say that DG is a third binomial straight line.

Let the same construction be made as before shown.

Then, since *AB* is a second bimedial divided at *C*, therefore *AC*, *CB* are medial straight lines commensurable in square only and containing a medial rectangle, [x. 38] so that the sum of the squares on *AC*, *CB* is also medial. [x. 15 and 23 Por.]

And it is equal to *DL*; therefore *DL* is also medial.

And it is applied to the rational straight line *DE*; therefore *MD* is also rational and incommensurable in length with *DE*. [x. 22]

For the same reason, *MG* is also rational and incommensurable in length with *ML*, that is, with *DE*; therefore each of the straight lines *DM*, *MG* is rational and incommensurable in length with *DE*.

And, since *AC* is incommensurable in length with *CB*, and, as *AC* is to *CB*, so is the square on *AC* to the rectangle *AC*, *CB*, therefore the square on *AC* is also incommensurable with the rectangle *AC*, *CB*. [x. 11]

Hence the sum of the squares on *AC*, *CB* is incommensurable with twice the rectangle *AC*, *CB*, [x. 12, 13] that is, *DL* is incommensurable with *MF*, so that *DM* is also incommensurable with *MG*. [vi. 1, x. 11]

And they are rational; therefore *DG* is binomial. [x. 36]

It is to be proved that it is also a third binomial straight line.

In manner similar to the foregoing we may conclude that *DM* is greater than *MG*, and that *DK* is commensurable with *KM*.

And the rectangle *DK*, *KM* is equal to the square on *MN*; therefore the square on *DM* is greater than the square on *MG* by the square on a straight line commensurable with *DM*.

And neither of the straight lines *DM*, *MG* is commensurable in length with *DE*.

Therefore *DG* is a third binomial straight line. [x. Deff. ii. 3]

Q. E. D.

We have to prove that [cf. x. 38]

$$\frac{1}{\sigma}\left(k^{\frac{1}{4}}\rho + \frac{\lambda^{\frac{1}{2}}\rho}{k^{\frac{1}{4}}}\right)^2$$

is a *third binomial* straight line.

Take x, y, z such that

$$\sigma x = k^{\frac{1}{2}}\rho^2,$$
$$\sigma y = \frac{\lambda\rho^2}{k^{\frac{1}{2}}},$$
$$\sigma \,.\, 2z = 2\sqrt{\lambda} \,.\, \rho^2.$$

(α) Now $k^{\frac{1}{4}}\rho$, $\frac{\lambda^{\frac{1}{2}}\rho}{k^{\frac{1}{4}}}$ are medial straight lines commensurable in square only and containing a *medial* rectangle. [x. 38]

The sum of the squares on them, or $\sigma(x+y)$, is *medial*;

therefore $(x+y)$ is rational and $\smile \sigma$ .... (1).

And $\sigma \,.\, 2z$ being medial also,

$2z$ is rational and $\smile \sigma$ ....(2).

Now $$k^{\frac{1}{4}}\rho : \frac{\lambda^{\frac{1}{2}}\rho}{k^{\frac{1}{4}}} = (k^{\frac{1}{4}}\rho)^2 : k^{\frac{1}{4}}\rho \,.\, \frac{\lambda^{\frac{1}{2}}\rho}{k^{\frac{1}{4}}}$$
$$= \sigma x : \sigma z,$$

whence $\sigma x \smile \sigma z$.

But $(k^{\frac{1}{4}}\rho)^2 \frown \left\{(k^{\frac{1}{4}}\rho)^2 + \left(\frac{\lambda^{\frac{1}{2}}\rho}{k^{\frac{1}{4}}}\right)^2\right\}$, or $\sigma x \frown \sigma(x+y)$, and $\sigma z \frown \sigma \,.\, 2z$;

therefore $\sigma(x+y) \smile \sigma \,.\, 2z$,

or $(x+y) \smile 2z$ ....(3).

Hence [(1), (2), (3)] $(x+y) + 2z$ is a binomial straight line............(4).

(β) As before, $(x+y) > 2z$,

and $x \frown y$.

Also $xy = z^2$.

Therefore [x. 17] $\sqrt{(x+y)^2 - (2z)^2} \frown (x+y)$.

And [(1), (2)] neither $(x+y)$ nor $2z$ is $\frown \sigma$.

Therefore $(x+y) + 2z$ is a *third binomial* straight line.

Obviously $$(x+y) + 2z = \frac{\rho^2}{\sigma}\left\{\frac{k+\lambda}{\sqrt{k}} + 2\sqrt{\lambda}\right\}.$$

PROPOSITION 63.

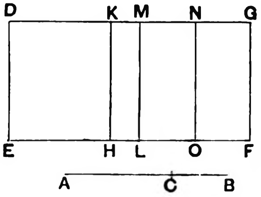
The square on the major straight line applied to a rational straight line produces as breadth the fourth binomial.

Let AB be a major straight line divided at C, so that AC is greater than CB;

let DE be a rational straight line,

and to *DE* let there be applied the parallelogram *DF* equal to the square on *AB* and producing *DG* as its breadth;

I say that *DG* is a fourth binomial straight line.



Let the same construction be made as before shown.

Then, since *AB* is a major straight line divided at *C*,

AC, *CB* are straight lines incommensurable in square which make the sum of the squares on them rational, but the rectangle contained by them medial. [x. 39]

Since then the sum of the squares on *AC*, *CB* is rational, therefore *DL* is rational;

therefore *DM* is also rational and commensurable in length with *DE*. [x. 20]

Again, since twice the rectangle *AC*, *CB*, that is, *MF*, is medial,

and it is applied to the rational straight line *ML*,

therefore *MG* is also rational and incommensurable in length with *DE*; [x. 22]

therefore *DM* is also incommensurable in length with *MG*. [x. 13]

Therefore *DM*, *MG* are rational straight lines commensurable in square only;

therefore *DG* is binomial. [x. 36]

It is to be proved that it is also a fourth binomial straight line.

In manner similar to the foregoing we can prove that *DM* is greater than *MG*,

and that the rectangle *DK*, *KM* is equal to the square on *MN*.

Since then the square on *AC* is incommensurable with the square on *CB*,

therefore *DH* is also incommensurable with *KL*,

so that *DK* is also incommensurable with *KM*. [vi. 1, x. 11]

But, if there be two unequal straight lines, and to the greater there be applied a parallelogram equal to the fourth part of the square on the less and deficient by a square figure, and if it divide it into incommensurable parts, then the square on the greater will be greater than the square on the less by the square on a straight line incommensurable in length with the greater; [x. 18]

therefore the square on *DM* is greater than the square on *MG* by the square on a straight line incommensurable with *DM*.

And *DM*, *MG* are rational straight lines commensurable in square only,

and *DM* is commensurable with the rational straight line *DE* set out.

Therefore *DG* is a fourth binomial straight line. [x. Deff. II. 4]

Q. E. D.

We have to prove that [cf. x. 39]

$$\frac{1}{\sigma}\left\{\frac{\rho}{\sqrt{2}}\sqrt{1+\frac{k}{\sqrt{1+k^2}}}+\frac{\rho}{\sqrt{2}}\sqrt{1-\frac{k}{\sqrt{1+k^2}}}\right\}^2$$

is a *fourth binomial* straight line.

For brevity we must call this expression

$$\frac{1}{\sigma}(u+v)^2.$$

Take x, y, z such that

$$\left.\begin{aligned}\sigma x &= u^2\\ \sigma y &= v^2\\ \sigma \,.\, 2z &= 2uv\end{aligned}\right\},$$

wherein it has to be remembered [x. 39] that u, v are incommensurable in square, (u^2+v^2) is rational, and uv is medial.

(α) (u^2+v^2), and therefore $\sigma(x+y)$, is rational;

therefore $(x+y)$ is rational and $\frown \sigma$ ....(1).

$2uv$, and therefore $\sigma \,.\, 2z$, is medial;

therefore $2z$ is rational and $\smile \sigma$ ....(2).

Thus $(x+y)$, $2z$ are rational and $\frown\!\!-$ ....(3),

so that $(x+y)+2z$ is a binomial straight line.

(β) As before, $x+y > 2z$,

and $xy = z^2$.

Now, since $u^2 \smile v^2$,

$\sigma x \smile \sigma y$, or $x \smile y$.

Hence [x. 18] $\sqrt{(x+y)^2-(2z)^2} \smile (x+y)$....................(4).

And $(x+y) \frown \sigma$, by (1).

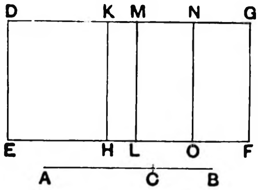
Therefore [(3), (4)] $(x+y)+2z$ is a *fourth binomial* straight line.

It is of course $\dfrac{\rho^2}{\sigma}\left\{1+\dfrac{1}{\sqrt{1+k^2}}\right\}$.

PROPOSITION 64.

The square on the side of a rational plus a medial area applied to a rational straight line produces as breadth the fifth binomial.

Let *AB* be the side of a rational plus a medial area, divided into its straight lines at *C*, so that *AC* is the greater; let a rational straight line *DE* be set out, and let there be applied to *DE* the parallelogram *DF* equal to the square on *AB*, producing *DG* as its breadth; I say that *DG* is a fifth binomial straight line.



Let the same construction as before be made.

Since then *AB* is the side of a rational plus a medial area, divided at *C*,

therefore *AC*, *CB* are straight lines incommensurable in square which make the sum of the squares on them medial, but the rectangle contained by them rational. [x. 40]

Since then the sum of the squares on *AC*, *CB* is medial,

therefore *DL* is medial,

so that *DM* is rational and incommensurable in length with *DE*. [x. 22]

Again, since twice the rectangle *AC*, *CB*, that is *MF*, is rational,

therefore *MG* is rational and commensurable with *DE*. [x. 20]

Therefore *DM* is incommensurable with *MG*; [x. 13]

therefore *DM*, *MG* are rational straight lines commensurable in square only;

therefore *DG* is binomial. [x. 36]

I say next that it is also a fifth binomial straight line.

For it can be proved similarly that the rectangle *DK*, *KM* is equal to the square on *MN*,

and that *DK* is incommensurable in length with *KM*;

therefore the square on *DM* is greater than the square on *MG* by the square on a straight line incommensurable with *DM*. [x. 18]

And *DM*, *MG* are commensurable in square only, and the less, *MG*, is commensurable in length with *DE*.

Therefore *DG* is a fifth binomial.

Q. E. D.

To prove that [cf. x. 40]

$$\frac{1}{\sigma}\left\{\frac{\rho}{\sqrt{2(1+k^2)}}\sqrt{\sqrt{1+k^2}+k}+\frac{\rho}{\sqrt{2(1+k^2)}}\sqrt{\sqrt{1+k^2}-k}\right\}^2$$

is a *fifth binomial* straight line.

For brevity denote it by $\frac{1}{\sigma}(u+v)^2$, and put

$$\sigma x = u^2,$$
$$\sigma y = v^2,$$
$$\sigma \,.\, 2z = 2uv.$$

Remembering that [x. 40] $u^2 \smile v^2$, (u^2+v^2) is medial, and $2uv$ is rational, we proceed thus.

(α) $\sigma(x+y)$ is medial;

therefore $(x+y)$ is rational and $\smile \sigma$ ....(1).

Next, $\sigma \,.\, 2z$ is rational;

therefore $2z$ is rational and $\frown \sigma$..........................(2).

Thus $(x+y)$, $2z$ are rational and $\frown\!\!-$ ....(3),

so that $(x+y)+2z$ is a binomial straight line.

(β) As before, $x+y>2z$,

$$xy = z^2,$$

and $x \smile y$.

Therefore [x. 18] $\sqrt{(x+y)^2-(2z)^2} \smile (x+y)$ ....(4).

Hence [(2), (3), (4)] $(x+y)+2z$ is a *fifth binomial* straight line.

It is of course $\frac{\rho^2}{\sigma}\left\{\frac{1}{\sqrt{1+k^2}}+\frac{1}{1+k^2}\right\}$.

PROPOSITION 65.

The square on the side of the sum of two medial areas applied to a rational straight line produces as breadth the sixth binomial.

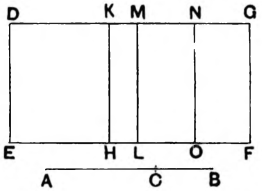
Let *AB* be the side of the sum of two medial areas, divided at *C*,

let *DE* be a rational straight line,

and let there be applied to *DE* the parallelogram *DF* equal to the square on *AB*, producing *DG* as its breadth;

I say that *DG* is a sixth binomial straight line.

For let the same construction be made as before.



Then, since *AB* is the side of the sum of two medial areas, divided at *C*,

therefore *AC*, *CB* are straight lines incommensurable in square which make the sum of the squares on them medial, the rectangle contained by them medial, and moreover the sum of the squares on them incommensurable with the rectangle contained by them, [X. 41]

so that, in accordance with what was before proved, each of the rectangles *DL*, *MF* is medial.

And they are applied to the rational straight line *DE*;

therefore each of the straight lines *DM*, *MG* is rational and incommensurable in length with *DE*. [X. 22]

And, since the sum of the squares on *AC*, *CB* is incommensurable with twice the rectangle *AC*, *CB*,

therefore *DL* is incommensurable with *MF*.

Therefore *DM* is also incommensurable with *MG*; [VI. 1, X. 11]

therefore *DM*, *MG* are rational straight lines commensurable in square only;

therefore *DG* is binomial. [X. 36]

I say next that it is also a sixth binomial straight line.

Similarly again we can prove that the rectangle *DK*, *KM* is equal to the square on *MN*,

and that *DK* is incommensurable in length with *KM*;

and, for the same reason, the square on *DM* is greater than the square on *MG* by the square on a straight line incommensurable in length with *DM*.

And neither of the straight lines *DM*, *MG* is commensurable in length with the rational straight line *DE* set out.

Therefore *DG* is a sixth binomial straight line.

Q. E. D.

To prove that [cf. x. 41]

$$\frac{1}{\sigma}\left\{\frac{\rho\lambda^{\frac{1}{4}}}{\sqrt{2}}\sqrt{1+\frac{k}{\sqrt{1+k^2}}}+\frac{\rho\lambda^{\frac{1}{4}}}{\sqrt{2}}\sqrt{1-\frac{k}{\sqrt{1+k^2}}}\right\}^2$$

is a ***sixth binomial*** straight line.

Denote it by $\frac{1}{\sigma}(u+v)^2$, and put

$$\sigma x = u^2,$$
$$\sigma y = v^2,$$
$$\sigma \,.\, 2z = 2uv.$$

Now, by x. 41, $u^2 \smile v^2$, (u^2+v^2) is medial, $2uv$ is medial, and $(u^2+v^2) \smile 2uv$.

(α) In this case $\sigma(x+y)$ is medial;

therefore $(x+y)$ is rational and $\smile \sigma$ ....(1).

In like manner, $2z$ is rational and $\smile \sigma$ ....(2).

And, since $\sigma(x+y) \smile \sigma \,.\, 2z$,

$(x+y) \smile 2z$ ....(3).

Therefore $(x+y)+2z$ is a binomial straight line.

(β) As before,

$$x+y > 2z,$$
$$xy = z^2,$$
$$x \smile y;$$

therefore [x. 18] $\sqrt{(x+y)^2-(2z)^2} \smile (x+y)$ ....(4).

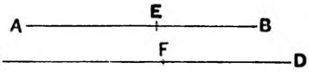
Hence [(1), (2), (3), (4)] $(x+y)+2z$ is a *sixth binomial* straight line.

It is obviously $\frac{\rho^2}{\sigma}\left\{\sqrt{\lambda}+\frac{\sqrt{\lambda}}{\sqrt{1+k^2}}\right\}$.

PROPOSITION 66.

A straight line commensurable in length with a binomial straight line is itself also binomial and the same in order.

Let AB be binomial, and let CD be commensurable in length with AB;



I say that CD is binomial and the same in order with AB.

For, since AB is binomial,
let it be divided into its terms at E,
and let AE be the greater term;

therefore *AE*, *EB* are rational straight lines commensurable in square only. [x. 36]

Let it be contrived that,

as *AB* is to *CD*, so is *AE* to *CF*; [vi. 12]

therefore also the remainder *EB* is to the remainder *FD* as *AB* is to *CD*. [v. 19]

But *AB* is commensurable in length with *CD*;

therefore *AE* is also commensurable with *CF*, and *EB* with *FD*. [x. 11]

And *AE*, *EB* are rational;

therefore *CF*, *FD* are also rational.

And, as *AE* is to *CF*, so is *EB* to *FD*. [v. 11]

Therefore, alternately, as *AE* is to *EB*, so is *CF* to *FD*. [v. 16]

But *AE*, *EB* are commensurable in square only;

therefore *CF*, *FD* are also commensurable in square only. [x. 11]

And they are rational;

therefore *CD* is binomial. [x. 36]

I say next that it is the same in order with *AB*.

For the square on *AE* is greater than the square on *EB* either by the square on a straight line commensurable with *AE* or by the square on a straight line incommensurable with it.

If then the square on *AE* is greater than the square on *EB* by the square on a straight line commensurable with *AE*, the square on *CF* will also be greater than the square on *FD* by the square on a straight line commensurable with *CF*. [x. 14]

And, if *AE* is commensurable with the rational straight line set out, *CF* will also be commensurable with it, [x. 12]

and for this reason each of the straight lines *AB*, *CD* is a first binomial, that is, the same in order. [x. Deff. ii. 1]

But, if *EB* is commensurable with the rational straight line set out, *FD* is also commensurable with it, [x. 12]

and for this reason again *CD* will be the same in order with *AB*,

for each of them will be a second binomial. [x. Deff. ii. 2]

But, if neither of the straight lines *AE*, *EB* is commensurable with the rational straight line set out, neither of the straight lines *CF*, *FD* will be commensurable with it, [x. 13]
and each of the straight lines *AB*, *CD* is a third binomial. [x. Deff. II. 3]

But, if the square on *AE* is greater than the square on *EB* by the square on a straight line incommensurable with *AE*,
the square on *CF* is also greater than the square on *FD* by the square on a straight line incommensurable with *CF*. [x. 14]

And, if *AE* is commensurable with the rational straight line set out, *CF* is also commensurable with it,
and each of the straight lines *AB*, *CD* is a fourth binomial. [x. Deff. II. 4]

But, if *EB* is so commensurable, so is *FD* also,
and each of the straight lines *AB*, *CD* will be a fifth binomial. [x. Deff. II. 5]

But, if neither of the straight lines *AE*, *EB* is so commensurable, neither of the straight lines *CF*, *FD* is commensurable with the rational straight line set out,
and each of the straight lines *AB*, *CD* will be a sixth binomial. [x. Deff. II. 6]

Hence a straight line commensurable in length with a binomial straight line is binomial and the same in order.

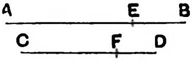
Q. E. D.

The proofs of this and the following propositions up to x. 70 inclusive are easy and require no elucidation. They are equivalent to saying that, if in each of the preceding irrational straight lines $\frac{m}{n}\rho$ is substituted for ρ, the resulting irrational is of the same kind as that from which it is altered.

Proposition 67.

A straight line commensurable in length with a bimedial straight line is itself also bimedial and the same in order.

Let *AB* be bimedial, and let *CD* be commensurable in length with *AB*;



I say that *CD* is bimedial and the same in order with *AB*.

For, since *AB* is bimedial,
let it be divided into its medials at *E*;

therefore *AE*, *EB* are medial straight lines commensurable in square only. [x. 37, 38]

And let it be contrived that,

as *AB* is to *CD*, so is *AE* to *CF*;

therefore also the remainder *EB* is to the remainder *FD* as *AB* is to *CD*. [v. 19]

But *AB* is commensurable in length with *CD*;

therefore *AE*, *EB* are also commensurable with *CF*, *FD* respectively. [x. 11]

But *AE*, *EB* are medial;

therefore *CF*, *FD* are also medial. [x. 23]

And since, as *AE* is to *EB*, so is *CF* to *FD*, [v. 11]

and *AE*, *EB* are commensurable in square only,

CF, *FD* are also commensurable in square only. [x. 11]

But they were also proved medial;

therefore *CD* is bimedial.

I say next that it is also the same in order with *AB*.

For since, as *AE* is to *EB*, so is *CF* to *FD*,

therefore also, as the square on *AE* is to the rectangle *AE*, *EB*, so is the square on *CF* to the rectangle *CF*, *FD*;

therefore, alternately,

as the square on *AE* is to the square on *CF*, so is the rectangle *AE*, *EB* to the rectangle *CF*, *FD*. [v. 16]

But the square on *AE* is commensurable with the square on *CF*;

therefore the rectangle *AE*, *EB* is also commensurable with the rectangle *CF*, *FD*.

If therefore the rectangle *AE*, *EB* is rational,

the rectangle *CF*, *FD* is also rational,

[and for this reason *CD* is a first bimedial]; [x. 37]

but if medial, medial, [x. 23, Por.]

and each of the straight lines *AB*, *CD* is a second bimedial. [x. 38]

And for this reason *CD* will be the same in order with *AB*.

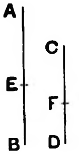
Q. E. D.

Proposition 68.

A straight line commensurable with a major straight line is itself also major.

Let AB be major, and let CD be commensurable with AB; I say that CD is major.

Let AB be divided at E;



therefore AE, EB are straight lines incommensurable in square which make the sum of the squares on them rational, but the rectangle contained by them medial. [x. 39]

Let the same construction be made as before.

Then since, as AB is to CD, so is AE to CF, and EB to FD,

therefore also, as AE is to CF, so is EB to FD. [v. 11]

But AB is commensurable with CD;

therefore AE, EB are also commensurable with CF, FD respectively. [x. 11]

And since, as AE is to CF, so is EB to FD,

alternately also,

as AE is to EB, so is CF to FD; [v. 16]

therefore also, *componendo*,

as AB is to BE, so is CD to DF; [v. 18]

therefore also, as the square on AB is to the square on BE, so is the square on CD to the square on DF. [vi. 20]

Similarly we can prove that, as the square on AB is to the square on AE, so also is the square on CD to the square on CF.

Therefore also, as the square on AB is to the squares on AE, EB, so is the square on CD to the squares on CF. FD;

therefore also, alternately,

as the square on AB is to the square on CD, so are the squares on AE, EB to the squares on CF, FD. [v. 16]

But the square on AB is commensurable with the square on CD;

therefore the squares on AE, EB are also commensurable with the squares on CF, FD.

And the squares on *AE*, *EB* together are rational; therefore the squares on *CF*, *FD* together are rational.

Similarly also twice the rectangle *AE*, *EB* is commensurable with twice the rectangle *CF*, *FD*.

And twice the rectangle *AE*, *EB* is medial; therefore twice the rectangle *CF*, *FD* is also medial. [x. 23, Por.]

Therefore *CF*, *FD* are straight lines incommensurable in square which make, at the same time, the sum of the squares on them rational, but the rectangle contained by them medial; therefore the whole *CD* is the irrational straight line called major. [x. 39]

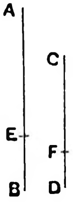
Therefore a straight line commensurable with the major straight line is major.

Q. E. D.

PROPOSITION 69.

A straight line commensurable with the side of a rational plus a medial area is itself also the side of a rational plus a medial area.

Let *AB* be the side of a rational plus a medial area, and let *CD* be commensurable with *AB*; it is to be proved that *CD* is also the side of a rational plus a medial area.



Let *AB* be divided into its straight lines at *E*; therefore *AE*, *EB* are straight lines incommensurable in square which make the sum of the squares on them medial, but the rectangle contained by them rational. [x. 40]

Let the same construction be made as before.

We can then prove similarly that

CF, *FD* are incommensurable in square,

and the sum of the squares on *AE*, *EB* is commensurable with the sum of the squares on *CF*, *FD*,

and the rectangle *AE*, *EB* with the rectangle *CF*, *FD*;

so that the sum of the squares on *CF*, *FD* is also medial, and the rectangle *CF*, *FD* rational.

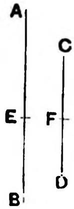
Therefore *CD* is the side of a rational plus a medial area.

Q. E. D.

PROPOSITION 70.

A straight line commensurable with the side of the sum of two medial areas is the side of the sum of two medial areas.

Let AB be the side of the sum of two medial areas, and CD commensurable with AB;



it is to be proved that CD is also the side of the sum of two medial areas.

For, since AB is the side of the sum of two medial areas,

let it be divided into its straight lines at E;

therefore AE, EB are straight lines incommensurable in square which make the sum of the squares on them medial, the rectangle contained by them medial, and furthermore the sum of the squares on AE, EB incommensurable with the rectangle AE, EB. [x. 41]

Let the same construction be made as before.

We can then prove similarly that

CF, FD are also incommensurable in square,

the sum of the squares on AE, EB is commensurable with the sum of the squares on CF, FD,

and the rectangle AE, EB with the rectangle CF, FD;

so that the sum of the squares on CF, FD is also medial,

the rectangle CF, FD is medial,

and moreover the sum of the squares on CF, FD is incommensurable with the rectangle CF, FD.

Therefore CD is the side of the sum of two medial areas.

Q. E. D.

PROPOSITION 71.

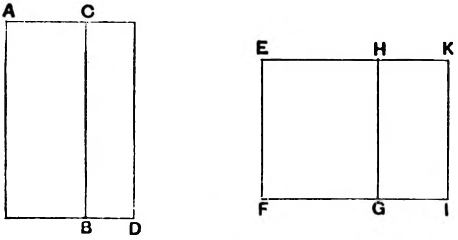
If a rational and a medial area be added together, four irrational straight lines arise, namely a binomial or a first bimedial or a major or a side of a rational plus a medial area.

Let AB be rational, and CD medial;

I say that the "side" of the area AD is a binomial or a first bimedial or a major or a side of a rational plus a medial area.

For *AB* is either greater or less than *CD*.

First, let it be greater;
let a rational straight line *EF* be set out,
let there be applied to *EF* the rectangle *EG* equal to *AB*, producing *EH* as breadth,
and let *HI*, equal to *DC*, be applied to *EF*, producing *HK* as breadth.



Then, since *AB* is rational and is equal to *EG*,
therefore *EG* is also rational.

And it has been applied to *EF*, producing *EH* as breadth;
therefore *EH* is rational and commensurable in length with *EF*. [x. 20]

Again, since *CD* is medial and is equal to *HI*,
therefore *HI* is also medial.

And it is applied to the rational straight line *EF*, producing *HK* as breadth;
therefore *HK* is rational and incommensurable in length with *EF* [x. 22]

And, since *CD* is medial,
while *AB* is rational,
therefore *AB* is incommensurable with *CD*,
so that *EG* is also incommensurable with *HI*.

But, as *EG* is to *HI*, so is *EH* to *HK*; [vi. 1]
therefore *EH* is also incommensurable in length with *HK*. [x. 11]

And both are rational;
therefore *EH*, *HK* are rational straight lines commensurable in square only;
therefore *EK* is a binomial straight line, divided at *H*. [x. 36]

And, since *AB* is greater than *CD*,
while *AB* is equal to *EG* and *CD* to *HI*,
therefore *EG* is also greater than *HI*;
therefore *EH* is also greater than *HK*.

The square, then, on *EH* is greater than the square on *HK* either by the square on a straight line commensurable in length with *EH* or by the square on a straight line incommensurable with it.

First, let the square on it be greater by the square on a straight line commensurable with itself.

Now the greater straight line *HE* is commensurable in length with the rational straight line *EF* set out;
therefore *EK* is a first binomial. [x. Deff. II. 1]

But *EF* is rational;
and, if an area be contained by a rational straight line and the first binomial, the side of the square equal to the area is binomial. [x. 54]

Therefore the "side" of *EI* is binomial;
so that the "side" of *AD* is also binomial.

Next, let the square on *EH* be greater than the square on *HK* by the square on a straight line incommensurable with *EH*.

Now the greater straight line *EH* is commensurable in length with the rational straight line *EF* set out;
therefore *EK* is a fourth binomial. [x. Deff. II. 4]

But *EF* is rational;
and, if an area be contained by a rational straight line and the fourth binomial, the "side" of the area is the irrational straight line called major. [x. 57]

Therefore the "side" of the area *EI* is major;
so that the "side" of the area *AD* is also major.

Next, let *AB* be less than *CD*;
therefore *EG* is also less than *HI*,
so that *EH* is also less than *HK*.

Now the square on *HK* is greater than the square on *EH* either by the square on a straight line commensurable with *HK* or by the square on a straight line incommensurable with it.

First, let the square on it be greater by the square on a straight line commensurable in length with itself.

Now the lesser straight line *EH* is commensurable in length with the rational straight line *EF* set out;
therefore *EK* is a second binomial. [x. Deff. II. 2]

But *EF* is rational,
and, if an area be contained by a rational straight line and the second binomial, the side of the square equal to it is a first bimedial; [x. 55]
therefore the "side" of the area *EI* is a first bimedial,
so that the "side" of *AD* is also a first bimedial.

Next, let the square on *HK* be greater than the square on *HE* by the square on a straight line incommensurable with *HK*.

Now the lesser straight line *EH* is commensurable with the rational straight line *EF* set out;
therefore *EK* is a fifth binomial. [x. Deff. II. 5]

But *EF* is rational;
and, if an area be contained by a rational straight line and the fifth binomial, the side of the square equal to the area is a side of a rational plus a medial area. [x. 58]

Therefore the "side" of the area *EI* is a side of a rational plus a medial area,
so that the "side" of the area *AD* is also a side of a rational plus a medial area.

Therefore etc. Q. E. D.

A *rational* area being of the form $k\rho^2$, and a *medial* area of the form $\sqrt{\lambda} \cdot \rho^2$, the problem is to classify

$$\sqrt{k\rho^2 + \sqrt{\lambda} \cdot \rho^2}$$

according to the different possible relations between k, λ.

Put
$$\sigma u = k\rho^2,$$
$$\sigma v = \sqrt{\lambda} \cdot \rho^2.$$

Then, since the former rectangle is rational, the latter medial,

u is rational and $\frown \sigma$,

v is rational and $\smile \sigma$.

Also the rectangles are incommensurable;
so that
$$u \smile v.$$

Hence u, v are rational and $\frown\!\!-$;
whence $(u + v)$ is a binomial straight line.

The possibilities now are as follows:

I. $u > v$.

Then either

(1) $\sqrt{u^2 - v^2} \frown u$,

or (2) $\sqrt{u^2 - v^2} \smile u$,

while in both cases $u \frown \sigma$.

In case (1) $(u + v)$ is a *first binomial* straight line,
and in case (2) $(u + v)$ is a *fourth binomial* straight line.

Thus $\sqrt{\sigma}\,(u + v)$ is either (1) a *binomial* straight line [x. 54] or (2) a *major* irrational straight line [x. 57].

II. $v > u$.

Then either

(1) $\sqrt{v^2 - u^2} \frown v$,

or (2) $\sqrt{v^2 - u^2} \smile v$,

while in both cases $v \smile \sigma$, but $u \frown \sigma$.

Hence, in case (1), $(v + u)$ is a *second binomial* straight line,
and, in case (2), $(v + u)$ is a *fifth binomial* straight line.

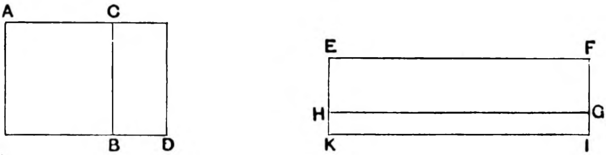
Thus $\sqrt{\sigma}\,(v + u)$ is either (1) a *first bimedial* straight line [x. 55], or (2) a *side of a rational plus a medial area* [x. 58].

PROPOSITION 72.

If two medial areas incommensurable with one another be added together, the remaining two irrational straight lines arise, namely either a second bimedial or a side of the sum of two medial areas.

For let two medial areas *AB*, *CD* incommensurable with one another be added together;

I say that the "side" of the area *AD* is either a second bimedial or a side of the sum of two medial areas.



For *AB* is either greater or less than *CD*.

First, if it so chance, let *AB* be greater than *CD*.

Let the rational straight line *EF* be set out,
and to *EF* let there be applied the rectangle *EG* equal to

AB and producing EH as breadth, and the rectangle HI equal to CD and producing HK as breadth.

Now, since each of the areas AB, CD is medial,
therefore each of the areas EG, HI is also medial.

And they are applied to the rational straight line FE, producing EH, HK as breadth;
therefore each of the straight lines EH, HK is rational and incommensurable in length with EF. [x. 22]

And, since AB is incommensurable with CD,
and AB is equal to EG, and CD to HI,
therefore EG is also incommensurable with HI.

But, as EG is to HI, so is EH to HK; [vi. 1]
therefore EH is incommensurable in length with HK. [x. 11]

Therefore EH, HK are rational straight lines commensurable in square only;
therefore EK is binomial. [x. 36]

But the square on EH is greater than the square on HK either by the square on a straight line commensurable with EH or by the square on a straight line incommensurable with it.

First, let the square on it be greater by the square on a straight line commensurable in length with itself.

Now neither of the straight lines EH, HK is commensurable in length with the rational straight line EF set out;
therefore EK is a third binomial. [x. Deff. ii. 3]

But EF is rational;
and, if an area be contained by a rational straight line and the third binomial, the "side" of the area is a second bimedial; [x. 56]
therefore the "side" of EI, that is, of AD, is a second bimedial.

Next, let the square on EH be greater than the square on HK by the square on a straight line incommensurable in length with EH.

Now each of the straight lines EH, HK is incommensurable in length with EF;
therefore EK is a sixth binomial. [x. Deff. ii. 6]

But, if an area be contained by a rational straight line and the sixth binomial, the "side" of the area is the side of the sum of two medial areas; [X. 59]

so that the "side" of the area AD is also the side of the sum of two medial areas.

Therefore etc.

Q. E. D.

We have to classify, according to the different possible relations between k, λ, the straight line

$$\sqrt{\sqrt{k} \cdot \rho^2 + \sqrt{\lambda} \cdot \rho^2},$$

where $\sqrt{k} \cdot \rho^2$ and $\sqrt{\lambda} \cdot \rho^2$ are incommensurable.

Suppose that
$$\sigma u = \sqrt{k} \cdot \rho^2,$$
$$\sigma v = \sqrt{\lambda} \cdot \rho^2.$$

It is immaterial whether $\sqrt{k} \cdot \rho^2$ or $\sqrt{\lambda} \cdot \rho^2$ is the greater. Suppose, e.g., that the former is.

Now, $\sqrt{k} \cdot \rho^2$, $\sqrt{\lambda} \cdot \rho^2$ being both *medial* areas, and σ rational,

u, v are both rational and $\smile \sigma$ ....(1).

Again, by hypothesis, $\sigma u \smile \sigma v$,

or $u \smile v$ ....(2).

Hence [(1), (2)] $(u + v)$ is a binomial straight line.

Next, $\sqrt{u^2 - v^2}$ is either commensurable or incommensurable in length with u.

(α) Suppose $\sqrt{u^2 - v^2} \frown u$.

In this case $(u + v)$ is a *third binomial* straight line, and therefore [X. 56]

$\sqrt{\sigma(u + v)}$ is a *second bimedial* straight line.

(β) If $\sqrt{u^2 - v^2} \smile u$,

$(u + v)$ is a *sixth binomial* straight line, and therefore [X. 59]

$\sqrt{\sigma(u + v)}$ is a *side of the sum of two medial areas.*

The binomial straight line and the irrational straight lines after it are neither the same with the medial nor with one another.

For the square on a medial, if applied to a rational straight line, produces as breadth a straight line rational and incommensurable in length with that to which it is applied. [X. 22]

But the square on the binomial, if applied to a rational straight line, produces as breadth the first binomial. [X. 60]

The square on the first bimedial, if applied to a rational straight line, produces as breadth the second binomial. [X. 61]

The square on the second bimedial, if applied to a rational straight line, produces as breadth the third binomial. [x. 62]

The square on the major, if applied to a rational straight line, produces as breadth the fourth binomial. [x. 63]

The square on the side of a rational plus a medial area, if applied to a rational straight line, produces as breadth the fifth binomial. [x. 64]

The square on the side of the sum of two medial areas, if applied to a rational straight line, produces as breadth the sixth binomial. [x. 65]

And the said breadths differ both from the first and from one another: from the first because it is rational, and from one another because they are not the same in order;

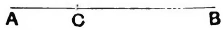
so that the irrational straight lines themselves also differ from one another.

The explanation after x. 72 is for the purpose of showing that all the irrational straight lines treated hitherto are different from one another, viz. the medial, the six irrational straight lines beginning with the binomial, and the six consisting of the first, second, third, fourth, fifth and sixth binomials.

PROPOSITION 73.

If from a rational straight line there be subtracted a rational straight line commensurable with the whole in square only, the remainder is irrational; and let it be called an **apotome.**

For from the rational straight line AB let the rational straight line BC, commensurable with the whole in square only, be subtracted;



I say that the remainder AC is the irrational straight line called **apotome.**

For, since AB is incommensurable in length with BC,

and, as AB is to BC, so is the square on AB to the rectangle AB, BC,

therefore the square on AB is incommensurable with the rectangle AB, BC. [x. 11]

But the squares on AB, BC are commensurable with the square on AB, [x. 15]

and twice the rectangle AB, BC is commensurable with the rectangle AB, BC. [x. 6]

And, inasmuch as the squares on AB, BC are equal to twice the rectangle AB, BC together with the square on CA, [II. 7]

therefore the squares on AB, BC are also incommensurable with the remainder, the square on AC. [X. 13, 16]

But the squares on AB, BC are rational;

therefore AC is irrational. [X. Def. 4]

And let it be called an **apotome**.

Q. E. D.

Euclid now passes to the irrational straight lines which are the *difference* and not, as before, the sum of two straight lines. *Apotome* ("portion cut off") accordingly takes the place of *binomial* and the other terms follow *mutatis mutandis*. The first hexad of propositions (73 to 78) exhibit the six irrational straight lines which are really the result of extracting the *square root* of the six irrationals in the later propositions 85 to 90 (or, strictly speaking, of finding the sides of squares equal to the rectangles formed by each of those six irrational straight lines respectively with a rational straight line). Thus, just as in the corresponding propositions about the irrational straight lines formed by addition, the further removed irrationals, so to speak, come first.

We shall denote the *apotome* etc. by $(x-y)$, which is formed by subtracting a certain lesser straight line y from a greater x. In X. 79 and later propositions y is called by Euclid the *annex* (ἡ προσαρμόζουσα), being the straight line which, when added to the apotome or other irrational formed by subtraction, makes up the greater x.

The methods of proof are exactly the same as in the preceding propositions about the irrational straight lines formed by *addition*.

In this proposition x, y are rational straight lines commensurable in square only, and we have to prove that $(x-y)$, the *apotome*, is irrational.

$x \frown\!\!- y$, so that $x \smile y$:

therefore, since $\qquad x : y = x^2 : xy,$

$$x^2 \smile xy.$$

But $x^2 \frown (x^2+y^2)$, and $xy \frown 2xy$;

therefore $\qquad x^2+y^2 \smile 2xy,$

whence $\qquad (x-y)^2 \smile (x^2+y^2).$

But (x^2+y^2) is rational,

therefore $(x-y)^2$, and consequently $(x-y)$, is irrational.

The *apotome* $(x-y)$ is of the form $\rho \sim \sqrt{k} \cdot \rho$, just as the binomial straight line is of the form $\rho + \sqrt{k} \cdot \rho$.

PROPOSITION 74.

If from a medial straight line there be subtracted a medial straight line which is commensurable with the whole in square only, and which contains with the whole a rational rectangle, the remainder is irrational. And let it be called a **first apotome of a medial** *straight line.*

For from the medial straight line AB let there be subtracted the medial straight line BC which is commensurable with AB in square only and with AB makes the rectangle AB, BC rational;

I say that the remainder AC is irrational; and let it be called a **first apotome of a medial** straight line.

For, since AB, BC are medial,

the squares on AB, BC are also medial.

But twice the rectangle AB, BC is rational;

therefore the squares on AB, BC are incommensurable with twice the rectangle AB, BC;

therefore twice the rectangle AB, BC is also incommensurable with the remainder, the square on AC, [cf. II. 7]

since, if the whole is incommensurable with one of the magnitudes, the original magnitudes will also be incommensurable. [X. 16]

But twice the rectangle AB, BC is rational;

therefore the square on AC is irrational;

therefore AC is irrational. [X. Def. 4]

And let it be called a **first apotome of a medial** straight line.

The *first apotome of a medial* straight line is the difference between straight lines of the form $k^{\frac{1}{4}}\rho$, $k^{\frac{3}{4}}\rho$, which are medial straight lines commensurable in square only and forming a rational rectangle.

By hypothesis, x^2, y^2 are *medial* areas.

And, since xy is rational, $(x^2 + y^2) \smile xy$

$\smile 2xy$,

whence $(x - y)^2 \smile 2xy$.

But $2xy$ is rational;

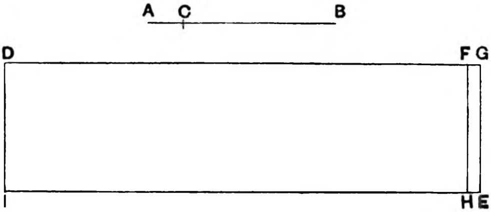
therefore $(x - y)^2$, and consequently $(x - y)$, is irrational.

This irrational, which is of the form $(k^{\frac{1}{4}}\rho \sim k^{\frac{3}{4}}\rho)$, is the *first apotome of a medial* straight line; the term corresponding of course to *first bimedial*, which applies where the sign is positive.

PROPOSITION 75.

If from a medial straight line there be subtracted a medial straight line which is commensurable with the whole in square only, and which contains with the whole a medial rectangle, the remainder is irrational; and let it be called a **second apotome of a medial** *straight line.*

For from the medial straight line *AB* let there be subtracted the medial straight line *CB* which is commensurable with the whole *AB* in square only and such that the rectangle *AB*, *BC*, which it contains with the whole *AB*, is medial; [x. 28] I say that the remainder *AC* is irrational; and let it be called a **second apotome of a medial** straight line.



For let a rational straight line *DI* be set out,
let *DE* equal to the squares on *AB*, *BC* be applied to *DI*, producing *DG* as breadth,
and let *DH* equal to twice the rectangle *AB*, *BC* be applied to *DI*, producing *DF* as breadth;
therefore the remainder *FE* is equal to the square on *AC*. [II. 7]

Now, since the squares on *AB*, *BC* are medial and commensurable,
therefore *DE* is also medial. [x. 15 and 23, Por.]

And it is applied to the rational straight line *DI*, producing *DG* as breadth;
therefore *DG* is rational and incommensurable in length with *DI*. [x. 22]

Again, since the rectangle *AB*, *BC* is medial,
therefore twice the rectangle *AB*, *BC* is also medial. [x. 23, Por.]

And it is equal to *DH*;
therefore *DH* is also medial.

And it has been applied to the rational straight line *DI*, producing *DF* as breadth;
therefore *DF* is rational and incommensurable in length with *DI*. [x. 22]

And, since *AB*, *BC* are commensurable in square only,
therefore *AB* is incommensurable in length with *BC*;
therefore the square on *AB* is also incommensurable with the rectangle *AB*, *BC*. [x. 11]

But the squares on *AB*, *BC* are commensurable with the square on *AB*, [x. 15]
and twice the rectangle *AB*, *BC* is commensurable with the rectangle *AB*, *BC*; [x. 6]
therefore twice the rectangle *AB*, *BC* is incommensurable with the squares on *AB*, *BC*. [x. 13]

But *DE* is equal to the squares on *AB*, *BC*,
and *DH* to twice the rectangle *AB*, *BC*;
therefore *DE* is incommensurable with *DH*.

But, as *DE* is to *DH*, so is *GD* to *DF*; [vi. 1]
therefore *GD* is incommensurable with *DF*. [x. 11]

And both are rational;
therefore *GD*, *DF* are rational straight lines commensurable in square only;
therefore *FG* is an apotome. [x. 73]

But *DI* is rational,
and the rectangle contained by a rational and an irrational straight line is irrational, [deduction from x. 20]
and its "side" is irrational.

And *AC* is the "side" of *FE*;
therefore *AC* is irrational.

And let it be called a **second apotome of a medial straight line**.

Q. E. D.

We have here the difference between $k^{\frac{1}{4}}\rho$, $\sqrt{\lambda}\,.\,\rho/k^{\frac{1}{4}}$, two medial straight lines commensurable in square only and containing a medial rectangle.

Apply each of the areas (x^2+y^2), $2xy$ to a rational straight line σ, i.e. suppose that

$$x^2+y^2=\sigma u,$$

$$2xy=\sigma v.$$

Then σu, σv are medial areas,
so that u, v are both rational and $\smallsmile \sigma$ ....(1).

Again, $x \smallsmile y$;
therefore $x^2 \smallsmile xy$,
and consequently $x^2 + y^2 \smallsmile 2xy$,
or $\sigma u \smallsmile \sigma v$,
and $u \smallsmile v$ ....(2).

Thus [(1), (2)] u, v are rational and $\frown$;
therefore [x. 73] $(u - v)$ is an *apotome*,
and, $(u - v)$ being thus irrational,

$(u - v)\,\sigma$ is an irrational area.

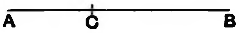
Hence $(x - y)^2$, and consequently $(x - y)$, is irrational.

The irrational straight line $k^{\frac{1}{4}}\rho \sim \frac{\sqrt{\lambda}\,.\,\rho}{k^{\frac{1}{4}}}$ is called a *second apotome of a medial* straight line.

PROPOSITION 76.

If from a straight line there be subtracted a straight line which is incommensurable in square with the whole and which with the whole makes the squares on them added together rational, but the rectangle contained by them medial, the remainder is irrational; and let it be called **minor.**

For from the straight line AB let there be subtracted the straight line BC which is incommensurable in square with the whole and fulfils the given conditions. [x. 33]



I say that the remainder AC is the irrational straight line called **minor.**

For, since the sum of the squares on AB, BC is rational, while twice the rectangle AB, BC is medial,
therefore the squares on AB, BC are incommensurable with twice the rectangle AB, BC;
and, *convertendo*, the squares on AB, BC are incommensurable with the remainder, the square on AC. [II. 7, x. 16]

But the squares on AB, BC are rational;
therefore the square on AC is irrational;
therefore AC is irrational.

And let it be called **minor.**

x, y are here of the form found in x. 33, viz.

$$\frac{\rho}{\sqrt{2}}\sqrt{1+\frac{k}{\sqrt{1+k^2}}},\quad \frac{\rho}{\sqrt{2}}\sqrt{1-\frac{k}{\sqrt{1+k^2}}}.$$

By hypothesis (x^2+y^2) is a *rational*, xy a *medial*, area.

Therefore $$(x^2+y^2) \smile 2xy,$$

whence $$(x-y)^2 \smile (x^2+y^2).$$

Therefore $(x-y)^2$, and consequently $(x-y)$, is irrational.

The *minor* (irrational) straight line is thus of the form

$$\frac{\rho}{\sqrt{2}}\sqrt{1+\frac{k}{\sqrt{1+k^2}}}-\frac{\rho}{\sqrt{2}}\sqrt{1-\frac{k}{\sqrt{1+k^2}}}.$$

Observe the use of *convertendo* (ἀναστρέψαντι) for the inference that, since $(x^2+y^2) \smile 2xy$, $(x^2+y^2) \smile (x-y)^2$. The use of the word corresponds exactly to its use in proportions.

PROPOSITION 77.

If from a straight line there be subtracted a straight line which is incommensurable in square with the whole, and which with the whole makes the sum of the squares on them medial, but twice the rectangle contained by them rational, the remainder is irrational: and let it be called **that which produces with a rational area a medial whole.**

For from the straight line AB let there be subtracted the straight line BC which is incommensurable in square with AB and fulfils the given conditions; [x. 34]

I say that the remainder AC is the irrational straight line aforesaid.

For, since the sum of the squares on AB, BC is medial,

while twice the rectangle AB, BC is rational,

therefore the squares on AB, BC are incommensurable with twice the rectangle AB, BC;

therefore the remainder also, the square on AC, is incommensurable with twice the rectangle AB, BC. [II. 7, x. 16]

And twice the rectangle AB, BC is rational;

therefore the square on AC is irrational;

therefore AC is irrational.

And let it be called **that which produces with a rational area a medial whole.**

Q. E. D.

Here x, y are of the form [cf. x. 34]

$$\frac{\rho}{\sqrt{2(1+k^2)}}\sqrt{\sqrt{1+k^2}+k},\quad \frac{\rho}{\sqrt{2(1+k^2)}}\sqrt{\sqrt{1+k^2}-k}.$$

By hypothesis, (x^2+y^2) is a *medial*, xy a *rational*, area;

thus $$(x^2+y^2) \smile 2xy,$$

and therefore $$(x-y)^2 \smile 2xy,$$

whence $(x-y)^2$, and consequently $(x-y)$, is irrational.

The irrational straight line

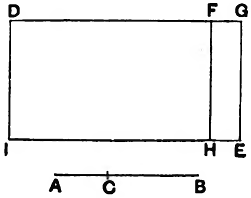
$$\frac{\rho}{\sqrt{2(1+k^2)}}\sqrt{\sqrt{1+k^2}+k} - \frac{\rho}{\sqrt{2(1+k^2)}}\sqrt{\sqrt{1+k^2}-k}$$

is called *that which produces with a rational area a medial whole* or more literally *that which with a rational area makes the whole medial* (ἡ μετὰ ῥητοῦ μέσον τὸ ὅλον ποιοῦσα). Here "produces" means "produces when a square is described on it." A clearer way of expressing the meaning would be to call this straight line the "*side*" *of a medial minus a rational area* corresponding to the "*side*" *of a rational plus a medial area* [x. 40].

PROPOSITION 78.

If from a straight line there be subtracted a straight line which is incommensurable in square with the whole and which with the whole makes the sum of the squares on them medial, twice the rectangle contained by them medial, and further the squares on them incommensurable with twice the rectangle contained by them, the remainder is irrational; and let it be called **that which produces with a medial area a medial whole.**

For from the straight line AB let there be subtracted the straight line BC incommensurable in square with AB and fulfilling the given conditions; [x. 35]



I say that the remainder AC is the irrational straight line called **that which produces with a medial area a medial whole.**

For let a rational straight line DI be set out,

to DI let there be applied DE equal to the squares on AB, BC, producing DG as breadth,

and let DH equal to twice the rectangle AB, BC be subtracted.

Therefore the remainder *FE* is equal to the square on *AC*, [II. 7]

so that *AC* is the "side" of *FE*.

Now, since the sum of the squares on *AB*, *BC* is medial and is equal to *DE*,

therefore *DE* is medial.

And it is applied to the rational straight line *DI*, producing *DG* as breadth;

therefore *DG* is rational and incommensurable in length with *DI*. [X. 22]

Again, since twice the rectangle *AB*, *BC* is medial and is equal to *DH*,

therefore *DH* is medial.

And it is applied to the rational straight line *DI*, producing *DF* as breadth;

therefore *DF* is also rational and incommensurable in length with *DI*. [X. 22]

And, since the squares on *AB*, *BC* are incommensurable with twice the rectangle *AB*, *BC*,

therefore *DE* is also incommensurable with *DH*.

But, as *DE* is to *DH*, so also is *DG* to *DF*; [VI. 1]

therefore *DG* is incommensurable with *DF*. [X. 11]

And both are rational;

therefore *GD*, *DF* are rational straight lines commensurable in square only.

Therefore *FG* is an apotome. [X. 73]

And *FH* is rational;

but the rectangle contained by a rational straight line and an apotome is irrational, [deduction from X. 20]

and its "side" is irrational.

And *AC* is the "side" of *FE*;

therefore *AC* is irrational.

And let it be called **that which produces with a medial area a medial whole.**

Q. E. D.

In this case x, y have respectively the forms [cf. X. 35]

$$\frac{\rho\lambda^{\frac{1}{4}}}{\sqrt{2}}\sqrt{1+\frac{k}{\sqrt{1+k^2}}},\quad \frac{\rho\lambda^{\frac{1}{4}}}{\sqrt{2}}\sqrt{1-\frac{k}{\sqrt{1+k^2}}}.$$

Suppose that $x^2+y^2=\sigma u,$

$$2xy=\sigma v.$$

By hypothesis, the areas σu, σv are medial;
therefore u, v are both rational and $\smile\ \sigma$ ....(1).

Further $\sigma u \smile \sigma v,$

so that $u \smile v$ ....(2).

Hence [(1), (2)] u, v are rational and $\frown\!\!-$,
so that $(u-v)$ is the irrational straight line called *apotome* [X. 73].

Thus $\sigma(u-v)$ is an irrational area,
so that $(x-y)^2$, and consequently $(x-y)$, is irrational.

The irrational straight line

$$\frac{\rho\lambda^{\frac{1}{4}}}{\sqrt{2}}\sqrt{1+\frac{k}{\sqrt{1+k^2}}}-\frac{\rho\lambda^{\frac{1}{4}}}{\sqrt{2}}\sqrt{1-\frac{k}{\sqrt{1+k^2}}}$$

is called *that which produces* [i.e. when a square is described on it] *with a medial area a medial whole*, more literally *that which with a medial area makes the whole medial* (ἡ μετὰ μέσου μέσον τὸ ὅλον ποιοῦσα). A clearer phrase (to us) would be the "*side*" *of the difference between two medial areas*, corresponding to the "*side*" *of* (*the sum of*) *two medial areas* [X. 41].

PROPOSITION 79.

To an apotome only one rational straight line can be annexed which is commensurable with the whole in square only.

Let AB be an apotome, and BC an annex to it;
therefore AC, CB are rational straight lines commensurable in square only. [X. 73]

I say that no other rational straight line can be annexed to AB which is commensurable with the whole in square only.

For, if possible, let BD be so annexed;
therefore AD, DB are also rational straight lines commensurable in square only. [X. 73]

Now, since the excess of the squares on AD, DB over twice the rectangle AD, DB is also the excess of the squares on AC, CB over twice the rectangle AC, CB,
for both exceed by the same, the square on AB, [II. 7]

therefore, alternately, the excess of the squares on AD, DB over the squares on AC, CB is the excess of twice the rectangle AD, DB over twice the rectangle AC, CB.

But the squares on AD, DB exceed the squares on AC, CB by a rational area,
for both are rational;
therefore twice the rectangle AD, DB also exceeds twice the rectangle AC, CB by a rational area:
which is impossible,
for both are medial [x. 21], and a medial area does not exceed a medial by a rational area. [x. 26]

Therefore no other rational straight line can be annexed to AB which is commensurable with the whole in square only.

Therefore only one rational straight line can be annexed to an apotome which is commensurable with the whole in square only.

Q. E. D.

This proposition proves the equivalent of the well-known theorem of surds that,
if $a - \sqrt{b} = x - \sqrt{y}$, then $a = x$, $b = y$;
and, if $\sqrt{a} - \sqrt{b} = \sqrt{x} - \sqrt{y}$, then $a = x$, $b = y$.

The method of proof corresponds to that of x. 42 for positive signs.

Suppose, if possible, that an *apotome* can be expressed as $(x - y)$ and also as $(x' - y')$, where x, y are rational straight lines commensurable in square only, and x', y' are so also.

Of x, x', let x be the greater.

Now, since
$$x - y = x' - y',$$
$$x^2 + y^2 - (x'^2 + y'^2) = 2xy - 2x'y'.$$

But $(x^2 + y^2)$, $(x'^2 + y'^2)$ are both rational, so that their difference is a rational area.

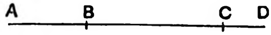
On the other hand, $2xy$, $2x'y'$ are both *medial* areas, being of the form $\sqrt{k} \cdot \rho^2$;
therefore the difference between two medial areas is rational:
which is impossible [x. 26].

Therefore etc.

PROPOSITION 80.

To a first apotome of a medial straight line only one medial straight line can be annexed which is commensurable with the whole in square only and which contains with the whole a rational rectangle.

For let *AB* be a first apotome of a medial straight line, and let *BC* be an annex to *AB*;



therefore *AC*, *CB* are medial straight lines commensurable in square only and such that the rectangle *AC*, *CB* which they contain is rational; [x. 74]

I say that no other medial straight line can be annexed to *AB* which is commensurable with the whole in square only and which contains with the whole a rational area.

For, if possible, let *DB* also be so annexed;

therefore *AD*, *DB* are medial straight lines commensurable in square only and such that the rectangle *AD*, *DB* which they contain is rational. [x. 74]

Now, since the excess of the squares on *AD*, *DB* over twice the rectangle *AD*, *DB* is also the excess of the squares on *AC*, *CB* over twice the rectangle *AC*, *CB*,

for they exceed by the same, the square on *AB*, [II. 7]

therefore, alternately, the excess of the squares on *AD*, *DB* over the squares on *AC*, *CB* is also the excess of twice the rectangle *AD*, *DB* over twice the rectangle *AC*, *CB*.

But twice the rectangle *AD*, *DB* exceeds twice the rectangle *AC*, *CB* by a rational area,

for both are rational.

Therefore the squares on *AD*, *DB* also exceed the squares on *AC*, *CB* by a rational area:

which is impossible,

for both are medial [x. 15 and 23, Por.], and a medial area does not exceed a medial by a rational area. [x. 26]

Therefore etc.

Q. E. D.

Suppose, if possible, that the same *first apotome of a medial* straight line can be expressed in terms of the required character in two ways, so that

$$x - y = x' - y',$$

and suppose that $x > x'$.

In this case $x^2 + y^2$, $(x'^2 + y'^2)$ are both *medial* areas, and $2xy$, $2x'y'$ are both *rational* areas;

and $$x^2 + y^2 - (x'^2 + y'^2) = 2xy - 2x'y'.$$

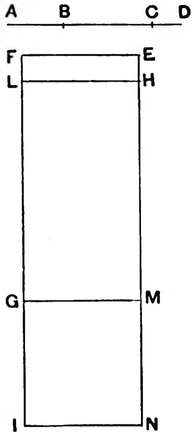
Hence x. 26 is contradicted again;

therefore etc.

PROPOSITION 81.

To a second apotome of a medial straight line only one medial straight line can be annexed which is commensurable with the whole in square only and which contains with the whole a medial rectangle.

Let AB be a second apotome of a medial straight line and BC an annex to AB;



therefore AC, CB are medial straight lines commensurable in square only and such that the rectangle AC, CB which they contain is medial. [x. 75]

I say that no other medial straight line can be annexed to AB which is commensurable with the whole in square only and which contains with the whole a medial rectangle.

For, if possible, let BD also be so annexed;

therefore AD, DB are also medial straight lines commensurable in square only and such that the rectangle AD, DB which they contain is medial. [x. 75]

Let a rational straight line EF be set out,

let EG equal to the squares on AC, CB be applied to EF, producing EM as breadth,

and let HG equal to twice the rectangle AC, CB be subtracted, producing HM as breadth;

therefore the remainder EL is equal to the square on AB, [II. 7]

so that AB is the "side" of EL.

Again, let EI equal to the squares on AD, DB be applied to EF, producing EN as breadth.

But EL is also equal to the square on AB;

therefore the remainder HI is equal to twice the rectangle AD, DB. [II. 7]

Now, since AC, CB are medial straight lines, therefore the squares on AC, CB are also medial.

And they are equal to *EG*;
therefore *EG* is also medial. [x. 15 and 23, Por.]

And it is applied to the rational straight line *EF*, producing *EM* as breadth;
therefore *EM* is rational and incommensurable in length with *EF*. [x. 22]

Again, since the rectangle *AC*, *CB* is medial,
twice the rectangle *AC*, *CB* is also medial. [x. 23, Por.]

And it is equal to *HG*;
therefore *HG* is also medial.

And it is applied to the rational straight line *EF*, producing *HM* as breadth;
therefore *HM* is also rational and incommensurable in length with *EF*. [x. 22]

And, since *AC*, *CB* are commensurable in square only,
therefore *AC* is incommensurable in length with *CB*.

But, as *AC* is to *CB*, so is the square on *AC* to the rectangle *AC*, *CB*;
therefore the square on *AC* is incommensurable with the rectangle *AC*, *CB*. [x. 11]

But the squares on *AC*, *CB* are commensurable with the square on *AC*,
while twice the rectangle *AC*, *CB* is commensurable with the rectangle *AC*, *CB*; [x. 6]
therefore the squares on *AC*, *CB* are incommensurable with twice the rectangle *AC*, *CB*. [x. 13]

And *EG* is equal to the squares on *AC*, *CB*,
while *GH* is equal to twice the rectangle *AC*, *CB*;
therefore *EG* is incommensurable with *HG*.

But, as *EG* is to *HG*, so is *EM* to *HM*; [vi. 1]
therefore *EM* is incommensurable in length with *MH*. [x. 11]

And both are rational;
therefore *EM*, *MH* are rational straight lines commensurable in square only;
therefore *EH* is an apotome, and *HM* an annex to it. [x. 73]

Similarly we can prove that *HN* is also an annex to it; therefore to an apotome different straight lines are annexed which are commensurable with the wholes in square only: which is impossible. [x. 79]

Therefore etc.

Q. E. D.

As the irrationality of the *second apotome of a medial* straight line was deduced [x. 75] from the irrationality of an apotome, so the present theorem is reduced to x. 79.

Suppose, if possible, that $(x-y)$, $(x'-y')$ are the *same* second apotome of a medial straight line;

and let (say) x be greater than x'.

Apply (x^2+y^2), $2xy$ and also $(x'^2+y'^2)$, $2x'y'$ to a rational straight line σ, i.e. put

$$\left.\begin{aligned} x^2+y^2 &= \sigma u \\ 2xy &= \sigma v \end{aligned}\right\} \text{ and } \left.\begin{aligned} x'^2+y'^2 &= \sigma u' \\ 2x'y' &= \sigma v' \end{aligned}\right\}.$$

Dealing with $(x-y)$ first, we have:

(x^2+y^2) is a medial area, and $2xy$ is also a medial area.

Therefore u, v are both rational and $\smile \sigma$ ....(1).

Also, since $x \frown y$, $\quad x \smile y$,

so that $\quad x^2 \smile xy$,

whence, as usual, $\quad x^2+y^2 \smile 2xy$,

that is, $\quad \sigma u \smile \sigma v$,

and therefore $\quad u \smile v$ ....(2).

Thus [(1) and (2)] u, v are rational and $\frown$,

so that $(u-v)$ is an *apotome*.

Similarly $(u'-v')$ is proved to be the *same* apotome.

Hence this apotome is formed in two ways:

which contradicts x. 79.

Therefore the original hypothesis is false, and a *second apotome of a medial* straight line is uniquely formed.

PROPOSITION 82.

To a minor straight line only one straight line can be annexed which is incommensurable in square with the whole and which makes, with the whole, the sum of the squares on them rational but twice the rectangle contained by them medial.

Let *AB* be the minor straight line, and let *BC* be an annex to *AB*;

therefore *AC*, *CB* are straight lines incommensurable in square

which make the sum of the squares on them rational, but twice the rectangle contained by them medial. [X. 76]

I say that no other straight line can be annexed to *AB* fulfilling the same conditions.

For, if possible, let *BD* be so annexed;

therefore *AD*, *DB* are also straight lines incommensurable in square which fulfil the aforesaid conditions. [X. 76]

Now, since the excess of the squares on *AD*, *DB* over the squares on *AC*, *CB* is also the excess of twice the rectangle *AD*, *DB* over twice the rectangle *AC*, *CB*,

while the squares on *AD*, *DB* exceed the squares on *AC*, *CB* by a rational area,

for both are rational,

therefore twice the rectangle *AD*, *DB* also exceeds twice the rectangle *AC*, *CB* by a rational area:

which is impossible, for both are medial. [X. 26]

Therefore to a minor straight line only one straight line can be annexed which is incommensurable in square with the whole and which makes the squares on them added together rational, but twice the rectangle contained by them medial.

Q. E. D.

Suppose, if possible, that, with the usual notation,

$$x - y = x' - y';$$

and let x (say) be greater than x'.

In this case $(x^2 + y^2)$, $(x'^2 + y'^2)$ are both *rational* areas, and $2xy$, $2x'y'$ are both *medial* areas.

But, as before, $(x^2 + y^2) - (x'^2 + y'^2) = 2xy - 2x'y'$,

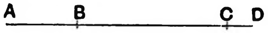
so that the difference between two medial areas is rational: which is impossible [X. 26].

Therefore etc.

PROPOSITION 83.

To a straight line which produces with a rational area a medial whole only one straight line can be annexed which is incommensurable in square with the whole straight line and which with the whole straight line makes the sum of the squares on them medial, but twice the rectangle contained by them rational.

Let *AB* be the straight line which produces with a rational area a medial whole,



and let *BC* be an annex to *AB*;

therefore *AC*, *CB* are straight lines incommensurable in square which fulfil the given conditions. [x. 77]

I say that no other straight line can be annexed to *AB* which fulfils the same conditions.

For, if possible, let *BD* be so annexed;

therefore *AD*, *DB* are also straight lines incommensurable in square which fulfil the given conditions. [x. 77]

Since then, as in the preceding cases,

the excess of the squares on *AD*, *DB* over the squares on *AC*, *CB* is also the excess of twice the rectangle *AD*, *DB* over twice the rectangle *AC*, *CB*,

while twice the rectangle *AD*, *DB* exceeds twice the rectangle *AC*, *CB* by a rational area,

for both are rational,

therefore the squares on *AD*, *DB* also exceed the squares on *AC*, *CB* by a rational area:

which is impossible, for both are medial. [x. 26]

Therefore no other straight line can be annexed to *AB* which is incommensurable in square with the whole and which with the whole fulfils the aforesaid conditions;

therefore only one straight line can be so annexed.

Q. E. D.

Suppose, with the same notation, that

$$x - y = x' - y'. \qquad (x > x')$$

Here, $(x^2 + y^2)$, $(x'^2 + y'^2)$ being both medial areas, and $2xy$, $2x'y'$ both rational areas,

while $$(x^2 + y^2) - (x'^2 + y'^2) = 2xy - 2x'y',$$

x. 26 is contradicted again.

Therefore etc.

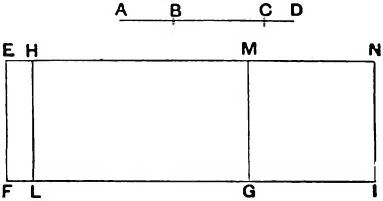
PROPOSITION 84.

To a straight line which produces with a medial area a medial whole only one straight line can be annexed which is incommensurable in square with the whole straight line and which with the whole straight line makes the sum of the squares *on them medial and twice the rectangle contained by them both medial and also incommensurable with the sum of the squares on them.*

Let AB be the straight line which produces with a medial area a medial whole,

and BC an annex to it;

therefore AC, CB are straight lines incommensurable in square which fulfil the aforesaid conditions. [x. 78]



I say that no other straight line can be annexed to AB which fulfils the aforesaid conditions.

For, if possible, let BD be so annexed,

so that AD, DB are also straight lines incommensurable in square which make the squares on AD, DB added together medial, twice the rectangle AD, DB medial, and also the squares on AD, DB incommensurable with twice the rectangle AD, DB. [x. 78]

Let a rational straight line EF be set out,

let EG equal to the squares on AC, CB be applied to EF, producing EM as breadth,

and let HG equal to twice the rectangle AC, CB be applied to EF, producing HM as breadth;

therefore the remainder, the square on AB [II. 7], is equal to EL;

therefore AB is the "side" of EL.

Again, let EI equal to the squares on AD, DB be applied to EF, producing EN as breadth.

But the square on AB is also equal to EL;

therefore the remainder, twice the rectangle AD, DB [II. 7], is equal to HI.

Now, since the sum of the squares on *AC*, *CB* is medial and is equal to *EG*,

therefore *EG* is also medial.

And it is applied to the rational straight line *EF*, producing *EM* as breadth;

therefore *EM* is rational and incommensurable in length with *EF*. [x. 22]

Again, since twice the rectangle *AC*, *CB* is medial and is equal to *HG*,

therefore *HG* is also medial.

And it is applied to the rational straight line *EF*, producing *HM* as breadth;

therefore *HM* is rational and incommensurable in length with *EF*. [x. 22]

And, since the squares on *AC*, *CB* are incommensurable with twice the rectangle *AC*, *CB*,

EG is also incommensurable with *HG*;

therefore *EM* is also incommensurable in length with *MH*. [vi. 1, x. 11]

And both are rational;

therefore *EM*, *MH* are rational straight lines commensurable in square only;

therefore *EH* is an apotome, and *HM* an annex to it. [x. 73]

Similarly we can prove that *EH* is again an apotome and *HN* an annex to it.

Therefore to an apotome different rational straight lines are annexed which are commensurable with the wholes in square only:

which was proved impossible. [x. 79]

Therefore no other straight line can be so annexed to *AB*.

Therefore to *AB* only one straight line can be annexed which is incommensurable in square with the whole and which with the whole makes the squares on them added together medial, twice the rectangle contained by them medial, and also the squares on them incommensurable with twice the rectangle contained by them.

Q. E. D.

With the usual notation, suppose that

$$x - y = x' - y'. \qquad (x > x')$$

Let $\left.\begin{array}{r} x^2 + y^2 = \sigma u \\ 2xy = \sigma v \end{array}\right\}$ and $\left.\begin{array}{r} x'^2 + y'^2 = \sigma u' \\ 2x'y' = \sigma v' \end{array}\right\}$.

Consider $(x - y)$ first;
it follows, since $(x^2 + y^2)$, $2xy$ are both medial areas, that
u, v are both rational and $\smile \sigma$ ....(1).

But $x^2 + y^2 \smile 2xy$,
that is, $\sigma u \smile \sigma v$,
and therefore $u \smile v$ ....(2).

Therefore [(1) and (2)] u, v are rational and $\frown\!-$;
hence $(u - v)$ is an *apotome*.

Similarly $(u' - v')$ is proved to be the *same* apotome.

Thus the same apotome is formed as such in two ways:
which is impossible [x. 79].

Therefore, etc.

DEFINITIONS III.

1. Given a rational straight line and an apotome, if the square on the whole be greater than the square on the annex by the square on a straight line commensurable in length with the whole, and the whole be commensurable in length with the rational straight line set out, let the apotome be called a **first apotome.**

2. But if the annex be commensurable in length with the rational straight line set out, and the square on the whole be greater than that on the annex by the square on a straight line commensurable with the whole, let the apotome be called a **second apotome.**

3. But if neither be commensurable in length with the rational straight line set out, and the square on the whole be greater than the square on the annex by the square on a straight line commensurable with the whole, let the apotome be called a **third apotome.**

4. Again, if the square on the whole be greater than the square on the annex by the square on a straight line incommensurable with the whole, then, if the whole be commensurable in length with the rational straight line set out, let the apotome be called a **fourth apotome**;

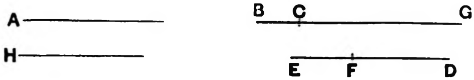
5. if the annex be so commensurable, a **fifth**;

6. and, if neither, a **sixth.**

PROPOSITION 85.

To find the first apotome.

Let a rational straight line A be set out,
and let BG be commensurable in length with A;
therefore BG is also rational.



Let two square numbers DE, EF be set out, and let their difference FD not be square;
therefore neither has ED to DF the ratio which a square number has to a square number.

Let it be contrived that,
as ED is to DF, so is the square on BG to the square on GC; [x. 6, Por.]
therefore the square on BG is commensurable with the square on GC. [x. 6]

But the square on BG is rational;
therefore the square on GC is also rational;
therefore GC is also rational.

And, since ED has not to DF the ratio which a square number has to a square number,
therefore neither has the square on BG to the square on GC the ratio which a square number has to a square number;
therefore BG is incommensurable in length with GC. [x. 9]

And both are rational;
therefore BG, GC are rational straight lines commensurable in square only;
therefore BC is an apotome. [x. 73]

I say next that it is also a first apotome.

For let the square on H be that by which the square on BG is greater than the square on GC.

Now since, as ED is to FD, so is the square on BG to the square on GC,
therefore also, *convertendo*, [v. 19, Por.]
as DE is to EF, so is the square on GB to the square on H.

But *DE* has to *EF* the ratio which a square number has to a square number,

for each is square;

therefore the square on *GB* also has to the square on *H* the ratio which a square number has to a square number;

therefore *BG* is commensurable in length with *H*. [x. 9]

And the square on *BG* is greater than the square on *GC* by the square on *H*;

therefore the square on *BG* is greater than the square on *GC* by the square on a straight line commensurable in length with *BG*.

And the whole *BG* is commensurable in length with the rational straight line *A* set out.

Therefore *BC* is a first apotome. [x. Deff. III. 1]

Therefore the first apotome *BC* has been found.

(Being) that which it was required to find.

Take $k\rho$ commensurable in length with ρ, the given rational straight line.

Let m^2, n^2 be square numbers such that $(m^2 - n^2)$ is not square.

Take x such that $m^2 : (m^2 - n^2) = k^2\rho^2 : x^2$ ....(1),

so that
$$x = k\rho\,\frac{\sqrt{m^2 - n^2}}{m}$$
$$= k\rho\,\sqrt{1 - \lambda^2},\quad \text{say.}$$

Then shall $k\rho - x$, or $k\rho - k\rho\,\sqrt{1-\lambda^2}$, be a *first apotome*.

For (α) it follows from (1) that x is rational but incommensurable with $k\rho$,

whence $k\rho$, x are rational and $\frown\!\!-$,

so that $(k\rho - x)$ is an apotome.

(β) If $y^2 = k^2\rho^2 - x^2$, then, by (1), *convertendo*,
$$m^2 : n^2 = k^2\rho^2 : y^2,$$
whence y, that is, $\sqrt{k^2\rho^2 - x^2}$, is commensurable in length with $k\rho$.

And $k\rho \frown \rho$;

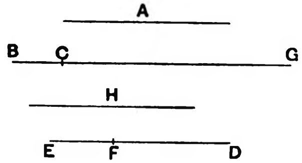
therefore $k\rho - x$ is a *first apotome*.

As explained in the note to x. 48, the first apotome
$$k\rho - k\rho\sqrt{1 - \lambda^2}$$
is one of the roots of the equation
$$x^2 - 2k\rho \,.\, x + \lambda^2k^2\rho^2 = 0.$$

Proposition 86.

To find the second apotome.

Let a rational straight line A be set out, and GC commensurable in length with A;
therefore GC is rational.



Let two square numbers DE, EF be set out, and let their difference DF not be square.

Now let it be contrived that, as FD is to DE, so is the square on CG to the square on GB. [x. 6, Por.]

Therefore the square on CG is commensurable with the square on GB. [x. 6]

But the square on CG is rational;
therefore the square on GB is also rational;
therefore BG is rational.

And, since the square on GC has not to the square on GB the ratio which a square number has to a square number,
CG is incommensurable in length with GB. [x. 9]

And both are rational;
therefore CG, GB are rational straight lines commensurable in square only;
therefore BC is an apotome. [x. 73]

I say next that it is also a second apotome.

For let the square on H be that by which the square on BG is greater than the square on GC.

Since then, as the square on BG is to the square on GC, so is the number ED to the number DF,
therefore, *convertendo*,
as the square on BG is to the square on H, so is DE to EF. [v. 19, Por.]

And each of the numbers DE, EF is square;
therefore the square on BG has to the square on H the ratio which a square number has to a square number;
therefore BG is commensurable in length with H. [x. 9]

And the square on BG is greater than the square on GC by the square on H;
therefore the square on BG is greater than the square on GC

by the square on a straight line commensurable in length with *BG*.

And *CG*, the annex, is commensurable with the rational straight line *A* set out.

Therefore *BC* is a second apotome. [x. Deff. iii. 2]

Therefore the second apotome *BC* has been found.

Q. E. D.

Take, as before, $k\rho$ commensurable in length with ρ.

Let m^2, n^2 be again square numbers, but $(m^2 - n^2)$ not square.

Take x such that $(m^2 - n^2) : m^2 = k^2\rho^2 : x^2$ ....(1),

whence
$$x = k\rho \frac{m}{\sqrt{m^2 - n^2}}$$
$$= \frac{k\rho}{\sqrt{1 - \lambda^2}}, \text{ say.}$$

Thus x is greater than $k\rho$.

Then $x - k\rho$, or $\frac{k\rho}{\sqrt{1 - \lambda^2}} - k\rho$, is a *second apotome*.

For (α), as before, x is rational and $\backsim k\rho$.

(β) If $x^2 - k^2\rho^2 = y^2$, we have, from (1),
$$m^2 : n^2 = x^2 : y^2.$$

Thus y, or $\sqrt{x^2 - k^2\rho^2}$, is commensurable in length with x.

And $k\rho$ is $\frown \rho$.

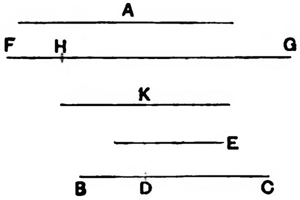
Therefore $x - k\rho$ is a *second apotome*.

As explained in the note on x. 49, the *second apotome*
$$\frac{k\rho}{\sqrt{1 - \lambda^2}} - k\rho$$
is the lesser root of the equation
$$x^2 - \frac{2k\rho}{\sqrt{1 - \lambda^2}} . x + \frac{\lambda^2}{1 - \lambda^2} k^2\rho^2 = 0.$$

Proposition 87.

To find the third apotome.

Let a rational straight line *A* be set out, let three numbers *E*, *BC*, *CD* be set out which have not to one another the ratio which a square number has to a square number, but let *CB* have to *BD* the ratio which a square number has to a square number.



Let it be contrived that, as *E* is to *BC*, so is the square on *A* to the square on *FG*, and, as BC is to CD, so is the square on FG to the square on GH. [x. 6, Por.]

Since then, as E is to BC, so is the square on A to the square on FG,
therefore the square on A is commensurable with the square on FG. [x. 6]

But the square on A is rational;
therefore the square on FG is also rational;
therefore FG is rational.

And, since E has not to BC the ratio which a square number has to a square number,
therefore neither has the square on A to the square on FG the ratio which a square number has to a square number;
therefore A is incommensurable in length with FG. [x. 9]

Again, since, as BC is to CD, so is the square on FG to the square on GH,
therefore the square on FG is commensurable with the square on GH. [x. 6]

But the square on FG is rational;
therefore the square on GH is also rational;
therefore GH is rational.

And, since BC has not to CD the ratio which a square number has to a square number,
therefore neither has the square on FG to the square on GH the ratio which a square number has to a square number;
therefore FG is incommensurable in length with GH. [x. 9]

And both are rational;
therefore FG, GH are rational straight lines commensurable in square only;
therefore FH is an apotome. [x. 73]

I say next that it is also a third apotome.

For since, as E is to BC, so is the square on A to the square on FG,
and, as BC is to CD, so is the square on FG to the square on HG,
therefore, *ex aequali*, as E is to CD, so is the square on A to the square on HG. [v. 22]

But *E* has not to *CD* the ratio which a square number has to a square number;

therefore neither has the square on *A* to the square on *GH* the ratio which a square number has to a square number;

therefore *A* is incommensurable in length with *GH*. [x. 9]

Therefore neither of the straight lines *FG*, *GH* is commensurable in length with the rational straight line *A* set out.

Now let the square on *K* be that by which the square on *FG* is greater than the square on *GH*.

Since then, as *BC* is to *CD*, so is the square on *FG* to the square on *GH*,

therefore, *convertendo*, as *BC* is to *BD*, so is the square on *FG* to the square on *K*. [v. 19, Por.]

But *BC* has to *BD* the ratio which a square number has to a square number;

therefore the square on *FG* also has to the square on *K* the ratio which a square number has to a square number.

Therefore *FG* is commensurable in length with *K*, [x. 9]

and the square on *FG* is greater than the square on *GH* by the square on a straight line commensurable with *FG*.

And neither of the straight lines *FG*, *GH* is commensurable in length with the rational straight line *A* set out;

therefore *FH* is a third apotome. [x. Deff. iii. 3]

Therefore the third apotome *FH* has been found.

Q. E. D.

Let ρ be a rational straight line.

Take numbers p, qm^2, $q(m^2 - n^2)$ which have not to one another the ratio of square to square.

Now let x, y be such that

$$p : qm^2 = \rho^2 : x^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

and

$$qm^2 : q(m^2 - n^2) = x^2 : y^2 \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Then shall $(x - y)$ be a *third apotome*.

For (a), from (1),

x is rational but $\smile \rho$ $\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$(3).

And, from (2), y is rational but $\smile x$.

Therefore x, y are rational and $\frown$,

so that $(x - y)$ is an apotome.

(β) By (1), (2), *ex aequali*,

$$p : q(m^2 - n^2) = \rho^2 : y^2,$$

whence $y \smile \rho$.

Thus, by this and (3), x, y are both $\smile \rho$ ....(4).

Lastly, let $z^2 = x^2 - y^2$, so that, from (2), *convertendo*,

$$qm^2 : qn^2 = x^2 : z^2;$$

therefore z, or $\sqrt{x^2 - y^2}$, $\frown x$ ....(5).

Thus [(4) and (5)] $(x - y)$ is a *third apotome*.

To find its form, we have, from (1) and (2),

$$x = \rho \, . \, \frac{m\sqrt{q}}{\sqrt{p}},$$

$$y = \rho \, . \, \frac{\sqrt{m^2 - n^2} \, . \, \sqrt{q}}{\sqrt{p}},$$

so that

$$x - y = \sqrt{\frac{q}{p}} \, . \, \rho \left(m - \sqrt{m^2 - n^2}\right).$$

This may be written in the form

$$m\sqrt{k} \, . \, \rho - m\sqrt{k} \, . \, \rho\sqrt{1 - \lambda^2}.$$

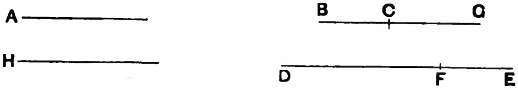
As explained in the note on x. 50, this is the lesser root of the equation

$$x^2 - 2m\sqrt{k} \, . \, \rho x + \lambda^2 m^2 k \rho^2 = 0.$$

PROPOSITION 88.

To find the fourth apotome.

Let a rational straight line A be set out, and BG commensurable in length with it;
therefore BG is also rational.



Let two numbers DF, FE be set out such that the whole DE has not to either of the numbers DF, EF the ratio which a square number has to a square number.

Let it be contrived that, as DE is to EF, so is the square on BG to the square on GC; [x. 6, Por.]
therefore the square on BG is commensurable with the square on GC. [x. 6]

But the square on BG is rational;
therefore the square on GC is also rational;
therefore GC is rational.

Now, since *DE* has not to *EF* the ratio which a square number has to a square number,

therefore neither has the square on *BG* to the square on *GC* the ratio which a square number has to a square number;

therefore *BG* is incommensurable in length with *GC*. [X. 9]

And both are rational;

therefore *BG*, *GC* are rational straight lines commensurable in square only;

therefore *BC* is an apotome. [X. 73]

Now let the square on *H* be that by which the square on *BG* is greater than the square on *GC*.

Since then, as *DE* is to *EF*, so is the square on *BG* to the square on *GC*,

therefore also, *convertendo*, as *ED* is to *DF*, so is the square on *GB* to the square on *H*. [V. 19, Por.]

But *ED* has not to *DF* the ratio which a square number has to a square number;

therefore neither has the square on *GB* to the square on *H* the ratio which a square number has to a square number;

therefore *BG* is incommensurable in length with *H*. [X. 9]

And the square on *BG* is greater than the square on *GC* by the square on *H*;

therefore the square on *BG* is greater than the square on *GC* by the square on a straight line incommensurable with *BG*.

And the whole *BG* is commensurable in length with the rational straight line *A* set out.

Therefore *BC* is a fourth apotome. [X. Deff. III. 4]

Therefore the fourth apotome has been found.

Q. E. D.

Beginning with ρ, $k\rho$, as in X. 85, 86, we take numbers m, n such that $(m+n)$ has not to either of the numbers m, n the ratio of a square number to a square number.

Take x such that $(m+n) : n = k^2\rho^2 : x^2$ ....(1),

whence
$$x = k\rho\sqrt{\frac{n}{m+n}}$$
$$= \frac{k\rho}{\sqrt{1+\lambda}}, \text{ say.}$$

Then shall $(k\rho - x)$, or $\left(k\rho - \frac{k\rho}{\sqrt{1+\lambda}}\right)$, be a *fourth apotome*.

For, by (1), x is rational and $\smile\!\!\!- \, k\rho$.

Also $\sqrt{k^2\rho^2 - x^2}$ is incommensurable with $k\rho$, since

$$(m + n) : m = k^2\rho^2 : (k^2\rho^2 - x^2),$$

and the ratio $(m + n) : m$ is not that of a square number to a square number.

And $k\rho \frown \rho$.

As explained in the note on x. 51, the *fourth apotome*

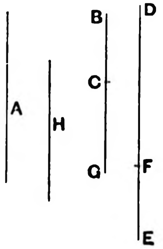
$$k\rho - \frac{k\rho}{\sqrt{1 + \lambda}}$$

is the lesser root of the quadratic equation

$$x^2 - 2k\rho \, . \, x + \frac{\lambda}{1 + \lambda} k^2\rho^2 = 0.$$

PROPOSITION 89.

To find the fifth apotome.



Let a rational straight line A be set out, and let CG be commensurable in length with A;

therefore CG is rational.

Let two numbers DF, FE be set out such that DE again has not to either of the numbers DF, FE the ratio which a square number has to a square number;

and let it be contrived that, as FE is to ED, so is the square on CG to the square on GB.

Therefore the square on GB is also rational; [x. 6]

therefore BG is also rational.

Now since, as DE is to EF, so is the square on BG to the square on GC,

while DE has not to EF the ratio which a square number has to a square number,

therefore neither has the square on BG to the square on GC the ratio which a square number has to a square number;

therefore BG is incommensurable in length with GC. [x. 9]

And both are rational;

therefore BG, GC are rational straight lines commensurable in square only;

therefore BC is an apotome. [x. 73]

I say next that it is also a fifth apotome.

For let the square on *H* be that by which the square on *BG* is greater than the square on *GC*.

Since then, as the square on *BG* is to the square on *GC*, so is *DE* to *EF*,

therefore, *convertendo*, as *ED* is to *DF*, so is the square on *BG* to the square on *H*. [v. 19, Por.]

But *ED* has not to *DF* the ratio which a square number has to a square number;

therefore neither has the square on *BG* to the square on *H* the ratio which a square number has to a square number;

therefore *BG* is incommensurable in length with *H*. [x. 9]

And the square on *BG* is greater than the square on *GC* by the square on *H*;

therefore the square on *GB* is greater than the square on *GC* by the square on a straight line incommensurable in length with *GB*.

And the annex *CG* is commensurable in length with the rational straight line *A* set out;

therefore *BC* is a fifth apotome. [x. Deff. III. 5]

Therefore the fifth apotome *BC* has been found.

Q. E. D.

Let ρ, $k\rho$ and the numbers m, n of the last proposition be taken.

Take x such that $n : (m+n) = k^2\rho^2 : x^2$ ....(1).

In this case $x > k\rho$, and $x = k\rho\sqrt{\dfrac{m+n}{n}}$

$$= k\rho\sqrt{1+\lambda}, \text{ say.}$$

Then shall $(x - k\rho)$, or $(k\rho\sqrt{1+\lambda} - k\rho)$, be a *fifth apotome*.

For, by (1), x is rational and $\frown\!\!-\ k\rho$.

And since, by (1), $(m+n) : m = x^2 : (x^2 - k^2\rho^2)$,

$\sqrt{x^2 - k^2\rho^2}$ is incommensurable with x.

Also $k\rho \frown \rho$.

As explained in the note on x. 52, the *fifth apotome*

$$k\rho\sqrt{1+\lambda} - k\rho$$

is the lesser root of the quadratic

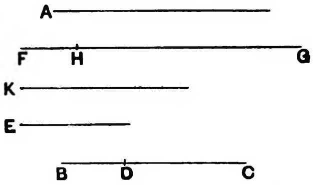
$$x^2 - 2k\rho\sqrt{1+\lambda}\,.\,x + \lambda k^2\rho^2 = 0.$$

PROPOSITION 90.

To find the sixth apotome.

Let a rational straight line A be set out, and three numbers E, BC, CD not having to one another the ratio which a square number has to a square number;

and further let CB also not have to BD the ratio which a square number has to a square number.



Let it be contrived that, as E is to BC, so is the square on A to the square on FG,

and, as BC is to CD, so is the square on FG to the square on GH. [x. 6, Por.]

Now since, as E is to BC, so is the square on A to the square on FG,

therefore the square on A is commensurable with the square on FG. [x. 6]

But the square on A is rational;

therefore the square on FG is also rational;

therefore FG is also rational.

And, since E has not to BC the ratio which a square number has to a square number,

therefore neither has the square on A to the square on FG the ratio which a square number has to a square number;

therefore A is incommensurable in length with FG. [x. 9]

Again, since, as BC is to CD, so is the square on FG to the square on GH,

therefore the square on FG is commensurable with the square on GH. [x. 6]

But the square on FG is rational;

therefore the square on GH is also rational;

therefore GH is also rational.

And, since BC has not to CD the ratio which a square number has to a square number, therefore neither has the square on *FG* to the square on *GH* the ratio which a square number has to a square number;

therefore *FG* is incommensurable in length with *GH*. [x. 9]

And both are rational;

therefore *FG*, *GH* are rational straight lines commensurable in square only;

therefore *FH* is an apotome. [x. 73]

I say next that it is also a sixth apotome.

For since, as *E* is to *BC*, so is the square on *A* to the square on *FG*,

and, as *BC* is to *CD*, so is the square on *FG* to the square on *GH*,

therefore, *ex aequali*, as *E* is to *CD*, so is the square on *A* to the square on *GH*. [v. 22]

But *E* has not to *CD* the ratio which a square number has to a square number;

therefore neither has the square on *A* to the square on *GH* the ratio which a square number has to a square number;

therefore *A* is incommensurable in length with *GH*; [x. 9]

therefore neither of the straight lines *FG*, *GH* is commensurable in length with the rational straight line *A*.

Now let the square on *K* be that by which the square on *FG* is greater than the square on *GH*.

Since then, as *BC* is to *CD*, so is the square on *FG* to the square on *GH*,

therefore, *convertendo*, as *CB* is to *BD*, so is the square on *FG* to the square on *K*. [v. 19, Por.]

But *CB* has not to *BD* the ratio which a square number has to a square number;

therefore neither has the square on *FG* to the square on *K* the ratio which a square number has to a square number;

therefore *FG* is incommensurable in length with *K*. [x. 9]

And the square on *FG* is greater than the square on *GH* by the square on *K*;

therefore the square on *FG* is greater than the square on *GH* by the square on a straight line incommensurable in length with *FG*.

And neither of the straight lines *FG*, *GH* is commensurable with the rational straight line *A* set out.

Therefore *FH* is a sixth apotome. [x. Deff. III. 6]

Therefore the sixth apotome *FH* has been found.

Q. E. D.

Let ρ be the given rational straight line.

Take numbers p, $(m+n)$, n which have not to one another the ratio of a square number to a square number, m, n being also chosen such that the ratio $(m+n):m$ is not that of square to square.

Take x, y such that $p:(m+n)=\rho^2:x^2$ .... (1),

$(m+n):n=x^2:y^2$ .... (2).

Then shall $(x-y)$ be a *sixth apotome.*

For, by (1), x is rational and $\smile \rho$ .... (3).

By (2), since x is rational,

y is rational and $\smile x$ .... (4).

Thus [(3), (4)] $(x-y)$ is an apotome.

Again, *ex aequali*, $p:n=\rho^2:y^2$,

whence $y \smile \rho$.

Thus x, y are both $\smile \rho$.

Lastly, *convertendo* from (2),

$$(m+n):m=x^2:(x^2-y^2),$$

whence $\sqrt{x^2-y^2} \smile x$.

Therefore $(x-y)$ is a *sixth apotome.*

From (1) and (2) we have

$$x=\rho\sqrt{\frac{m+n}{p}},$$

$$y=\rho\sqrt{\frac{n}{p}},$$

so that the *sixth apotome* may be written

$$\rho\sqrt{\frac{m+n}{p}}-\rho\sqrt{\frac{n}{p}},$$

or, more simply, $\sqrt{k}\,.\,\rho-\sqrt{\lambda}\,.\,\rho$.

As explained in the note on x. 53, the *sixth apotome* is the lesser root of the equation

$$x^2-2\sqrt{k}\,.\,\rho x+(k-\lambda)\rho^2=0.$$

PROPOSITION 91.

If an area be contained by a rational straight line and a first apotome, the "side" of the area is an apotome.

For let the area *AB* be contained by the rational straight line *AC* and the first apotome *AD*;

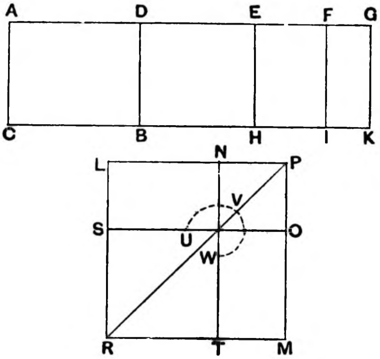
I say that the "side" of the area *AB* is an apotome.

For, since AD is a first apotome, let DG be its annex; therefore AG, GD are rational straight lines commensurable in square only. [x. 73]

And the whole AG is commensurable with the rational straight line AC set out,

and the square on AG is greater than the square on GD by the square on a straight line commensurable in length with AG; [x. Deff. III. 1]

if therefore there be applied to AG a parallelogram equal to the fourth part of the square on DG and deficient by a square figure, it divides it into commensurable parts. [x. 17]



Let DG be bisected at E,

let there be applied to AG a parallelogram equal to the square on EG and deficient by a square figure,

and let it be the rectangle AF, FG;

therefore AF is commensurable with FG.

And through the points E, F, G let EH, FI, GK be drawn parallel to AC.

Now, since AF is commensurable in length with FG,

therefore AG is also commensurable in length with each of the straight lines AF, FG. [x. 15]

But AG is commensurable with AC;

therefore each of the straight lines AF, FG is commensurable in length with AC. [x. 12]

And *AC* is rational;

therefore each of the straight lines *AF*, *FG* is also rational,

so that each of the rectangles *AI*, *FK* is also rational. [x. 19]

Now, since *DE* is commensurable in length with *EG*,

therefore *DG* is also commensurable in length with each of the straight lines *DE*, *EG*. [x. 15]

But *DG* is rational and incommensurable in length with *AC*;

therefore each of the straight lines *DE*, *EG* is also rational and incommensurable in length with *AC*; [x. 13]

therefore each of the rectangles *DH*, *EK* is medial. [x. 21]

Now let the square *LM* be made equal to *AI*, and let there be subtracted the square *NO* having a common angle with it, the angle *LPM*, and equal to *FK*;

therefore the squares *LM*, *NO* are about the same diameter. [vi. 26]

Let *PR* be their diameter, and let the figure be drawn.

Since then the rectangle contained by *AF*, *FG* is equal to the square on *EG*,

therefore, as *AF* is to *EG*, so is *EG* to *FG*. [vi. 17]

But, as *AF* is to *EG*, so is *AI* to *EK*,

and, as *EG* is to *FG*, so is *EK* to *KF*; [vi. 1]

therefore *EK* is a mean proportional between *AI*, *KF*. [v. 11]

But *MN* is also a mean proportional between *LM*, *NO*, as was before proved, [Lemma after x. 53]

and *AI* is equal to the square *LM*, and *KF* to *NO*;

therefore *MN* is also equal to *EK*.

But *EK* is equal to *DH*, and *MN* to *LO*;

therefore *DK* is equal to the gnomon *UVW* and *NO*.

But *AK* is also equal to the squares *LM*, *NO*;

therefore the remainder *AB* is equal to *ST*.

But *ST* is the square on *LN*;

therefore the square on *LN* is equal to *AB*;

therefore *LN* is the "side" of *AB*.

I say next that *LN* is an apotome.

For, since each of the rectangles *AI*, *FK* is rational, and they are equal to *LM*, *NO*,
therefore each of the squares *LM*, *NO*, that is, the squares on *LP*, *PN* respectively, is also rational;
therefore each of the straight lines *LP*, *PN* is also rational.

Again, since *DH* is medial and is equal to *LO*,
therefore *LO* is also medial.

Since then *LO* is medial,
while *NO* is rational,
therefore *LO* is incommensurable with *NO*.

But, as *LO* is to *NO*, so is *LP* to *PN*; [VI. 1]
therefore *LP* is incommensurable in length with *PN*. [X. 11]

And both are rational;
therefore *LP*, *PN* are rational straight lines commensurable in square only;
therefore *LN* is an apotome. [X. 73]

And it is the "side" of the area *AB*;
therefore the "side" of the area *AB* is an apotome.

Therefore etc.

This proposition corresponds to X. 54, and the problem solved in it is to find and to classify the *side of a square equal to the rectangle contained by a first apotome and* ρ, or (algebraically) to find

$$\sqrt{\rho\left(k\rho - k\rho\sqrt{1-\lambda^2}\right)}.$$

First find u, v from the equations

$$\left.\begin{aligned} u + v &= k\rho \\ uv &= \tfrac{1}{4}k^2\rho^2(1-\lambda^2) \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

If u, v represent the values so found, put

$$\left.\begin{aligned} x^2 &= \rho u \\ y^2 &= \rho v \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

and $(x - y)$ shall be the square root required.

To prove this Euclid argues thus.

By (1), $u : \tfrac{1}{2}k\rho\sqrt{1-\lambda^2} = \tfrac{1}{2}k\rho\sqrt{1-\lambda^2} : v$,

whence $\rho u : \tfrac{1}{2}k\rho^2\sqrt{1-\lambda^2} = \tfrac{1}{2}k\rho^2\sqrt{1-\lambda^2} : \rho v$,

or $x^2 : \tfrac{1}{2}k\rho^2\sqrt{1-\lambda^2} = \tfrac{1}{2}k\rho^2\sqrt{1-\lambda^2} : y^2$.

But [Lemma after X. 53]

$$x^2 : xy = xy : y^2,$$

so that $xy = \tfrac{1}{2}k\rho^2\sqrt{1-\lambda^2}$ ....(3).

Therefore
$$\begin{aligned}(x-y)^2 &= x^2 + y^2 - 2xy\\ &= \rho(u+v) - k\rho^2\sqrt{1-\lambda^2}\\ &= k\rho^2 - k\rho^2\sqrt{1-\lambda^2}.\end{aligned}$$

Thus $(x-y)$ is equal to $\sqrt{\rho(k\rho - k\rho\sqrt{1-\lambda^2})}$.

It has next to be proved that $(x-y)$ is an *apotome*.

From (1) it follows, by x. 17, that
$$u \frown v;$$
thus u, v are both commensurable with $(u+v)$ and therefore with ρ......(4).

Hence u, v are both rational,
so that ρu, ρv are rational areas;
therefore, by (2), x^2, y^2 are rational and commensurable ....(5),
whence also x, y are rational straight lines ....(6).

Next, $k\rho\sqrt{1-\lambda^2}$ is rational and $\smile \rho$;
therefore $\frac{1}{2}k\rho^2\sqrt{1-\lambda^2}$ is a *medial* area.

That is, by (3), xy is a medial area.

But [(5)] y^2 is a *rational* area;
therefore
$$xy \smile y^2,$$
or
$$x \smile y.$$

But [(6)] x, y are both rational.

Therefore x, y are rational and $\frown\!-$;
so that $(x-y)$ is an *apotome*.

To find the form of $(x-y)$ algebraically, we have, by solving (1),
$$u = \tfrac{1}{2}k\rho(1+\lambda),$$
$$v = \tfrac{1}{2}k\rho(1-\lambda),$$
whence, from (2),
$$x = \rho\sqrt{\frac{k}{2}(1+\lambda)},$$
$$y = \rho\sqrt{\frac{k}{2}(1-\lambda)},$$
and
$$x - y = \rho\sqrt{\frac{k}{2}(1+\lambda)} - \rho\sqrt{\frac{k}{2}(1-\lambda)}.$$

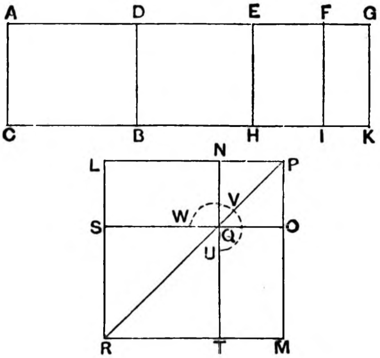
As explained in the note on x. 54, $(x-y)$ is the lesser positive root of the biquadratic equation
$$x^4 - 2k\rho^2 . x^2 + \lambda^2 k^2\rho^4 = 0.$$

PROPOSITION 92.

If an area be contained by a rational straight line and a second apotome, the "side" of the area is a first apotome of a medial straight line.

For let the area *AB* be contained by the rational straight line *AC* and the second apotome *AD*;

I say that the "side" of the area *AB* is a first apotome of a medial straight line.



For let *DG* be the annex to *AD*;

therefore *AG*, *GD* are rational straight lines commensurable in square only, [x. 73]

and the annex *DG* is commensurable with the rational straight line *AC* set out,

while the square on the whole *AG* is greater than the square on the annex *GD* by the square on a straight line commensurable in length with *AG*. [x. Deff. III. 2]

Since then the square on *AG* is greater than the square on *GD* by the square on a straight line commensurable with *AG*,

therefore, if there be applied to *AG* a parallelogram equal to the fourth part of the square on *GD* and deficient by a square figure, it divides it into commensurable parts. [x. 17]

Let then *DG* be bisected at *E*,

let there be applied to *AG* a parallelogram equal to the square on *EG* and deficient by a square figure,

and let it be the rectangle *AF*, *FG*;

therefore *AF* is commensurable in length with *FG*.

Therefore *AG* is also commensurable in length with each of the straight lines *AF*, *FG*. [x. 15]

But *AG* is rational and incommensurable in length with *AC*;

therefore each of the straight lines *AF*, *FG* is also rational and incommensurable in length with *AC*; [x. 13]

therefore each of the rectangles *AI*, *FK* is medial. [x. 21]

Again, since *DE* is commensurable with *EG*,

therefore *DG* is also commensurable with each of the straight lines *DE*, *EG*. [x. 15]

But *DG* is commensurable in length with *AC*.

Therefore each of the rectangles *DH*, *EK* is rational. [x. 19]

Let then the square *LM* be constructed equal to *AI*,

and let there be subtracted *NO* equal to *FK* and being about the same angle with *LM*, namely the angle *LPM*;

therefore the squares *LM*, *NO* are about the same diameter. [vi. 26]

Let *PR* be their diameter, and let the figure be drawn.

Since then *AI*, *FK* are medial and are equal to the squares on *LP*, *PN*,

the squares on *LP*, *PN* are also medial;

therefore *LP*, *PN* are also medial straight lines commensurable in square only.

And, since the rectangle *AF*, *FG* is equal to the square on *EG*,

therefore, as *AF* is to *EG*, so is *EG* to *FG*, [vi. 17]

while, as *AF* is to *EG*, so is *AI* to *EK*,

and, as *EG* is to *FG*, so is *EK* to *FK*; [vi. 1]

therefore *EK* is a mean proportional between *AI*, *FK*. [v. 11]

But *MN* is also a mean proportional between the squares *LM*, *NO*,

and *AI* is equal to *LM*, and *FK* to *NO*;

therefore *MN* is also equal to *EK*.

But *DH* is equal to *EK*, and *LO* equal to *MN*;

therefore the whole *DK* is equal to the gnomon *UVW* and *NO*.

Since then the whole *AK* is equal to *LM*, *NO*,

and, in these, *DK* is equal to the gnomon *UVW* and *NO*,

therefore the remainder *AB* is equal to *TS*.

But *TS* is the square on *LN*;
therefore the square on *LN* is equal to the area *AB*;
therefore *LN* is the "side" of the area *AB*.

I say that *LN* is a first apotome of a medial straight line.

For, since *EK* is rational and is equal to *LO*,
therefore *LO*, that is, the rectangle *LP*, *PN*, is rational.

But *NO* was proved medial;
therefore *LO* is incommensurable with *NO*.

But, as *LO* is to *NO*, so is *LP* to *PN*; [VI. 1]
therefore *LP*, *PN* are incommensurable in length. [X. 11]

Therefore *LP*, *PN* are medial straight lines commensurable in square only which contain a rational rectangle;
therefore *LN* is a first apotome of a medial straight line. [X. 74]

And it is the "side" of the area *AB*.

Therefore the "side" of the area *AB* is a first apotome of a medial straight line.

Q. E. D.

There is an evident flaw in the text in the place (Heiberg, p. 282, ll. 17—20: translation p. 196 above) where it is said that "since then *AI*, *FK* are medial and are equal to the squares on *LP*, *PN*, the squares on *LP*, *PN* are also medial; *therefore* LP, PN *are also medial straight lines commensurable in square only*." It is not till the last lines of the proposition (Heiberg, p. 284, ll. 17, 18) that it is proved that *LP*, *PN* are *incommensurable in length*. What should have been proved in the former passage is that the *squares* on *LP*, *PN* are commensurable, so that *LP*, *PN* are commensurable in square (not commensurable in square *only*). I have supplied the step in the note below: "Also $x^2 \frown y^2$, since $u \frown v$." Theon seems to have observed the omission and to have put "and commensurable with one another" after "medial" in the passage quoted, though even this does not show *why* the squares on *LP*, *PN* are commensurable. One MS. (V) also has "only" (μόνον) erased after "commensurable in square."

This proposition amounts to finding and classifying

$$\sqrt{\rho\left(\frac{k\rho}{\sqrt{1-\lambda^2}} - k\rho\right)}.$$

The method is that of the last proposition. Euclid solves, first, the equations

$$\left.\begin{aligned} u + v &= \frac{k\rho}{\sqrt{1-\lambda^2}} \\ uv &= \tfrac{1}{4}k^2\rho^2 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Then, using the values of u, v so found, he puts

$$\left.\begin{array}{l} x^2 = \rho u \\ y^2 = \rho v \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

and $(x - y)$ is the square root required.

That $$(x - y) = \sqrt{\rho\left(\frac{k\rho}{\sqrt{1 - \lambda^2}} - k\rho\right)}$$

is proved in the same way as is the corresponding fact in X. 91.

From (1) $$u : \tfrac{1}{2}k\rho = \tfrac{1}{2}k\rho : v,$$

so that $$\rho u : \tfrac{1}{2}k\rho^2 = \tfrac{1}{2}k\rho^2 : \rho v.$$

But $$x^2 : xy = xy : y^2,$$

whence, by (2), $$xy = \tfrac{1}{2}k\rho^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Therefore $$\begin{aligned}(x - y)^2 &= x^2 + y^2 - 2xy \\ &= \rho(u + v) - k\rho^2 \\ &= \rho\left(\frac{k\rho}{\sqrt{1 - \lambda^2}} - k\rho\right).\end{aligned}$$

Next, we have to prove that $(x - y)$ is a *first apotome of a medial* straight line.

From (1) it follows, by X. 17, that

$$u \frown v \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4),$$

therefore u, v are both $\frown (u + v)$.

But [(1)] $(u + v)$ is rational and $\smile \rho$;

therefore u, v are both rational and $\smile \rho$ $\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$ (5).

Therefore ρu, ρv, or x^2, y^2, are both *medial* areas, and x, y are medial straight lines $\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$ (6).

Also $x^2 \frown y^2$, since $u \frown v$ [(4)] $\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$ (7).

Now xy, or $\frac{1}{2}k\rho^2$, is a *rational* area;

therefore $$xy \smile y^2,$$

and $$x \smile y.$$

Hence [(6), (7), (3)] x, y are medial straight lines commensurable in square only and containing a rational rectangle;

therefore $(x - y)$ is a *first apotome of a medial* straight line.

Algebraical solution of the equations gives

$$u = \frac{1}{2}\frac{1 + \lambda}{\sqrt{1 - \lambda^2}}k\rho,$$

$$v = \frac{1}{2}\frac{1 - \lambda}{\sqrt{1 - \lambda^2}}k\rho,$$

and $$x - y = \rho\sqrt{\frac{k}{2}\left(\frac{1 + \lambda}{1 - \lambda}\right)^{\frac{1}{2}}} - \rho\sqrt{\frac{k}{2}\left(\frac{1 - \lambda}{1 + \lambda}\right)^{\frac{1}{2}}}$$

As explained in the note on X. 55, this is the lesser positive root of the equation

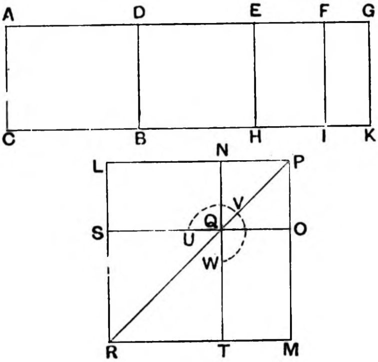
$$x^4 - \frac{2k\rho^2}{\sqrt{1 - \lambda^2}}x^2 + \frac{\lambda^2}{1 - \lambda^2}k^2\rho^4 = 0.$$

PROPOSITION 93.

If an area be contained by a rational straight line and a third apotome, the "side" of the area is a second apotome of a medial straight line.

For let the area AB be contained by the rational straight line AC and the third apotome AD;
I say that the "side" of the area AB is a second apotome of a medial straight line.

For let DG be the annex to AD;
therefore AG, GD are rational straight lines commensurable in square only,
and neither of the straight lines AG, GD is commensurable in length with the rational straight line AC set out,
while the square on the whole AG is greater than the square on the annex DG by the square on a straight line commensurable with AG. [x. Deff. III. 3]



Since then the square on AG is greater than the square on GD by the square on a straight line commensurable with AG,
therefore, if there be applied to AG a parallelogram equal to the fourth part of the square on DG and deficient by a square figure, it will divide it into commensurable parts. [x. 17]

Let then DG be bisected at E,
let there be applied to AG a parallelogram equal to the square on EG and deficient by a square figure,
and let it be the rectangle AF, FG.

Let *EH*, *FI*, *GK* be drawn through the points *E*, *F*, *G* parallel to *AC*.

Therefore *AF*, *FG* are commensurable;

therefore *AI* is also commensurable with *FK*. [vi. 1, x. 11]

And, since *AF*, *FG* are commensurable in length,

therefore *AG* is also commensurable in length with each of the straight lines *AF*, *FG*. [x. 15]

But *AG* is rational and incommensurable in length with *AC*;

so that *AF*, *FG* are so also. [x. 13]

Therefore each of the rectangles *AI*, *FK* is medial. [x. 21]

Again, since *DE* is commensurable in length with *EG*,

therefore *DG* is also commensurable in length with each of the straight lines *DE*, *EG*. [x. 15]

But *GD* is rational and incommensurable in length with *AC*;

therefore each of the straight lines *DE*, *EG* is also rational and incommensurable in length with *AC*; [x. 13]

therefore each of the rectangles *DH*, *EK* is medial. [x. 21]

And, since *AG*, *GD* are commensurable in square only,

therefore *AG* is incommensurable in length with *GD*.

But *AG* is commensurable in length with *AF*, and *DG* with *EG*;

therefore *AF* is incommensurable in length with *EG*. [x. 13]

But, as *AF* is to *EG*, so is *AI* to *EK*; [vi. 1]

therefore *AI* is incommensurable with *EK*. [x. 11]

Now let the square *LM* be constructed equal to *AI*,

and let there be subtracted *NO* equal to *FK* and being about the same angle with *LM*;

therefore *LM*, *NO* are about the same diameter. [vi. 26]

Let *PR* be their diameter, and let the figure be drawn.

Now, since the rectangle *AF*, *FG* is equal to the square on *EG*,

therefore, as *AF* is to *EG*, so is *EG* to *FG*. [vi. 17]

But, as *AF* is to *EG*, so is *AI* to *EK*,
and, as *EG* is to *FG*, so is *EK* to *FK*; [VI. 1]
therefore also, as *AI* is to *EK*, so is *EK* to *FK*; [V. 11]
therefore *EK* is a mean proportional between *AI*, *FK*.

But *MN* is also a mean proportional between the squares *LM*, *NO*,
and *AI* is equal to *LM*, and *FK* to *NO*;
therefore *EK* is also equal to *MN*.

But *MN* is equal to *LO*, and *EK* equal to *DH*;
therefore the whole *DK* is also equal to the gnomon *UVW* and *NO*.

But *AK* is also equal to *LM*, *NO*;
therefore the remainder *AB* is equal to *ST*, that is, to the square on *LN*;
therefore *LN* is the "side" of the area *AB*.

I say that *LN* is a second apotome of a medial straight line.

For, since *AI*, *FK* were proved medial, and are equal to the squares on *LP*, *PN*,
therefore each of the squares on *LP*, *PN* is also medial;
therefore each of the straight lines *LP*, *PN* is medial.

And, since *AI* is commensurable with *FK*, [VI. 1, X. 11]
therefore the square on *LP* is also commensurable with the square on *PN*.

Again, since *AI* was proved incommensurable with *EK*,
therefore *LM* is also incommensurable with *MN*,
that is, the square on *LP* with the rectangle *LP*, *PN*;
so that *LP* is also incommensurable in length with *PN*; [VI. 1, X. 11]
therefore *LP*, *PN* are medial straight lines commensurable in square only.

I say next that they also contain a medial rectangle.

For, since *EK* was proved medial, and is equal to the rectangle *LP*, *PN*,
therefore the rectangle *LP*, *PN* is also medial,
so that *LP*, *PN* are medial straight lines commensurable in square only which contain a medial rectangle.

Therefore LN is a second apotome of a medial straight line; [x. 75]

and it is the "side" of the area AB.

Therefore the "side" of the area AB is a second apotome of a medial straight line.

Q. E. D.

Here we are to find and classify the irrational straight line

$$\sqrt{\rho(\sqrt{k}.\rho - \sqrt{k}.\rho\sqrt{1-\lambda^2})}.$$

Following the same method, we put

$$\left.\begin{aligned} u + v &= \sqrt{k}\,.\,\rho \\ uv &= \tfrac{1}{4}k\rho^2(1-\lambda^2) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Next, u, v being found, let

$$\left.\begin{aligned} x^2 &= \rho u \\ y^2 &= \rho v \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2);$$

then $(x - y)$ is the square root required and is a *second apotome of a medial* straight line.

That $(x - y)$ is the square root required and that x^2, y^2 are medial areas, so that x, y are medial straight lines, is proved exactly as in the last proposition.

The rectangle xy, being equal to $\frac{1}{2}\sqrt{k}\,.\,\rho^2\sqrt{1-\lambda^2}$, is also medial.

Now, from (1), by x. 17, $\qquad u \frown v,$

whence $\qquad u + v \frown u.$

But $\qquad (u + v)$, or $\sqrt{k}\,.\,\rho$, $\smile \frac{1}{2}\sqrt{k}\,.\,\rho\sqrt{1-\lambda^2}$;

therefore $\qquad u \smile \frac{1}{2}\sqrt{k}\,.\,\rho\sqrt{1-\lambda^2},$

and consequently $\qquad \rho u \smile \frac{1}{2}\sqrt{k}\,.\,\rho^2\sqrt{1-\lambda^2},$

or $\qquad x^2 \smile xy,$

whence $\qquad x \smile y.$

And, since $u \frown v$, $\qquad \rho u \frown \rho v,$

or $\qquad x^2 \frown y^2$

Thus x, y are medial straight lines commensurable in square only.

And xy is a medial area.

Therefore $(x - y)$ is a *second apotome of a medial* straight line.

Its actual form is found by solving equations (1), (2);

thus
$$u = \tfrac{1}{2}(\sqrt{k}\,.\,\rho + \lambda\sqrt{k}\,.\,\rho),$$
$$v = \tfrac{1}{2}(\sqrt{k}\,.\,\rho - \lambda\sqrt{k}\,.\,\rho),$$

and
$$x - y = \rho\sqrt{\frac{\sqrt{k}}{2}(1+\lambda)} - \rho\sqrt{\frac{\sqrt{k}}{2}(1-\lambda)}.$$

As explained in the note on x. 56, this is the lesser positive root of the equation

$$x^4 - 2\sqrt{k}\,.\,\rho^2x^2 + \lambda^2k\rho^4 = 0.$$

PROPOSITION 94.

If an area be contained by a rational straight line and a fourth apotome, the "side" of the area is minor.

For let the area AB be contained by the rational straight line AC and the fourth apotome AD;

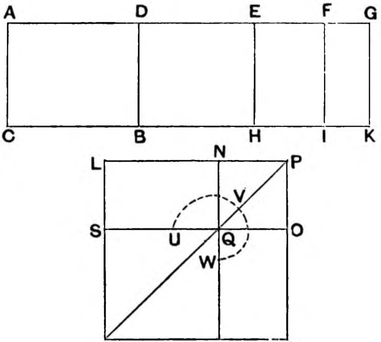
I say that the "side" of the area AB is minor.

For let DG be the annex to AD;

therefore AG, GD are rational straight lines commensurable in square only,

AG is commensurable in length with the rational straight line AC set out,

and the square on the whole AG is greater than the square on the annex DG by the square on a straight line incommensurable in length with AG, [x. Deff. III. 4]



Since then the square on AG is greater than the square on GD by the square on a straight line incommensurable in length with AG,

therefore, if there be applied to AG a parallelogram equal to the fourth part of the square on DG and deficient by a square figure, it will divide it into incommensurable parts. [x. 18]

Let then DG be bisected at E,

let there be applied to AG a parallelogram equal to the square on EG and deficient by a square figure,

and let it be the rectangle AF, FG;

therefore AF is incommensurable in length with FG.

Let *EH*, *FI*, *GK* be drawn through *E*, *F*, *G* parallel to *AC*, *BD*.

Since then *AG* is rational and commensurable in length with *AC*,
therefore the whole *AK* is rational. [x. 19]

Again, since *DG* is incommensurable in length with *AC*, and both are rational,
therefore *DK* is medial. [x. 21]

Again, since *AF* is incommensurable in length with *FG*,
therefore *AI* is also incommensurable with *FK*. [vi. 1, x. 11]

Now let the square *LM* be constructed equal to *AI*,
and let there be subtracted *NO* equal to *FK* and about the same angle, the angle *LPM*.

Therefore the squares *LM*, *NO* are about the same diameter. [vi. 26]

Let *PR* be their diameter, and let the figure be drawn.

Since then the rectangle *AF*, *FG* is equal to the square on *EG*,
therefore, proportionally, as *AF* is to *EG*, so is *EG* to *FG*. [vi. 17]

But, as *AF* is to *EG*, so is *AI* to *EK*,
and, as *EG* is to *FG*, so is *EK* to *FK*; [vi. 1]
therefore *EK* is a mean proportional between *AI*, *FK*. [v. 11]

But *MN* is also a mean proportional between the squares *LM*, *NO*,
and *AI* is equal to *LM*, and *FK* to *NO*;
therefore *EK* is also equal to *MN*.

But *DH* is equal to *EK*, and *LO* is equal to *MN*;
therefore the whole *DK* is equal to the gnomon *UVW* and *NO*.

Since, then, the whole *AK* is equal to the squares *LM*, *NO*,
and, in these, *DK* is equal to the gnomon *UVW* and the square *NO*,
therefore the remainder *AB* is equal to *ST*, that is, to the square on *LN*;
therefore *LN* is the "side" of the area *AB*.

I say that *LN* is the irrational straight line called minor.

For, since *AK* is rational and is equal to the squares on *LP*, *PN*,

therefore the sum of the squares on *LP*, *PN* is rational.

Again, since *DK* is medial,

and *DK* is equal to twice the rectangle *LP*, *PN*,

therefore twice the rectangle *LP*, *PN* is medial.

And, since *AI* was proved incommensurable with *FK*,

therefore the square on *LP* is also incommensurable with the square on *PN*.

Therefore *LP*, *PN* are straight lines incommensurable in square which make the sum of the squares on them rational, but twice the rectangle contained by them medial.

Therefore *LN* is the irrational straight line called minor; [X. 76]

and it is the "side" of the area *AB*.

Therefore the "side" of the area *AB* is minor.

Q. E. D.

We have here to find and classify the straight line

$$\sqrt{\rho\left(k\rho - \frac{k\rho}{\sqrt{1+\lambda}}\right)}.$$

As usual, we find u, v from the equations

$$\left.\begin{aligned} u + v &= k\rho \\ uv &= \tfrac{1}{4}\frac{k^2\rho^2}{1+\lambda} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

and then, giving u, v their values, we put

$$\left.\begin{aligned} x^2 &= \rho u \\ y^2 &= \rho v \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Then $(x - y)$ is the required square root.

This is proved in the same way as before, and, as before, it is proved that

$$xy = \tfrac{1}{2}\frac{k\rho^2}{\sqrt{1+\lambda}}.$$

Now, from (1), by X. 18, $u \smile v$;

therefore $\rho u \smile \rho v$,

or $x^2 \smile y^2$,

so that x, y are incommensurable in square.

And $x^2 + y^2$, or $\rho(u + v)$, is a *rational* area ($k\rho^2$).

But $2xy = \dfrac{k\rho^2}{\sqrt{1+\lambda}}$, which is a *medial* area.

Hence [X. 76] $(x - y)$ is the irrational straight line called *minor*.

Algebraical solution gives

$$u = \tfrac{1}{2}k\rho\left(1 + \sqrt{\frac{\lambda}{1+\lambda}}\right),$$

$$v = \tfrac{1}{2}k\rho\left(1 - \sqrt{\frac{\lambda}{1+\lambda}}\right),$$

whence $$x - y = \rho\sqrt{\frac{k}{2}\left(1 + \sqrt{\frac{\lambda}{1+\lambda}}\right)} - \rho\sqrt{\frac{k}{2}\left(1 - \sqrt{\frac{\lambda}{1+\lambda}}\right)}.$$

As explained in the note on x. 57, this is the lesser positive root of the equation

$$x^4 - 2k\rho^2 \,.\, x^2 + \frac{\lambda}{1+\lambda}\,k^2\rho^4 = 0.$$

PROPOSITION 95.

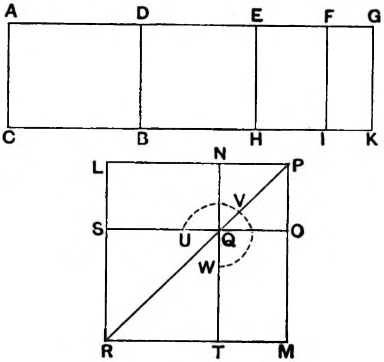
If an area be contained by a rational straight line and a fifth apotome, the "side" of the area is a straight line which produces with a rational area a medial whole.

For let the area *AB* be contained by the rational straight line *AC* and the fifth apotome *AD*;

I say that the "side" of the area *AB* is a straight line which produces with a rational area a medial whole.

For let *DG* be the annex to *AD*;

therefore *AG*, *GD* are rational straight lines commensurable in square only,



the annex *GD* is commensurable in length with the rational straight line *AC* set out,

and the square on the whole *AG* is greater than the square on the annex *DG* by the square on a straight line incommensurable with *AG*. [x. Deff. III. 5]

Therefore, if there be applied to *AG* a parallelogram equal to the fourth part of the square on *DG* and deficient by a square figure, it will divide it into incommensurable parts. [x. 18]

Let then *DG* be bisected at the point *E*,

let there be applied to *AG* a parallelogram equal to the square on *EG* and deficient by a square figure, and let it be the rectangle *AF*, *FG*;

therefore *AF* is incommensurable in length with *FG*.

Now, since *AG* is incommensurable in length with *CA*, and both are rational,

therefore *AK* is medial. [x. 21]

Again, since *DG* is rational and commensurable in length with *AC*,

DK is rational. [x. 19]

Now let the square *LM* be constructed equal to *AI*, and let the square *NO* equal to *FK* and about the same angle, the angle *LPM*, be subtracted;

therefore the squares *LM*, *NO* are about the same diameter. [VI. 26]

Let *PR* be their diameter, and let the figure be drawn.

Similarly then we can prove that *LN* is the "side" of the area *AB*.

I say that *LN* is the straight line which produces with a rational area a medial whole.

For, since *AK* was proved medial and is equal to the squares on *LP*, *PN*,

therefore the sum of the squares on *LP*, *PN* is medial.

Again, since *DK* is rational and is equal to twice the rectangle *LP*, *PN*,

the latter is itself also rational.

And, since *AI* is incommensurable with *FK*,

therefore the square on *LP* is also incommensurable with the square on *PN*;

therefore *LP*, *PN* are straight lines incommensurable in square which make the sum of the squares on them medial but twice the rectangle contained by them rational.

Therefore the remainder *LN* is the irrational straight line called that which produces with a rational area a medial whole; [x. 77]

and it is the "side" of the area *AB*.

Therefore the "side" of the area *AB* is a straight line which produces with a rational area a medial whole.

Q. E. D.

Here the problem is to find and classify

$$\sqrt{\rho\,(k\rho\sqrt{1+\lambda} - k\rho)}.$$

As usual, we put

$$\left.\begin{aligned} u + v &= k\rho\sqrt{1+\lambda} \\ uv &= \tfrac{1}{4}k^2\rho^2 \end{aligned}\right\} \quad \dots\dots\dots\dots\dots(1),$$

and, u, v being found, we take

$$\left.\begin{aligned} x^2 &= \rho u \\ y^2 &= \rho v \end{aligned}\right\} \quad \dots\dots\dots\dots\dots(2).$$

Then $(x - y)$ so found is our required square root.

This fact is proved as before, and, as before, we see that

$$xy = \tfrac{1}{2}k\rho^2.$$

Now from (1), by x. 18, $u \smile v$,

whence $\rho u \smile \rho v$,

or $x^2 \smile y^2$,

and x, y are incommensurable in square.

Next $(x^2 + y^2) = \rho\,(u + v) = k\rho^2\sqrt{1+\lambda}$, which is a *medial* area.

And $2xy = k\rho^2$, which is a *rational* area.

Hence $(x - y)$ is the "*side*" *of a medial, minus a rational, area.* [x. 77]

Algebraical solution gives

$$u = \frac{k\rho}{2}\left(\sqrt{1+\lambda} + \sqrt{\lambda}\right),$$

$$v = \frac{k\rho}{2}\left(\sqrt{1+\lambda} - \sqrt{\lambda}\right),$$

and therefore

$$x - y = \rho\sqrt{\frac{k}{2}\left(\sqrt{1+\lambda} + \sqrt{\lambda}\right)} - \rho\sqrt{\frac{k}{2}\left(\sqrt{1+\lambda} - \sqrt{\lambda}\right)},$$

which is, as explained in the note to x. 58, the lesser positive root of the equation

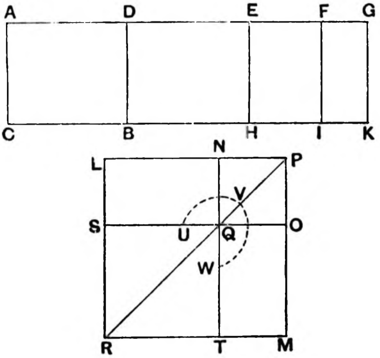
$$x^4 - 2k\rho^2\sqrt{1+\lambda}\,.\,x^2 + \lambda k^2\rho^4 = 0.$$

PROPOSITION 96.

If an area be contained by a rational straight line and a sixth apotome, the "side" of the area is a straight line which produces with a medial area a medial whole.

For let the area AB be contained by the rational straight line AC and the sixth apotome AD;

I say that the "side" of the area AB is a straight line which produces with a medial area a medial whole.



For let DG be the annex to AD;

therefore AG, GD are rational straight lines commensurable in square only,

neither of them is commensurable in length with the rational straight line AC set out,

and the square on the whole AG is greater than the square on the annex DG by the square on a straight line incommensurable in length with AG. [x. Deff. III. 6]

Since then the square on AG is greater than the square on GD by the square on a straight line incommensurable in length with AG,

therefore, if there be applied to AG a parallelogram equal to the fourth part of the square on DG and deficient by a square figure, it will divide it into incommensurable parts. [x. 18]

Let then DG be bisected at E,

let there be applied to AG a parallelogram equal to the square on EG and deficient by a square figure, and let it be the rectangle AF, FG;

therefore AF is incommensurable in length with FG.

But, as AF is to FG, so is AI to FK; [VI. 1]

therefore AI is incommensurable with FK. [X. 11]

And, since AG, AC are rational straight lines commensurable in square only,

AK is medial. [X. 21]

Again, since AC, DG are rational straight lines and incommensurable in length,

DK is also medial. [X. 21]

Now, since AG, GD are commensurable in square only, therefore AG is incommensurable in length with GD.

But, as AG is to GD, so is AK to KD; [VI. 1]

therefore AK is incommensurable with KD. [X. 11]

Now let the square LM be constructed equal to AI, and let NO equal to FK, and about the same angle, be subtracted;

therefore the squares LM, NO are about the same diameter. [VI. 26]

Let PR be their diameter, and let the figure be drawn.

Then in manner similar to the above we can prove that LN is the "side" of the area AB.

I say that LN is a straight line which produces with a medial area a medial whole.

For, since AK was proved medial and is equal to the squares on LP, PN,

therefore the sum of the squares on LP, PN is medial.

Again, since DK was proved medial and is equal to twice the rectangle LP, PN,

twice the rectangle LP, PN is also medial.

And, since AK was proved incommensurable with DK, the squares on LP, PN are also incommensurable with twice the rectangle LP, PN.

And, since AI is incommensurable with FK,

therefore the square on LP is also incommensurable with the square on PN;

therefore *LP*, *PN* are straight lines incommensurable in square which make the sum of the squares on them medial, twice the rectangle contained by them medial, and further the squares on them incommensurable with twice the rectangle contained by them.

Therefore *LN* is the irrational straight line called that which produces with a medial area a medial whole; [x. 78] and it is the "side" of the area *AB*.

Therefore the "side" of the area is a straight line which produces with a medial area a medial whole.

Q. E. D.

We have to find and classify

$$\sqrt{\rho\left(\sqrt{k}\,.\,\rho-\sqrt{\lambda}\,.\,\rho\right)}.$$

Put, as usual,

$$\left.\begin{aligned} u+v &= \sqrt{k}\,.\,\rho \\ uv &= \tfrac{1}{4}\lambda\rho^2 \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

and, *u*, *v* being thus found, let

$$\left.\begin{aligned} x^2 &= \rho u \\ y^2 &= \rho v \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Then, as before, $(x-y)$ is the square root required.

For, from (1), by x. 18, $u \smile v$,

whence $\rho u \smile \rho v$,

or $x^2 \smile y^2$,

and x, y are incommensurable in square.

Next, $x^2+y^2=\rho\,(u+v)=\sqrt{k}\,.\,\rho^2$, which is a *medial* area.

Also $2xy=\sqrt{\lambda}\,.\,\rho^2$, which is again a *medial* area.

Lastly, $\sqrt{k}\,.\,\rho$, $\sqrt{\lambda}\,.\,\rho$ are by hypothesis $\frown$, so that

$$\sqrt{k}\,.\,\rho \smile \sqrt{\lambda}\,.\,\rho,$$

whence $\sqrt{k}\,.\,\rho^2 \smile \sqrt{\lambda}\,.\,\rho^2$,

or $(x^2+y^2) \smile 2xy$.

Thus $(x-y)$ is the "*side*" *of a medial, minus a medial, area* [x. 78].

Algebraical solution gives

$$u=\frac{\rho}{2}\left(\sqrt{k}+\sqrt{k-\lambda}\right),$$

$$v=\frac{\rho}{2}\left(\sqrt{k}-\sqrt{k-\lambda}\right),$$

whence $x-y=\rho\sqrt{\tfrac{1}{2}\left(\sqrt{k}+\sqrt{k-\lambda}\right)}-\rho\sqrt{\tfrac{1}{2}\left(\sqrt{k}-\sqrt{k-\lambda}\right)}$.

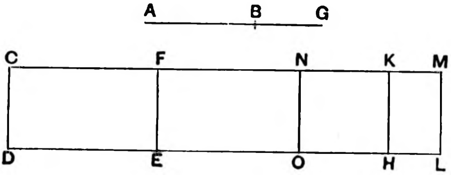
This, as explained in the note on x. 59, is the lesser positive root of the equation

$$x^4-2\sqrt{k}\,.\,\rho^2x^2+(k-\lambda)\,\rho^4=0.$$

PROPOSITION 97.

The square on an apotome applied to a rational straight line produces as breadth a first apotome.

Let AB be an apotome, and CD rational, and to CD let there be applied CE equal to the square on AB and producing CF as breadth;
I say that CF is a first apotome.



For let BG be the annex to AB;
therefore AG, GB are rational straight lines commensurable in square only. [x. 73]

To CD let there be applied CH equal to the square on AG, and KL equal to the square on BG.

Therefore the whole CL is equal to the squares on AG, GB, and, in these, CE is equal to the square on AB;
therefore the remainder FL is equal to twice the rectangle AG, GB. [II. 7]

Let FM be bisected at the point N,
and let NO be drawn through N parallel to CD;
therefore each of the rectangles FO, LN is equal to the rectangle AG, GB.

Now, since the squares on AG, GB are rational,
and DM is equal to the squares on AG, GB,
therefore DM is rational.

And it has been applied to the rational straight line CD, producing CM as breadth;
therefore CM is rational and commensurable in length with CD. [x. 20]

Again, since twice the rectangle AG, GB is medial, and FL is equal to twice the rectangle AG, GB,
therefore FL is medial.

And it is applied to the rational straight line *CD*, producing *FM* as breadth;

therefore *FM* is rational and incommensurable in length with *CD*. [x. 22]

And, since the squares on *AG*, *GB* are rational,

while twice the rectangle *AG*, *GB* is medial,

therefore the squares on *AG*, *GB* are incommensurable with twice the rectangle *AG*, *GB*.

And *CL* is equal to the squares on *AG*, *GB*,

and *FL* to twice the rectangle *AG*, *GB*;

therefore *DM* is incommensurable with *FL*.

But, as *DM* is to *FL*, so is *CM* to *FM*; [vi. 1]

therefore *CM* is incommensurable in length with *FM*. [x. 11]

And both are rational;

therefore *CM*, *MF* are rational straight lines commensurable in square only;

therefore *CF* is an apotome. [x. 73]

I say next that it is also a first apotome.

For, since the rectangle *AG*, *GB* is a mean proportional between the squares on *AG*, *GB*,

and *CH* is equal to the square on *AG*,

KL equal to the square on *BG*,

and *NL* equal to the rectangle *AG*, *GB*,

therefore *NL* is also a mean proportional between *CH*, *KL*;

therefore, as *CH* is to *NL*, so is *NL* to *KL*.

But, as *CH* is to *NL*, so is *CK* to *NM*,

and, as *NL* is to *KL*, so is *NM* to *KM*; [vi. 1]

therefore the rectangle *CK*, *KM* is equal to the square on *NM* [vi. 17], that is, to the fourth part of the square on *FM*.

And, since the square on *AG* is commensurable with the square on *GB*,

CH is also commensurable with *KL*.

But, as *CH* is to *KL*, so is *CK* to *KM*; [vi. 1]

therefore *CK* is commensurable with *KM*. [x. 11]

Since then *CM*, *MF* are two unequal straight lines, and to *CM* there has been applied the rectangle *CK*, *KM* equal to the fourth part of the square on *FM* and deficient by a square figure,

while *CK* is commensurable with *KM*,

therefore the square on *CM* is greater than the square on *MF* by the square on a straight line commensurable in length with *CM*. [x. 17]

And *CM* is commensurable in length with the rational straight line *CD* set out;

therefore *CF* is a first apotome. [x. Deff. III. 1]

Therefore etc.

Q. E. D.

Here begins the hexad of propositions solving the problems which are the converse of those in the hexad just concluded. Props. 97 to 102 correspond of course to Props. 60 to 65 relating to the binomials etc.

We have in x. 97 to prove that, $(\rho - \sqrt{k} . \rho)$ being an *apotome*,

$$\frac{(\rho \sim \sqrt{k} . \rho)^2}{\sigma}$$

is a *first apotome*, and we have to find it geometrically.

Euclid's procedure may be represented thus.

Take x, y, z such that

$$\left.\begin{aligned} \sigma x &= \rho^2 \\ \sigma y &= k\rho^2 \\ \sigma . 2z &= 2\sqrt{k} . \rho^2 \end{aligned}\right\} \quad \text{.............................(1)}.$$

Thus $$(x+y) - 2z = \frac{(\rho - \sqrt{k} . \rho)^2}{\sigma},$$

and we have to prove that $(x+y) - 2z$ is a *first apotome*.

(α) Now $\rho^2 + k\rho^2$, or $\sigma(x+y)$, is rational;

therefore $(x+y)$ is rational and $\frown \sigma$ ....(2).

And $2\sqrt{k} . \rho^2$, or $\sigma . 2z$, is medial:

therefore $2z$ is rational and $\smile \sigma$..(3).

But, $\sigma(x+y)$ being *rational*, and $\sigma . 2z$ *medial*,

$$\sigma(x+y) \smile \sigma . 2z,$$

whence $$(x+y) \smile 2z.$$

Therefore, since $(x+y)$, $2z$ are both rational [(2), (3)],

$(x+y)$, $2z$ are rational and $\frown\!\!\!\!-$ ....(4).

Hence $(x+y) - 2z$ is an *apotome*.

(β) Since $\sqrt{k} . \rho^2$ is a mean proportional between ρ^2, $k\rho^2$,

σz is a mean proportional between σx, σy [by (1)].

That is, $$\sigma x : \sigma z = \sigma z : \sigma y,$$

or $$x : z = z : y,$$

and $$xy = z^2, \text{ or } \tfrac{1}{4}(2z)^2 \text{ ....(5)}.$$

And, since $\rho^2 \frown k\rho^2$, $\sigma x \frown \sigma y$,

or $x \frown y \quad \dots\dots\dots\dots\dots\dots \text{(6)}.$

Hence [(5), (6)], by x. 17,

$$\sqrt{(x+y)^2 - (2z)^2} \frown (x+y).$$

And [(4)] $(x+y)$, $2z$ are rational and $\frown\!\!-$,

while [(2)] $(x+y) \frown \sigma$;

therefore $(x+y) - 2z$ is a *first apotome.*

The actual value of $(x+y) - 2z$ is of course

$$\frac{\rho^2}{\sigma}\{(1+k) - 2\sqrt{k}\}.$$

PROPOSITION 98.

The square on a first apotome of a medial straight line applied to a rational straight line produces as breadth a second apotome.

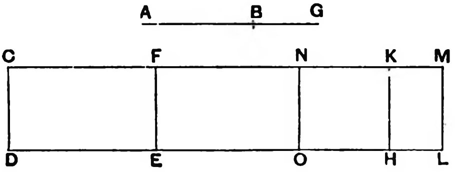
Let AB be a first apotome of a medial straight line and CD a rational straight line,

and to CD let there be applied CE equal to the square on AB, producing CF as breadth;

I say that CF is a second apotome.

For let BG be the annex to AB;

therefore AG, GB are medial straight lines commensurable in square only which contain a rational rectangle. [x. 74]



To CD let there be applied CH equal to the square on AG, producing CK as breadth, and KL equal to the square on GB, producing KM as breadth;

therefore the whole CL is equal to the squares on AG, GB;

therefore CL is also medial. [x. 15 and 23, Por.]

And it is applied to the rational straight line CD, producing CM as breadth;

therefore CM is rational and incommensurable in length with CD. [x. 22]

Now, since *CL* is equal to the squares on *AG*, *GB*,
and, in these, the square on *AB* is equal to *CE*,
therefore the remainder, twice the rectangle *AG*, *GB*, is equal to *FL*. [II. 7]

But twice the rectangle *AG*, *GB* is rational;
therefore *FL* is rational.

And it is applied to the rational straight line *FE*, producing *FM* as breadth;
therefore *FM* is also rational and commensurable in length with *CD*. [X. 20]

Now, since the sum of the squares on *AG*, *GB*, that is, *CL*, is medial, while twice the rectangle *AG*, *GB*, that is, *FL*, is rational,
therefore *CL* is incommensurable with *FL*.

But, as *CL* is to *FL*, so is *CM* to *FM*; [VI. 1]
therefore *CM* is incommensurable in length with *FM*. [X. 11]

And both are rational;
therefore *CM*, *MF* are rational straight lines commensurable in square only;
therefore *CF* is an apotome. [X. 73]

I say next that it is also a second apotome.

For let *FM* be bisected at *N*,
and let *NO* be drawn through *N* parallel to *CD*;
therefore each of the rectangles *FO*, *NL* is equal to the rectangle *AG*, *GB*.

Now, since the rectangle *AG*, *GB* is a mean proportional between the squares on *AG*, *GB*,
and the square on *AG* is equal to *CH*,
the rectangle *AG*, *GB* to *NL*,
and the square on *BG* to *KL*,
therefore *NL* is also a mean proportional between *CH*, *KL*;
therefore, as *CH* is to *NL*, so is *NL* to *KL*.

But, as *CH* is to *NL*, so is *CK* to *NM*,
and, as *NL* is to *KL*, so is *NM* to *MK*; [VI. 1]
therefore, as *CK* is to *NM*, so is *NM* to *KM*; [V. 11]
therefore the rectangle *CK*, *KM* is equal to the square on *NM* [VI. 17], that is, to the fourth part of the square on *FM*.

Since then *CM*, *MF* are two unequal straight lines, and the rectangle *CK*, *KM* equal to the fourth part of the square on *MF* and deficient by a square figure has been applied to the greater, *CM*, and divides it into commensurable parts, therefore the square on *CM* is greater than the square on *MF* by the square on a straight line commensurable in length with *CM*. [x. 17]

And the annex *FM* is commensurable in length with the rational straight line *CD* set out;
therefore *CF* is a second apotome. [x. Deff. III. 2]

Therefore etc.

Q. E. D.

In this case we have to find and classify

$$\frac{(k^{\frac{1}{4}}\rho \sim k^{\frac{3}{4}}\rho)^2}{\sigma}$$

Take x, y, z such that

$$\left.\begin{aligned} \sigma x &= k^{\frac{1}{2}}\rho^2 \\ \sigma y &= k^{\frac{3}{2}}\rho^2 \\ \sigma \,.\, 2z &= 2k\rho^2 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

(α) Now $k^{\frac{1}{2}}\rho^2$, $k^{\frac{3}{2}}\rho^2$ are medial areas;
therefore $\sigma(x+y)$ is medial,
whence $(x+y)$ is rational and $\smile \sigma$ $\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$.

But $2k\rho^2$, and therefore $\sigma \,.\, 2z$, is rational,
whence $2z$ is rational and $\frown \sigma$ $\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$.

And, $\sigma(x+y)$ being medial, and $\sigma \,.\, 2z$ rational,

$$\sigma(x+y) \smile \sigma \,.\, 2z,$$

or

$$(x+y) \smile 2z.$$

Hence $(x+y)$, $2z$ are rational straight lines commensurable in square only, and therefore $(x+y)-2z$ is an *apotome*.

(β) We prove, as before, that

$$xy = \tfrac{1}{4}(2z)^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

Also $k^{\frac{1}{2}}\rho^2 \frown k^{\frac{3}{2}}\rho^2$, or $\sigma x \frown \sigma y$,
so that

$$x \frown y \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

[This step is omitted in P, and Heiberg accordingly brackets it. The result is, however, assumed.]

Therefore [(4), (5)], by x. 17,

$$\sqrt{(x+y)^2-(2z)^2} \frown (x+y).$$

And $2z \frown \sigma$.
Therefore $(x+y)-2z$ is a *second apotome*.

Obviously $$(x+y)-2z = \frac{\rho^2}{\sigma}\{\sqrt{k}\,(1+k)-2k\}.$$

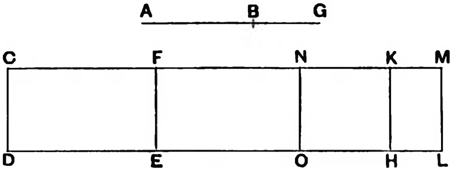
PROPOSITION 99.

The square on a second apotome of a medial straight line applied to a rational straight line produces as breadth a third apotome.

Let AB be a second apotome of a medial straight line, and CD rational,

and to CD let there be applied CE equal to the square on AB, producing CF as breadth;

I say that CF is a third apotome.



For let BG be the annex to AB;

therefore AG, GB are medial straight lines commensurable in square only which contain a medial rectangle. [x. 75]

Let CH equal to the square on AG be applied to CD, producing CK as breadth,

and let KL equal to the square on BG be applied to KH, producing KM as breadth;

therefore the whole CL is equal to the squares on AG, GB;

therefore CL is also medial. [x. 15 and 23, Por.]

And it is applied to the rational straight line CD, producing CM as breadth;

therefore CM is rational and incommensurable in length with CD. [x. 22]

Now, since the whole CL is equal to the squares on AG, GB, and, in these, CE is equal to the square on AB,

therefore the remainder LF is equal to twice the rectangle AG, GB. [II. 7]

Let then FM be bisected at the point N,

and let NO be drawn parallel to CD;

therefore each of the rectangles FO, NL is equal to the rectangle AG, GB.

But the rectangle *AG*, *GB* is medial;
therefore *FL* is also medial.

And it is applied to the rational straight line *EF*, producing *FM* as breadth;
therefore *FM* is also rational and incommensurable in length with *CD*. [x. 22]

And, since *AG*, *GB* are commensurable in square only,
therefore *AG* is incommensurable in length with *GB*;
therefore the square on *AG* is also incommensurable with the rectangle *AG*, *GB*. [vi. 1, x. 11]

But the squares on *AG*, *GB* are commensurable with the square on *AG*,
and twice the rectangle *AG*, *GB* with the rectangle *AG*, *GB*;
therefore the squares on *AG*, *GB* are incommensurable with twice the rectangle *AG*, *GB*. [x. 13]

But *CL* is equal to the squares on *AG*, *GB*,
and *FL* is equal to twice the rectangle *AG*, *GB*;
therefore *CL* is also incommensurable with *FL*.

But, as *CL* is to *FL*, so is *CM* to *FM*; [vi. 1]
therefore *CM* is incommensurable in length with *FM*. [x. 11]

And both are rational;
therefore *CM*, *MF* are rational straight lines commensurable in square only;
therefore *CF* is an apotome. [x. 73]

I say next that it is also a third apotome.

For, since the square on *AG* is commensurable with the square on *GB*,
therefore *CH* is also commensurable with *KL*,
so that *CK* is also commensurable with *KM*. [vi. 1, x. 11]

And, since the rectangle *AG*, *GB* is a mean proportional between the squares on *AG*, *GB*,
and *CH* is equal to the square on *AG*,
KL equal to the square on *GB*,
and *NL* equal to the rectangle *AG*, *GB*,
therefore *NL* is also a mean proportional between *CH*, *KL*;
therefore, as *CH* is to *NL*, so is *NL* to *KL*.

But, as *CH* is to *NL*, so is *CK* to *NM*,
and, as *NL* is to *KL*, so is *NM* to *KM*; [vi. 1]
therefore, as *CK* is to *MN*, so is *MN* to *KM*; [v. 11]
therefore the rectangle *CK*, *KM* is equal to [the square on *MN*, that is, to] the fourth part of the square on *FM*.

Since then *CM*, *MF* are two unequal straight lines, and a parallelogram equal to the fourth part of the square on *FM* and deficient by a square figure has been applied to *CM*, and divides it into commensurable parts,
therefore the square on *CM* is greater than the square on *MF* by the square on a straight line commensurable with *CM*. [x. 17]

And neither of the straight lines *CM*, *MF* is commensurable in length with the rational straight line *CD* set out;
therefore *CF* is a third apotome. [x. Deff. iii. 3]

Therefore etc.

Q. E. D.

We have to find and classify

$$\frac{1}{\sigma}\left(k^{\frac{1}{4}}\rho \sim \frac{\sqrt{\lambda}\,.\,\rho}{k^{\frac{1}{4}}}\right)^2.$$

Take x, y, z such that

$$\left.\begin{aligned}\sigma x &= \sqrt{k}\,.\,\rho^2\\ \sigma y &= \frac{\lambda}{\sqrt{k}}\,.\,\rho^2\\ \sigma\,.\,2z &= 2\sqrt{\lambda}\,.\,\rho^2\end{aligned}\right\}.$$

(a) Then $\sigma(x+y)$ is a medial area,
whence $(x+y)$ is rational and $\smile \sigma$ ....(1).

Also $\sigma\,.\,2z$ is medial,
whence $2z$ is rational and $\smile \sigma$ ....(2).

Again $$k^{\frac{1}{4}}\rho \smile \frac{\sqrt{\lambda}\,.\,\rho}{k^{\frac{1}{4}}},$$

whence $$\sqrt{k}\,.\,\rho^2 \smile \sqrt{\lambda}\,.\,\rho^2.$$

And $$\sqrt{k}\,.\,\rho^2 \frown \left(\sqrt{k}\,.\,\rho^2 + \frac{\lambda}{\sqrt{k}}\rho^2\right),$$

while $$\sqrt{\lambda}\,.\,\rho^2 \frown 2\sqrt{\lambda}\,.\,\rho^2;$$

therefore $$\left(\sqrt{k}\,.\,\rho^2 + \frac{\lambda}{\sqrt{k}}\rho^2\right) \smile 2\sqrt{\lambda}\,.\,\rho^2,$$

or $$\sigma(x+y) \smile \sigma\,.\,2z,$$

and $(x+y) \smile 2z$ ....(3).

Thus [(1), (2), (3)] $(x+y)$, $2z$ are rational and $\frown\!-$, so that $(x+y)-2z$ is an *apotome*.

(β) $\sigma x \frown \sigma y$, so that $x \frown y$.

And, as before, $$xy = \tfrac{1}{4}(2z)^2.$$

Therefore [x. 17] $$\sqrt{(x+y)^2-(2z)^2} \frown (x+y).$$

And neither $(x+y)$ nor $2z$ is $\frown \sigma$.

Therefore $(x+y)-2z$ is a *third apotome.*

It is of course equal to

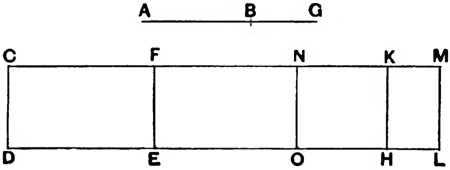
$$\frac{\rho^2}{\sigma}\left\{\frac{k+\lambda}{\sqrt{k}} - 2\sqrt{\lambda}\right\}.$$

PROPOSITION 100.

The square on a minor straight line applied to a rational straight line produces as breadth a fourth apotome.

Let AB be a minor and CD a rational straight line, and to the rational straight line CD let CE be applied equal to the square on AB and producing CF as breadth;

I say that CF is a fourth apotome.



For let BG be the annex to AB;

therefore AG, GB are straight lines incommensurable in square which make the sum of the squares on AG, GB rational, but twice the rectangle AG, GB medial. [x. 76]

To CD let there be applied CH equal to the square on AG and producing CK as breadth,

and KL equal to the square on BG, producing KM as breadth;

therefore the whole CL is equal to the squares on AG, GB.

And the sum of the squares on AG, GB is rational;

therefore CL is also rational.

And it is applied to the rational straight line CD, producing CM as breadth;

therefore CM is also rational and commensurable in length with CD. [x. 20]

And, since the whole CL is equal to the squares on AG, GB, and, in these, CE is equal to the square on AB,

therefore the remainder FL is equal to twice the rectangle AG, GB. [II. 7]

Let then FM be bisected at the point N,

and let NO be drawn through N parallel to either of the straight lines CD, ML;

therefore each of the rectangles FO, NL is equal to the rectangle AG, GB.

And, since twice the rectangle AG, GB is medial and is equal to FL,

therefore FL is also medial.

And it is applied to the rational straight line FE, producing FM as breadth;

therefore FM is rational and incommensurable in length with CD. [X. 22]

And, since the sum of the squares on AG, GB is rational, while twice the rectangle AG, GB is medial,

the squares on AG, GB are incommensurable with twice the rectangle AG, GB.

But CL is equal to the squares on AG, GB,

and FL equal to twice the rectangle AG, GB;

therefore CL is incommensurable with FL.

But, as CL is to FL, so is CM to MF; [VI. 1]

therefore CM is incommensurable in length with MF. [X. 11]

And both are rational;

therefore CM, MF are rational straight lines commensurable in square only;

therefore CF is an apotome. [X. 73]

I say that it is also a fourth apotome.

For, since AG, GB are incommensurable in square,

therefore the square on AG is also incommensurable with the square on GB.

And CH is equal to the square on AG,

and KL equal to the square on GB;

therefore CH is incommensurable with KL.

But, as *CH* is to *KL*, so is *CK* to *KM*; [VI. 1]
therefore *CK* is incommensurable in length with *KM*. [X. 11]

And, since the rectangle *AG*, *GB* is a mean proportional between the squares on *AG*, *GB*,
and the square on *AG* is equal to *CH*,
the square on *GB* to *KL*,
and the rectangle *AG*, *GB* to *NL*,
therefore *NL* is a mean proportional between *CH*, *KL*;
therefore, as *CH* is to *NL*, so is *NL* to *KL*.

But, as *CH* is to *NL*, so is *CK* to *NM*,
and, as *NL* is to *KL*, so is *NM* to *KM*; [VI. 1]
therefore, as *CK* is to *MN*, so is *MN* to *KM*; [V. 11]
therefore the rectangle *CK*, *KM* is equal to the square on *MN* [VI. 17], that is, to the fourth part of the square on *FM*.

Since then *CM*, *MF* are two unequal straight lines, and the rectangle *CK*, *KM* equal to the fourth part of the square on *MF* and deficient by a square figure has been applied to *CM* and divides it into incommensurable parts,
therefore the square on *CM* is greater than the square on *MF* by the square on a straight line incommensurable with *CM*. [X. 18]

And the whole *CM* is commensurable in length with the rational straight line *CD* set out;
therefore *CF* is a fourth apotome. [X. Deff. III. 4]

Therefore etc.

Q. E. D.

We have to find and classify

$$\frac{1}{\sigma}\left\{\frac{\rho}{\sqrt{2}}\sqrt{1+\frac{k}{\sqrt{1+k^2}}}-\frac{\rho}{\sqrt{2}}\sqrt{1-\frac{k}{\sqrt{1+k^2}}}\right\}^2.$$

We will call this, for brevity,

$$\frac{1}{\sigma}(u-v)^2.$$

Take x, y, z such that

$$\left.\begin{aligned}\sigma x &= u^2\\ \sigma y &= v^2\\ \sigma\,.\,2z &= 2uv\end{aligned}\right\},$$

where it has to be remembered that u^2, v^2 are incommensurable, (u^2+v^2) is rational, and $2uv$ medial.

It follows that $\sigma(x+y)$ is rational and $\sigma\,.\,2z$ medial,
so that $(x+y)$ is rational and $\frown \sigma$ ....(1),
while $2z$ is rational and $\smile \sigma$ ....(2),
and $\sigma(x+y) \smile \sigma\,.\,2z$,
so that $(x+y) \smile 2z$ ....(3).

Thus [(1), (2), (3)] $(x+y)$, $2z$ are rational and $\frown\!-$,
so that $(x+y)-2z$ is an *apotome*.

Next, since

$$u^2 \smile v^2,$$

$$\sigma x \smile \sigma y,$$

or

$$x \smile y.$$

And it is proved, as usual, that

$$xy = z^2 = \tfrac{1}{4}(2z)^2.$$

Therefore [x. 18] $\sqrt{(x+y)^2-(2z)^2} \smile (x+y)$.

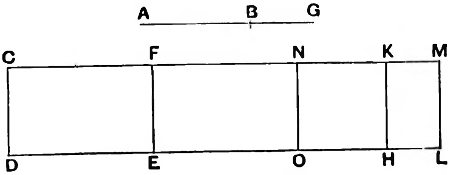
But $(x+y) \frown \sigma$,
therefore $x+y-2z$ is a *fourth apotome*.

Its value is of course $\dfrac{\rho^2}{\sigma}\left(1-\dfrac{1}{\sqrt{1+k^2}}\right)$.

PROPOSITION 101.

The square on the straight line which produces with a rational area a medial whole, if applied to a rational straight line, produces as breadth a fifth apotome.

Let *AB* be the straight line which produces with a rational area a medial whole, and *CD* a rational straight line, and to *CD* let *CE* be applied equal to the square on *AB* and producing *CF* as breadth;
I say that *CF* is a fifth apotome.



For let *BG* be the annex to *AB*;
therefore *AG*, *GB* are straight lines incommensurable in square which make the sum of the squares on them medial but twice the rectangle contained by them rational. [x. 77]

To *CD* let there be applied *CH* equal to the square on *AG*, and *KL* equal to the square on *GB*;
therefore the whole *CL* is equal to the squares on *AG*, *GB*.

But the sum of the squares on *AG*, *GB* together is medial;

therefore *CL* is medial.

And it is applied to the rational straight line *CD*, producing *CM* as breadth;

therefore *CM* is rational and incommensurable with *CD*. [x. 22]

And, since the whole *CL* is equal to the squares on *AG*, *GB*,

and, in these, *CE* is equal to the square on *AB*,

therefore the remainder *FL* is equal to twice the rectangle *AG*, *GB*. [II. 7]

Let then *FM* be bisected at *N*,

and through *N* let *NO* be drawn parallel to either of the straight lines *CD*, *ML*;

therefore each of the rectangles *FO*, *NL* is equal to the rectangle *AG*, *GB*.

And, since twice the rectangle *AG*, *GB* is rational and equal to *FL*,

therefore *FL* is rational.

And it is applied to the rational straight line *EF*, producing *FM* as breadth;

therefore *FM* is rational and commensurable in length with *CD*. [x. 20]

Now, since *CL* is medial, and *FL* rational,

therefore *CL* is incommensurable with *FL*.

But, as *CL* is to *FL*, so is *CM* to *MF*; [VI. 1]

therefore *CM* is incommensurable in length with *MF*. [x. 11]

And both are rational;

therefore *CM*, *MF* are rational straight lines commensurable in square only;

therefore *CF* is an apotome. [x. 73]

I say next that it is also a fifth apotome.

For we can prove similarly that the rectangle *CK*, *KM* is equal to the square on *NM*, that is, to the fourth part of the square on *FM*.

And, since the square on *AG* is incommensurable with the square on *GB*,

while the square on *AG* is equal to *CH*,
and the square on *GB* to *KL*,
therefore *CH* is incommensurable with *KL*.

But, as *CH* is to *KL*, so is *CK* to *KM*; [VI. 1]
therefore *CK* is incommensurable in length with *KM*. [X. 11]

Since then *CM*, *MF* are two unequal straight lines,
and a parallelogram equal to the fourth part of the square on *FM* and deficient by a square figure has been applied to *CM*, and divides it into incommensurable parts,
therefore the square on *CM* is greater than the square on *MF* by the square on a straight line incommensurable with *CM*. [X. 18]

And the annex *FM* is commensurable with the rational straight line *CD* set out;
therefore *CF* is a fifth apotome. [X. Deff. III. 5]

Q. E. D.

We have to find and classify

$$\frac{1}{\sigma}\left\{\frac{\rho}{\sqrt{2(1+k^2)}}\sqrt{\sqrt{1+k^2}+k}-\frac{\rho}{\sqrt{2(1+k^2)}}\sqrt{\sqrt{1+k^2}-k}\right\}^2.$$

Call this $\frac{1}{\sigma}(u-v)^2$, and take x, y, z such that

$$\left.\begin{aligned}\sigma x &= u^2\\ \sigma y &= v^2\\ \sigma\,.\,2z &= 2uv\end{aligned}\right\}.$$

In this case u^2. v^2 are incommensurable, (u^2+v^2) is a medial area and $2uv$ a rational area.

Since $\sigma(x+y)$ is medial and $\sigma\,.\,2z$ rational,
$(x+y)$ is rational and $\smile\sigma$,
$2z$ is rational and $\frown\sigma$,
while $(x+y)\smile 2z$.

It follows that $(x+y)$, $2z$ are rational and $\frown\!-$,
so that $(x+y)-2z$ is an apotome.

Again, as before, $xy=z^2=\frac{1}{4}(2z)^2$,
and, since $u^2\smile v^2$, $\sigma x\smile\sigma y$,
or $x\smile y$.

Hence [X. 18] $\sqrt{(x+y)^2-(2z)^2}\smile(x+y)$.

And $2z\frown\sigma$.

Therefore $(x+y)-2z$ is a *fifth apotome*.

It is of course equal to

$$\frac{\rho^2}{\sigma}\left(\frac{1}{\sqrt{1+k^2}}-\frac{1}{1+k^2}\right).$$

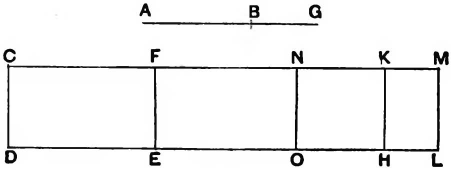
PROPOSITION 102.

The square on the straight line which produces with a medial area a medial whole, if applied to a rational straight line, produces as breadth a sixth apotome.

Let AB be the straight line which produces with a medial area a medial whole, and CD a rational straight line,

and to CD let CE be applied equal to the square on AB and producing CF as breadth;

I say that CF is a sixth apotome.



For let BG be the annex to AB;

therefore AG, GB are straight lines incommensurable in square which make the sum of the squares on them medial, twice the rectangle AG, GB medial, and the squares on AG, GB incommensurable with twice the rectangle AG, GB. [X. 78]

Now to CD let there be applied CH equal to the square on AG and producing CK as breadth,

and KL equal to the square on BG;

therefore the whole CL is equal to the squares on AG, GB;

therefore CL is also medial.

And it is applied to the rational straight line CD, producing CM as breadth;

therefore CM is rational and incommensurable in length with CD. [X. 22]

Since now CL is equal to the squares on AG, GB,

and, in these, CE is equal to the square on AB,

therefore the remainder FL is equal to twice the rectangle AG, GB. [II. 7]

And twice the rectangle AG, GB is medial;

therefore FL is also medial.

And it is applied to the rational straight line *FE*, producing *FM* as breadth;
therefore *FM* is rational and incommensurable in length with *CD*. [X. 22]

And, since the squares on *AG*, *GB* are incommensurable with twice the rectangle *AG*, *GB*,
and *CL* is equal to the squares on *AG*, *GB*,
and *FL* equal to twice the rectangle *AG*, *GB*,
therefore *CL* is incommensurable with *FL*.

But, as *CL* is to *FL*, so is *CM* to *MF*; [VI. 1]
therefore *CM* is incommensurable in length with *MF*. [X. 11]

And both are rational.

Therefore *CM*, *MF* are rational straight lines commensurable in square only;
therefore *CF* is an apotome. [X. 73]

I say next that it is also a sixth apotome.

For, since *FL* is equal to twice the rectangle *AG*, *GB*, let *FM* be bisected at *N*,
and let *NO* be drawn through *N* parallel to *CD*;
therefore each of the rectangles *FO*, *NL* is equal to the rectangle *AG*, *GB*.

And, since *AG*, *GB* are incommensurable in square,
therefore the square on *AG* is incommensurable with the square on *GB*.

But *CH* is equal to the square on *AG*,
and *KL* is equal to the square on *GB*;
therefore *CH* is incommensurable with *KL*.

But, as *CH* is to *KL*, so is *CK* to *KM*; [VI. 1]
therefore *CK* is incommensurable with *KM*. [X. 11]

And, since the rectangle *AG*, *GB* is a mean proportional between the squares on *AG*, *GB*,
and *CH* is equal to the square on *AG*,
KL equal to the square on *GB*,
and *NL* equal to the rectangle *AG*, *GB*,
therefore *NL* is also a mean proportional between *CH*, *KL*;
therefore, as *CH* is to *NL*, so is *NL* to *KL*.

And for the same reason as before the square on *CM* is greater than the square on *MF* by the square on a straight line incommensurable with *CM*. [X. 18]

And neither of them is commensurable with the rational straight line *CD* set out;

therefore *CF* is a sixth apotome. [X. Deff. III. 6]

Q. E. D.

We have to find and classify

$$\frac{1}{\sigma}\left\{\frac{\rho\lambda^{\frac{1}{4}}}{\sqrt{2}}\sqrt{1+\frac{k}{\sqrt{1+k^2}}}-\frac{\rho\lambda^{\frac{1}{4}}}{\sqrt{2}}\sqrt{1-\frac{k}{\sqrt{1+k^2}}}\right\}^2.$$

Call this $\frac{1}{\sigma}(u-v)^2$, and put

$$\sigma x = u^2,$$
$$\sigma y = v^2,$$
$$\sigma \,.\, 2z = 2uv.$$

Here u^2, v^2 are incommensurable,

(u^2+v^2), $2uv$ are both medial areas,

and $$(u^2+v^2) \smile 2uv.$$

Since $\sigma(x+y)$, $\sigma\,.\,2z$ are medial and incommensurable,

$(x+y)$ is rational and $\smile \sigma$,

$2z$ is rational and $\smile \sigma$,

and $$(x+y) \smile 2z.$$

Hence $(x+y)$, $2z$ are rational and $\frown\!\!-$,

so that $(x+y)-2z$ is an apotome.

Again, since u^2, v^2, or σx, σy, are incommensurable,

$$x \smile y.$$

And, as before, $$xy = z^2 = \tfrac{1}{4}(2z)^2.$$

Therefore [X. 18] $$\sqrt{(x+y)^2-(2z)^2} \smile (x+y).$$

And neither $(x+y)$ nor $2z$ is $\frown z$;

therefore $(x+y)-2z$ is a *sixth apotome*.

It is of course $$\frac{\rho^2}{\sigma}\left(\sqrt{\lambda}-\frac{\sqrt{\lambda}}{\sqrt{1+k^2}}\right).$$

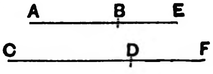
PROPOSITION 103.

A straight line commensurable in length with an apotome is an apotome and the same in order.

Let *AB* be an apotome,

and let *CD* be commensurable in length with *AB*;

I say that *CD* is also an apotome and the same in order with *AB*.



For, since AB is an apotome, let BE be the annex to it; therefore AE, EB are rational straight lines commensurable in square only. [X. 73]

Let it be contrived that the ratio of BE to DF is the same as the ratio of AB to CD; [VI. 12]

therefore also, as one is to one, so are all to all; [V. 12]

therefore also, as the whole AE is to the whole CF, so is AB to CD.

But AB is commensurable in length with CD.

Therefore AE is also commensurable with CF, and BE with DF. [X. 11]

And AE, EB are rational straight lines commensurable in square only;

therefore CF, FD are also rational straight lines commensurable in square only. [X. 13]

Now since, as AE is to CF, so is BE to DF,

alternately therefore, as AE is to EB, so is CF to FD. [V. 16]

And the square on AE is greater than the square on EB either by the square on a straight line commensurable with AE or by the square on a straight line incommensurable with it.

If then the square on AE is greater than the square on EB by the square on a straight line commensurable with AE, the square on CF will also be greater than the square on FD by the square on a straight line commensurable with CF. [X. 14]

And, if AE is commensurable in length with the rational straight line set out,

CF is so also, [X. 12]

if BE, then DF also, [*id.*]

and, if neither of the straight lines AE, EB, then neither of the straight lines CF, FD. [X. 13]

But, if the square on AE is greater than the square on EB by the square on a straight line incommensurable with AE,

the square on CF will also be greater than the square on FD by the square on a straight line incommensurable with CF. [X. 14]

And, if AE is commensurable in length with the rational straight line set out,

CF is so also,

if BE, then DF also, [x. 12]

and, if neither of the straight lines AE, EB, then neither of the straight lines CF, FD. [x. 13]

Therefore CD is an apotome and the same in order with AB.

Q. E. D.

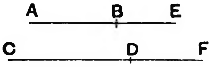
This and the following propositions to 107 inclusive (like the corresponding theorems x. 66 to 70) are easy and require no elucidation. They are equivalent to saying that, if in any of the preceding irrational straight lines $\frac{m}{n}\rho$ is substituted for ρ, the resulting irrational is of the same kind and order as that from which it is altered.

PROPOSITION 104.

A straight line commensurable with an apotome of a medial straight line is an apotome of a medial straight line and the same in order.

Let AB be an apotome of a medial straight line, and let CD be commensurable in length with AB;

I say that CD is also an apotome of a medial straight line and the same in order with AB.



For, since AB is an apotome of a medial straight line, let EB be the annex to it.

Therefore AE, EB are medial straight lines commensurable in square only. [x. 74, 75]

Let it be contrived that, as AB is to CD, so is BE to DF; [vi. 12]

therefore AE is also commensurable with CF, and BE with DF. [v. 12, x. 11]

But AE, EB are medial straight lines commensurable in square only;

therefore CF, FD are also medial straight lines [x. 23] commensurable in square only; [x. 13]

therefore CD is an apotome of a medial straight line. [x. 74, 75]

I say next that it is also the same in order with AB.

Since, as AE is to EB, so is CF to FD,
therefore also, as the square on AE is to the rectangle AE, EB, so is the square on CF to the rectangle CF, FD.

But the square on AE is commensurable with the square on CF;
therefore the rectangle AE, EB is also commensurable with the rectangle CF, FD. [v. 16, x. 11]

Therefore, if the rectangle AE, EB is rational, the rectangle CF, FD will also be rational, [x. Def. 4]
and if the rectangle AE, EB is medial, the rectangle CF, FD is also medial. [x. 23, Por.]

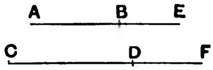
Therefore CD is an apotome of a medial straight line and the same in order with AB. [x. 74, 75]

Q. E. D.

Proposition 105.

A straight line commensurable with a minor straight line is minor.

Let AB be a minor straight line, and CD commensurable with AB;
I say that CD is also minor.



Let the same construction be made as before;
then, since AE, EB are incommensurable in square, [x. 76]
therefore CF, FD are also incommensurable in square. [x. 13]

Now since, as AE is to EB, so is CF to FD, [v. 12, v. 16]
therefore also, as the square on AE is to the square on EB, so is the square on CF to the square on FD. [vi. 22]

Therefore, *componendo*, as the squares on AE, EB are to the square on EB, so are the squares on CF, FD to the square on FD. [v. 18]

But the square on BE is commensurable with the square on DF;
therefore the sum of the squares on AE, EB is also commensurable with the sum of the squares on CF, FD. [v. 16, x. 11]

But the sum of the squares on AE, EB is rational; [x. 76]
therefore the sum of the squares on CF, FD is also rational. [x. Def. 4]

Again, since, as the square on *AE* is to the rectangle *AE*, *EB*, so is the square on *CF* to the rectangle *CF*, *FD*,

while the square on *AE* is commensurable with the square on *CF*,

therefore the rectangle *AE*, *EB* is also commensurable with the rectangle *CF*, *FD*.

But the rectangle *AE*, *EB* is medial; [x. 76]

therefore the rectangle *CF*, *FD* is also medial; [x. 23, Por.]

therefore *CF*, *FD* are straight lines incommensurable in square which make the sum of the squares on them rational, but the rectangle contained by them medial.

Therefore *CD* is minor. [x. 76]

Q. E. D.

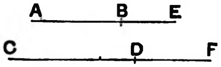
PROPOSITION 106.

A straight line commensurable with that which produces with a rational area a medial whole is a straight line which produces with a rational area a medial whole.

Let *AB* be a straight line which produces with a rational area a medial whole,

and *CD* commensurable with *AB*;

I say that *CD* is also a straight line which produces with a rational area a medial whole.



For let *BE* be the annex to *AB*;

therefore *AE*, *EB* are straight lines incommensurable in square which make the sum of the squares on *AE*, *EB* medial, but the rectangle contained by them rational. [x. 77]

Let the same construction be made.

Then we can prove, in manner similar to the foregoing, that *CF*, *FD* are in the same ratio as *AE*, *EB*,

the sum of the squares on *AE*, *EB* is commensurable with the sum of the squares on *CF*, *FD*,

and the rectangle *AE*, *EB* with the rectangle *CF*, *FD*;

so that *CF*, *FD* are also straight lines incommensurable in square which make the sum of the squares on *CF*, *FD* medial, but the rectangle contained by them rational.

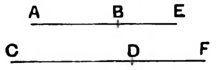
Therefore *CD* is a straight line which produces with a rational area a medial whole. [x. 77]

Q. E. D.

Proposition 107.

A straight line commensurable with that which produces with a medial area a medial whole is itself also a straight line which produces with a medial area a medial whole.

Let *AB* be a straight line which produces with a medial area a medial whole,



and let *CD* be commensurable with *AB*;

I say that *CD* is also a straight line which produces with a medial area a medial whole.

For let *BE* be the annex to *AB*,

and let the same construction be made;

therefore *AE*, *EB* are straight lines incommensurable in square which make the sum of the squares on them medial, the rectangle contained by them medial, and further the sum of the squares on them incommensurable with the rectangle contained by them. [x. 78]

Now, as was proved, *AE*, *EB* are commensurable with *CF*, *FD*,

the sum of the squares on *AE*, *EB* with the sum of the squares on *CF*, *FD*,

and the rectangle *AE*, *EB* with the rectangle *CF*, *FD*;

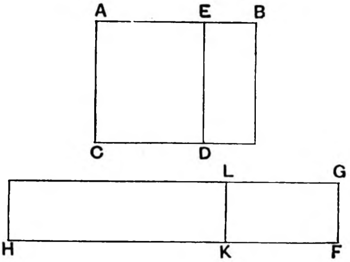
therefore *CF*, *FD* are also straight lines incommensurable in square which make the sum of the squares on them medial, the rectangle contained by them medial, and further the sum of the squares on them incommensurable with the rectangle contained by them.

Therefore *CD* is a straight line which produces with a medial area a medial whole. [x. 78]

PROPOSITION 108.

If from a rational area a medial area be subtracted, the "side" of the remaining area becomes one of two irrational straight lines, either an apotome or a minor straight line.

For from the rational area *BC* let the medial area *BD* be subtracted;



I say that the "side" of the remainder *EC* becomes one of two irrational straight lines, either an apotome or a minor straight line.

For let a rational straight line *FG* be set out, to *FG* let there be applied the rectangular parallelogram *GH* equal to *BC*,

and let *GK* equal to *DB* be subtracted;

therefore the remainder *EC* is equal to *LH*.

Since then *BC* is rational, and *BD* medial,

while *BC* is equal to *GH*, and *BD* to *GK*,

therefore *GH* is rational, and *GK* medial.

And they are applied to the rational straight line *FG*;

therefore *FH* is rational and commensurable in length with *FG*, [x. 20]

while *FK* is rational and incommensurable in length with *FG*; [x. 22]

therefore *FH* is incommensurable in length with *FK*. [x. 13]

Therefore *FH*, *FK* are rational straight lines commensurable in square only;

therefore *KH* is an apotome [x. 73], and *KF* the annex to it.

Now the square on *HF* is greater than the square on *FK* by the square on a straight line either commensurable with *HF* or not commensurable.

First, let the square on it be greater by the square on a straight line commensurable with it.

Now the whole *HF* is commensurable in length with the rational straight line *FG* set out;

therefore *KH* is a first apotome. [x. Deff. III. 1]

But the "side" of the rectangle contained by a rational straight line and a first apotome is an apotome. [x. 91]

Therefore the "side" of *LH*, that is, of *EC*, is an apotome.

But, if the square on *HF* is greater than the square on *FK* by the square on a straight line incommensurable with *HF*,

while the whole *FH* is commensurable in length with the rational straight line *FG* set out,

KH is a fourth apotome. [x. Deff. III. 4]

But the "side" of the rectangle contained by a rational straight line and a fourth apotome is minor. [x. 94]

Q. E. D.

A rational area being of the form $k\rho^2$, and a medial area of the form $\sqrt{\lambda}\,.\,\rho^2$, the problem is to classify

$$\sqrt{k\rho^2 - \sqrt{\lambda}\,.\,\rho^2}$$

according to the different possible relations between k, λ.

Suppose that $$\sigma u = k\rho^2,$$ $$\sigma v = \sqrt{\lambda}\,.\,\rho^2.$$

Since σu is rational and σv medial,

u is rational and $\frown \sigma$,

while v is rational and $\smile \sigma$.

Therefore $$u \smile v;$$

thus u, v are rational and $\frown\!\!-$,

whence $(u - v)$ is an apotome.

The possibilities are now as follows.

(1) $\sqrt{u^2 - v^2} \frown u$,

(2) $\sqrt{u^2 - v^2} \smile u$.

In both cases $u \frown \sigma$,

so that $(u - v)$ is either (1) a *first apotome*,

or (2) a *fourth apotome*.

In case (1) $\sqrt{\sigma(u - v)}$ is an *apotome* [x. 91],

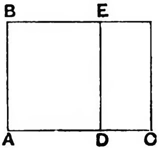
but in case (2) $\sqrt{\sigma(u - v)}$ is a *minor* irrational straight line [x. 94].

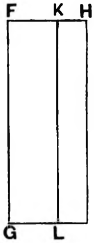
PROPOSITION 109.

If from a medial area a rational area be subtracted, there arise two other irrational straight lines, either a first apotome of a medial straight line or a straight line which produces with a rational area a medial whole.

For from the medial area *BC* let the rational area *BD* be subtracted.

I say that the "side" of the remainder *EC* becomes one of two irrational straight lines, either a first apotome of a medial straight line or a straight line which produces with a rational area a medial whole.





For let a rational straight line *FG* be set out,
and let the areas be similarly applied.

It follows then that *FH* is rational and incommensurable in length with *FG*,
while *KF* is rational and commensurable in length with *FG*;
therefore *FH*, *FK* are rational straight lines commensurable in square only; [x. 13]
therefore *KH* is an apotome, and *FK* the annex to it. [x. 73]

Now the square on *HF* is greater than the square on *FK* either by the square on a straight line commensurable with *HF* or by the square on a straight line incommensurable with it.

If then the square on *HF* is greater than the square on *FK* by the square on a straight line commensurable with *HF*,
while the annex *FK* is commensurable in length with the rational straight line *FG* set out,
KH is a second apotome. [x. Deff. III. 2]

But *FG* is rational;
so that the "side" of *LH*, that is, of *EC*, is a first apotome of a medial straight line. [x. 92]

But, if the square on *HF* is greater than the square on *FK* by the square on a straight line incommensurable with *HF*,
while the annex *FK* is commensurable in length with the rational straight line *FG* set out,
KH is a fifth apotome; [x. Deff. III. 5]
so that the "side" of *EC* is a straight line which produces with a rational area a medial whole. [x. 95]

In this case we have to classify

$$\sqrt{\sqrt{k}\,.\,\rho^2 - \lambda\rho^2}.$$

Suppose that $\sigma u = \sqrt{k}\,.\,\rho^2,$

$$\sigma v = \lambda\rho^2.$$

Thus, σu being medial and σv rational,
u is rational and $\smile \sigma$,
while v is rational and $\frown \sigma$.

Thus, as before, u, v are rational and $\frown\!\!\!\!-$,
so that $(u - v)$ is an apotome.

Now either

(1) $\sqrt{u^2 - v^2} \frown u,$

or (2) $\sqrt{u^2 - v^2} \smile u,$

while in both cases v is commensurable with σ.

Therefore $(u - v)$ is either (1) a second apotome,
or (2) a fifth apotome,

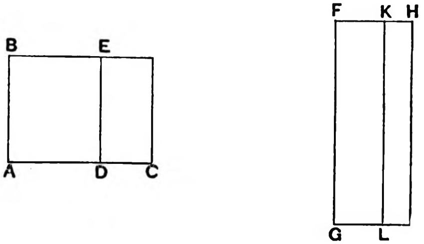
and hence in case (1) $\sqrt{\sigma (u - v)}$ is the *first apotome of a medial* straight line, [x. 92]

and in case (2) $\sqrt{\sigma (u - v)}$ is the "*side*" *of a medial, minus a rational, area.* [x. 95]

PROPOSITION 110.

If from a medial area there be subtracted a medial area incommensurable with the whole, the two remaining irrational straight lines arise, either a second apotome of a medial straight line or a straight line which produces with a medial area a medial whole.

For, as in the foregoing figures, let there be subtracted from the medial area *BC* the medial area *BD* incommensurable with the whole;



I say that the "side" of *EC* is one of two irrational straight lines, either a second apotome of a medial straight line or a straight line which produces with a medial area a medial whole.

For, since each of the rectangles *BC*, *BD* is medial, and *BC* is incommensurable with *BD*, it follows that each of the straight lines *FH*, *FK* will be rational and incommensurable in length with *FG*. [x. 22]

And, since *BC* is incommensurable with *BD*, that is, *GH* with *GK*, *HF* is also incommensurable with *FK*; [vi. 1, x. 11] therefore *FH*, *FK* are rational straight lines commensurable in square only; therefore *KH* is an apotome. [x. 73]

If then the square on *FH* is greater than the square on *FK* by the square on a straight line commensurable with *FH*, while neither of the straight lines *FH*, *FK* is commensurable in length with the rational straight line *FG* set out, *KH* is a third apotome. [x. Deff. iii. 3]

But *KL* is rational, and the rectangle contained by a rational straight line and a third apotome is irrational, and the "side" of it is irrational, and is called a second apotome of a medial straight line; [x. 93] so that the "side" of *LH*, that is, of *EC*, is a second apotome of a medial straight line.

But, if the square on *FH* is greater than the square on *FK* by the square on a straight line incommensurable with *FH*, while neither of the straight lines *HF*, *FK* is commensurable in length with *FG*, *KH* is a sixth apotome. [x. Deff. iii. 6]

But the "side" of the rectangle contained by a rational straight line and a sixth apotome is a straight line which produces with a medial area a medial whole. [x. 96]

Therefore the "side" of *LH*, that is, of *EC*, is a straight line which produces with a medial area a medial whole.

Q. E. D.

We have to classify $\sqrt{\sqrt{k}\,.\,\rho^2 - \sqrt{\lambda}\,.\,\rho^2}$,
where $\sqrt{k}\,.\,\rho^2$ is incommensurable with $\sqrt{\lambda}\,.\,\rho^2$.

Put $\sigma u = \sqrt{k}\,.\,\rho^2$,

$\sigma v = \sqrt{\lambda}\,.\,\rho^2$.

Then u is rational and $\smile \sigma$,
v is rational and $\smile \sigma$,
and $u \smile v$.

Therefore u, v are rational and $\frown\!\!-$,
so that $(u - v)$ is an apotome.

Now either

(1) $\sqrt{u^2 - v^2} \frown u$,

or (2) $\sqrt{u^2 - v^2} \smile u$,

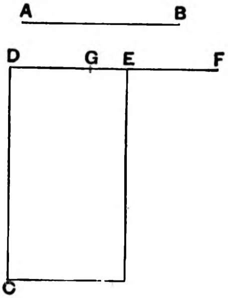
while in both cases both u and v are $\smile \sigma$.

In case (1) $(u - v)$ is a *third apotome*,
and in case (2) $(u - v)$ is a *sixth apotome*,
so that $\sqrt{\sigma(u - v)}$ is either (1) a *second apotome of a medial* straight line [X. 93], or (2) a "*side*" *of the difference between two medial areas* [X. 96].

PROPOSITION 111.

The apotome is not the same with the binomial straight line.

Let AB be an apotome;
I say that AB is not the same with the binomial straight line.



For, if possible, let it be so;
let a rational straight line DC be set out, and to CD let there be applied the rectangle CE equal to the square on AB and producing DE as breadth.

Then, since AB is an apotome,
DE is a first apotome. [X. 97]

Let EF be the annex to it;
therefore DF, FE are rational straight lines commensurable in square only,
the square on DF is greater than the square on FE by the square on a straight line commensurable with DF,
and DF is commensurable in length with the rational straight line DC set out. [X. Deff. III. 1]

Again, since AB is binomial,
therefore DE is a first binomial straight line. [X. 60]

Let it be divided into its terms at G,
and let DG be the greater term;
therefore DG, GE are rational straight lines commensurable in square only,

the square on *DG* is greater than the square on *GE* by the square on a straight line commensurable with *DG*, and the greater term *DG* is commensurable in length with the rational straight line *DC* set out. [X. Deff. II. 1]

Therefore *DF* is also commensurable in length with *DG*; [X. 12]

therefore the remainder *GF* is also commensurable in length with *DF*. [X. 15]

But *DF* is incommensurable in length with *EF*;

therefore *FG* is also incommensurable in length with *EF*. [X. 13]

Therefore *GF*, *FE* are rational straight lines commensurable in square only;

therefore *EG* is an apotome. [X. 73]

But it is also rational:

which is impossible.

Therefore the apotome is not the same with the binomial straight line.

Q. E. D.

This proposition proves the equivalent of the fact that

$\sqrt{x} + \sqrt{y}$ cannot be equal to $\sqrt{x'} - \sqrt{y'}$, and

$x + \sqrt{y}$ cannot be equal to $x' - \sqrt{y'}$.

We should prove these results by squaring the respective expressions; and Euclid's procedure corresponds to this exactly.

He has to prove that

$$\rho + \sqrt{k} \,.\, \rho \text{ cannot be equal to } \rho' - \sqrt{\lambda} \,.\, \rho'.$$

For, if possible, let this be so.

Take the straight lines $\dfrac{(\rho + \sqrt{k} \,.\, \rho)^2}{\sigma}$, $\dfrac{(\rho' - \sqrt{\lambda} \,.\, \rho')^2}{\sigma}$;

these must be equal, and therefore

$$\frac{\rho^2}{\sigma}(1 + k + 2\sqrt{k}) = \frac{\rho'^2}{\sigma}(1 + \lambda - 2\sqrt{\lambda}) \quad \ldots\ldots\ldots\ldots\ldots (1).$$

Now $\dfrac{\rho^2}{\sigma}(1 + k)$, $\dfrac{\rho'^2}{\sigma}(1 + \lambda)$ are rational and $\frown$;

therefore

$$\left\{\frac{\rho'^2}{\sigma}(1 + \lambda) - \frac{\rho^2}{\sigma}(1 + k)\right\} \frown \frac{\rho'^2}{\sigma}(1 + \lambda)$$

$$\smile \frac{\rho'^2}{\sigma} \,.\, 2\sqrt{\lambda}.$$

And, since both sides are rational, it follows that

$$\left\{\frac{\rho'^2}{\sigma}(1 + \lambda) - \frac{\rho^2}{\sigma}(1 + k)\right\} - \frac{\rho'^2}{\sigma} \,.\, 2\sqrt{\lambda} \text{ is an } \textit{apotome}.$$

But, by (1), this expression is equal to $\frac{\rho^2}{\sigma} \cdot 2\sqrt{k}$, which is *rational.*

Hence an *apotome*, which is *irrational*, is also *rational*: which is impossible.

This proposition is the connecting link which enables Euclid to prove that *all* the compound irrationals with positive signs above discussed are different from *all* the corresponding compound irrationals with negative signs, while the two sets are all different from one another and from the *medial* straight line. The recapitulation following makes this clear.

The apotome and the irrational straight lines following it are neither the same with the medial straight line nor with one another.

For the square on a medial straight line, if applied to a rational straight line, produces as breadth a straight line rational and incommensurable in length with that to which it is applied, [X. 22]

while the square on an apotome, if applied to a rational straight line, produces as breadth a first apotome, [X. 97]

the square on a first apotome of a medial straight line, if applied to a rational straight line, produces as breadth a second apotome, [X. 98]

the square on a second apotome of a medial straight line, if applied to a rational straight line, produces as breadth a third apotome, [X. 99]

the square on a minor straight line, if applied to a rational straight line, produces as breadth a fourth apotome, [X. 100]

the square on the straight line which produces with a rational area a medial whole, if applied to a rational straight line, produces as breadth a fifth apotome, [X. 101]

and the square on the straight line which produces with a medial area a medial whole, if applied to a rational straight line, produces as breadth a sixth apotome. [X. 102]

Since then the said breadths differ from the first and from one another, from the first because it is rational, and from one another since they are not the same in order,

it is clear that the irrational straight lines themselves also differ from one another.

And, since the apotome has been proved not to be the same as the binomial straight line, [X. 111]

but, if applied to a rational straight line, the straight lines following the apotome produce, as breadths, each according to its own order, apotomes, and those following the binomial straight line themselves also, according to their order, produce the binomials as breadths,

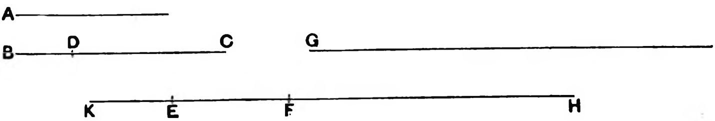
therefore those following the apotome are different, and those following the binomial straight line are different, so that there are, in order, thirteen irrational straight lines in all,

Medial,
Binomial,
First bimedial,
Second bimedial,
Major,
"Side" of a rational plus a medial area,
"Side" of the sum of two medial areas,
Apotome,
First apotome of a medial straight line,
Second apotome of a medial straight line,
Minor,
Producing with a rational area a medial whole,
Producing with a medial area a medial whole.

PROPOSITION 112.

The square on a rational straight line applied to the binomial straight line produces as breadth an apotome the terms of which are commensurable with the terms of the binomial and moreover in the same ratio; and further the apotome so arising will have the same order as the binomial straight line.

Let A be a rational straight line,
let BC be a binomial, and let DC be its greater term;
let the rectangle BC, EF be equal to the square on A;



I say that EF is an apotome the terms of which are commensurable with CD, DB, and in the same ratio, and further EF will have the same order as BC.

For again let the rectangle *BD*, *G* be equal to the square on *A*.

Since then the rectangle *BC*, *EF* is equal to the rectangle *BD*, *G*,

therefore, as *CB* is to *BD*, so is *G* to *EF*. [vi. 16]

But *CB* is greater than *BD*;

therefore *G* is also greater than *EF*. [v. 16, v. 14]

Let *EH* be equal to *G*;

therefore, as *CB* is to *BD*, so is *HE* to *EF*;

therefore, *separando*, as *CD* is to *BD*, so is *HF* to *FE*. [v. 17]

Let it be contrived that, as *HF* is to *FE*, so is *FK* to *KE*;

therefore also the whole *HK* is to the whole *KF* as *FK* is to *KE*;

for, as one of the antecedents is to one of the consequents, so are all the antecedents to all the consequents. [v. 12]

But, as *FK* is to *KE*, so is *CD* to *DB*; [v. 11]

therefore also, as *HK* is to *KF*, so is *CD* to *DB*. [*id.*]

But the square on *CD* is commensurable with the square on *DB*; [x. 36]

therefore the square on *HK* is also commensurable with the square on *KF*. [vi. 22, x. 11]

And, as the square on *HK* is to the square on *KF*, so is *HK* to *KE*, since the three straight lines *HK*, *KF*, *KE* are proportional. [v. Def. 9]

Therefore *HK* is commensurable in length with *KE*,

so that *HE* is also commensurable in length with *EK*. [x. 15]

Now, since the square on *A* is equal to the rectangle *EH*, *BD*,

while the square on *A* is rational,

therefore the rectangle *EH*, *BD* is also rational.

And it is applied to the rational straight line *BD*;

therefore *EH* is rational and commensurable in length with *BD*; [x. 20]

so that *EK*, being commensurable with it, is also rational and commensurable in length with *BD*.

Since, then, as *CD* is to *DB*, so is *FK* to *KE*,

while *CD*, *DB* are straight lines commensurable in square only,

therefore *FK*, *KE* are also commensurable in square only. [x. 11]

But *KE* is rational;

therefore *FK* is also rational.

Therefore *FK*, *KE* are rational straight lines commensurable in square only;

therefore *EF* is an apotome. [x. 73]

Now the square on *CD* is greater than the square on *DB* either by the square on a straight line commensurable with *CD* or by the square on a straight line incommensurable with it.

If then the square on *CD* is greater than the square on *DB* by the square on a straight line commensurable with *CD*, the square on *FK* is also greater than the square on *KE* by the square on a straight line commensurable with *FK*. [x. 14]

And, if *CD* is commensurable in length with the rational straight line set out,

so also is *FK*; [x. 11, 12]

if *BD* is so commensurable,

so also is *KE*; [x. 12]

but, if neither of the straight lines *CD*, *DB* is so commensurable,

neither of the straight lines *FK*, *KE* is so.

But, if the square on *CD* is greater than the square on *DB* by the square on a straight line incommensurable with *CD*,

the square on *FK* is also greater than the square on *KE* by the square on a straight line incommensurable with *FK*. [x. 14]

And, if *CD* is commensurable with the rational straight line set out,

so also is *FK*;

if *BD* is so commensurable,

so also is *KE*;

but, if neither of the straight lines *CD*, *DB* is so commensurable,

neither of the straight lines *FK*, *KE* is so;

so that *FE* is an apotome, the terms of which *FK*, *KE* are commensurable with the terms *CD*, *DB* of the binomial straight line and in the same ratio, and it has the same order as *BC*.

Q. E. D.

Heiberg considers that this proposition and the succeeding ones are interpolated, though the interpolation must have taken place before Theon's time. His argument is that X. 112—115 are nowhere used, but that X. 111 rounds off the complete discussion of the 13 irrationals (as indicated in the recapitulation), thereby giving what was necessary for use in connexion with the investigation of the five regular solids. For besides X. 73 (used in XIII. 6, 11) X. 94 and 97 are used in XIII. 11, 6 respectively; and Euclid could not have stopped at X. 97 without leaving the discussion of irrationals imperfect, for X. 98—102 are closely connected with X. 97, and X. 103—111 add, as it were, the coping-stone to the whole doctrine. On the other hand, X. 112—115 are not connected with the rest of the treatise on the 13 irrationals and are not used in the stereometric books. They are rather the germ of a new study and a more abstruse investigation of irrationals *in themselves*. Prop. 115 in particular extends the number of the different kinds of irrationals. As however X. 112—115 are old and serviceable theorems, Heiberg thinks that, though Euclid did not give them, they may have been taken from Apollonius.

I will only point out what seems to me open to doubt in the above, namely that X. 112—114 (excluding 115) are not connected with the rest of the exposition of the 13 irrationals. It seems to me that they *are* so connected. X. 111 has shown us that a *binomial* straight line cannot also be an *apotome*. But X. 112—114 show us *how either of them can be used to rationalise the other*, thus giving what is surely an important relation between them.

X. 112 is the equivalent of rationalising the denominators of the fractions

$$\frac{c^2}{\sqrt{A}+\sqrt{B}}, \quad \frac{c^2}{a+\sqrt{B}},$$

by multiplying numerator and denominator by $\sqrt{A}-\sqrt{B}$ and $a-\sqrt{B}$ respectively.

Euclid proves that $\dfrac{\sigma^2}{\rho+\sqrt{k}\,.\,\rho}=\lambda\rho-\sqrt{k}\,.\,\lambda\rho\ (k<1)$, and his method enables us to see that $\lambda=\sigma^2/(\rho^2-k\rho^2)$.

The proof is a remarkable instance of the dexterity of the Greeks in using geometry as the equivalent of our algebra. Like so many proofs in Archimedes and Apollonius, it leaves us completely in the dark as to how it was evolved. That the Greeks must have had some analytical method which suggested the steps of such proofs seems certain; but *what* it was must remain apparently an insoluble mystery.

I will reproduce by means of algebraical symbols the exact course of Euclid's proof.

He has to prove that $\dfrac{\sigma^2}{\rho+\sqrt{k}\,.\,\rho}$ is an apotome related in a certain way to the binomial straight line $\rho + \sqrt{k} \cdot \rho$. If u be the straight line required, $(u + w) - w$ is shown to be an apotome of the kind described, where w is determined in the following manner.

$$\left.\begin{array}{ll} \text{We have} & (\rho + \sqrt{k} \cdot \rho)\, u = \sigma^2 = \sqrt{k} \cdot \rho \cdot x, \text{ say,} \\ \text{whence} & x > u. \\ \text{Let} & x = u + v. \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Then $(\rho + \sqrt{k} \cdot \rho) : \sqrt{k} \cdot \rho = (u + v) : u$,

and hence $\rho : \sqrt{k} \cdot \rho = v : u$ ....(2).

Let w be taken such that

$$v : u = (u + w) : w \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Thus $v : u = (u + v + w) : (u + w)$ ....(4),

and therefore $\rho : \sqrt{k} \cdot \rho = (u + v + w) : (u + w)$.

From the last proportion,

$$(u + v + w)^2 \frown (u + w)^2,$$

and, from the two preceding, $(u + w)$ is a mean proportional between $(u + v + w)$, w, so that

$$(u + v + w)^2 : (u + w)^2 = (u + v + w) : w.$$

Therefore $(u + v + w) \frown w$,

whence $(u + v) \frown w$.

Now $\sqrt{k} \cdot \rho\, (u + v) = \sigma^2$, which is rational;

therefore $(u + v)$ is rational and $\frown \sqrt{k} \cdot \rho$;

hence w is also rational and $\frown \sqrt{k} \cdot \rho$ ....(5).

Next, by (2), (3), since ρ, $\sqrt{k} \cdot \rho$ are $\frown\!\!-$,

$$(u + w) \frown\!\!- w,$$

and w is rational;

therefore $(u + w)$ is rational,

and $(u + w)$, w are rational and $\frown\!\!-$.

Hence $(u + w) - w$ is an *apotome*.

Now either (I) $\sqrt{\rho^2 - k\rho^2} \frown \rho$,

or (II) $\sqrt{\rho^2 - k\rho^2} \smile \rho$.

In case (I) $\sqrt{(u + w)^2 - w^2} \frown (u + w)$, [(2), (3) and X. 14]

and in case (II) $\sqrt{(u + w)^2 - w^2} \smile (u + w)$. [*id.*]

Then, since [(5)] $w \frown \sqrt{k} \cdot \rho$,

by X. 11 and (2), (3), $(u + w) \frown \rho$ ....(6).

[This step is omitted in Euclid, but the result is assumed.]

If therefore $\rho \frown \sigma$, $(u + w) \frown \sigma$;

if $\sqrt{k} \cdot \rho \frown \sigma$, $w \frown \sigma$; [(5)]

and, if neither ρ nor $\sqrt{k} \cdot \rho$ is $\frown \sigma$, neither $(u + w)$ nor w will be $\frown \sigma$.

Thus the order of the apotome $(u + w) - w$ is the same as that of the binomial straight line $\rho + \sqrt{k} \cdot \rho$; while [(2), (3)] the terms are proportional and [(5), (6)] commensurable respectively.

We find $(u+w)$, w algebraically thus.

By (1), $$u = \frac{\sigma^2}{\rho + \sqrt{k}\,.\,\rho};$$

and, by (2), (3), $$\frac{u+w}{w} = \frac{\rho}{\sqrt{k}\,.\,\rho},$$

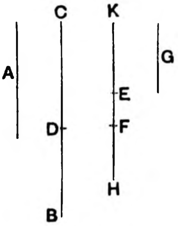
whence $$w = \frac{u\,.\,\sqrt{k}\,.\,\rho}{\rho - \sqrt{k}\,.\,\rho}$$
$$= \frac{\sigma^2\,.\,\sqrt{k}\,.\,\rho}{\rho^2 - k\rho^2}.$$

Thus $$u + w = w\,.\,\frac{1}{\sqrt{k}} = \frac{\sigma^2\,.\,\rho}{\rho^2 - k\rho^2}.$$

Therefore $$(u+w) - w = \sigma^2\,.\,\frac{\rho - \sqrt{k}\,.\,\rho}{\rho^2 - k\rho^2}.$$

PROPOSITION 113.

The square on a rational straight line, if applied to an apotome, produces as breadth the binomial straight line the terms of which are commensurable with the terms of the apotome and in the same ratio; and further the binomial so arising has the same order as the apotome.



Let *A* be a rational straight line and *BD* an apotome, and let the rectangle *BD*, *KH* be equal to the square on *A*, so that the square on the rational straight line *A* when applied to the apotome *BD* produces *KH* as breadth;

I say that *KH* is a binomial straight line the terms of which are commensurable with the terms of *BD* and in the same ratio; and further *KH* has the same order as *BD*.

For let *DC* be the annex to *BD*;

therefore *BC*, *CD* are rational straight lines commensurable in square only. [x. 73]

Let the rectangle *BC*, *G* be also equal to the square on *A*.

But the square on *A* is rational;

therefore the rectangle *BC*, *G* is also rational.

And it has been applied to the rational straight line *BC*;

therefore *G* is rational and commensurable in length with *BC*. [x. 20]

Since now the rectangle BC, G is equal to the rectangle BD, KH,

therefore, proportionally, as CB is to BD, so is KH to G. [vi. 16]

But BC is greater than BD;

therefore KH is also greater than G. [v. 16, v. 14]

Let KE be made equal to G;

therefore KE is commensurable in length with BC.

And since, as CB is to BD, so is HK to KE,

therefore, *convertendo*, as BC is to CD, so is KH to HE. [v. 19, Por.]

Let it be contrived that, as KH is to HE, so is HF to FE;

therefore also the remainder KF is to FH as KH is to HE, that is, as BC is to CD. [v. 19]

But BC, CD are commensurable in square only;

therefore KF, FH are also commensurable in square only. [x. 11]

And since, as KH is to HE, so is KF to FH,

while, as KH is to HE, so is HF to FE,

therefore also, as KF is to FH, so is HF to FE, [v. 11]

so that also, as the first is to the third, so is the square on the first to the square on the second; [v. Def. 9]

therefore also, as KF is to FE, so is the square on KF to the square on FH.

But the square on KF is commensurable with the square on FH,

for KF, FH are commensurable in square;

therefore KF is also commensurable in length with FE, [x. 11]

so that KF is also commensurable in length with KE. [x. 15]

But KE is rational and commensurable in length with BC;

therefore KF is also rational and commensurable in length with BC. [x. 12]

And, since, as BC is to CD, so is KF to FH,

alternately, as BC is to KF, so is DC to FH. [v. 16]

But BC is commensurable with KF;

therefore FH is also commensurable in length with CD. [x. 11]

But *BC*, *CD* are rational straight lines commensurable in square only;
therefore *KF*, *FH* are also rational straight lines [x. Def. 3] commensurable in square only;
therefore *KH* is binomial. [x. 36]

If now the square on *BC* is greater than the square on *CD* by the square on a straight line commensurable with *BC*,
the square on *KF* will also be greater than the square on *FH* by the square on a straight line commensurable with *KF*. [x 14]

And, if *BC* is commensurable in length with the rational straight line set out,
so also is *KF*;
if *CD* is commensurable in length with the rational straight line set out,
so also is *FH*,
but, if neither of the straight lines *BC*, *CD*,
then neither of the straight lines *KF*, *FH*.

But, if the square on *BC* is greater than the square on *CD* by the square on a straight line incommensurable with *BC*,
the square on *KF* is also greater than the square on *FH* by the square on a straight line incommensurable with *KF*. [x. 14]

And, if *BC* is commensurable with the rational straight line set out,
so also is *KF*;
if *CD* is so commensurable,
so also is *FH*;
but, if neither of the straight lines *BC*, *CD*,
then neither of the straight lines *KF*, *FH*.

Therefore *KH* is a binomial straight line, the terms of which *KF*, *FH* are commensurable with the terms *BC*, *CD* of the apotome and in the same ratio,
and further *KH* has the same order as *BD*.

Q. E. D.

This proposition, which is companion to the preceding, gives us the equivalent of the rationalisation of the denominator of

$$\frac{c^2}{\sqrt{A} - \sqrt{B}} \quad \text{or} \quad \frac{c^2}{a \sim \sqrt{B}}.$$

Euclid (or the writer) proves that

$$\frac{\sigma^2}{\rho - \sqrt{k} \cdot \rho} = \lambda\rho + \lambda\sqrt{k} \cdot \rho, \qquad (k < 1)$$

and his method enables us to see that $\lambda = \sigma^2/(\rho^2 - k\rho^2)$.

Let $\dfrac{\sigma^2}{\rho - \sqrt{k} \cdot \rho} = u$;

and it is proved that u is the binomial straight line $(u - w) + w$, where w is determined as shown below.

$$u(\rho - \sqrt{k} \cdot \rho) = \sigma^2 = \rho x, \text{ say,}$$

whence $\rho : (\rho - \sqrt{k} \cdot \rho) = u : x$ ....(1),

so that $x < u$.

Let then $x = u - v$.

Since $(u - v)\rho = \sigma^2$, a rational area,

$(u - v)$ is rational and $\frown \rho$ ....(2).

And [(1)] $\rho : (\rho - \sqrt{k} \cdot \rho) = u : (u - v)$,

so that, *convertendo*, $\rho : \sqrt{k} \cdot \rho = u : v$.

Suppose that $u : v = w : (v - w)$,

so that [v. 19] $(u - w) : w = u : v = w : (v - w)$.

Thus, w being a mean proportional between $(u - w)$, $(v - w)$,

$$(u - w)^2 : w^2 = (u - w) : (v - w).$$

But $(u - w)^2 : w^2 = u^2 : v^2$

$= \rho^2 : k\rho^2$ ....(3),

so that $(u - w)^2 \frown w^2$.

Therefore $(u - w) \frown (v - w)$

$\frown \{(u - w) - (v - w)\}$

$\frown (u - v)$.

Therefore [(2)] $(u - w)$ is rational and $\frown \rho$ ....(4).

And, since $\rho : \sqrt{k} \cdot \rho = (u - w) : w$,

w is rational and $\frown \sqrt{k} \cdot \rho$ ....(5).

Hence [(4), (5)] $(u - w)$, w are rational and $\frown\!\!-$,

so that $(u - w) + w$ is a *binomial* straight line.

Now either (I) $\sqrt{\rho^2 - k\rho^2} \frown \rho$,

or (II) $\sqrt{\rho^2 - k\rho^2} \smile \rho$.

In case (I) $\sqrt{(u - w)^2 - w^2} \frown (u - w)$,

and in case (II) $\sqrt{(u - w)^2 - w^2} \smile (u - w)$. [(3) and x. 14]

And, if $\rho \frown \sigma$, $(u - w) \frown \sigma$; [(4)]

if $\sqrt{k} \cdot \rho \frown \sigma$, $w \frown \sigma$; [5]

while, if neither ρ nor $\sqrt{k} \cdot \rho$ is $\frown \sigma$, neither $(u - w)$ nor w is $\frown \sigma$.

Hence $(u - w) + w$ is a binomial straight line of the same order as the apotome $\rho - \sqrt{k} \cdot \rho$, its terms are proportional to those of the apotome [(3)], and commensurable with them respectively [(4), (5)].

To find $(u - w)$, w algebraically we have

$$u = \frac{\sigma^2}{\rho - \sqrt{k} \cdot \rho},$$

$$\frac{u - w}{w} = \frac{\rho}{\sqrt{k} \cdot \rho}.$$

From the latter $$w = \frac{u \cdot \sqrt{k} \cdot \rho}{\rho + \sqrt{k} \cdot \rho}$$

$$= \frac{\sigma^2 \cdot \sqrt{k} \cdot \rho}{\rho^2 - k\rho^2}.$$

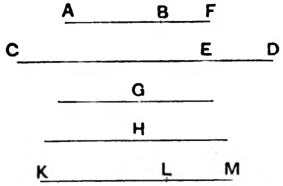
Thus $$u - w = w \cdot \frac{1}{\sqrt{k}} = \frac{\sigma^2 \rho}{\rho^2 - k\rho^2}.$$

Therefore $$(u - w) + w = \sigma^2 \cdot \frac{\rho + \sqrt{k} \cdot \rho}{\rho^2 - k\rho^2}.$$

PROPOSITION 114.

If an area be contained by an apotome and the binomial straight line the terms of which are commensurable with the terms of the apotome and in the same ratio, the "side" of the area is rational.

For let an area, the rectangle *AB*, *CD*, be contained by the apotome *AB* and the binomial straight line *CD*, and let *CE* be the greater term of the latter; let the terms *CE*, *ED* of the binomial straight line be commensurable with the terms *AF*, *FB* of the apotome and in the same ratio; and let the "side" of the rectangle *AB*, *CD* be *G*;



I say that *G* is rational.

For let a rational straight line *H* be set out, and to *CD* let there be applied a rectangle equal to the square on *H* and producing *KL* as breadth.

Therefore *KL* is an apotome.

Let its terms be *KM*, *ML* commensurable with the terms *CE*, *ED* of the binomial straight line and in the same ratio. [X. 112]

But *CE*, *ED* are also commensurable with *AF*, *FB* and in the same ratio;

therefore, as *AF* is to *FB*, so is *KM* to *ML*.

Therefore, alternately, as *AF* is to *KM*, so is *BF* to *LM*;
therefore also the remainder *AB* is to the remainder *KL* as *AF* is to *KM*. [V. 19]

But *AF* is commensurable with *KM*; [X. 12]
therefore *AB* is also commensurable with *KL*. [X. 11]

And, as *AB* is to *KL*, so is the rectangle *CD*, *AB* to the rectangle *CD*, *KL*; [VI. 1]
therefore the rectangle *CD*, *AB* is also commensurable with the rectangle *CD*, *KL*. [X. 11]

But the rectangle *CD*, *KL* is equal to the square on *H*;
therefore the rectangle *CD*, *AB* is commensurable with the square on *H*.

But the square on *G* is equal to the rectangle *CD*, *AB*;
therefore the square on *G* is commensurable with the square on *H*.

But the square on *H* is rational;
therefore the square on *G* is also rational;
therefore *G* is rational.

And it is the "side" of the rectangle *CD*, *AB*.

Therefore etc.

PORISM. And it is made manifest to us by this also that it is possible for a rational area to be contained by irrational straight lines.

Q. E. D.

This theorem is equivalent to the proof of the fact that

$$\sqrt{(\sqrt{A} - \sqrt{B})(\lambda\sqrt{A} + \lambda\sqrt{B})} = \sqrt{\lambda(A - B)},$$

and

$$\sqrt{(a \sim \sqrt{B})(\lambda a + \lambda\sqrt{B})} = \sqrt{\lambda(a^2 \sim B)}.$$

The result of the theorem X. 112 is used for the purpose thus.

We have to prove that

$$\sqrt{(\rho - \sqrt{k} \cdot \rho)(\lambda\rho + \lambda\sqrt{k} \cdot \rho)}$$

is *rational*.

By X. 112 we have, if σ is a rational straight line,

$$\frac{\sigma^2}{\lambda\rho + \lambda\sqrt{k} \cdot \rho} = \lambda'\rho - \lambda'\sqrt{k} \cdot \rho \quad \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Now $\rho : \lambda'\rho = \sqrt{k} \cdot \rho : \lambda' \sqrt{k} \cdot \rho = (\rho - \sqrt{k} \cdot \rho) : (\lambda'\rho - \lambda' \sqrt{k} \cdot \rho)$,
so that $(\rho - \sqrt{k} \cdot \rho) \frown (\lambda'\rho - \lambda' \sqrt{k} \cdot \rho)$.

Multiplying each by $(\lambda\rho + \lambda \sqrt{k} \cdot \rho)$, we have

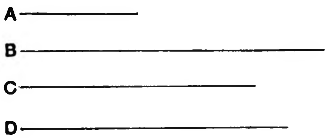
$$(\rho - \sqrt{k} \cdot \rho)(\lambda\rho + \lambda \sqrt{k} \cdot \rho) \frown (\lambda\rho + \lambda \sqrt{k} \cdot \rho)(\lambda'\rho - \lambda' \sqrt{k} \cdot \rho)$$
$$\frown \sigma^2, \qquad \text{by (1)}.$$

That is, $(\rho - \sqrt{k} \cdot \rho)(\lambda\rho + \lambda \sqrt{k} \cdot \rho)$ is a rational area,
and therefore $\sqrt{(\rho - \sqrt{k} \cdot \rho)(\lambda\rho + \lambda \sqrt{k} \cdot \rho)}$ is rational.

PROPOSITION 115.

From a medial straight line there arise irrational straight lines infinite in number, and none of them is the same as any of the preceding.

Let *A* be a medial straight line;
I say that from *A* there arise irrational straight lines infinite in number, and none of them is the same as any of the preceding.



Let a rational straight line *B* be set out,
and let the square on *C* be equal to the rectangle *B*, *A*;
therefore *C* is irrational; [x. Def. 4]
for that which is contained by an irrational and a rational straight line is irrational. [deduction from x. 20]

And it is not the same with any of the preceding;
for the square on none of the preceding, if applied to a rational straight line produces as breadth a medial straight line.

Again, let the square on *D* be equal to the rectangle *B*, *C*;
therefore the square on *D* is irrational. [deduction from x. 20]

Therefore *D* is irrational; [x. Def. 4]
and it is not the same with any of the preceding, for the square on none of the preceding, if applied to a rational straight line, produces *C* as breadth.

Similarly, if this arrangement proceeds *ad infinitum*, it is manifest that from the medial straight line there arise irrational straight lines infinite in number, and none is the same with any of the preceding.

Q. E. D.

Heiberg is clearly right in holding that this proposition, at all events, is alien to the general scope of Book x, and is therefore probably an interpolation, made however before Theon's time. It is of the same character as a *scholium* at the end of the Book, which is (along with the interpolated proposition proving, in two ways, the incommensurability of the diagonal of a square with its side) relegated by August as well as Heiberg to an Appendix.

The proposition amounts to this.

The straight line $k^{\frac{1}{4}}\rho$ being medial, if σ be a rational straight line, $\sqrt{k^{\frac{1}{4}}\rho\sigma}$ is a new irrational straight line. So is the mean proportional between this and another rational straight line σ', and so on indefinitely.

ANCIENT EXTENSIONS OF THE THEORY OF BOOK X.

From the hints given by the author of the commentary found in Arabic by Woepcke (cf. pp. 3—4 above) it would seem probable that Apollonius' extensions of the theory of irrationals took two directions: (1) generalising the *medial* straight line of Euclid, and (2) forming compound irrationals by the addition and subtraction of more than two terms of the sort composing the binomials, apotomes, etc. The commentator writes (Woepcke's article, pp. 694 sqq.):

"It is also necessary that we should know that, not only when we join together two straight lines rational and commensurable in square do we obtain the binomial straight line, but three or four lines produce in an analogous manner the same thing. In the first case, we obtain the trinomial straight line, since the whole line is irrational; and in the second case we obtain the quadrinomial, and so on *ad infinitum*. The proof of the (irrationality of the) line composed of three lines rational and commensurable in square is exactly the same as the proof relating to the combination of two lines.

"But we must start afresh and remark that not only can we take one sole medial line between two lines commensurable in square, but we can take three or four of them and so on *ad infinitum*, since we can take, between any two given straight lines, as many lines as we wish in continued proportion.

"Likewise, in the lines formed by addition not only can we construct the binomial straight line, but we can also construct the trinomial, as well as the first and second trimedial; and, further, the line composed of three straight lines incommensurable in square and such that the one of them gives with each of the two others a sum of squares (which is) rational, while the rectangle contained by the two lines is medial, so that there results a *major* (irrational) composed of three lines.

"And, in an analogous manner, we obtain the straight line which is the 'side' of a rational plus a medial area, composed of three straight lines, and, likewise, that which is the 'side' of (the sum of) two medials."

The generalisation of the medial is apparently after the following manner. Let x, y be two straight lines rational and commensurable in square only and suppose that m means are interposed, so that

$$x : x_1 = x_1 : x_2 = x_2 : x_3 = \ldots = x_{m-1} : x_m = x_m : y.$$

We easily derive herefrom $\dfrac{x}{x_r} = \left(\dfrac{x}{x_1}\right)^r,$

$$\frac{x}{y} = \left(\frac{x}{x_1}\right)^{m+1},$$

and hence
$$x_1^{\,r} = x_r \,.\, x^{r-1},$$
$$x_1^{\,m+1} = y \,.\, x^m,$$
so that
$$(x_r \,.\, x^{r-1})^{m+1} = (y \,.\, x^m)^r,$$
and therefore
$$x_r^{\,m+1} = x^{m-r+1} \,.\, y^r,$$
or
$$x_r = (x^{m-r+1} \,.\, y^r)^{\frac{1}{m+1}},$$
which is the generalised medial.

We now pass to the trinomial etc., with the commentator's further remarks about them.

(1) *The trinomial.* "Suppose three rational straight lines commensurable in square only. The line composed of two of these lines, that is, the binomial straight line, is irrational, and, in consequence, the area contained by this line and the remaining line is irrational, and, likewise, the double of the area contained by these two lines will be irrational. Thus the square on the whole line composed of three lines is irrational and consequently the line is irrational, and it is called a trinomial straight line."

It is easy to see that this "proof" is not conclusive as stated. Nor does Woepcke seem to show how the proposition can be proved on Euclidean lines. But I think it would be somewhat as follows.

Suppose x, y, z to be rational and $\frown$.
Then x^2, y^2, z^2 are rational, and $2yz$, $2zx$, $2xy$ are all medial.
First, $(2yz + 2zx + 2xy)$ cannot be rational.
For suppose this sum equal to a rational area, say σ^2.

Since
$$2yz + 2zx + 2xy = \sigma^2,$$
$$2zx + 2xy = \sigma^2 - 2yz,$$
or the sum of two medial areas incommensurable with one another is equal to the difference between a rational area and a medial area.

But the "side" of the sum of the two medial areas must [x. 72] be one of two irrationals with a positive sign; and the "side" of the difference between a rational area and a medial area must [x. 108] be one of two irrationals with a negative sign.

And the first "side" cannot be the same as the second [x. 111 and explanation following].

Therefore
$$2zx + 2xy \neq \sigma^2 - 2yz,$$
and $2yz + 2zx + 2xy$ is consequently *irrational.*

Therefore
$$(x^2 + y^2 + z^2) \smile (2yz + 2zx + 2xy),$$
whence
$$(x + y + z)^2 \smile (x^2 + y^2 + z^2),$$
so that $(x + y + z)^2$, and therefore also $(x + y + z)$, is irrational.

The commentator goes on:

"And, if we have four lines commensurable in square, as we have said, the procedure will be exactly the same; and we shall treat the succeeding lines in an analogous manner."

Without speculating further as to how the extension was made to the *quadrinomial* etc., we may suppose with Woepcke that Apollonius probably investigated the multinomial
$$\rho + \sqrt{\kappa} \,.\, \rho + \sqrt{\lambda} \,.\, \rho + \sqrt{\mu} \,.\, \rho + \ldots$$

(2) The *first trimedial* straight line.

The commentator here says: "Suppose we have three medial lines commensurable in square [only], one of which contains with each of the two others a rational rectangle; then the straight line composed of the two lines is irrational and is called the first bimedial; the remaining line is medial, and the area contained by these two lines is irrational. Consequently the square on the whole line is irrational."

To begin with, the conditions here given are incompatible. If x, y, z be medial straight lines such that xy, xz are both rational,

$$y : z = xy : xz = m : n,$$

and y, z are commensurable *in length* and not *in square only*.

Hence it seems that we must, with Woepcke, understand "three medial straight lines such that *one is commensurable with each of the other two* in square only and makes with it a rational rectangle."

If x, y, z be the three medial straight lines,

$$(x^2 + y^2 + z^2) \frown x^2,$$

so that $(x^2 + y^2 + z^2)$ is medial.

Also we have $2xy$, $2xz$ both rational and $2yz$ medial.

Now $(x^2 + y^2 + z^2) + 2yz + 2xy + 2xz$ cannot be rational, for, if it were, the sum of two medial areas, $(x^2 + y^2 + z^2)$, $2yz$, would be rational: which is impossible. [Cf. x. 72.]

Hence $(x + y + z)$ is irrational.

(3) The *second trimedial* straight line.

Suppose x, y, z to be medial straight lines commensurable in square only and containing with each other medial rectangles.

Then $(x^2 + y^2 + z^2) \frown x^2$, and is medial.

Also $2yz$, $2zx$, $2xy$ are all medial areas.

To prove the irrationality in this case I presume that the method would be like that of x. 38 about the *second bimedial.*

Suppose σ to be a rational straight line and let

$$\left.\begin{aligned} (x^2 + y^2 + z^2) &= \sigma t \\ 2yz &= \sigma u \\ 2zx &= \sigma v \\ 2xy &= \sigma w \end{aligned}\right\}$$

Here, since, e.g., $xz : xy = v : w$,

or $z : y = v : w$,

and similarly $x : z = w : u$,

u, v, w are *commensurable in square only.*

Also, since

$$\begin{aligned}(x^2 + y^2 + z^2) &\frown x^2 \\ &\smile xy,\end{aligned}$$

t is incommensurable with w.

Similarly t is incommensurable with u, v.

But t, u, v, w are all rational and $\frown \sigma$.

Therefore $(t+u+v+w)$ is a quadrinomial and therefore irrational.

Therefore $\sigma(t+u+v+w)$, or $(x+y+z)^2$, is irrational,

whence $(x+y+z)$ is irrational.

(4) The *major* made up of three straight lines.

The commentator describes this as "the line composed of three straight lines incommensurable in square and such that one of them gives with each of the other two a sum of squares (which is) rational, while the rectangle contained by the two lines is medial."

If x, y, z are the three straight lines, this would indicate

$$\begin{array}{r}(x^2+y^2)\text{ rational,}\\ (x^2+z^2)\text{ rational,}\\ 2yz\text{ medial.}\end{array}$$

Woepcke points out (pp. 696—8, note) the difficulties connected with this supposition or the supposition of

$$\begin{array}{r}(x^2+y^2)\text{ rational,}\\ (x^2+z^2)\text{ rational,}\\ 2xy\text{ (or }2xz\text{) medial,}\end{array}$$

and concludes that what is meant is the supposition

$$\left.\begin{array}{r}(x^2+y^2)\text{ rational}\\ xy\text{ medial}\\ xz\text{ medial}\end{array}\right\}.$$

(though the text is against this).

The assumption of (x^2+y^2) and (x^2+z^2) being concurrently rational is certainly further removed from Euclid, for x. 33 only enables us to find *one pair* of lines having the property, as x, y.

But we will not pursue these speculations further.

As regards further irrationals formed by *subtraction* the commentator writes as follows.

"Again, it is not necessary that, in the irrational straight lines formed by means of subtraction, we should confine ourselves to making one subtraction only, so as to obtain the apotome, or the first apotome of the medial, or the second apotome of the medial, or the minor, or the straight line which produces with a rational area a medial whole, or that which produces with a medial area a medial whole; but we shall be able here to make two or three or four subtractions.

"When we do that, we show in manner analogous to the foregoing that the lines which remain are irrational and that each of them is one of the lines formed by subtraction. That is to say that, if from a rational line we cut off another rational line commensurable with the whole line in square, we obtain, for remainder, an apotome; and, if we subtract from this line (which is) cut off and rational—that which Euclid calls the *annex* (προσαρμόζουσα)—another rational line which is commensurable with it in square, we obtain, as the remainder, an apotome; likewise, if we cut off from the rational line cut off from this line (i.e. the annex of the apotome last arrived at) another line which is commensurable with it in square, the remainder is an apotome. The same thing occurs in the subtraction of the other lines."

As Woepcke remarks, the idea is the formation of the successive apotomes $\sqrt{a}-\sqrt{b}$, $\sqrt{b}-\sqrt{c}$, $\sqrt{c}-\sqrt{d}$, etc. We should naturally have expected to see the writer form and discuss the following expressions

$$(\sqrt{a}-\sqrt{b})-\sqrt{c},$$

$$\{(\sqrt{a}-\sqrt{b})-\sqrt{c}\}-\sqrt{d}, \text{ etc.}$$

BOOK XI.

DEFINITIONS.

1. A **solid** is that which has length, breadth, and depth.

2. An extremity of a solid is a surface.

3. A **straight line** is **at right angles to a plane,** when it makes right angles with all the straight lines which meet it and are in the plane.

4. A **plane** is **at right angles to a plane** when the straight lines drawn, in one of the planes, at right angles to the common section of the planes are at right angles to the remaining plane.

5. The **inclination of a straight line to a plane** is, assuming a perpendicular drawn from the extremity of the straight line which is elevated above the plane to the plane, and a straight line joined from the point thus arising to the extremity of the straight line which is in the plane, the angle contained by the straight line so drawn and the straight line standing up.

6. The **inclination of a plane to a plane** is the acute angle contained by the straight lines drawn at right angles to the common section at the same point, one in each of the planes.

7. A plane is said to be **similarly inclined** to a plane as another is to another when the said angles of the inclinations are equal to one another.

8. **Parallel planes** are those which do not meet.

9. **Similar solid figures** are those contained by similar planes equal in multitude.

10. **Equal and similar solid figures** are those contained by similar planes equal in multitude and in magnitude.

11. A **solid angle** is the inclination constituted by more than two lines which meet one another and are not in the same surface, towards all the lines.

Otherwise: A **solid angle** is that which is contained by more than two plane angles which are not in the same plane and are constructed to one point.

12. A **pyramid** is a solid figure, contained by planes, which is constructed from one plane to one point.

13. A **prism** is a solid figure contained by planes two of which, namely those which are opposite, are equal, similar and parallel, while the rest are parallelograms.

14. When, the diameter of a semicircle remaining fixed, the semicircle is carried round and restored again to the same position from which it began to be moved, the figure so comprehended is a **sphere.**

15. The **axis of the sphere** is the straight line which remains fixed and about which the semicircle is turned.

16. The **centre of the sphere** is the same as that of the semicircle.

17. A **diameter of the sphere** is any straight line drawn through the centre and terminated in both directions by the surface of the sphere.

18. When, one side of those about the right angle in a right-angled triangle remaining fixed, the triangle is carried round and restored again to the same position from which it began to be moved, the figure so comprehended is a **cone.**

And, if the straight line which remains fixed be equal to the remaining side about the right angle which is carried round, the cone will be **right-angled**; if less, **obtuse-angled**; and if greater, **acute-angled.**

19. The **axis of the cone** is the straight line which remains fixed and about which the triangle is turned.

20. And the **base** is the circle described by the straight line which is carried round.

21. When, one side of those about the right angle in a rectangular parallelogram remaining fixed, the parallelogram is carried round and restored again to the same position from which it began to be moved, the figure so comprehended is a **cylinder**.

22. The **axis of the cylinder** is the straight line which remains fixed and about which the parallelogram is turned.

23. And the **bases** are the circles described by the two sides opposite to one another which are carried round.

24. **Similar cones and cylinders** are those in which the axes and the diameters of the bases are proportional.

25. A **cube** is a solid figure contained by six equal squares.

26. An **octahedron** is a solid figure contained by eight equal and equilateral triangles.

27. An **icosahedron** is a solid figure contained by twenty equal and equilateral triangles.

28. A **dodecahedron** is a solid figure contained by twelve equal, equilateral, and equiangular pentagons.

DEFINITION 1.

Στερεόν ἐστι τὸ μῆκος καὶ πλάτος καὶ βάθος ἔχον.

This definition was evidently traditional, as may be inferred from a number of passages in Plato and Aristotle. Thus Plato speaks (*Sophist*, 235 D) of making an imitation of a model (παράδειγμα) "in length and breadth and depth" and (*Laws*, 817 E) of "the art of measuring length, surface and depth" as one of three μαθήματα. *Depth*, the third dimension, is used alone as a description of "body" by Aristotle, the term being regarded as connoting the other two dimensions; thus (*Metaph.* 1020 a 13, 11) "length is a line, breadth a surface, and depth body"; "that which is continuous in one direction is length, in two directions breadth, and in three depth." Similarly Plato (*Rep.* 528 B, D), when reconsidering his classification of astronomy as next to (plane) geometry: "although the science dealing with the additional dimension of depth is next in order, yet, owing to the fact that it is studied absurdly, I passed it over and put next to geometry astronomy, the *motion* of (bodies having) depth." In Aristotle (*Topics* VI. 5, 142 b 24) we find "the definition of body, that which has three dimensions (διαστάσεις)"; elsewhere he speaks of it as "that which has all the dimensions" (*De caelo* I. 1, 268 b 6), "that which has dimension every way" (τὸ πάντῃ διάστασιν ἔχον, *Metaph.* 1066 b 32) etc. In the *Physics*

(IV. 1, 208 b 13 sqq.) he speaks of the "dimensions" as *six*, dividing each of the three into two opposites, "up and down, before and behind, right and left," though of course, as he explains, these terms are relative.

Heron, as might be expected, combines the two forms of the definition. "A solid body is that which has length, breadth, and depth: or that which possesses the three dimensions." (Def. 11.)

Similarly Theon of Smyrna (p. 111, 19, ed. Hiller): "that which is extended (διαστατόν) and divisible in three directions is solid, having length, breadth and depth."

DEFINITION 2.

Στερεοῦ δὲ πέρας ἐπιφάνεια.

In like manner Aristotle says (*Metaph.* 1066 b 23) that the notion (λόγος) of body is "that which is bounded by surfaces" (ἐπιπέδοις in this case) and (*Metaph.* 1060 b 15) "surfaces (ἐπιφάνειαι) are divisions of bodies."

So Heron (Def. 11): "Every solid is bounded (περατοῦται) by surfaces, and is produced when a surface is moved from a forward position in a backward direction."

DEFINITION 3.

Εὐθεῖα πρὸς ἐπίπεδον ὀρθή ἐστιν, ὅταν πρὸς πάσας τὰς ἁπτομένας αὐτῆς εὐθείας καὶ οὔσας ἐν τῷ ἐπιπέδῳ ὀρθὰς ποιῇ γωνίας.

This definition and the next are given almost word for word by Heron (Def. 115).

That a straight line *can* be so related to a plane as described in Def. 3 is established in XI. 4. The fact has been made the basis of a definition of a *plane* which is attributed by Crelle to Fourier, and is as follows. "A plane is formed by the totality of all the straight lines which, passing through one and the same point of a straight line in space, stand perpendicular to it." Stated in this form, the definition is open to the objection that the conception of a right angle, involving the measurement of angles, presupposes a plane, inasmuch as the measurement of angles depends ultimately upon the superposition of two planes and their coincidence throughout when two lines in one coincide with two lines in the other respectively. Cf. my note on I. Def. 7, Vol. I. pp. 173—5.

DEFINITION 4.

Ἐπίπεδον πρὸς ἐπίπεδον ὀρθόν ἐστιν, ὅταν αἱ τῇ κοινῇ τομῇ τῶν ἐπιπέδων πρὸς ὀρθὰς ἀγόμεναι εὐθεῖαι ἐν ἑνὶ τῶν ἐπιπέδων τῷ λοιπῷ ἐπιπέδῳ πρὸς ὀρθὰς ὦσιν.

Both this definition and Def. 6 use the *common section* of two planes, though it is not till XI. 3 that this common section is proved to be a straight line. The definition however, just like Def. 3, is legitimate, because the object is to explain the meaning of terms, not to prove anything

The definition of perpendicular planes is made by Legendre a particular case of Def. 6, the limiting case, namely, where the angle representing the "inclination of a plane to a plane" is a right angle.

DEFINITION 5.

Εὐθείας πρὸς ἐπίπεδον κλίσις ἐστίν, ὅταν ἀπὸ τοῦ μετεώρου πέρατος τῆς εὐθείας ἐπὶ τὸ ἐπίπεδον κάθετος ἀχθῇ, καὶ ἀπὸ τοῦ γενομένου σημείου ἐπὶ τὸ ἐν τῷ ἐπιπέδῳ πέρας τῆς εὐθείας εὐθεῖα ἐπιζευχθῇ, ἡ περιεχομένη γωνία ὑπὸ τῆς ἀχθείσης καὶ τῆς ἐφεστώσης.

In other words, the inclination of a straight line to a plane is the angle between the straight line and its *projection* on the plane. This angle is of course less than the angle between the straight line and any other straight line in the plane through the intersection of the straight line and plane; and the fact is sometimes made the subject of a proposition in modern text-books. It is easily proved by means of the propositions XI. 4, I. 19 and 18.

DEFINITION 6.

Ἐπιπέδου πρὸς ἐπίπεδον κλίσις ἐστὶν ἡ περιεχομένη ὀξεῖα γωνία ὑπὸ τῶν πρὸς ὀρθὰς τῇ κοινῇ τομῇ ἀγομένων πρὸς τῷ αὐτῷ σημείῳ ἐν ἑκατέρῳ τῶν ἐπιπέδων.

When two planes meet in a straight line, they form what is called in modern text-books a *dihedral angle*, which is defined as the *opening* or *angular opening* between the two planes. This *dihedral angle* is an "angle" altogether different in kind from a plane angle, as again it is different from a *solid angle* as defined by Euclid (i.e. a trihedral, tetrahedral, etc. angle). Adopting for the moment Apollonius' conception of an angle as the "bringing together of a surface or solid towards one point under a broken line or surface" (Proclus, p. 123, 16), we may regard a dihedral angle as the bringing together of the broken surface formed by two intersecting planes not to a *point* but to a *straight line*, namely the intersection of the planes. Legendre, in a proposition on the subject, applied provisionally the term *corner* to describe the dihedral angle between two planes; and this would be a better word, I think, than *opening* to use in the definition.

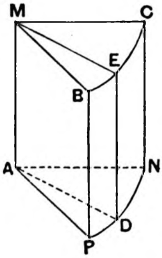
The distinct species of "angle" which we call dihedral is, however, *measured* by a certain plane angle, namely that which Euclid describes in the present definition and calls the *inclination of a plane to a plane*, and which in some modern text-books is called the *plane angle of the dihedral angle*.

It is necessary to show that this plane angle is a proper measure of the dihedral angle, and accordingly Legendre has a proposition to this effect. In order to prove it, it is necessary to show that, given two planes meeting in a straight line,

(1) the plane angle in question is the same at all points of the straight line forming the common section;

(2) if the dihedral angle between two planes increases or diminishes in a certain ratio, the plane angle in question will increase or diminish in the same ratio.

(1) If MAN, MAP be two planes intersecting in MA, and if AN, AP be drawn in the planes respectively and at right angles to MA, the angle NAP is the *inclination of the plane to the plane* or the *plane angle of the dihedral angle*.



Let MC, MB be also drawn in the respective planes at right angles to MA.

Then since, in the plane MAN, MC and AN are drawn at right angles to the same straight line MA,

MC, AN are parallel.

For the same reason, MB, AP are parallel.

Therefore [XI. 10] the angle BMC is equal to the angle PAN.

And M may be any point on MA. Therefore the plane angle described in the definition is the same at all points of AM.

(2) In the plane *NAP* draw the arc *NDP* of any circle with centre *A*, and draw the radius *AD*.

Now the planes *NAP*, *CMB*, being both at right angles to the straight line *MA*, are parallel; [XI. 14]

therefore the intersections *AD*, *ME* of these planes with the plane *MAD* are parallel, [XI. 16]

and consequently the angles *BME*, *PAD* are equal. [XI. 10]

If now the plane angle *NAD* were equal to the plane angle *DAP*, the dihedral angle *NAMD* would be equal to the dihedral angle *DAMP*;

for, if the angle *PAD* were applied to the angle *DAN*, *AM* remaining the same, the corresponding dihedral angles would coincide.

Successive applications of this result show that, if the angles *NAD*, *DAP* each contain a certain angle a certain number of times, the dihedral angles *NAMD*, *DAMP* will contain the corresponding dihedral angle the same number of times respectively.

Hence, where the angles *NAD*, *DAP* are *commensurable*, the dihedral angles corresponding to them are in the same ratio.

Legendre then extends the proof to the case where the plane angles are incommensurable by reference to an exactly similar extension in his proposition corresponding to Euclid VI. 1, for which see the note on that proposition.

Modern text-books make the extension by an appeal to *limits*.

DEFINITION 7.

Ἐπίπεδον πρὸς ἐπίπεδον ὁμοίως κεκλίσθαι λέγεται καὶ ἕτερον πρὸς ἕτερον, ὅταν αἱ εἰρημέναι τῶν κλίσεων γωνίαι ἴσαι ἀλλήλαις ὦσιν.

DEFINITION 8.

Παράλληλα ἐπίπεδά ἐστι τὰ ἀσύμπτωτα.

Heron has the same definition of parallel planes (Def. 115). The Greek word which is translated "which do not meet" is ἀσύμπτωτα, the term which has been adopted for the *asymptotes* of a curve.

DEFINITION 9.

Ὅμοια στερεὰ σχήματά ἐστι τὰ ὑπὸ ὁμοίων ἐπιπέδων περιεχόμενα ἴσων τὸ πλῆθος.

DEFINITION 10.

Ἴσα δὲ καὶ ὅμοια στερεὰ σχήματά ἐστι τὰ ὑπὸ ὁμοίων ἐπιπέδων περιεχόμενα ἴσων τῷ πλήθει καὶ τῷ μεγέθει.

These definitions, the second of which practically only substitutes the words "equal and similar" for the word "similar" in the first, have been the mark of much criticism.

Simson holds that the equality of solid figures is a thing which ought to be *proved*, by the method of superposition, or otherwise, and hence that Def. 10 is not a definition but a *theorem* which ought not to have been placed among the definitions. Secondly, he gives an example to show that the definition or theorem is not universally true. He takes a pyramid and then erects on the base, on opposite sides of it, two equal pyramids smaller than the first. The addition and subtraction of these pyramids respectively from the first give two solid figures which satisfy the definition but are clearly not equal (the smaller having a re-entrant angle); whence it also appears that two unequal solid angles may be contained by the same number of equal plane angles.

Maintaining then that Def. 10 is an interpolation by "an unskilful hand," Simson transfers to a place before Def. 9 the definition of a solid angle, and then defines similar solid figures as follows:

Similar solid figures are such as have all their solid angles equal, each to each, and which are contained by the same number of similar planes.

Legendre has an invaluable discussion of the whole subject of these definitions (Note XII., pp. 323—336, of the 14th edition of his *Éléments de Géométrie*). He remarks in the first place that, as Simson said, Def. 10 is not properly a definition, but a theorem which it is necessary to prove; for it is not evident that two solids are equal for the sole reason that they have an equal number of equal faces, and, if true, the fact should be proved by superposition or otherwise. The fault of Def. 10 is also common to Def. 9. For, if Def. 10 is not proved, one might suppose that there exist two unequal and dissimilar solids with equal faces; but, in that case, according to Definition 9, a solid having faces similar to those of the two first would be similar to both of them, i.e. to two solids of different form: a conclusion implying a contradiction or at least not according with the natural meaning of the word "similar."

What then is to be said in defence of the two definitions as given by Euclid? It is to be observed that the figures which Euclid actually proves equal or similar by reference to Deff. 9, 10 are such that their solid angles do not consist of more than *three* plane angles; and he proves sufficiently clearly that, if three plane angles forming one solid angle be respectively equal to three plane angles forming another solid angle, the two solid angles are equal. If now two polyhedra have their faces equal respectively, the corresponding solid angles will be made up of the same number of plane angles, and the plane angles forming each solid angle in one polyhedron will be respectively equal to the plane angles forming the corresponding solid angle in the other. Therefore, if the plane angles in each solid angle are not more than three in number, the corresponding solid angles will be equal. But if the corresponding faces are equal, and the corresponding solid angles equal, the solids must be equal; for they can be superposed, or at least they will be symmetrical with one another. Hence the statement of Deff. 9, 10 is true and admissible at all events in the case of figures with trihedral angles, which is the only case taken by Euclid.

Again, the example given by Simson to prove the incorrectness of Def. 10 introduces a solid with a re-entrant angle. But it is more than probable that Euclid deliberately intended to exclude such solids and to take cognizance of *convex* polyhedra only; hence Simson's example is not conclusive against the definition.

Legendre observes that Simson's own definition, though true, has the disadvantage that it contains a number of superfluous conditions. To get over the difficulties, Legendre himself divides the definition of similar solids into two, the first of which defines similar *triangular pyramids* only, and the second (which defines similar polyhedra in general) is based on the first.

Two triangular pyramids are similar when they have pairs of faces respectively similar, similarly placed and equally inclined to one another.

Then, having formed a triangle with the vertices of three angles taken on the same face or base of a polyhedron, we may imagine the vertices of the different solid angles of the polyhedron situated outside of the plane of this base to be the vertices of as many triangular pyramids which have the triangle for common base, and each of these pyramids will determine the position of one solid angle of the polyhedron. This being so,

Two polyhedra are similar when they have similar bases, and the vertices of their corresponding solid angles outside the bases are determined by triangular pyramids similar each to each.

As a matter of fact, Cauchy proved that two *convex* solid figures are equal if they are contained by equal plane figures similarly arranged. Legendre gives a proof which, he says, is nearly the same as Cauchy's, depending on two lemmas which lead to the theorem that, *Given a convex polyhedron in which all the solid angles are made up of more than three plane angles, it is impossible to vary the inclinations of the planes of this solid so as to produce a second polyhedron formed by the same planes arranged in the same manner as in the given polyhedron.* The convex polyhedron *in which all the solid angles are made up of more than three plane angles* is obtained by cutting off from *any* given polyhedron all the triangular pyramids forming trihedral angles (if one and the same edge is common to *two* trihedral angles, only one of these angles is suppressed in the first operation). This is legitimate because trihedral angles are invariable from their nature.

Hence it would appear that Heron's definition of equal solid figures, which adds "similarly situated" to Euclid's "similar" is correct, if it be understood to apply to *convex* polyhedra only: *Equal solid figures are those which are contained by equal and similarly situated planes, equal in number and magnitude:* where, however, the words "equal and" before "similarly situated" might be dispensed with.

Heron (Def. 118) defines *similar solid figures* as *those which are contained by planes similar and similarly situated.* If understood of *convex* polyhedra, there would not appear to be any objection to this, in view of the truth of Cauchy's proposition about equal solid figures.

DEFINITION 11.

Στερεὰ γωνία ἐστὶν ἡ ὑπὸ πλειόνων ἢ δύο γραμμῶν ἁπτομένων ἀλλήλων καὶ μὴ ἐν τῇ αὐτῇ ἐπιφανείᾳ οὐσῶν πρὸς πάσαις ταῖς γραμμαῖς κλίσις. Ἄλλως· στερεὰ γωνία ἐστὶν ἡ ὑπὸ πλειόνων ἢ δύο γωνιῶν ἐπιπέδων περιεχομένη μὴ οὐσῶν ἐν τῷ αὐτῷ ἐπιπέδῳ πρὸς ἑνὶ σημείῳ συνισταμένων.

Heiberg conjectures that the first of these two definitions, which is not in Euclid's manner, was perhaps taken by him from some earlier *Elements*.

The phraseology of the second definition is exactly that of Plato when he is speaking of solid angles in the *Timaeus* (p. 55). Thus he speaks (1) of four equilateral triangles so put together (ξυνιστάμενα) that each set of three plane angles makes one solid angle, (2) of eight equilateral triangles put together so that each set of four plane angles makes one solid angle, and (3) of six squares making eight solid angles, each composed of three plane right angles.

As we know, Apollonius defined an angle as the "bringing together of a surface or solid to one point under a broken line or surface." Heron (Def. 22) even omits the word "broken" and says that *A solid angle is in general* (κοινῶς) *the bringing together of a surface which has its concavity in one and the same direction to one point.* It is clear from an allusion in Proclus (p. 123, 1—6) to the half of a cone cut off by a triangle through the axis, and from a scholium to this definition, that there was controversy as to the correctness of describing as a solid angle the "angle" enclosed by fewer than three surfaces (including curved surfaces). Thus the scholiast says that Euclid's definition of a solid angle as made up of three or more plane angles is deficient because it does not e.g. cover the case of the angle of a "fourth part of a sphere," which is contained by more than two surfaces, though not all plane. But he declines to admit that the half-cone forms a solid angle at the vertex, for in that case the vertex of the cone would itself be an angle, and a solid angle would then be formed both by two surfaces and by one surface: "which is not true." Heron on the other hand (Def. 22) distinctly speaks of solid angles which are not contained by plane rectilineal angles, "e.g. the angles of cones." The conception of the latter "angles" as the *limit* of solid angles with an infinite number of infinitely small constituent plane angles does not appear in the Greek geometers so far as I know.

In modern text-books a polyhedral angle is usually spoken of as *formed* (or *bounded*) *by three or more planes meeting at a point*, or it is *the angular opening between such planes at the point where they meet.*

DEFINITION 12.

Πυραμίς ἐστι σχῆμα στερεὸν ἐπιπέδοις περιεχόμενον ἀπὸ ἑνὸς ἐπιπέδου πρὸς ἑνὶ σημείῳ συνεστώς.

This definition is by no means too clear, nor is the slightly amplified definition added to it by Heron (Def. 99). *A pyramid is the figure brought together to one point, by putting together triangles, from a triangular, quadrilateral or polygonal, that is, any rectilineal, base.*

As we might expect, there is great variety in the definitions given in modern text-books. Legendre says *a pyramid is the solid formed when several triangular planes start from one point and are terminated at the different sides of one polygonal plane.*

Mr H. M. Taylor and Smith and Bryant call it *a polyhedron all but one of whose faces meet in a point.*

Mehler reverses Legendre's form and gives the content of Euclid's in clearer language. "*An* n*-sided pyramid is bounded by an* n*-sided polygon as base and* n *triangles which connect its sides with one and the same point outside it.*"

Rausenberger points out that a pyramid is the figure cut off from a solid angle formed of any number of plane angles by a plane which intersects the solid angle.

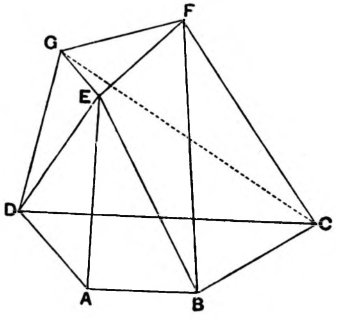
DEFINITION 13.

Πρίσμα ἐστὶ σχῆμα στερεὸν ἐπιπέδοις περιεχόμενον, ὧν δύο τὰ ἀπεναντίον ἴσα τε καὶ ὅμοιά ἐστι καὶ παράλληλα, τὰ δὲ λοιπὰ παραλληλόγραμμα.

Mr H. M. Taylor, followed by Smith and Bryant, defines a prism as *a polyhedron all but two of the faces of which are parallel to one straight line.*

Mehler calls an *n*-sided prism *a body contained between two parallel planes and enclosed by* n *other planes with parallel lines of intersection.*

Heron's definition of a prism is much wider (Def. 105). *Prisms are those figures which are connected* (συνάπτοντα) *from a rectilineal base to a rectilineal area by rectilineal collocation* (κατ' εὐθύγραμμον σύνθεσιν). By this Heron must apparently mean any convex solid formed by connecting the sides and angles of two polygons in different planes, and each having any number of sides, by straight lines forming triangular faces (where of course two adjacent triangles may be in one plane and so form one quadrilateral face) in the manner shown in the annexed figure, where *ABCD*, *EFG* represent the base and its opposite.



Heron goes on to explain that, if the face opposite to the base reduces to a straight line, and a solid is formed by connecting the base to its extremities by straight lines, as in the other case, the resulting figure is neither a pyramid nor a prism.

Further, he defines *parallelogrammic* (in the body of the definition *parallel-sided*) *prisms* as being those prisms which have six faces and have their opposite planes parallel.

DEFINITION 14.

Σφαῖρά ἐστιν, ὅταν ἡμικυκλίου μενούσης τῆς διαμέτρου περιενεχθὲν τὸ ἡμικύκλιον εἰς τὸ αὐτὸ πάλιν ἀποκατασταθῇ, ὅθεν ἤρξατο φέρεσθαι, τὸ περιληφθὲν σχῆμα.

The scholiast observes that this definition is not properly a definition ot a sphere but a description of the mode of generating it. But it will be seen, in the last propositions of Book XIII., why, Euclid put the definition in this form. It is because it is this particular view of a sphere which he uses to prove that the vertices of the regular solids which he wishes to "comprehend" in certain spheres do lie on the surfaces of those spheres. He proves in fact that the said vertices lie on *semicircles described on certain diameters of the spheres.* For the real definition the scholiast refers to Theodosius' *Sphaerica*. But of course the proper definition was given much earlier. In Aristotle the characteristic of a sphere is that *its extremity is equally distant from its centre* (τὸ ἴσον ἀπέχειν τοῦ μέσου τὸ ἔσχατον, *De caelo* II. 14, 297 a 24). Heron (Def. 76) uses the same form as that in which Euclid defines the circle: *A sphere is a solid figure bounded by one surface, such that all the straight lines falling on it from one point of those which lie within the figure are equal to one another.* So the usual definition in the text-books: *A sphere is a closed surface such that all points of it are equidistant from a fixed point within it.*

DEFINITION 15.

Ἄξων δὲ τῆς σφαίρας ἐστὶν ἡ μένουσα εὐθεῖα, περὶ ἣν τὸ ἡμικύκλιον στρέφεται.

That *any* diameter of a sphere may be called an axis is made clear by Heron (Def. 78). *The diameter of the sphere is called an axis, and is any straight line drawn through the centre and bounded in both directions by the sphere, immovable, about which the sphere is moved and turned.* Cf. Euclid's Def. 17.

DEFINITION 16.

Κέντρον δὲ τῆς σφαίρας ἐστὶ τὸ αὐτό, ὃ καὶ τοῦ ἡμικυκλίου.

Heron, Def. 77. *The middle* (*point*) *of the sphere is called its centre; and this same point is also the centre of the hemisphere.*

DEFINITION 17.

Διάμετρος δὲ τῆς σφαίρας ἐστὶν εὐθεῖά τις διὰ τοῦ κέντρου ἠγμένη καὶ περατουμένη ἐφ᾽ ἑκάτερα τὰ μέρη ὑπὸ τῆς ἐπιφανείας τῆς σφαίρας.

DEFINITION 18.

Κῶνός ἐστιν, ὅταν ὀρθογωνίου τριγώνου μενούσης μιᾶς πλευρᾶς τῶν περὶ τὴν ὀρθὴν γωνίαν περιενεχθὲν τὸ τρίγωνον εἰς τὸ αὐτὸ πάλιν ἀποκατασταθῇ, ὅθεν ἤρξατο φέρεσθαι, τὸ περιληφθὲν σχῆμα. κἂν μὲν ἡ μένουσα εὐθεῖα ἴση ᾖ τῇ λοιπῇ [τῇ] περὶ τὴν ὀρθὴν περιφερομένῃ, ὀρθογώνιος ἔσται ὁ κῶνος, ἐὰν δὲ ἐλάττων, ἀμβλυγώνιος, ἐὰν δὲ μείζων, ὀξυγώνιος.

This definition, or rather description of the genesis, of a (right) cone is interesting on account of the second sentence distinguishing between *right-angled*, *obtuse-angled* and *acute-angled* cones. This distinction is quite unnecessary for Euclid's purpose and is not used by him in Book XII.; it is no doubt a relic of the method, still in use in Euclid's time, by which the earlier Greek geometers produced conic sections, namely, by cutting right cones only by sections always perpendicular to an edge. With this system the parabola was a *section of a right-angled cone*, the hyperbola a *section of an obtuse-angled cone*, and the ellipse a *section of an acute-angled cone*. The conic sections were so called by Archimedes, and generally until Apollonius, who was the first to give the complete theory of their generation by means of sections not perpendicular to an edge, and from cones which are in general *oblique circular* cones. Thus Apollonius begins his *Conics* with the more scientific definition of a cone. If, he says, a straight line infinite in length, and passing always through a fixed point, be made to move round the circumference of a circle which is not in the same plane with the point, so as to pass successively through every point of that circumference, the moving straight line will trace out the surface of a *double cone*, or two similar cones lying in opposite directions and meeting in the fixed point, which is the *apex* of each cone. The circle about which the straight line moves is called the *base* of the cone lying between the said circle and the fixed point, and the *axis* is defined as the straight line drawn from the fixed point, or the apex, to the centre of the circle forming the base. Apollonius goes on to say that the cone is a *scalene* or *oblique* cone except in the particular case where the axis is perpendicular to the base. In this latter case it is a *right* cone.

Archimedes called the right cone an *isosceles* cone. This fact, coupled with the appearance in his treatise *On Conoids and Spheroids* (7, 8, 9) of *sections of acute-angled cones* (ellipses) as sections of conical surfaces which are proved to be oblique circular cones by finding their circular sections, makes it sufficiently clear that Archimedes, if he had defined a cone, would have defined it in the same way as Apollonius does.

DEFINITION 19.

Ἄξων δὲ τοῦ κώνου ἐστὶν ἡ μένουσα εὐθεῖα, περὶ ἣν τὸ τρίγωνον στρέφεται.

DEFINITION 20.

Βάσις δὲ ὁ κύκλος ὁ ὑπὸ τῆς περιφερομένης εὐθείας γραφόμενος.

DEFINITION 21.

Κύλινδρός ἐστιν, ὅταν ὀρθογωνίου παραλληλογράμμου μενούσης μιᾶς πλευρᾶς τῶν περὶ τὴν ὀρθὴν γωνίαν περιενεχθὲν τὸ παραλληλόγραμμον εἰς τὸ αὐτὸ πάλιν ἀποκατασταθῇ, ὅθεν ἤρξατο φέρεσθαι, τὸ περιληφθὲν σχῆμα.

DEFINITION 22.

Ἄξων δὲ τοῦ κυλίνδρου ἐστὶν ἡ μένουσα εὐθεῖα, περὶ ἣν τὸ παραλληλόγραμμον στρέφεται.

DEFINITION 23.

Βάσεις δὲ οἱ κύκλοι οἱ ὑπὸ τῶν ἀπεναντίον περιαγομένων δύο πλευρῶν γραφόμενοι.

DEFINITION 24.

Ὅμοιοι κῶνοι καὶ κύλινδροί εἰσιν, ὧν οἵ τε ἄξονες καὶ αἱ διάμετροι τῶν βάσεων ἀνάλογόν εἰσιν.

DEFINITION 25.

Κύβος ἐστὶ σχῆμα στερεὸν ὑπὸ ἓξ τετραγώνων ἴσων περιεχόμενον.

DEFINITION 26.

Ὀκτάεδρόν ἐστι σχῆμα στερεὸν ὑπὸ ὀκτὼ τριγώνων ἴσων καὶ ἰσοπλεύρων περιεχόμενον.

DEFINITION 27.

Εἰκοσάεδρόν ἐστι σχῆμα στερεὸν ὑπὸ εἴκοσι τριγώνων ἴσων καὶ ἰσοπλεύρων περιεχόμενον.

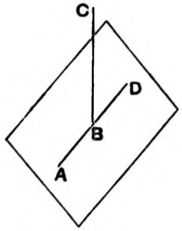
DEFINITION 28.

Δωδεκάεδρόν ἐστι σχῆμα στερεὸν ὑπὸ δώδεκα πενταγώνων ἴσων καὶ ἰσοπλεύρων καὶ ἰσογωνίων περιεχόμενον.

BOOK XI. PROPOSITIONS.

PROPOSITION 1.

A part of a straight line cannot be in the plane of reference and a part in a plane more elevated.



For, if possible, let a part *AB* of the straight line *ABC* be in the plane of reference, and a part *BC* in a plane more elevated.

There will then be in the plane of reference some straight line continuous with *AB* in a straight line.

Let it be *BD*;

therefore *AB* is a common segment of the two straight lines *ABC*, *ABD*:

which is impossible, inasmuch as, if we describe a circle with centre *B* and distance *AB*, the diameters will cut off unequal circumferences of the circle.

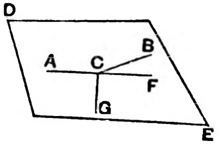
Therefore a part of a straight line cannot be in the plane of reference, and a part in a plane more elevated.

Q. E. D.

1. **the plane of reference,** *τὸ ὑποκείμενον ἐπίπεδον*, the plane laid down or assumed.
2. **more elevated,** *μετεωροτέρῳ*.

There is no doubt that the proofs of the first three propositions are unsatisfactory owing to the fact that Euclid is not able to make any use of his definition of a plane for the purpose of these proofs, and they really depend upon truths which can only be assumed as axiomatic. The definition of a plane as *that surface which lies evenly with the straight lines on itself*, whatever its exact meaning may be, is nowhere appealed to as a criterion to show whether a particular surface is or is not a plane. If the meaning of it is what I conjecture in the note on Book I., Def. 7 (Vol. I. p. 171), if, namely, it only tries to express without an appeal to sight what Plato meant by the "middle covering the extremities" (i.e. apparently, in the case of a plane, the fact that a plane looked at edgewise takes the form of a straight line), then it is perhaps possible to connect the definition with a method of generating a plane which has commended itself to many writers as giving a better definition. Thus, if we conceive a straight line in space and a point outside it placed so that, in Plato's words, the line "covers" the point as we look at them, the line will also "cover" every straight line which passes through the given point and some one point on the given straight line. Hence, if a straight line passing always through a fixed point moves in such a way as to pass successively through every point of a given straight line which does not contain the given point, the moving straight line describes a surface which satisfies the Euclidean definition of a plane as I have interpreted it. But if we adopt the definition of a plane as *the surface described by a straight line which, passing through a given point, turns about it in such a way as always to intersect a given straight line not passing through the given point*, this definition, though it would help us to prove Eucl. XI. 2, does not give us the fundamental properties of a plane; some postulate is necessary in addition. The same is true even if we take a definition which gives *more* than is required to determine a plane, the definition known as Simson's, though it is at least as early as the time of Theon of Smyrna, who says (p. 112, 5) that *a plane is a surface such that, if a straight line meet it in two points, the straight line lies wholly in it* (ὅλη αὐτῷ ἐφαρμόζεται). This is also called the *axiom of the plane*. (For some attempts to *prove* this on the basis of other definitions of a plane see my note on the definition of a *plane surface*, I. Def. 7.) If this definition or axiom be assumed, Prop. 1 becomes evident, for, as Legendre says, "In accordance with the definition of the plane, when a straight line has two points common with a plane, it lies wholly in the plane."

Euclid practically assumes the axiom when he says in this proposition "there will be in the plane of reference some straight line continuous with *AB*." Clavius tries, unsuccessfully, to deduce this from Euclid's own definition of a plane; and he seems to admit his failure, because he proceeds to try another tack. Draw, he says, in the plane *DE*, the straight line *CG* at right angles to *AC*, and, again in the plane *DE*, *CF* at right angles to *CG* [I. 11]. Then *AC*, *CF* make right angles with *CG* in the same plane; therefore (I. 14) *ACF* is a straight line. But this does not really help, because Euclid assumes tacitly, in Book I. as well as Book XI., that a straight line joining two points in a plane lies wholly in that plane.



A curious point in Euclid's proof is the reason given why two straight lines cannot have a common segment. The argument is precisely that of the "proof" of the same thing given by Proclus on I. 1 (see note on Book I. Post. 2, Vol. I. p. 197) and is of course inconclusive. The fact that two straight lines cannot have a common segment must be taken to be involved in the definition of, and the postulates relating to, the straight line; and the "proof" given here can hardly, I should say, be Euclid's, though the interpolation, if it be such, must have been made very early.

The proof assumes too that a circle *can* be described so as to cut *BA*, *BC* and *BD*, or, in other words, it assumes that *AD*, *BC* are in one plane; that is, Prop. 1 as we have it really assumes the result of Prop. 2. There is therefore ground for Simson's alteration of the proof (after the point where *BD* has been taken in the given plane in a straight line with *AB*) to the following:

"Let any plane pass through the straight line *AD* and be turned about it until it pass through the point *C*.

And, because the points B, C are in this plane, the straight line BC is in it. [Simson's def.]

Therefore there are two straight lines ABC, ABD in the same plane that have a common segment AB:
which is impossible."

Simson, of course, justifies the last inference by reference to his Corollary to I. 11, which, however, as we have seen, is not a valid proof of the assumption, which is really implied in I. Post. 2.

An alternative reading, perhaps due to Theon, says, after the words "which is impossible" in the Greek text, "for a straight line does not meet a straight line in more points than one; otherwise the straight lines will coincide." Simson (who however does not seem to have had the second clause beginning "otherwise" in the text which he used) attacks this alternative reading in a rather confused note chiefly directed against a criticism by Thomas Simpson, without (as it seems to me) sufficient reason. It contains surely a legitimate argument. The supposed straight lines ABC, ABD meet in more than two points, namely in all the points between A and B. But two straight lines cannot have two points common without coinciding altogether; therefore ABC must coincide with ABD.

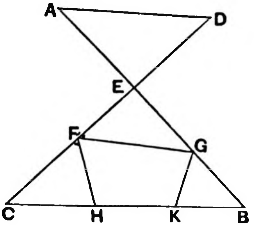
PROPOSITION 2.

If two straight lines cut one another, they are in one plane, and every triangle is in one plane.

For let the two straight lines AB, CD cut one another at the point E;

I say that AB, CD are in one plane, and every triangle is in one plane.

For let points F, G be taken at random on EC, EB,
let CB, FG be joined,
and let FH, GK be drawn across;
I say first that the triangle ECB is in one plane.



For, if part of the triangle ECB, either FHC or GBK, is in the plane of reference, and the rest in another,
a part also of one of the straight lines EC, EB will be in the plane of reference, and a part in another.

But, if the part $FCBG$ of the triangle ECB be in the plane of reference, and the rest in another,
a part also of both the straight lines EC, EB will be in the plane of reference and a part in another:
which was proved absurd. [XI. 1]

Therefore the triangle *ECB* is in one plane.

But, in whatever plane the triangle *ECB* is, in that plane also is each of the straight lines *EC*, *EB*,

and, in whatever plane each of the straight lines *EC*, *EB* is, in that plane are *AB*, *CD* also. [XI. 1]

Therefore the straight lines *AB*, *CD* are in one plane, and every triangle is in one plane.

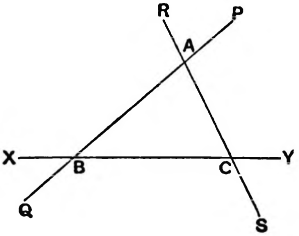
Q. E. D.

It must be admitted that the "proof" of this proposition is not of any value. For one thing, Euclid only takes certain triangles and a certain quadrilateral respectively forming part of the original triangle, and argues about these. But, for anything we are supposed to know, there may be some part of the triangle bounded (let us say) by some curve which is not in the same plane with the triangle.

We may agree with Simson that it would be preferable to enunciate the proposition as follows.

Two straight lines which intersect are in one plane, and three straight lines which intersect two and two are in one plane.

Adopting Smith and Bryant's figure in preference to Simson's, we suppose three straight lines *PQ*, *RS*, *XY* to intersect two and two in *A*, *B*, *C*.



Then Simson's proof (adopted by Legendre also) proceeds thus.

Let any plane pass through the straight line *PQ*, and let this plane be turned about *PQ* (produced indefinitely) as axis until it passes through the point *C*.

Then, since the points *A*, *C* are in this plane, the straight line *AC* (and therefore the straight line *RS* produced indefinitely) lies wholly in the plane. [Simson's def.]

For the same reason, since the points *B*, *C* are in the plane, the straight line *XY* lies wholly in the plane.

Hence all three straight lines *PQ*, *RS*, *XY* (and of course any pair of them) lie in one plane.

But it has still to be proved that there is *only* one plane passing through the three straight lines.

This may be done, as in Mr Taylor's Euclid, thus.

Suppose, if possible, that there are *two* different planes through *A*, *B*, *C*.

The straight lines *BC*, *CA*, *AB* then lie wholly in each of the two planes.

Now any straight line in one of the two planes must intersect at least two of the straight lines (produced if necessary);

let it intersect two of them in *K*, *L*.

Then, since *K*, *L* are also in the second plane, the line *KL* lies wholly in that plane.

Hence every straight line in either of the planes lies wholly in the other also; and therefore the planes are coincident throughout their whole surface.

It follows from the above that

A plane is determined (i.e. uniquely *determined*) *by any of the following data:*

(1) *by three straight lines meeting one another two and two,*

(2) *by three points not in a straight line,*

(3) *by two straight lines meeting one another,*

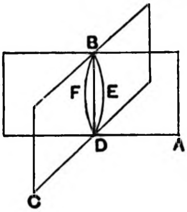
(4) *by a straight line and a point without it.*

PROPOSITION 3.

If two planes cut one another, their common section is a straight line.

For let the two planes *AB*, *BC* cut one another, and let the line *DB* be their common section;

I say that the line *DB* is a straight line.



For, if not, from *D* to *B* let the straight line *DEB* be joined in the plane *AB*, and in the plane *BC* the straight line *DFB*.

Then the two straight lines *DEB*, *DFB* will have the same extremities, and will clearly enclose an area:

which is absurd.

Therefore *DEB*, *DFB* are not straight lines.

Similarly we can prove that neither will there be any other straight line joined from *D* to *B* except *DB* the common section of the planes *AB*, *BC*.

Therefore etc.

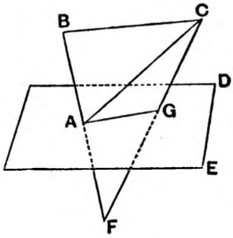
Q. E. D.

I think Simson is right in objecting to the words after "which is absurd," to the effect that *DEB*, *DFB* are not straight lines, and that neither can there be any other straight line joined from *D* to *B* except *DB*, as being unnecessary. It is right to conclude at once from the absurdity that *BD* cannot *but* be a straight line.

Legendre makes his proof depend on Prop. 2. "For, if, among the points common to the two planes, three should be found which are not in a straight line, the two planes in question, each passing through three points, would only amount to one and the same plane." [This of course assumes that three points determine one and *only* one plane, which, strictly speaking, involves more than Prop. 2 itself, as shown in the last note.]

A favourite proposition in modern text-books is the following. The proof seems to be due to von Staudt (Killing, *Grundlagen der Geometrie*, Vol. II. p. 43).

If two planes meet in a point, they meet in a straight line.



Let ABC, ADE be two given planes meeting at A.

Take any points B, C lying on the plane ABC, and not on the plane ADE but on the same side of it.

Join AB, AC, and produce BA to F.

Join CF.

Then, since B, F are on opposite sides of the plane ADE,

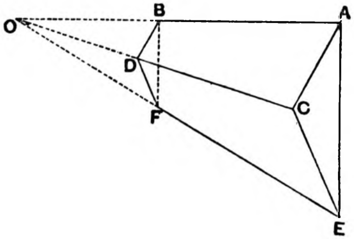
C, F are also on opposite sides of it.

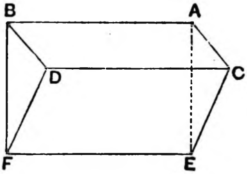
Therefore CF must meet the plane ADE in some point, say G.

Then, since A, G are both in each of the planes ABC, ADE, the straight line AG is in both planes. [Simson's def.]

This is also the place to insert the proposition that, *If three planes intersect two and two, their lines of intersection either meet in a point or are parallel two and two.*

Let there be three planes intersecting in the straight lines AB, CD, EF.





Now AB, EF are in a plane; therefore they either meet in a point or are parallel.

(1) Let them meet in O.

Then O, being a point in AB, lies in the plane AD, and, being also a point in EF, lies also in the plane ED.

Therefore O, being common to the planes AD, DE, must lie on CD, the line of their intersection;

i.e. CD, if produced, passes through O.

(2) Let AB, EF not meet, but let them be parallel.

Then CD cannot meet AB; for, if it did, it must necessarily meet EF, by the first case.

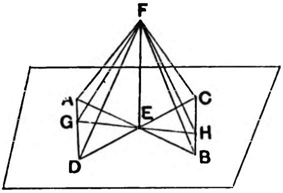
Therefore CD, AB, being in one plane, are parallel.

Similarly CD, EF are parallel.

PROPOSITION 4.

If a straight line be set up at right angles to two straight lines which cut one another, at their common point of section, it will also be at right angles to the plane through them.

For let a straight line EF be set up at right angles to the two straight lines AB, CD, which cut one another at the point E, from E;



I say that EF is also at right angles to the plane through AB, CD.

For let AE, EB, CE, ED be cut off equal to one another,
and let any straight line GEH be drawn across through E, at random;
let AD, CB be joined,
and further let FA, FG, FD, FC, FH, FB be joined from the point F taken at random <on EF>.

Now, since the two straight lines AE, ED are equal to the two straight lines CE, EB, and contain equal angles, [I. 15]
therefore the base AD is equal to the base CB,
and the triangle AED will be equal to the triangle CEB; [I. 4]
so that the angle DAE is also equal to the angle EBC.

But the angle AEG is also equal to the angle BEH; [I. 15]
therefore AGE, BEH are two triangles which have two angles equal to two angles respectively, and one side equal to one side, namely that adjacent to the equal angles, that is to say, AE to EB;
therefore they will also have the remaining sides equal to the remaining sides. [I. 26]

Therefore GE is equal to EH, and AG to BH.

And, since AE is equal to EB,
while FE is common and at right angles,
therefore the base FA is equal to the base FB. [I. 4]

For the same reason
FC is also equal to FD.

And, since AD is equal to CB,
and FA is also equal to FB,
the two sides FA, AD are equal to the two sides FB, BC respectively;
and the base FD was proved equal to the base FC;
therefore the angle FAD is also equal to the angle FBC. [I. 8]

And since, again, *AG* was proved equal to *BH*, and further *FA* also equal to *FB*, the two sides *FA*, *AG* are equal to the two sides *FB* *BH*.

And the angle *FAG* was proved equal to the angle *FBH*; therefore the base *FG* is equal to the base *FH*. [I. 4]

Now since, again, *GE* was proved equal to *EH*, and *EF* is common, the two sides *GE*, *EF* are equal to the two sides *HE*, *EF*; and the base *FG* is equal to the base *FH*; therefore the angle *GEF* is equal to the angle *HEF*. [I. 8]

Therefore each of the angles *GEF*, *HEF* is right.

Therefore *FE* is at right angles to *GH* drawn at random through *E*.

Similarly we can prove that *FE* will also make right angles with all the straight lines which meet it and are in the plane of reference.

But a straight line is at right angles to a plane when it makes right angles with all the straight lines which meet it and are in that same plane; [XI. Def. 3] therefore *FE* is at right angles to the plane of reference.

But the plane of reference is the plane through the straight lines *AB*, *CD*.

Therefore *FE* is at right angles to the plane through *AB*, *CD*.

Therefore etc.

Q. E. D.

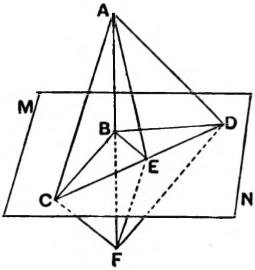
The steps to be successively proved in order to establish this proposition by Euclid's method are

(1) triangles *AED*, *BEC* equal in all respects, [by I. 4]
(2) triangles *AEG*, *BEH* equal in all respects, [by I. 26]
so that *AG* is equal to *BH*, and *GE* to *EH*,
(3) triangles *AEF*, *BEF* equal in all respects, [I. 4]
so that *AF* is equal to *BF*,
(4) likewise triangles *CEF*, *DEF*,
so that *CF* is equal to *DF*,
(5) triangles *FAD*, *FBC* equal in all respects, [I. 8]
so that the angles *FAG*, *FBH* are equal,
(6) triangles *FAG*, *FBH* equal in all respects, [by (2), (3), (5) and I. 4]
so that *FG* is equal to *FH*,

(7) triangles *FEG*, *FEH* equal in all respects, [by (2), (6) and I. 8]
so that the angles *FEG*, *FEH* are equal,
and therefore *FE* is at right angles to *GH*.

In consequence of the length of the above proof others have been suggested, and the proof which now finds most general acceptance is that of Cauchy, which is as follows.

Let *AB* be perpendicular to two straight lines *BC*, *BD* in the plane *MN* at their point of intersection *B*.



In the plane *MN* draw *BE*, *any* straight line through *B*.

Join *CD*, and let *CD* meet *BE* in *E*.

Produce *AB* to *F* so that *BF* is equal to *AB*.

Join *AC*, *AE*, *AD*, *CF*, *EF*, *DF*.

Since *BC* is perpendicular to *AF* at its middle point *B*,

AC is equal to *CF*.

Similarly *AD* is equal to *DF*.

Since in the triangles *ACD*, *FCD* the two sides *AC*, *CD* are respectively equal to the two sides *FC*, *CD*, and the third sides *AD*, *FD* are also equal,

the angles *ACD*, *FCD* are equal. [I. 8]

The triangles *ACE*, *FCE* thus have two sides and the included angle equal, whence

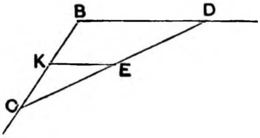
EA is equal to *EF*. [I. 4]

The triangles *ABE*, *FBE* have now all their sides equal respectively; therefore the angles *ABE*, *FBE* are equal, [I. 8]

and *AB* is perpendicular to *BE*.

And *BE* is in *any* straight line through *B* in the plane *MN*.

Legendre's proof is not so easy, but it is interesting. We are first required to draw through any point *E* within the angle *CBD* a straight line *CD* bisected at *E*.



To do this we draw *EK* parallel to *DB* meeting *BC* in *K*, and then mark off *KC* equal to *BK*.

CE is then joined and produced to *D*; and *CD* is the straight line required.

Now, joining *AC*, *AE*, *AD* in the figure above, we have, since *CD* is bisected at *E*,

(1) in the triangle *ACD*,

$$AC^2 + AD^2 = 2AE^2 + 2ED^2,$$

and also (2) in the triangle *BCD*,

$$BC^2 + BD^2 = 2BE^2 + 2ED^2.$$

Subtracting, and remembering that the triangles *ABC*, *ABD* are right-angled, so that

$$AC^2 - BC^2 = AB^2,$$

and $$AD^2 - BD^2 = AB^2,$$

we have $$2AB^2 = 2AE^2 - 2BE^2,$$

or $$AE^2 = AB^2 + BE^2,$$

whence [I. 48] the angle ABE is a right angle, and AB is perpendicular to BE.

It follows of course from this proposition that the perpendicular AB is the *shortest distance from* A *to the plane* MN.

And it can readily be proved that,

If from a point without a plane oblique straight lines be drawn to the plane, (1) *those meeting the plane at equal distances from the foot of the perpendicular are equal, and*

(2) *of two straight lines meeting the plane at unequal distances from the foot of the perpendicular, the more remote is the greater.*

Lastly, it is easily seen that

From a point outside a plane only one perpendicular can be drawn to that plane.

For, if possible, let there be two perpendiculars. Then a plane can be drawn through them, and this will cut the original plane in a straight line.

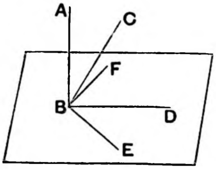
This straight line and the two perpendiculars will form a plane triangle which has two right angles: which is impossible.

PROPOSITION 5.

If a straight line be set up at right angles to three straight lines which meet one another, at their common point of section, the three straight lines are in one plane.

For let a straight line AB be set up at right angles to the three straight lines BC, BD, BE, at their point of meeting at B;

I say that BC, BD, BE are in one plane.



For suppose they are not, but, if possible, let BD, BE be in the plane of reference and BC in one more elevated;

let the plane through AB, BC be produced;

it will thus make, as common section in the plane of reference, a straight line. [XI. 3]

Let it make BF.

Therefore the three straight lines AB, BC, BF are in one plane, namely that drawn through AB, BC.

Now, since AB is at right angles to each of the straight lines BD, BE,

therefore AB is also at right angles to the plane through BD, BE. [XI. 4]

But the plane through BD, BE is the plane of reference; therefore AB is at right angles to the plane of reference.

Thus AB will also make right angles with all the straight lines which meet it and are in the plane of reference. [XI. Def. 3]

But BF which is in the plane of reference meets it; therefore the angle ABF is right.

But, by hypothesis, the angle ABC is also right; therefore the angle ABF is equal to the angle ABC.

And they are in one plane: which is impossible.

Therefore the straight line BC is not in a more elevated plane; therefore the three straight lines BC, BD, BE are in one plane.

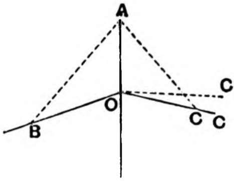
Therefore, if a straight line be set up at right angles to three straight lines, at their point of meeting, the three straight lines are in one plane. Q. E. D.

It follows that, *if a right angle be turned about one of the straight lines containing it the other will describe a plane.*

At any point in a straight line it is possible to draw *only one* plane which is at right angles to the straight line.

One such plane can be found by taking any two planes through the given straight line, drawing perpendiculars to the straight line in the respective planes, e.g. BO, CO in the planes AOB, AOC, each perpendicular to AO, and then drawing a plane (BOC) through the perpendiculars.

If there were another plane through O perpendicular to AO, it must meet the plane through AO and some perpendicular to it as OC in a straight line OC' different from OC.



Then, by XI. 4, AOC' is a right angle, and in the same plane with the right angle AOC: which is impossible.

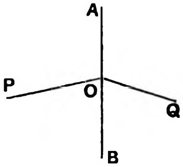
Next, *one plane and only one can be drawn through a point outside a straight line at right angles to that line.*

Let P be the given point, AB the given straight line.

In the plane through P and AB, draw PO perpendicular to AB, and through O draw another straight line OQ at right angles to AB.

Then the plane through OP, OQ is perpendicular to AB.

If there were another plane through P perpendicular to AB, either



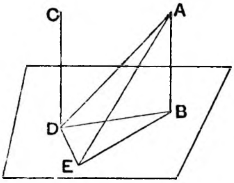
(1) it would intersect AB at O but not pass through OQ, or
(2) it would intersect AB at a point different from O.

In either case, an absurdity would result.

PROPOSITION 6.

If two straight lines be at right angles to the same plane, the straight lines will be parallel.

For let the two straight lines AB, CD be at right angles to the plane of reference;
I say that AB is parallel to CD.



For let them meet the plane of reference at the points B, D,
let the straight line BD be joined,
let DE be drawn, in the plane of reference, at right angles to BD,
let DE be made equal to AB,
and let BE, AE, AD be joined.

Now, since AB is at right angles to the plane of reference, it will also make right angles with all the straight lines which meet it and are in the plane of reference. [XI. Def. 3]

But each of the straight lines BD, BE is in the plane of reference and meets AB;
therefore each of the angles ABD, ABE is right.

For the same reason
each of the angles CDB, CDE is also right.

And, since AB is equal to DE,
and BD is common,
the two sides AB, BD are equal to the two sides ED, DB;
and they include right angles;
therefore the base AD is equal to the base BE. [I. 4]

And, since AB is equal to DE,
while AD is also equal to BE,
the two sides AB, BE are equal to the two sides ED, DA;
and AE is their common base;
therefore the angle ABE is equal to the angle EDA. [I. 8]

But the angle ABE is right;
therefore the angle EDA is also right;
therefore ED is at right angles to DA.

But it is also at right angles to each of the straight lines BD, DC;
therefore ED is set up at right angles to the three straight lines BD, DA, DC at their point of meeting;
therefore the three straight lines BD, DA, DC are in one plane. [XI. 5]

But, in whatever plane DB, DA are, in that plane is AB also,
for every triangle is in one plane; [XI. 2]
therefore the straight lines AB, BD, DC are in one plane.

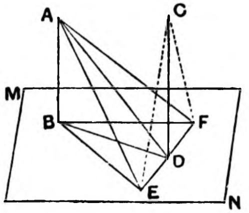
And each of the angles ABD, BDC is right;
therefore AB is parallel to CD. [I. 28]

Therefore etc. Q. E. D.

If anyone wishes to convince himself of the real necessity for some general agreement as to the order in which propositions in elementary geometry should be taken, let him contemplate the hopeless result of too much independence on the part of editors in the matter of this proposition and its converse, XI. 8.

Legendre adopts a different, and elegant, method of proof; but he applies it to XI. 8, which he gives first, and then deduces XI. 6 from it by *reductio ad absurdum*. Dr Mehler uses Legendre's method of proof but applies it to XI. 6, and then gives XI. 8 as a deduction from it. Lardner follows Legendre. Holgate, the editor of a recent American book, gives Euclid's proof of XI. 6 and deduces XI. 8 by *reductio ad absurdum*. His countrymen, Schultze and Sevenoak, give XI. 8 first, but put it after, and deduce it from, Eucl. XI. 10; they then give XI. 6, practically as a deduction from XI. 8 by *reductio ad absurdum*, after a proposition corresponding to Eucl. XI. 11 and 12, and a corollary to the effect that through a given point one and only one perpendicular can be drawn to a given plane.

We will now give the proof of XI. 6 by Legendre's method (adopted by Smith and Bryant as well as by Mehler).



Let AB, CD be both perpendicular to the same plane MN.

Join BD.

Now, since BD meets AB, CD, both of which are perpendicular to the plane MN in which BD is,
the angles ABD, CDB are right angles.

AB, CD will therefore be parallel *provided that they are in the same plane.*

Through D draw EDF, in the plane MN, at right angles to BD, and make ED equal to DF.

Join BE, BF, AE, AD, AF.

Then the triangles BDE, BDF are equal in all respects (by I. 4), so that

BE is equal to BF.

It follows, since the angles ABE, ABF are right, that the triangles ABE, ABF are equal in all respects, and

AE is equal to AF.

[Mehler now argues elegantly thus. If CE, CF be also joined, it is clear that

CE is equal to CF.

Hence each of the four points A, B, C, D is equidistant from the two points E, F.

Therefore the points A, B, C, D *are in one plane*, so that AB, CD are parallel.

If, however, we do not use the locus of points equidistant from two fixed points, we proceed as follows.]

The triangles AED, AFD have their sides equal respectively;
hence [I. 8] the angles ADE, ADF are equal,
so that ED is at right angles to AD.

Thus ED is at right angles to BD, AD, CD;
therefore CD is in the plane through AD, BD. [XI. 5]

But AB is in that same plane; [XI. 2]
therefore AB, CD are in the same plane.

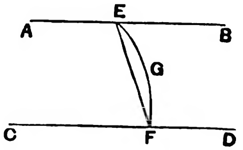
And the angles ABD, CDB are right;
therefore AB, CD are parallel.

PROPOSITION 7.

If two straight lines be parallel and points be taken at random on each of them, the straight line joining the points is in the same plane with the parallel straight lines.

Let AB, CD be two parallel straight lines, and let points E, F be taken at random on them respectively;

I say that the straight line joining the points E, F is in the same plane with the parallel straight lines.



For suppose it is not, but, if possible, let it be in a more elevated plane as EGF,

and let a plane be drawn through EGF;

it will then make, as section in the plane of reference, a straight line. [XI. 3]

Let it make it, as EF;
therefore the two straight lines EGF, EF will enclose an area:
which is impossible.

Therefore the straight line joined from E to F is not in a plane more elevated;
therefore the straight line joined from E to F is in the plane through the parallel straight lines AB, CD.

Therefore etc.

Q. E. D.

It is true that this proposition, in the form in which Euclid enunciates it, is hardly necessary if the plane is defined as a surface such that, if any two points be taken in it, the straight line joining them lies wholly in the surface. But Euclid did not give this definition; and, moreover, Prop. 2 would be usefully supplemented by a proposition which should prove that *two parallel straight lines determine a plane* (i.e. one plane and one *only*) *which also contains all the straight lines which join a point on one of the parallels to a point on the other.* That there cannot be *two* planes through a pair of parallels would be proved in the same way as we prove that two or three intersecting straight lines cannot be in two different planes, inasmuch as each transversal lying in one of the two supposed planes through the parallels would lie wholly in the other also, so that the two supposed planes must coincide throughout (cf. note on Prop. 2 above).

But, whatever be the value of the proposition as it is, Simson seems to have spoilt it completely. He leaves out the construction of a plane through EGF, which, as Euclid says, must cut the plane containing the parallels in a straight line; and, instead, he says, "In the plane $ABCD$ in which the parallels are draw the straight line EHF from E to F." Now, although we can easily draw a straight line from E to F, to claim that we can draw it *in the plane in which the parallels are* is surely to assume the very result which is to be proved. All that we could properly say is that the straight line joining E to F is in *some* plane which contains the parallels; we do not know that there is no more than *one* such plane, or that the parallels determine a plane *uniquely*, without some such argument as that which Euclid gives.

Nor can I subscribe to the remarks in Simson's note on the proposition. He says (1) "This proposition has been put into this book by some unskilful editor, as is evident from this, that straight lines which are drawn from one point to another in a plane are, in the preceding books, supposed to be in that plane; and if they were not, some demonstrations in which one straight line is supposed to meet another would not be conclusive. For instance, in Prop. 30, Book I, the straight line GK would not meet EF, if GK were not in the plane in which are the parallels AB, CD, and in which, by hypothesis, the straight line EF is." But the subject-matter of Book I. and Book XI. is quite different; in Book I. everything is in one plane, and when Euclid, in defining parallels, says they are straight lines *in the same plane* etc., he only does so because he must, in order to exclude non-intersecting straight lines which are *not* parallel. Thus in I. 30 there is nothing wrong in assuming that there may be three parallels in one plane, and that the straight line GHK cuts all three.

But in Book XI. it becomes a question whether there can be *more* than one plane through parallel straight lines.

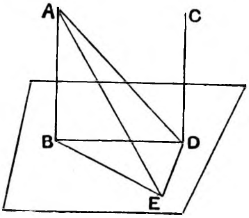
Simson goes on to say (2) "Besides, this 7th Proposition is demonstrated by the preceding 3rd; in which the very same thing which is proposed to be demonstrated in the 7th is twice assumed, viz., that the straight line drawn from one point to another in a plane is in that plane." But there is nothing in Prop. 3 about a plane in which two parallel straight lines are; therefore there is no assumption of the result of Prop. 7. What is assumed is that, given two points in *a plane*, they can be joined by a straight line in the plane: a legitimate assumption.

Lastly, says Simson, "And the same thing is assumed in the preceding 6th Prop. in which the straight line which joins the points B, D that are in the plane to which AB and CD are at right angles is supposed to be in that plane." Here again there is no question of a *plane in which two parallels are*; so that the criticism here, as with reference to Prop. 3, appears to rest on a misapprehension.

PROPOSITION 8.

If two straight lines be parallel, and one of them be at right angles to any plane, the remaining one will also be at right angles to the same plane.

Let AB, CD be two parallel straight lines, and let one of them, AB, be at right angles to the plane of reference;



I say that the remaining one, CD, will also be at right angles to the same plane.

For let AB, CD meet the plane of reference at the points B, D, and let BD be joined;

therefore AB, CD, BD are in one plane. [XI. 7]

Let DE be drawn, in the plane of reference, at right angles to BD,

let DE be made equal to AB,

and let BE, AE, AD be joined.

Now, since AB is at right angles to the plane of reference, therefore AB is also at right angles to all the straight lines which meet it and are in the plane of reference; [XI. Def. 3]

therefore each of the angles ABD, ABE is right.

And, since the straight line BD has fallen on the parallels AB, CD, therefore the angles *ABD*, *CDB* are equal to two right angles. [I. 29]

But the angle *ABD* is right;
therefore the angle *CDB* is also right;
therefore *CD* is at right angles to *BD*.

And, since *AB* is equal to *DE*,
and *BD* is common,
the two sides *AB*, *BD* are equal to the two sides *ED*, *DB*;
and the angle *ABD* is equal to the angle *EDB*,
for each is right;
therefore the base *AD* is equal to the base *BE*.

And, since *AB* is equal to *DE*,
and *BE* to *AD*,
the two sides *AB*, *BE* are equal to the two sides *ED*, *DA* respectively,
and *AE* is their common base;
therefore the angle *ABE* is equal to the angle *EDA*.

But the angle *ABE* is right;
therefore the angle *EDA* is also right;
therefore *ED* is at right angles to *AD*.

But it is also at right angles to *DB*;
therefore *ED* is also at right angles to the plane through *BD*, *DA*. [XI. 4]

Therefore *ED* will also make right angles with all the straight lines which meet it and are in the plane through *BD*, *DA*.

But *DC* is in the plane through *BD*, *DA*, inasmuch as *AB*, *BD* are in the plane through *BD*, *DA*, [XI. 2]
and *DC* is also in the plane in which *AB*, *BD* are.

Therefore *ED* is at right angles to *DC*,
so that *CD* is also at right angles to *DE*.

But *CD* is also at right angles to *BD*.

Therefore *CD* is set up at right angles to the two straight lines *DE*, *DB* which cut one another, from the point of section at *D*;

so that *CD* is also at right angles to the plane through *DE*, *DB*. [XI. 4]

But the plane through *DE*, *DB* is the plane of reference; therefore *CD* is at right angles to the plane of reference.

Therefore etc.

Q. E. D.

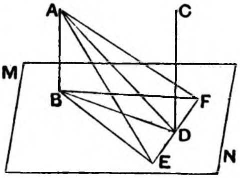
Simson objects to the words which explain why *DC* is in the plane through *BD*, *DA*, viz. "inasmuch as *AB*, *BD* are in the plane through *BD*, *DA*, and *DC* is also in the plane in which *AB*, *BD* are," as being too roundabout. He concludes that they are corrupt or interpolated, and that we ought only to have the words "because all three are in the plane in which are the parallels *AB*, *CD*" (by Prop. 7 preceding). But I think Euclid's words can be defended. Prop. 7 says nothing of a plane determined by *two* transversals as *BD*, *DA* are. Hence it is natural to say that *DC* is in the same plane in which *AB*, *BD* are [Prop. 7], and *AB*, *BD* are in the same plane as *BD*, *DA* [Prop. 2], so that *DC* is in the plane through *BD*, *DA*.

Legendre's alternative proof is split by him into two propositions.

(1) *Let* AB *be a perpendicular to the plane* MN *and* EF *a line situated in that plane; if from* B, *the foot of the perpendicular*, BD *be drawn perpendicular to* EF, *and* AD *be joined, I say that* AD *will be perpendicular to* EF.

(2) *If* AB *is perpendicular to the plane* MN, *every straight line* CD *parallel to* AB *will be perpendicular to the same plane.*

To prove both propositions together we suppose *CD* given, join *BD*, and draw *EF* perpendicular to *BD* in the plane *MN*.



(1) As before, we make *DE* equal to *DF* and join *BE*, *BF*, *AE*, *AF*.

Then, since the angles *BDE*, *BDF* are right, and *DE*, *DF* equal,

BE is equal to *BF*. [I. 4]

And, since *AB* is perpendicular to the plane,

the angles *ABE*, *ABF* are both right.

Therefore, in the triangles *ABE*, *ABF*,

AE is equal to *AF*. [I. 4]

Lastly, in the triangles *ADE*, *ADF*, since *AE* is equal to *AF*, and *DE* to *DF*, while *AD* is common,

the angle *ADE* is equal to the angle *ADF*, [I. 8]

so that *AD* is perpendicular to *EF*.

(2) *ED* being thus perpendicular to *DA*, and also (by construction) perpendicular to *DB*,

ED is perpendicular to the plane *ADB*. [XI. 4]

But *CD*, being parallel to *AB*, is in the plane *ABD*;

therefore *ED* is perpendicular to *CD*. [XI. Def. 3]

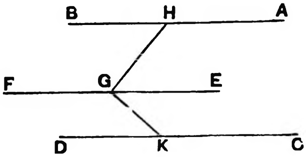
Also, since AB, CD are parallel,
and ABD is a right angle,
CDB is also a right angle.

Thus CD is perpendicular to both DE and DB, and therefore to the plane MN through DE, DB.

PROPOSITION 9.

Straight lines which are parallel to the same straight line and are not in the same plane with it are also parallel to one another.

For let each of the straight lines AB, CD be parallel to EF, not being in the same plane with it;
I say that AB is parallel to CD.



For let a point G be taken at random on EF,
and from it let there be drawn GH, in the plane through EF, AB, at right angles to EF, and GK in the plane through FE, CD again at right angles to EF.

Now, since EF is at right angles to each of the straight lines GH, GK,
therefore EF is also at right angles to the plane through GH, GK. [XI. 4]

And EF is parallel to AB;
therefore AB is also at right angles to the plane through HG, GK. [XI. 8]

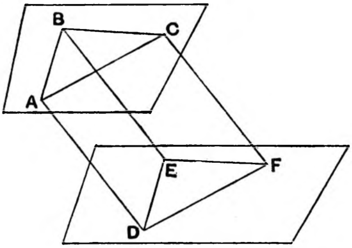
For the same reason
CD is also at right angles to the plane through HG, GK;
therefore each of the straight lines AB, CD is at right angles to the plane through HG,. GK.

But, if two straight lines be at right angles to the same plane, the straight lines are parallel; [XI. 6]
therefore AB is parallel to CD.

PROPOSITION 10.

If two straight lines meeting one another be parallel to two straight lines meeting one another not in the same plane, they will contain equal angles.

For let the two straight lines AB, BC meeting one another be parallel to the two straight lines DE, EF meeting one another, not in the same plane;
I say that the angle ABC is equal to the angle DEF.



For let BA, BC, ED, EF be cut off equal to one another, and let AD, CF, BE, AC, DF be joined.

Now, since BA is equal and parallel to ED,
therefore AD is also equal and parallel to BE. [I. 33]

For the same reason
CF is also equal and parallel to BE.

Therefore each of the straight lines AD, CF is equal and parallel to BE.

But straight lines which are parallel to the same straight line and are not in the same plane with it are parallel to one another; [XI. 9]
therefore AD is parallel and equal to CF.

And AC, DF join them;
therefore AC is also equal and parallel to DF. [I. 33]

Now, since the two sides AB, BC are equal to the two sides DE, EF,
and the base AC is equal to the base DF,
therefore the angle ABC is equal to the angle DEF. [I. 8]

Therefore etc.

The result of this proposition does not appear to be quoted in Euclid until XII. 3; but Euclid no doubt inserted it here advisedly, because it has the effect of incidentally proving that the "inclination of two planes to one another," as defined in XI. Def. 6, is one and the same angle at whatever point of the common section the plane angle measuring it is drawn.

PROPOSITION 11.

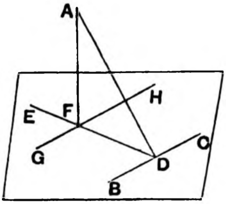
From a given elevated point to draw a straight line perpendicular to a given plane.

Let *A* be the given elevated point, and the plane of reference the given plane;

thus it is required to draw from the point *A* a straight line perpendicular to the plane of reference.

Let any straight line *BC* be drawn, at random, in the plane of reference, and let *AD* be drawn from the point *A* perpendicular to *BC*. [I. 12]

If then *AD* is also perpendicular to the plane of reference, that which was enjoined will have been done.



But, if not, let *DE* be drawn from the point *D* at right angles to *BC* and in the plane of reference, [I. 11]

let *AF* be drawn from *A* perpendicular to *DE*, [I. 12]

and let *GH* be drawn through the point *F* parallel to *BC*. [I. 31]

Now, since *BC* is at right angles to each of the straight lines *DA*, *DE*,

therefore *BC* is also at right angles to the plane through *ED*, *DA*. [XI. 4]

And *GH* is parallel to it;

but, if two straight lines be parallel, and one of them be at right angles to any plane, the remaining one will also be at right angles to the same plane; [XI. 8]

therefore *GH* is also at right angles to the plane through *ED*, *DA*.

Therefore *GH* is also at right angles to all the straight lines which meet it and are in the plane through *ED*, *DA*. [XI. Def. 3]

But *AF* meets it and is in the plane through *ED*, *DA*; therefore *GH* is at right angles to *FA*, so that *FA* is also at right angles to *GH*.

But *AF* is also at right angles to *DE*; therefore *AF* is at right angles to each of the straight lines *GH*, *DE*.

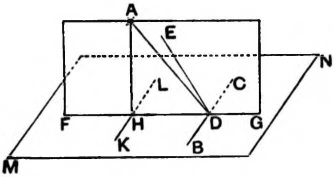
But, if a straight line be set up at right angles to two straight lines which cut one another, at the point of section, it will also be at right angles to the plane through them; [XI. 4] therefore *FA* is at right angles to the plane through *ED*, *GH*.

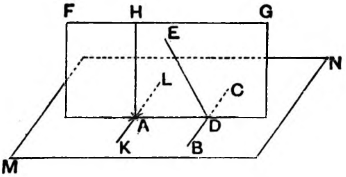
But the plane through *ED*, *GH* is the plane of reference; therefore *AF* is at right angles to the plane of reference.

Therefore from the given elevated point *A* the straight line *AF* has been drawn perpendicular to the plane of reference.

Q. E. F.

The text-books differ in the *form* which they give to this proposition rather than in substance. They commonly assume the construction of a *plane* through the point *A* at right angles to any straight line *BC* in the given plane (the construction being effected in the manner shown at the end of the note on XI. 5 above). The advantage of this method is that it enables a perpendicular to be drawn from a point *in* the plane also, by the same construction. (Where the letters for the two figures differ, those referring to the second figure are put in brackets.)





We can include the construction of the plane through *A* perpendicular to *BC*, and make the whole into one proposition, thus.

BC being any straight line in the given plane *MN*, draw *AD* perpendicular to *BC*.

In any plane passing through *BC* but not through *A* draw *DE* at right angles to *BC*.

Through *DA*, *DE* draw a plane; this will intersect the given plane *MN* in a straight line, as *FD* (*AD*).

In the plane *AG* draw *AH* perpendicular to *FG* (*AD*).

Then *AH* is the perpendicular required.

In the plane *MN*, through *H* in the first figure and *A* in the second, draw *KL* parallel to *BC*.

Now, since *BC* is perpendicular to both *DA* and *DE*, *BC* is perpendicular to the plane *AG*. [XI. 4]

Therefore *KL*, being parallel to *BC*, is also perpendicular to the plane *AG* [XI. 8], and therefore to *AH* which meets it and is in that plane.

Therefore *AH* is perpendicular to both *FD* (*AD*) and *KL* at their point of intersection.

Therefore *AH* is perpendicular to the plane *MN*.

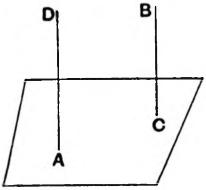
Thus we have solved the problem in XI. 12 as well as that in XI. 11; and this direct method of drawing a perpendicular to a plane from a point *in* it is obviously preferable to Euclid's method by which the construction of a perpendicular to a plane from a point *without* it is assumed, and a line is merely drawn from a point in the plane parallel to the perpendicular obtained in XI. 11.

PROPOSITION 12.

To set up a straight line at right angles to a given plane from a given point in it.

Let the plane of reference be the given plane, and *A* the point in it;

thus it is required to set up from the point *A* a straight line at right angles to the plane of reference.



Let any elevated point *B* be conceived, from *B* let *BC* be drawn perpendicular to the plane of reference, [XI. 11]

and through the point *A* let *AD* be drawn parallel to *BC*. [I. 31]

Then, since *AD*, *CB* are two parallel straight lines, while one of them, *BC*, is at right angles to the plane of reference,

therefore the remaining one, *AD*, is also at right angles to the plane of reference. [XI. 8]

Therefore *AD* has been set up at right angles to the given plane from the point *A* in it.

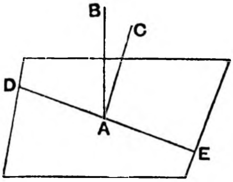
PROPOSITION 13.

From the same point two straight lines cannot be set up at right angles to the same plane on the same side.

For, if possible, from the same point *A* let the two straight lines *AB*, *AC* be set up at right angles to the plane of reference and on the same side,

and let a plane be drawn through *BA*, *AC*;

it will then make, as section through *A* in the plane of reference, a straight line. [XI. 3]



Let it make *DAE*;

therefore the straight lines *AB*, *AC*, *DAE* are in one plane.

And, since *CA* is at right angles to the plane of reference, it will also make right angles with all the straight lines which meet it and are in the plane of reference. [XI. Def. 3]

But *DAE* meets it and is in the plane of reference;

therefore the angle *CAE* is right.

For the same reason

the angle *BAE* is also right;

therefore the angle *CAE* is equal to the angle *BAE*.

And they are in one plane:

which is impossible.

Therefore etc.

Q. E. D.

Simson added words to this as follows:

"Also, from a point above a plane there can be but one perpendicular to that plane; for, if there could be two, they would be parallel to one another [XI. 6], which is absurd."

Euclid does not give this result, but we have already had it in the note above to XI. 4 (*ad fin.*).

PROPOSITION 14.

Planes to which the same straight line is at right angles will be parallel.

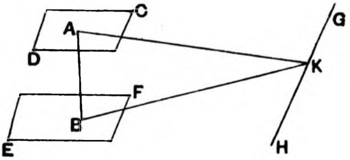
For let any straight line *AB* be at right angles to each of the planes *CD*, *EF*;

I say that the planes are parallel.

For, if not, they will meet when produced.

Let them meet;

they will then make, as common section, a straight line. [XI. 3]



Let them make *GH*;

let a point *K* be taken at random on *GH*,

and let *AK*, *BK* be joined.

Now, since *AB* is at right angles to the plane *EF*,

therefore *AB* is also at right angles to *BK* which is a straight line in the plane *EF* produced; [XI. Def. 3]

therefore the angle *ABK* is right.

For the same reason

the angle *BAK* is also right.

Thus, in the triangle *ABK*, the two angles *ABK*, *BAK* are equal to two right angles:

which is impossible. [I. 17]

Therefore the planes *CD*, *EF* will not meet when produced;

therefore the planes *CD*, *EF* are parallel. [XI. Def. 8]

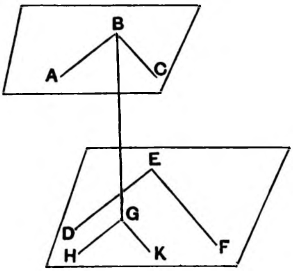
Therefore planes to which the same straight line is at right angles are parallel.

Q. E. D.

PROPOSITION 15.

If two straight lines meeting one another be parallel to two straight lines meeting one another, not being in the same plane, the planes through them are parallel.

For let the two straight lines *AB*, *BC* meeting one another be parallel to the two straight lines *DE*, *EF* meeting one another, not being in the same plane;



I say that the planes produced through *AB*, *BC* and *DE*, *EF* will not meet one another.

For let *BG* be drawn from the point *B* perpendicular to the plane through *DE*, *EF* [XI. 11], and let it meet the plane at the point *G*;

through *G* let *GH* be drawn parallel to *ED*, and *GK* parallel to *EF*. [I. 31]

Now, since *BG* is at right angles to the plane through *DE*, *EF*,

therefore it will also make right angles with all the straight lines which meet it and are in the plane through *DE*, *EF*. [XI. Def. 3]

But each of the straight lines *GH*, *GK* meets it and is in the plane through *DE*, *EF*;

therefore each of the angles *BGH*, *BGK* is right.

And, since *BA* is parallel to *GH*, [XI. 9]

therefore the angles *GBA*, *BGH* are equal to two right angles. [I. 29]

But the angle *BGH* is right;

therefore the angle *GBA* is also right;

therefore *GB* is at right angles to *BA*.

For the same reason

GB is also at right angles to *BC*.

Since then the straight line *GB* is set up at right angles to the two straight lines *BA*, *BC* which cut one another, therefore *GB* is also at right angles to the plane through *BA*, *BC*. [XI. 4]

But planes to which the same straight line is at right angles are parallel; [XI. 14]
therefore the plane through *AB*, *BC* is parallel to the plane through *DE*, *EF*.

Therefore, if two straight lines meeting one another be parallel to two straight lines meeting one another, not in the same plane, the planes through them are parallel.

Q. E. D.

This result is arrived at in the American text-books already quoted by starting from the relation between a plane and a straight line parallel to it. The series of propositions is worth giving. A straight line and a plane being parallel if they do not meet however far they may be produced, we have the following propositions.

1. *Any plane containing one, and only one, of two parallel straight lines is parallel to the other.*

For suppose *AB*, *CD* to be parallel and *CD* to lie in the plane *MN*.

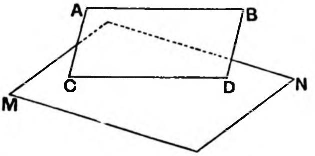
Then *AB*, *CD* determine a plane intersecting *MN* in the straight line *CD*

Thus, if *AB* meets *MN*, it must meet it at some point in *CD*.

But this is impossible, since *AB* is parallel to *CD*.

Therefore *AB* will not meet the plane *MN*, and is therefore parallel to it.

[This proposition and the proof are in Legendre.]



The following theorems follow as corollaries.

2. *Through a given straight line a plane can be drawn parallel to any other given straight line; and, if the lines are not parallel, only one such plane can be drawn.*

We have simply to draw through any point on the first line a straight line parallel to the second line and then pass a plane through these two intersecting lines. This plane is then, by the above proposition, parallel to the second given straight line.

3. *Through a given point a plane can be drawn parallel to any two straight lines in space; and, if the latter are not parallel, only one such plane can be drawn.*

Here we draw through the point straight lines parallel respectively to the given straight lines and then draw a plane through the lines so drawn.

Next we have the partial converse of the first proposition above.

4. *If a straight line is parallel to a plane, it is also parallel to the intersection of any plane through it with the given plane.*

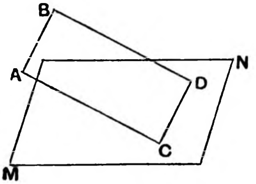
Let *AB* be parallel to the plane *MN*, and let any plane through *AB* intersect *MN* in *CD*.

Now *AB* and *CD* cannot meet, because, if they did, *AB* would meet the plane *MN*.

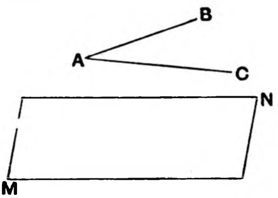
And *AB*, *CD* are in one plane.

Therefore *AB*, *CD* are parallel.

From this follows as a corollary:



5. *If each of two intersecting straight lines is parallel to a given plane, the plane containing them is parallel to the given plane.*

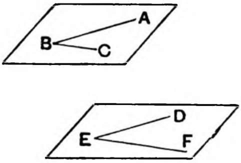


Let AB, AC be parallel to the plane MN.

Then, if the plane ABC were to meet the plane MN, the intersection would be parallel both to AB and to AC: which is impossible.

Lastly, we have Euclid's proposition.

6. *If two straight lines forming an angle are respectively parallel to two other straight lines forming an angle, the plane of the first angle is parallel to the plane of the second.*



Let ABC, DEF be the angles formed by straight lines parallel to one another respectively.

Then, since AB is parallel to DE, the plane of DEF is parallel to AB [(1) above].

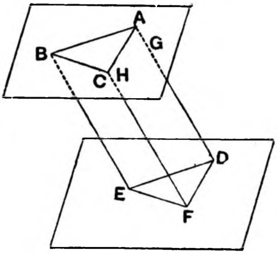
Similarly the plane of DEF is parallel to BC.

Hence the plane of DEF is parallel to the plane of ABC [(5)].

Legendre arrives at the result by yet another method. He first proves Eucl. XI. 16 to the effect that, *if two parallel planes are cut by a third, the lines of intersection are parallel*, and then deduces from this that, *if two parallel straight lines are terminated by two parallel planes, the straight lines are equal in length.*

(The latter inference is obvious because the plane through the parallels cuts the parallel planes in parallel lines; which therefore, with the given parallel lines, form a parallelogram.)

Legendre is now in a position to prove Euclid's proposition XI. 15.



If ABC, DEF be the angles, make AB equal to DE, and BC equal to EF, and join CA, FD, BE, CF, AD.

Then, as in Eucl. XI. 10, the triangles ABC, DEF are equal in all respects;

and AD, BE, CF are all equal.

It is now proved that the planes are parallel by *reductio ad absurdum* from the last preceding result. For, if the plane ABC is not parallel to the plane DEF, let the plane drawn through B parallel to the plane DEF meet CF, AD in H, G respectively.

Then, by the last result BE, HF, GD will all be equal.

But BE, CF, AD are all equal:

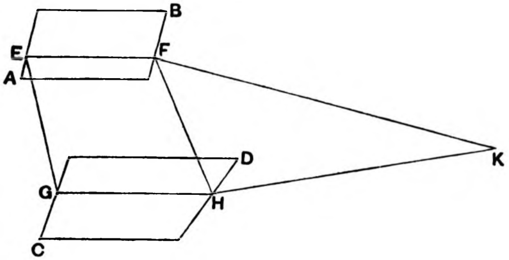
which is impossible.

Therefore etc.

PROPOSITION 16.

If two parallel planes be cut by any plane, their common sections are parallel.

For let the two parallel planes *AB*, *CD* be cut by the plane *EFGH*,
and let *EF*, *GH* be their common sections;
I say that *EF* is parallel to *GH*.



For, if not, *EF*, *GH* will, when produced, meet either in the direction of *F*, *H* or of *E*, *G*.

Let them be produced, as in the direction of *F*, *H*, and let them, first, meet at *K*.

Now, since *EFK* is in the plane *AB*,
therefore all the points on *EFK* are also in the plane *AB*. [XI. 1]

But *K* is one of the points on the straight line *EFK*;
therefore *K* is in the plane *AB*.

For the same reason
K is also in the plane *CD*;
therefore the planes *AB*, *CD* will meet when produced.

But they do not meet, because they are, by hypothesis, parallel;
therefore the straight lines *EF*, *GH* will not meet when produced in the direction of *F*, *H*.

Similarly we can prove that neither will the straight lines *EF*, *GH* meet when produced in the direction of *E*, *G*.

But straight lines which do not meet in either direction are parallel. [I. Def. 23]

Therefore *EF* is parallel to *GH*.

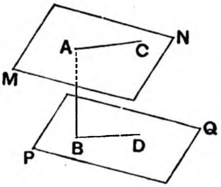
Therefore etc. Q. E. D.

Simson points out that, in here quoting I. Def. 23, Euclid should have said "But straight lines *in one plane* which do not meet in either direction are parallel."

From this proposition is deduced the converse of XI. 14.

If a straight line is perpendicular to one of two parallel planes, it is perpendicular to the other also.

For suppose that *MN*, *PQ* are two parallel planes, and that *AB* is perpendicular to *MN*.



Through *AB* draw any plane, and let it intersect the planes *MN*, *PQ* in *AC*, *BD* respectively.

Therefore *AC*, *BD* are parallel. [XI. 16]

But *AC* is perpendicular to *AB*;

therefore *AB* is also perpendicular to *BD*.

That is, *AB* is perpendicular to any line in *PQ* passing through *B*;

therefore *AB* is perpendicular to *PQ*.

It follows as a corollary that

Through a given point one plane, and only one, can be drawn parallel to a given plane.

In the above figure let *A* be the given point and *PQ* the given plane.

Draw *AB* perpendicular to *PQ*.

Through *A* draw a plane *MN* at right angles to *AB* (see note on XI. 5 above).

Then *MN* is parallel to *PQ*. [XI. 14]

If there could pass through *A* a second plane parallel to *PQ*, *AB* would also be perpendicular to it.

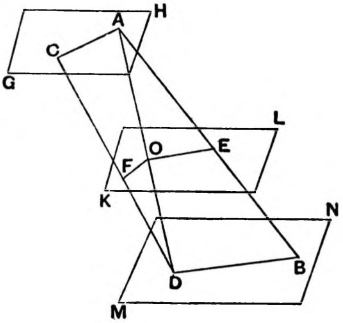
That is, *AB* would be perpendicular to two different planes through *A*: which is impossible (see the same note).

Also it is readily proved that,

If two planes are parallel to a third plane, they are parallel to one another.

PROPOSITION 17.

If two straight lines be cut by parallel planes, they will be cut in the same ratios.



For let the two straight lines *AB*, *CD* be cut by the parallel planes *GH*, *KL*, *MN* at the points *A*, *E*, *B* and *C*, *F*, *D*;

I say that, as the straight line *AE* is to *EB*, so is *CF* to *FD*.

For let *AC*, *BD*, *AD* be joined,

let *AD* meet the plane *KL* at the point *O*,

and let *EO*, *OF* be joined.

Now, since the two parallel planes *KL*, *MN* are cut by the plane *EBDO*,

their common sections *EO*, *BD* are parallel. [XI. 16]

For the same reason, since the two parallel planes *GH*, *KL* are cut by the plane *AOFC*,

their common sections *AC*, *OF* are parallel. [*id.*]

And, since the straight line *EO* has been drawn parallel to *BD*, one of the sides of the triangle *ABD*,

therefore, proportionally, as *AE* is to *EB*, so is *AO* to *OD*. [VI. 2]

Again, since the straight line *OF* has been drawn parallel to *AC*, one of the sides of the triangle *ADC*,

proportionally, as *AO* is to *OD*, so is *CF* to *FD*. [*id.*]

But it was also proved that, as *AO* is to *OD*, so is *AE* to *EB*;

therefore also, as *AE* is to *EB*, so is *CF* to *FD*. [V. 11]

Therefore etc.

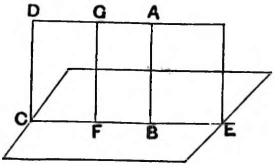
Q. E. D.

PROPOSITION 18.

If a straight line be at right angles to any plane, all the planes through it will also be at right angles to the same plane.

For let any straight line *AB* be at right angles to the plane of reference;

I say that all the planes through *AB* are also at right angles to the plane of reference.



For let the plane *DE* be drawn through *AB*,

let *CE* be the common section of the plane *DE* and the plane of reference,

let a point *F* be taken at random on *CE*,

and from *F* let *FG* be drawn in the plane *DE* at right angles to *CE*. [I. 11]

Now, since *AB* is at right angles to the plane of reference, *AB* is also at right angles to all the straight lines which meet it and are in the plane of reference; [XI. Def. 3]

so that it is also at right angles to *CE*;

therefore the angle *ABF* is right.

But the angle *GFB* is also right;

therefore *AB* is parallel to *FG*. [I. 28]

But *AB* is at right angles to the plane of reference;

therefore *FG* is also at right angles to the plane of reference. [XI. 8]

Now a plane is at right angles to a plane, when the straight lines drawn, in one of the planes, at right angles to the common section of the planes are at right angles to the remaining plane. [XI. Def. 4]

And *FG*, drawn in one of the planes *DE* at right angles to *CE*, the common section of the planes, was proved to be at right angles to the plane of reference;

therefore the plane *DE* is at right angles to the plane of reference.

Similarly also it can be proved that all the planes through *AB* are at right angles to the plane of reference.

Therefore etc.

Q. E. D.

Starting as Euclid does from the definition of perpendicular planes as planes such that all straight lines drawn in one of the planes at right angles to the common section are at right angles to the other plane, it is necessary for him to show that, if *F* be *any* point in *CE*, and *FG* be drawn in the plane *DE* at right angles to *CE*, *FG* will be perpendicular to the plane to which *AB* is perpendicular.

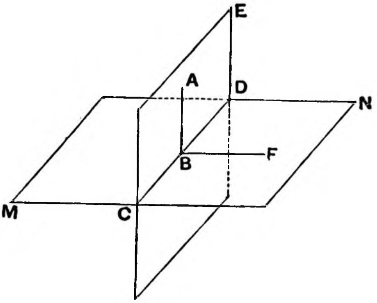
It is perhaps more scientific to make the definition, as Legendre makes it, a particular case of the definition of the *inclination of planes*. Perpendicular planes would thus be planes such that the angle which (when it is acute) Euclid calls the inclination of a plane to a plane is a right angle. When to this is added the fact incidentally proved in XI. 10 that the "inclination of a plane to a plane" is the same at whatever point in their common section it is drawn, it is sufficient to prove the perpendicularity of two planes if *one* straight line drawn, in one of them, perpendicular to their common section is perpendicular to the other.

If this point of view is taken, Props. 18, 19 are much simplified (cf. Legendre, H. M. Taylor, Smith and Bryant, Rausenberger, Schultze and Sevenoak, Holgate). The alternative proof is as follows.

Let *AB* be perpendicular to the plane *MN*, and *CE* any plane through *AB*, meeting the plane *MN* in the straight line *CD*.

In the plane *MN* draw *BF* at right angles to *CD*.

Then *ABF* is the angle which Euclid calls (in the case where it is acute) the "inclination of the plane to the plane."



But, since *AB* is perpendicular to the plane *MN*, it is perpendicular to *BF* in it.

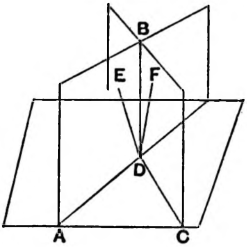
Therefore the angle *ABF* is a right angle;
whence the plane *CE* is perpendicular to the plane *MN*.

PROPOSITION 19.

If two planes which cut one another be at right angles to any plane, their common section will also be at right angles to the same plane.

For let the two planes *AB*, *BC* be at right angles to the plane of reference,
and let *BD* be their common section;
I say that *BD* is at right angles to the plane of reference.

For suppose it is not, and from the point *D* let *DE* be drawn in the plane *AB* at right angles to the straight line *AD*, and *DF* in the plane *BC* at right angles to *CD*.



Now, since the plane *AB* is at right angles to the plane of reference,
and *DE* has been drawn in the plane *AB* at right angles to *AD*, their common section,
therefore *DE* is at right angles to the plane of reference.
[XI. Def. 4]

Similarly we can prove that

DF is also at right angles to the plane of reference.

Therefore from the same point *D* two straight lines have been set up at right angles to the plane of reference on the same side:

which is impossible. [XI. 13]

Therefore no straight line except the common section *DB* of the planes *AB*, *BC* can be set up from the point *D* at right angles to the plane of reference.

Therefore etc.

Q. E. D.

Legendre, followed by other writers already quoted, uses a preliminary proposition equivalent to Euclid's definition of planes at right angles to one another.

If two planes are perpendicular to one another, a straight line drawn in one of them perpendicular to their common section will be perpendicular to the other.

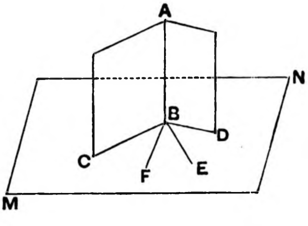
Let the perpendicular planes *CE*, *MN* (figure of last note) intersect in *CD*, and let *AB* be drawn in *CE* perpendicular to *CD*.

In the plane *MN* draw *BF* at right angles to *CD*.

Then, since the planes are perpendicular, the angle *ABF* (their *inclination*) is a right angle.

Therefore *AB* is perpendicular to both *CD* and *BF*, and therefore to the plane *MN*.

We are now in a position to prove XI. 19, viz. *If two planes be perpendicular to a third, their intersection is also perpendicular to that third plane.*



Let each of the two planes *AC*, *AD* intersecting in *AB* be perpendicular to the plane *MN*.

Let *AC*, *AD* intersect *MN* in *BC*, *BD* respectively.

In the plane *MN* draw *BE* at right angles to *BC* and *BF* at right angles to *BD*.

Now, since the planes *AC*, *MN* are at right angles, and *BE* is drawn in the latter perpendicular to *BC*, *BE* is perpendicular to the plane *AC*.

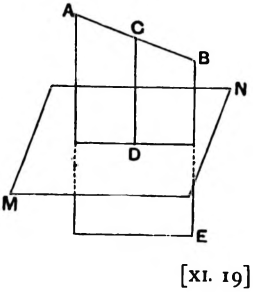
Hence *AB* is perpendicular to *BE*. [XI. 4]

Similarly *AB* is perpendicular to *BF*.

Therefore *AB* is perpendicular to the plane through *BE*, *BF*, i.e. to the plane *MN*.

An useful problem is that of drawing a common perpendicular to two straight lines not in one plane, and in connexion with this the following proposition may be given.

Given a plane and a straight line not perpendicular to it, one plane, and only one, can be drawn through the straight line perpendicular to the plane.



Let AB be the given straight line, MN the given plane.

From any point C in AB draw CD perpendicular to the plane MN.

Through AB and CD draw a plane AE.

Then the plane AE is perpendicular to the plane MN. [XI. 18]

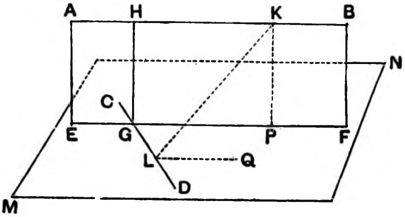
If any other plane could be drawn through AB perpendicular to MN, the intersection AB of the two planes perpendicular to MN would itself be perpendicular to MN: [XI. 19]
which contradicts the hypothesis.

To draw a common perpendicular to two straight lines not in the same plane.

Let AB, CD be the given straight lines.

Through CD draw the plane MN parallel to AB (Prop. 2 in note to XI. 15).

Through AB draw the plane AF perpendicular to the plane MN (see the last preceding proposition).



Let the planes AF, MN intersect in EF, and let EF meet CD in G.

From G, in the plane AF, draw GH at right angles to EF, meeting AB in H.

GH is then the required perpendicular.

For AB is parallel to EF (Prop. 4 in note to XI. 15); therefore GH, being perpendicular to EF, is also perpendicular to AB.

But, the plane AF being perpendicular to the plane MN, and GH being perpendicular to EF, their intersection,

GH is perpendicular to the plane MN, and therefore to CD.

Therefore GH is perpendicular to both AB and CD.

Only one common perpendicular can be drawn to two straight lines not in one plane.

For, if possible, let KL also be perpendicular to both AB and CD.

Let the plane through KL, AB meet the plane MN in LQ.

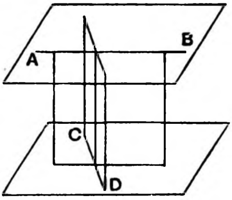
Then AB is parallel to LQ (Prop. 4 in note to XI. 15), so that KL, being perpendicular to AB, is also perpendicular to LQ.

Therefore KL is perpendicular to both CL and LQ, and consequently to the plane MN.

But, if KP be drawn in the plane AF perpendicular to EF, KP is also perpendicular to the plane MN.

Thus there are two perpendiculars from the point K to the plane MN: which is impossible.

Rausenberger's construction for the same problem is more elegant. Draw, he says, through each straight line a plane parallel to the other. Then draw through each straight line a plane perpendicular to the plane through the other. The two planes last drawn will intersect in a straight line, and this straight line is the common perpendicular required.

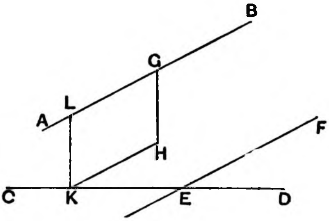


The form of the construction best suited for examination purposes, because the most self-contained, is doubtless that given by Smith and Bryant.

Let AB, CD be the two given straight lines.

Through any point E in CD draw EF parallel to AB.

From any point G in AB draw GH perpendicular to the plane CDF, meeting the plane in H.



Through H in the plane CDF draw HK parallel to FE or AB, to cut CD in K.

Then, since AB, HK are parallel, $AGHK$ is a plane.

Complete the parallelogram $GHKL$.

Now, since LK, GH are parallel, and GH is perpendicular to the plane CDF,

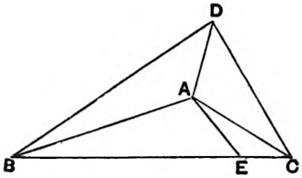
LK is perpendicular to the plane CDF.

Therefore LK is perpendicular to CD and KH, and therefore to AB which is parallel to KH.

PROPOSITION 20.

If a solid angle be contained by three plane angles, any two, taken together in any manner, are greater than the remaining one.

For let the solid angle at A be contained by the three plane angles BAC, CAD, DAB;



I say that any two of the angles BAC, CAD, DAB, taken together in any manner, are greater than the remaining one.

If now the angles BAC, CAD, DAB are equal to one another, it is manifest that any two are greater than the remaining one.

But, if not, let BAC be greater,

and on the straight line AB, and at the point A on it, let the

angle *BAE* be constructed, in the plane through *BA*, *AC*, equal to the angle *DAB*;
let *AE* be made equal to *AD*,
and let *BEC*, drawn across through the point *E*, cut the straight lines *AB*, *AC* at the points *B*, *C*;
let *DB*, *DC* be joined.

Now, since *DA* is equal to *AE*,
and *AB* is common,
two sides are equal to two sides;
and the angle *DAB* is equal to the angle *BAE*;
therefore the base *DB* is equal to the base *BE*. [I. 4]

And, since the two sides *BD*, *DC* are greater than *BC*, [I. 20]
and of these *DB* was proved equal to *BE*,
therefore the remainder *DC* is greater than the remainder *EC*.

Now, since *DA* is equal to *AE*,
and *AC* is common,
and the base *DC* is greater than the base *EC*,
therefore the angle *DAC* is greater than the angle *EAC*. [I. 25]

But the angle *DAB* was also proved equal to the angle *BAE*;
therefore the angles *DAB*, *DAC* are greater than the angle *BAC*.

Similarly we can prove that the remaining angles also, taken together two and two, are greater than the remaining one.

Therefore etc.

Q. E. D.

After excluding the obvious case in which all three angles are equal, Euclid goes on to say "If not, let the angle *BAC* be greater," without adding greater than *what*. Heiberg is clearly right in saying that he means greater than *BAD*, i.e. greater than *one* of the adjacent angles. This is proved by the words at the end "Similarly we can prove," etc. Euclid thus excludes as obvious the case where one of the three angles is not greater than either of the other two, but proves the remaining cases. This is scientific, but he might further have excluded as obvious the case in which one angle is greater than one of the others but equal to or less than the remaining one.

Simson remarks that the angle *BAC* may happen to be *equal* to one of the other two and writes accordingly "If they [all three angles] are not [equal], let *BAC* be that angle which is not less than either of the other two, and is greater than one of them *DAB*." He then proves, in the same way as Euclid does, that the angles *DAB*, *DAC* are greater than the angle *BAC*, adding finally: "But *BAC* is not less than either of the angles *DAB*, *DAC*; therefore *BAC*, with either of them, is greater than the other."

It would be better, as indicated by Legendre and Rausenberger, to begin by saying that, "If one of the three angles is either equal to or less than either of the other two, it is evident that the sum of those two is greater than the first. It is therefore only necessary to prove, *for the case in which one angle is greater than each of the others*, that the sum of the two latter is greater than the former.

Accordingly let *BAC* be greater than each of the other angles." We then proceed as in Euclid.

PROPOSITION 21.

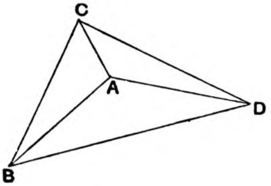
Any solid angle is contained by plane angles less than four right angles.

Let the angle at *A* be a solid angle contained by the plane angles *BAC*, *CAD*, *DAB*;

I say that the angles *BAC*, *CAD*, *DAB* are less than four right angles.

For let points *B*, *C*, *D* be taken at random on the straight lines *AB*, *AC*, *AD* respectively,

and let *BC*, *CD*, *DB* be joined.



Now, since the solid angle at *B* is contained by the three plane angles *CBA*, *ABD*, *CBD*,

any two are greater than the remaining one; [XI. 20]

therefore the angles *CBA*, *ABD* are greater than the angle *CBD*.

For the same reason

the angles *BCA*, *ACD* are also greater than the angle *BCD*,

and the angles *CDA*, *ADB* are greater than the angle *CDB*;

therefore the six angles *CBA*, *ABD*, *BCA*, *ACD*, *CDA*, *ADB* are greater than the three angles *CBD*, *BCD*, *CDB*.

But the three angles *CBD*, *BDC*, *BCD* are equal to two right angles; [I. 32]

therefore the six angles *CBA*, *ABD*, *BCA*, *ACD*, *CDA*, *ADB* are greater than two right angles.

And, since the three angles of each of the triangles *ABC*, *ACD*, *ADB* are equal to two right angles,

therefore the nine angles of the three triangles, the angles *CBA*, *ACB*, *BAC*, *ACD*, *CDA*, *CAD*, *ADB*, *DBA*, *BAD* are equal to six right angles;

and of them the six angles *ABC*, *BCA*, *ACD*, *CDA*, *ADB*, *DBA* are greater than two right angles;

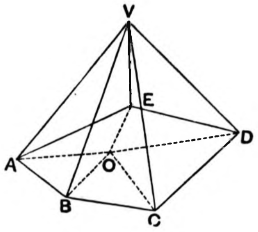
therefore the remaining three angles *BAC*, *CAD*, *DAB* containing the solid angle are less than four right angles.

Therefore etc.

Q. E. D.

It will be observed that, although Euclid enunciates this proposition for *any* solid angle, he only proves it for the particular case of a *trihedral* angle. This is in accordance with his manner of proving one case and leaving the others to the reader. The omission of the convex polyhedral angle here corresponds to the omission, after I. 32, of the proposition about the interior angles of a convex polygon given by Proclus and in most books. The proof of the present proposition for any convex polyhedral angle can of course be arranged so as not to assume the proposition that the interior angles of a convex polygon together with four right angles are equal to twice as many right angles as the figure has sides.

Let there be any convex polyhedral angle with *V* as vertex, and let it be cut by any plane meeting its faces in, say, the polygon *ABCDE*.



Take *O* any point within the polygon, and in its plane, and join *OA*, *OB*, *OC*, *OD*, *OE*.

Then all the angles of the triangles with vertex *O* are equal to twice as many right angles as the polygon has sides; [I. 32]

therefore the interior angles of the polygon together with all the angles round *O* are equal to twice as many right angles as the polygon has sides.

Also the sum of the angles of the triangles *VAB*, *VBC*, etc., with vertex *V* are equal to twice as many right angles as the polygon has sides;

and all the said angles are equal to the sum of (1) the plane angles at *V* forming the polyhedral angle and (2) the base angles of the triangles with vertex *V*.

This latter sum is therefore equal to the sum of (3) all the angles round *O* and (4) all the interior angles of the polygon.

Now, by Euclid's proposition, of the three angles forming the solid angle at *A*, the angles *VAE*, *VAB* are together greater than the angle *EAB*.

Similarly, at *B*, the angles *VBA*, *VBC* are together greater than the angle *ABC*.

And so on.

Therefore, by addition, the base angles of the triangles with vertex *V*

[(2) above] are together greater than the sum of the angles of the polygon [(4) above].

Hence, by way of compensation, the sum of the plane angles at V [(1) above] is less than the sum of the angles round O [(3) above].

But the latter sum is equal to four right angles; therefore the plane angles forming the polyhedral angle are together less than four right angles.

The proposition is only true of *convex* polyhedral angles, i.e. those in which the plane of any face cannot, if produced, ever cut the solid angle.

There are certain propositions relating to equal (and symmetrical) trihedral angles which are necessary to the consideration of the polyhedra dealt with by Euclid, all of which (as before remarked) have trihedral angles only.

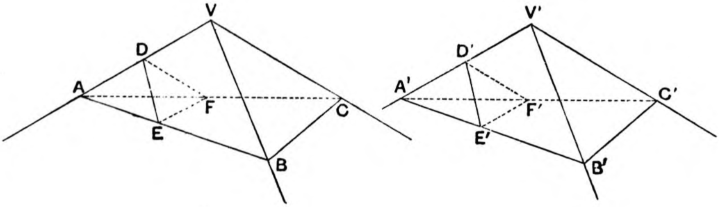
1. *Two trihedral angles are equal if two face angles and the included dihedral angle of the one are respectively equal to two face angles and the included dihedral angle of the other, the equal parts being arranged in the same order.*

2. *Two trihedral angles are equal if two dihedral angles and the included face angle of the one are respectively equal to two dihedral angles and the included face angle of the other, all equal parts being arranged in the same order.*

These propositions are proved immediately by superposition.

3. *Two trihedral angles are equal if the three face angles of the one are respectively equal to the three face angles of the other, and all are arranged in the same order.*

Let $V—ABC$ and $V'—A'B'C'$ be two trihedral angles such that the angle AVB is equal to the angle $A'V'B'$, the angle BVC to the angle $B'V'C'$, and the angle CVA to the angle $C'V'A'$.



We first prove that *corresponding pairs of face angles include equal dihedral angles.*

E.g., the dihedral angle formed by the plane angles CVA, AVB is equal to that formed by the plane angles $C'V'A'$, $A'V'B'$.

Take points A, B, C on VA, VB, VC and points A', B', C' on $V'A'$, $V'B'$, $V'C'$, such that VA, VB, VC, $V'A'$, $V'B'$, $V'C'$ are all equal.

Join BC, CA, AB, $B'C'$, $C'A'$, $A'B'$.

Take any point D on AV, and measure $A'D'$ along $A'V'$ equal to AD.

From D draw DE in the plane AVB, and DF in the plane CVA, perpendicular to AV. Then DE, DF will meet AB, AC respectively, the angles VAB, VAC, the base angles of two isosceles triangles, being less than right angles.

Join EF.

Draw the triangle $D'E'F'$ in the same way.

Now, by means of the hypothesis and construction, it appears that the triangles VAB, $V'A'B'$ are equal in all respects.

So are the triangles VAC, $V'A'C'$, and the triangles VBC, $V'B'C'$.

Thus BC, CA, AB are respectively equal to $B'C'$, $C'A'$, $A'B'$, and the triangles ABC, $A'B'C'$ are equal in all respects.

Now, in the triangles ADE, $A'D'E'$,
the angles ADE, DAE are equal to the angles $A'D'E'$, $D'A'E'$ respectively,
and AD is equal to $A'D'$.

Therefore the triangles ADE, $A'D'E'$ are equal in all respects.

Similarly the triangles ADF, $A'D'F'$ are equal in all respects.

Thus, in the triangles AEF, $A'E'F'$,
EA, AF are respectively equal to $E'A'$, $A'F'$,
and the angle EAF is equal to the angle $E'A'F'$ (from above);
therefore the triangles AEF, $A'E'F'$ are equal in all respects.

Lastly, in the triangles DEF, $D'E'F'$, the three sides are respectively equal to the three sides;
therefore the triangles are equal in all respects.

Therefore the angles EDF, $E'D'F'$ are equal.

But these angles are the measures of the dihedral angles formed by the planes CVA, AVB and by the planes $C'V'A'$, $A'V'B'$ respectively.

Therefore these dihedral angles are equal.

Similarly for the other two dihedral angles.

Hence the trihedral angles coincide if one is applied to the other;
that is, they are equal.

To understand what is implied by "taken in the same order" we may suppose ourselves to be placed at the vertices, and to take the faces in clockwise direction, or the reverse, for *both* angles.

If the face angles and dihedral angles are *taken in reverse directions*, i.e. in clockwise direction in one and in counterclockwise direction in the other, then, if the other conditions in the above three propositions are fulfilled, the trihedral angles are not equal but *symmetrical.*

If the faces of a trihedral angle be produced beyond the vertex, they form another trihedral angle. It is easily seen that these *vertical trihedral angles* are *symmetrical.*

PROPOSITION 22.

If there be three plane angles of which two, taken together in any manner, are greater than the remaining one, and they are contained by equal straight lines, it is possible to construct a triangle out of the straight lines joining the extremities of the equal straight lines.

Let there be three plane angles ABC, DEF, GHK, of which two, taken together in any manner, are greater than the remaining one, namely

the angles ABC, DEF greater than the angle GHK,

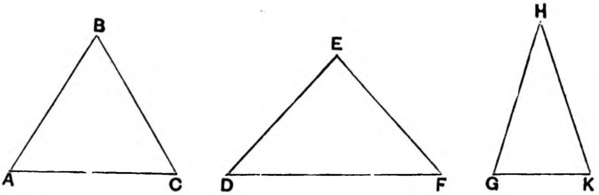
the angles DEF, GHK greater than the angle ABC,

and, further, the angles GHK, ABC greater than the angle DEF;

let the straight lines AB, BC, DE, EF, GH, HK be equal,

and let AC, DF, GK be joined;

I say that it is possible to construct a triangle out of straight lines equal to AC, DF, GK, that is, that any two of the straight lines AC, DF, GK are greater than the remaining one.



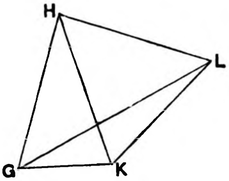
Now, if the angles ABC, DEF, GHK are equal to one another, it is manifest that, AC, DF, GK being equal also, it is possible to construct a triangle out of straight lines equal to AC, DF, GK.

But, if not, let them be unequal,

and on the straight line HK, and at the point H on it, let the angle KHL be constructed equal to the angle ABC;

let HL be made equal to one of the straight lines AB, BC, DE, EF, GH, HK,

and let KL, GL be joined.



Now, since the two sides AB, BC are equal to the two sides KH, HL,

and the angle at B is equal to the angle KHL,

therefore the base AC is equal to the base KL. [I. 4]

And, since the angles ABC, GHK are greater than the angle DEF,

while the angle ABC is equal to the angle KHL,
therefore the angle GHL is greater than the angle DEF.

And, since the two sides GH, HL are equal to the two sides DE, EF,
and the angle GHL is greater than the angle DEF,
therefore the base GL is greater than the base DF. [I. 24]

But GK, KL are greater than GL.
Therefore GK, KL are much greater than DF.

But KL is equal to AC;
therefore AC, GK are greater than the remaining straight line DF.

Similarly we can prove that
AC, DF are greater than GK,
and further DF, GK are greater than AC.

Therefore it is possible to construct a triangle out of straight lines equal to AC, DF, GK.

Q. E. D.

The Greek text gives an alternative proof, which is relegated by Heiberg to the Appendix. Simson selected the alternative proof in preference to that given above; he objected however to words near the beginning, "If not, let the angles at the points B, E, H be unequal and that at B greater than either of the angles at E, H," and altered the words so as to take account of the possibility that the angle at B might be equal to one of the other two.

As will be seen, Euclid takes no account of the relative magnitude of the angles except as regards the case when all three are equal. Having proved that *one* base is less than the sum of the two others, he says that "similarly we can prove" the same thing for the other two bases.

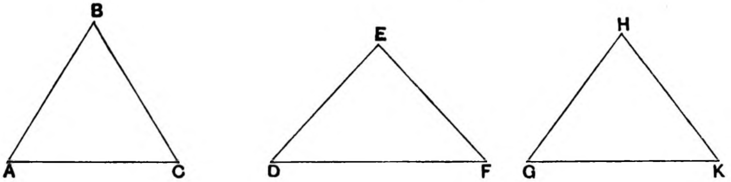
If a distinction is to be made according to the relative magnitude of the three angles, we may say, as in the corresponding place in XI. 21, that, if one of the three angles is either equal to or less than *either* of the other two, the bases subtending those two angles must obviously be together greater than the base subtending the first. Thus it is only necessary to prove, for the case in which one angle is *greater than either of the others*, that the sum of the bases subtending those others is greater than that subtending the first. This is practically the course taken in the interpolated alternative proof.

PROPOSITION 23.

To construct a solid angle out of three plane angles two of which, taken together in any manner, are greater than the remaining one: thus the three angles must be less than four right angles.

Let the angles *ABC*, *DEF*, *GHK* be the three given plane angles, and let two of these, taken together in any manner, be greater than the remaining one, while, further, the three are less than four right angles;

thus it is required to construct a solid angle out of angles equal to the angles *ABC*, *DEF*, *GHK*.



Let *AB*, *BC*, *DE*, *EF*, *GH*, *HK* be cut off equal to one another,

and let *AC*, *DF*, *GK* be joined;

it is therefore possible to construct a triangle out of straight lines equal to *AC*, *DF*, *GK*. [XI. 22]

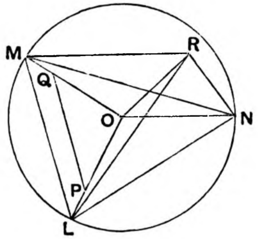
Let *LMN* be so constructed that *AC* is equal to *LM*, *DF* to *MN*, and further *GK* to *NL*,

let the circle *LMN* be described about the triangle *LMN*,

let its centre be taken, and let it be *O*;

let *LO*, *MO*, *NO* be joined;

I say that *AB* is greater than *LO*.



For, if not, *AB* is either equal to *LO*, or less.

First, let it be equal.

Then, since *AB* is equal to *LO*,

while *AB* is equal to *BC*, and *OL* to *OM*,

the two sides *AB*, *BC* are equal to the two sides *LO*, *OM* respectively;

and, by hypothesis, the base *AC* is equal to the base *LM*;

therefore the angle *ABC* is equal to the angle *LOM*. [I. 8]

For the same reason

the angle *DEF* is also equal to the angle *MON*,

and further the angle *GHK* to the angle *NOL*;

therefore the three angles *ABC*, *DEF*, *GHK* are equal to the three angles *LOM*, *MON*, *NOL*.

But the three angles *LOM*, *MON*, *NOL* are equal to four right angles;
therefore the angles *ABC*, *DEF*, *GHK* are equal to four right angles.

But they are also, by hypothesis, less than four right angles: which is absurd.

Therefore *AB* is not equal to *LO*.

I say next that neither is *AB* less than *LO*.

For, if possible, let it be so,
and let *OP* be made equal to *AB*, and *OQ* equal to *BC*,
and let *PQ* be joined.

Then, since *AB* is equal to *BC*,
OP is also equal to *OQ*,
so that the remainder *LP* is equal to *QM*.

Therefore *LM* is parallel to *PQ*, [VI. 2]
and *LMO* is equiangular with *PQO*; [I. 29]
therefore, as *OL* is to *LM*, so is *OP* to *PQ*; [VI. 4]
and alternately, as *LO* is to *OP*, so is *LM* to *PQ*. [V. 16]

But *LO* is greater than *OP*;
therefore *LM* is also greater than *PQ*.

But *LM* was made equal to *AC*;
therefore *AC* is also greater than *PQ*.

Since, then, the two sides *AB*, *BC* are equal to the two sides *PO*, *OQ*,
and the base *AC* is greater than the base *PQ*,
therefore the angle *ABC* is greater than the angle *POQ*. [I. 25]

Similarly we can prove that
the angle *DEF* is also greater than the angle *MON*,
and the angle *GHK* greater than the angle *NOL*.

Therefore the three angles *ABC*, *DEF*, *GHK* are greater than the three angles *LOM*, *MON*, *NOL*.

But, by hypothesis, the angles *ABC*, *DEF*, *GHK* are less than four right angles;
therefore the angles *LOM*, *MON*, *NOL* are much less than four right angles.

But they are also equal to four right angles: which is absurd.

Therefore *AB* is not less than *LO*.

And it was proved that neither is it equal; therefore *AB* is greater than *LO*.

Let then *OR* be set up from the point *O* at right angles to the plane of the circle *LMN*, [XI. 12]
and let the square on *OR* be equal to that area by which the square on *AB* is greater than the square on *LO*; [Lemma]
let *RL*, *RM*, *RN* be joined.

Then, since *RO* is at right angles to the plane of the circle *LMN*,
therefore *RO* is also at right angles to each of the straight lines *LO*, *MO*, *NO*.

And, since *LO* is equal to *OM*,
while *OR* is common and at right angles,
therefore the base *RL* is equal to the base *RM*. [I. 4]

For the same reason
RN is also equal to each of the straight lines *RL*, *RM*;
therefore the three straight lines *RL*, *RM*, *RN* are equal to one another.

Next, since by hypothesis the square on *OR* is equal to that area by which the square on *AB* is greater than the square on *LO*,
therefore the square on *AB* is equal to the squares on *LO*, *OR*.

But the square on *LR* is equal to the squares on *LO*, *OR*,
for the angle *LOR* is right; [I. 47]
therefore the square on *AB* is equal to the square on *RL*;
therefore *AB* is equal to *RL*.

But each of the straight lines *BC*, *DE*, *EF*, *GH*, *HK* is equal to *AB*,
while each of the straight lines *RM*, *RN* is equal to *RL*;
therefore each of the straight lines *AB*, *BC*, *DE*, *EF*, *GH*, *HK* is equal to each of the straight lines *RL*, *RM*, *RN*.

And, since the two sides *LR*, *RM* are equal to the two sides *AB*, *BC*,
and the base *LM* is by hypothesis equal to the base *AC*,
therefore the angle *LRM* is equal to the angle *ABC*. [I. 8]

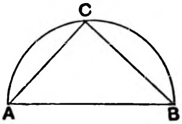
For the same reason
the angle *MRN* is also equal to the angle *DEF*,
and the angle *LRN* to the angle *GHK*.

Therefore, out of the three plane angles *LRM*, *MRN*, *LRN*, which are equal to the three given angles *ABC*, *DEF*, *GHK*, the solid angle at *R* has been constructed, which is contained by the angles *LRM*, *MRN*, *LRN*.

Q. E. F.

LEMMA.

But how it is possible to take the square on *OR* equal to that area by which the square on *AB* is greater than the square on *LO*, we can show as follows.



Let the straight lines *AB*, *LO* be set out,
and let *AB* be the greater;
let the semicircle *ABC* be described on *AB*,
and into the semicircle *ABC* let *AC* be fitted equal to the straight line *LO*, not being greater than the diameter *AB*; [IV. 1]
let *CB* be joined

Since then the angle *ACB* is an angle in the semicircle *ACB*,
therefore the angle *ACB* is right. [III. 31]

Therefore the square on *AB* is equal to the squares on *AC*, *CB*. [I. 47]

Hence the square on *AB* is greater than the square on *AC* by the square on *CB*.

But *AC* is equal to *LO*.

Therefore the square on *AB* is greater than the square on *LO* by the square on *CB*.

If then we cut off *OR* equal to *BC*, the square on *AB* will be greater than the square on *LO* by the square on *OR*.

Q. E. F.

The whole difficulty in this proposition is the proof of a fact which makes the construction *possible*, viz. the fact that, if *LMN* be a triangle with sides respectively equal to the bases of the isosceles triangles which have the given angles as vertical angles and the equal sides all of the same length, then one of these equal sides, as AB, is greater than the radius LO of the circle circumscribing the triangle LMN.

Assuming that AB is greater than LO, we have only to draw from O a perpendicular OR to the plane of the triangle LMN, to make OR of such a length that the sum of the squares on LO, OR is equal to the square on AB, and to join RL, RM, RN. (The manner of finding OR such that the square on it is equal to the difference between the squares on AB and LO is shown in the Lemma at the end of the text of the proposition. We have already had the same construction in the Lemma after X. 13.)

Then clearly RL, RM, RN are equal to AB and to one another [I. 4 and I. 47].

Therefore the triangles LRM, MRN, NRL have their three sides respectively equal to those of the triangles ABC, DEF, GHK respectively.

Hence their vertical angles are equal to the three given angles respectively; and the required solid angle is constructed.

We return now to the proposition to be proved as a preliminary to the construction, viz. that, in the figures, AB is greater than LO.

It will be observed that Euclid, as his manner is, proves it for one case only, that, namely, in which O, the centre of the circle circumscribing the triangle LMN, falls *within* the triangle, leaving the other cases for the reader to prove. As usual, however, the two other cases are found in the Greek text, after the formal conclusion of the proposition, as above, ending with the words ὅπερ ἔδει ποιῆσαι. This position for the proofs itself suggests that they are not Euclid's but are interpolated; and this is rendered certain by the fact that words distinguishing three cases at the point where the centre O of the circumscribing circle is found, "It [the centre] will then be either within the triangle LMN or on one of its sides or without. First let it be within," are found in the MSS. B and V only and are manifestly interpolated. Nevertheless the additional two cases must have been inserted very early, as they are found in all the best MSS.

In order to give a clear view of the proof of all three cases as given in the text, we will reproduce all three (Euclid's as well as the others) with abbreviations to make them catch the eye better.

In all three cases the proof is by *reductio ad absurdum*, and it is proved first that AB cannot be *equal* to LO, and secondly that AB cannot be *less* than LO.

Case I.

(1) Suppose, if possible, that $AB = LO$.

Then AB, BC are respectively equal to LO, OM;
and $AC = LM$ (by construction).

Therefore $\angle ABC = \angle LOM$.

Similarly $\angle DEF = \angle MON$,

$\angle GHK = \angle NOL$.

Adding, we have

$$\angle ABC + \angle DEF + \angle GHK = \angle LOM + \angle MON + \angle NOL$$
$$= \text{four right angles}:$$

which contradicts the hypothesis.

Therefore $AB \neq LO$.

(2) Suppose that $AB < LO$.

Make OP, OQ (measured along OL, OM) each equal to AB.

Thus, OL, OM being equal also, it follows that

$$PQ \text{ is } \parallel \text{ to } LM.$$

Hence $\qquad LM : PQ = LO : OP$;

and, since $LO > OP$,

LM, i.e. AC, $> PQ$.

Thus, in $\triangle$s POQ, ABC, two sides are equal to two sides, and base $AC >$ base PQ;

therefore $\qquad \angle ABC > \angle POQ$, $\quad$ i.e. $\angle LOM$.

Similarly $\qquad \angle DEF > \angle MON$,

$\qquad \angle GHK > \angle NOL$,

and it follows by addition that

$$\angle ABC + \angle DEF + \angle GHK > \text{(four right angles)}:$$

which again contradicts the hypothesis.

Case II.

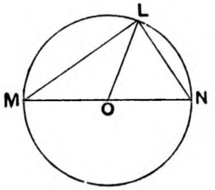
(1) Suppose, if possible, that $AB = LO$.

Then $(AB + BC)$, or $(DE + EF) = MO + OL$

$= MN$

$= DF$:

which contradicts the hypothesis.



(2) The supposition that $AB < LO$ is even more impossible; for in this case it would result that

$$DE + EF < DF.$$

Case III.

(1) Suppose, if possible, that $AB = LO$.

Then, in the triangles ABC, LOM, two sides AB, BC are respectively equal to two sides LO, OM, and the bases AC, LM are equal;

therefore $\qquad \angle ABC = \angle LOM$.

Similarly $\quad \angle GHK = \angle NOL$.

Therefore, by addition,

$$\angle MON = \angle ABC + \angle GHK$$

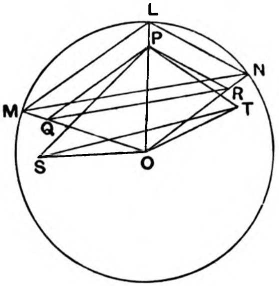
$$> \angle DEF \text{ (by hypothesis).}$$

But, in the triangles DEF, MON, which are equal in all respects,

$$\angle MON = \angle DEF.$$

But it was proved that $\angle MON > \angle DEF$:

which is impossible.



(2) Suppose, if possible, that $AB < LO$.

Along OL, OM measure OP, OQ each equal to AB.

Then LM, PQ are parallel, and

$$LM : PQ = LO : OP,$$

whence, since $LO > OP$,

$$LM, \text{ or } AC, > PQ.$$

Thus, in the triangles ABC, POQ,

$$\angle ABC > \angle POQ, \text{ i.e. } \angle LOM.$$

Similarly, by taking OR along ON equal to AB, we prove that

$$\angle GHK > \angle LON.$$

Now, at O, make $\angle POS$ equal to $\angle ABC$, and $\angle POT$ equal to $\angle GHK$.

Make OS, OT each equal to OP, and join ST, SP, TP.

Then, in the equal triangles ABC, POS,

$$AC = PS,$$

so that
$$LM = PS.$$

Similarly
$$LN = PT.$$

Therefore in the triangles MLN, SPT, since $\angle MLN > \angle SPT$ [this is assumed, but should have been explained],

$$MN > ST,$$

or
$$DF > ST.$$

Lastly, in $\triangle$s DEF, SOT, which have two sides equal to two sides, since $DF > ST$,

$$\angle DEF > \angle SOT$$

$$> \angle ABC + \angle GHK \text{ (by construction)}:$$

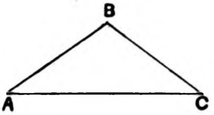
which contradicts the hypothesis.

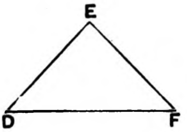
Simson gives rather different proofs for all three cases; but the essence of them can be put, I think, a little more shortly than in his text, as well as more clearly

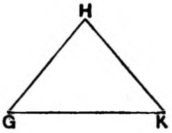
Case I. (O within $\triangle LMN$.)

(1) Let AB be, if possible, equal to LO.

Then the $\triangle$s ABC, DEF, GHK must be identically equal to the $\triangle$s LOM, MON, NOL respectively.





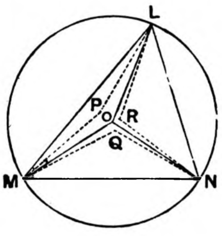


Therefore the vertical angles at O in the latter triangles are equal respectively to the angles at B, E, H.

The latter are therefore together equal to four right angles:

which is impossible.

(2) If AB be less than LO, construct on the bases LM, MN, NL triangles with vertices P, Q, R and identically equal to the $\triangle$s ABC, DEF, GHK respectively.



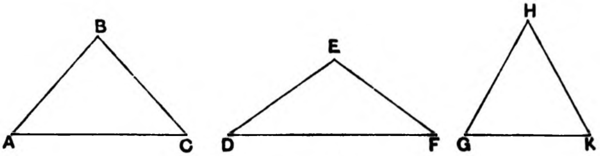
Then P, Q, R will fall within the respective angles at O, since $PL = PM$ and $< LO$, and similarly in the other cases.

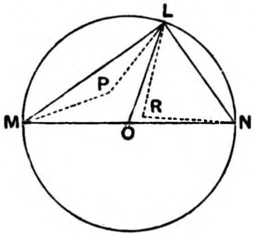
Thus [I. 21] the angles at P, Q, R are respectively greater than the angles at O in which they lie.

Therefore the sum of the angles at P, Q, R, i.e. the sum of the angles at B, E, H, is greater than four right angles:
which again contradicts the hypothesis.

Case II. (O lying on MN.)

In this case, whether (1) $AB = LO$, or (2) $AB < LO$, a triangle cannot be formed with MN as base and each of the other sides equal to AB. In other words, the triangle DEF either reduces to a straight line or is impossible.





Case III. (O lying outside the $\triangle LMN$.)

(1) Suppose, if possible, that $AB = LO$.

Then the triangles LOM, MON, NOL are identically equal to the triangles ABC, DEF, GHK.

Since $\angle LOM + \angle LON = \angle MON$,

$$\angle ABC + \angle GHK = \angle DEF:$$

which contradicts the hypothesis.

(2) Suppose that $AB < OL$.

Draw, as before, on LM, MN, NL as bases triangles with vertices P, Q, R and identically equal to the $\triangle$s ABC, DEF, GHK.

Next, at N on the straight line NR, make $\angle RNS$ equal to the angle PLM, cut off NS equal to LM and join RS, LS.

Then $\triangle NRS$ is identically equal to $\triangle LPM$ or $\triangle ABC$.

Now $(\angle LNR + \angle RNS) < (\angle NLO + \angle OLM)$,

that is, $\angle LNS < \angle NLM$.

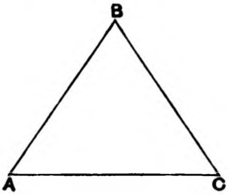
Thus, in $\triangle$s LNS, NLM, two sides are equal to two sides, and the included angle in the former is less than the included angle in the other.

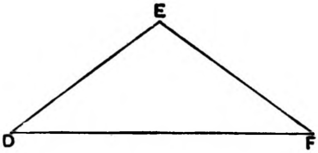
Therefore $LS < MN$.

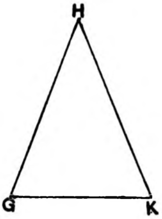
Hence, in the triangles MQN, LRS, two sides are equal to two sides, and $MN > LS$.

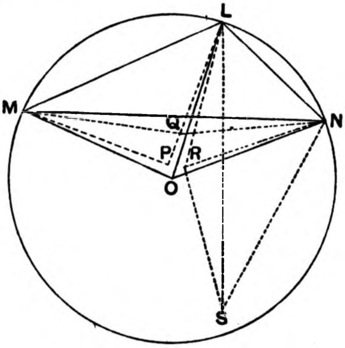
$$
\begin{aligned}
\text{Therefore}\qquad \angle MQN &> \angle LRS \\
&> (\angle LRN + \angle SRN) \\
&> (\angle LRN + \angle LPM). \\
\text{That is,}\qquad \angle DEF &> (\angle GHK + \angle ABC):
\end{aligned}
$$

which is impossible.







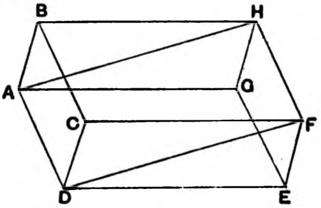


PROPOSITION 24.

If a solid be contained by parallel planes, the opposite planes in it are equal and parallelogrammic.

For let the solid $CDHG$ be contained by the parallel planes AC, GF, AH, DF, BF, AE;

I say that the opposite planes in it are equal and parallelogrammic.



For, since the two parallel planes BG, CE are cut by the plane AC,

their common sections are parallel. [XI. 16]

Therefore AB is parallel to DC.

Again, since the two parallel planes BF, AE are cut by the plane AC,
their common sections are parallel. [XI. 16]

Therefore BC is parallel to AD.

But AB was also proved parallel to DC;
therefore AC is a parallelogram.

Similarly we can prove that each of the planes DF, FG, GB, BF, AE is a parallelogram.

Let AH, DF be joined.

Then, since AB is parallel to DC, and BH to CF,
the two straight lines AB, BH which meet one another are parallel to the two straight lines DC, CF which meet one another, not in the same plane;
therefore they will contain equal angles; [XI. 10]
therefore the angle ABH is equal to the angle DCF.

And, since the two sides AB, BH are equal to the two sides DC, CF, [I. 34]
and the angle ABH is equal to the angle DCF,
therefore the base AH is equal to the base DF,
and the triangle ABH is equal to the triangle DCF. [I. 4]

And the parallelogram BG is double of the triangle ABH,
and the parallelogram CE double of the triangle DCF; [I. 34]
therefore the parallelogram BG is equal to the parallelogram CE.

Similarly we can prove that
AC is also equal to GF,
and AE to BF.

Therefore etc.

Q. E. D.

As Heiberg says, this proposition is carelessly enunciated. Euclid means a solid contained by *six* planes and not more, the planes are parallel *two and two*, and the opposite faces are equal in the sense of *identically* equal, or, as Simson puts it, equal *and similar*. The *similarity* is necessary in order to enable the equality of the parallelepipeds in the next proposition to be inferred from the 10th definition of Book XI. Hence a better enunciation would be:

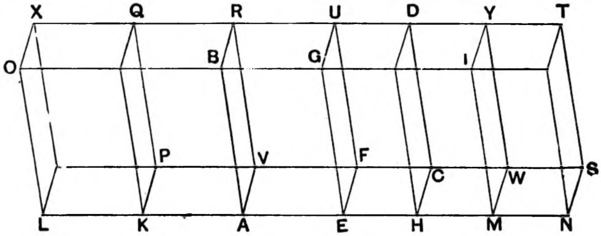
If a solid be contained by six planes parallel two and two, the opposite faces respectively are equal and similar parallelograms.

The proof is simple and requires no elucidation.

PROPOSITION 25.

If a parallelepipedal solid be cut by a plane which is parallel to the opposite planes, then, as the base is to the base, so will the solid be to the solid.

For let the parallelepipedal solid *ABCD* be cut by the plane *FG* which is parallel to the opposite planes *RA*, *DH*; I say that, as the base *AEFV* is to the base *EHCF*, so is the solid *ABFU* to the solid *EGCD*.



For let *AH* be produced in each direction,
let any number of straight lines whatever, *AK*, *KL*, be made equal to *AE*,
and any number whatever, *HM*, *MN*, equal to *EH*;
and let the parallelograms *LP*, *KV*, *HW*, *MS* and the solids *LQ*, *KR*, *DM*, *MT* be completed.

Then, since the straight lines *LK*, *KA*, *AE* are equal to one another,
the parallelograms *LP*, *KV*, *AF* are also equal to one another,
KO, *KB*, *AG* are equal to one another,
and further *LX*, *KQ*, *AR* are equal to one another, for they are opposite. [XI. 24]

For the same reason
the parallelograms *EC*, *HW*, *MS* are also equal to one another,
HG, *HI*, *IN* are equal to one another,
and further *DH*, *MY*, *NT* are equal to one another.

Therefore in the solids *LQ*, *KR*, *AU* three planes are equal to three planes.

But the three planes are equal to the three opposite; therefore the three solids *LQ*, *KR*, *AU* are equal to one another.

For the same reason
the three solids *ED*, *DM*, *MT* are also equal to one another.

Therefore, whatever multiple the base *LF* is of the base *AF*, the same multiple also is the solid *LU* of the solid *AU*.

For the same reason,
whatever multiple the base *NF* is of the base *FH*, the same multiple also is the solid *NU* of the solid *HU*.

And, if the base *LF* is equal to the base *NF*, the solid *LU* is also equal to the solid *NU*;
if the base *LF* exceeds the base *NF*, the solid *LU* also exceeds the solid *NU*;
and, if one falls short, the other falls short.

Therefore, there being four magnitudes, the two bases *AF*, *FH*, and the two solids *AU*, *UH*,
equimultiples have been taken of the base *AF* and the solid *AU*, namely the base *LF* and the solid *LU*,
and equimultiples of the base *HF* and the solid *HU*, namely the base *NF* and the solid *NU*,
and it has been proved that, if the base *LF* exceeds the base *FN*, the solid *LU* also exceeds the solid *NU*,
if the bases are equal, the solids are equal,
and if the base falls short, the solid falls short.

Therefore, as the base *AF* is to the base *FH*, so is the solid *AU* to the solid *UH*. [v. Def. 5]

Q. E. D.

It is to be observed that, as the word *parallelogrammic* was used in Book I. without any definition of its meaning, so παραλληλεπίπεδος, *parallelepipedal*, is here used without explanation. While it means simply "with parallel planes," i.e. "faces," the term is appropriated to the particular solid which has *six* plane faces parallel two and two. The proper translation of στερεὸν παραλληλεπίπεδον is *parallelepipedal solid*, not *solid parallelepiped*, as it is usually translated. Still less is the solid a parallel*o*piped, as the word is not uncommonly written.

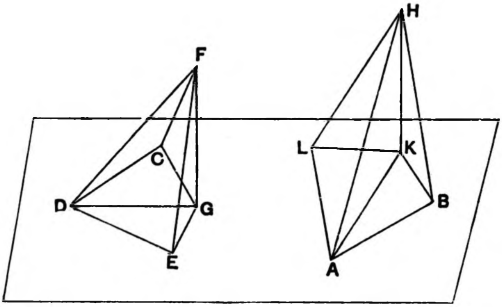
The opposite faces in each set of parallelepipedal solids in this proposition are not only equal but equal *and similar*. Euclid infers that the solids in each set are equal from Def. 10; but, as we have seen in the note on Deff. 9, 10, though it is true, where no solid angle in the figures is contained by more than three plane angles, that two solid figures are equal and similar which are contained by the same number of equal and similar faces, similarly arranged, the fact should have been proved. To do this, we have only to prove the proposition, given above in the note on XI. 21, that *two trihedral angles are equal if the three face angles of the one are respectively equal to the three face angles in the other, and all are arranged in the same order*, and then to prove equality by applying one figure to the other as is done by Simson in his proposition C.

Application will also, of course, establish what is assumed by Euclid of the solids formed by the multiples of the original solids, namely that, if $LF \gtrless NF$, the solid $LU \gtrless$ the solid NU.

PROPOSITION 26.

On a given straight line, and at a given point on it, to construct a solid angle equal to a given solid angle.

Let AB be the given straight line, A the given point on it, and the angle at D, contained by the angles EDC, EDF, FDC, the given solid angle;
thus it is required to construct on the straight line AB, and at the point A on it, a solid angle equal to the solid angle at D.



For let a point F be taken at random on DF,
let FG be drawn from F perpendicular to the plane through ED, DC, and let it meet the plane at G, [XI. 11]
let DG be joined,
let there be constructed on the straight line AB and at the point A on it the angle BAL equal to the angle EDC, and the angle BAK equal to the angle EDG, [I. 23]
let AK be made equal to DG,

let *KH* be set up from the point *K* at right angles to the plane through *BA*, *AL*, [XI. 12]
let *KH* be made equal to *GF*,
and let *HA* be joined;
I say that the solid angle at *A*, contained by the angles *BAL*, *BAH*, *HAL* is equal to the solid angle at *D* contained by the angles *EDC*, *EDF*, *FDC*.

For let *AB*, *DE* be cut off equal to one another,
and let *HB*, *KB*, *FE*, *GE* be joined.

Then, since *FG* is at right angles to the plane of reference, it will also make right angles with all the straight lines which meet it and are in the plane of reference; [XI. Def. 3]
therefore each of the angles *FGD*, *FGE* is right.

For the same reason
each of the angles *HKA*, *HKB* is also right.

And, since the two sides *KA*, *AB* are equal to the two sides *GD*, *DE* respectively,
and they contain equal angles,
therefore the base *KB* is equal to the base *GE*. [I. 4]

But *KH* is also equal to *GF*,
and they contain right angles;
therefore *HB* is also equal to *FE*. [I. 4]

Again, since the two sides *AK*, *KH* are equal to the two sides *DG*, *GF*,
and they contain right angles,
therefore the base *AH* is equal to the base *FD*. [I. 4]

But *AB* is also equal to *DE*;
therefore the two sides *HA*, *AB* are equal to the two sides *DF*, *DE*.

And the base *HB* is equal to the base *FE*;
therefore the angle *BAH* is equal to the angle *EDF*. [I. 8]

For the same reason
the angle *HAL* is also equal to the angle *FDC*.

And the angle *BAL* is also equal to the angle *EDC*.

Therefore on the straight line AB, and at the point A on it, a solid angle has been constructed equal to the given solid angle at D.

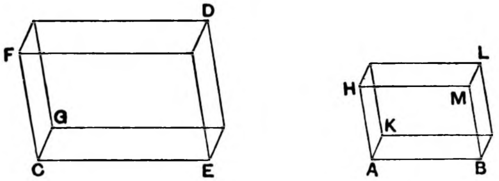
Q. E. F.

This proposition again assumes the equality of two trihedral angles which have the three plane angles of the one respectively equal to the three plane angles of the other taken in the same order.

PROPOSITION 27.

On a given straight line to describe a parallelepipedal solid similar and similarly situated to a given parallelepipedal solid.

Let AB be the given straight line and CD the given parallelepipedal solid;
thus it is required to describe on the given straight line AB a parallelepipedal solid similar and similarly situated to the given parallelepipedal solid CD.



For on the straight line AB and at the point A on it let the solid angle, contained by the angles BAH, HAK, KAB, be constructed equal to the solid angle at C, so that the angle BAH is equal to the angle ECF, the angle BAK equal to the angle ECG, and the angle KAH to the angle GCF;
and let it be contrived that,
as EC is to CG, so is BA to AK,
and, as GC is to CF, so is KA to AH. [VI. 12]

Therefore also, *ex aequali*,
as EC is to CF, so is BA to AH. [V. 22]

Let the parallelogram HB and the solid AL be completed.

Now since, as EC is to CG, so is BA to AK,
and the sides about the equal angles ECG, BAK are thus proportional,

therefore the parallelogram *GE* is similar to the parallelogram *KB*.

For the same reason
the parallelogram *KH* is also similar to the parallelogram *GF*, and further *FE* to *HB*;
therefore three parallelograms of the solid *CD* are similar to three parallelograms of the solid *AL*.

But the former three are both equal and similar to the three opposite parallelograms,
and the latter three are both equal and similar to the three opposite parallelograms;
therefore the whole solid *CD* is similar to the whole solid *AL*. [XI. Def. 9]

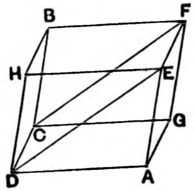
Therefore on the given straight line *AB* there has been described *AL* similar and similarly situated to the given parallelepipedal solid *CD*.

Q. E. F.

PROPOSITION 28.

If a parallelepipedal solid be cut by a plane through the diagonals of the opposite planes, the solid will be bisected by the plane.

For let the parallelepipedal solid *AB* be cut by the plane *CDEF* through the diagonals *CF*, *DE* of opposite planes;
I say that the solid *AB* will be bisected by the plane *CDEF*.



For, since the triangle *CGF* is equal to the triangle *CFB*, [I. 34]
and *ADE* to *DEH*,
while the parallelogram *CA* is also equal to the parallelogram *EB*, for they are opposite,
and *GE* to *CH*,
therefore the prism contained by the two triangles *CGF*, *ADE* and the three parallelograms *GE*, *AC*, *CE* is also equal to the prism contained by the two triangles *CFB*, *DEH* and the three parallelograms *CH*, *BE*, *CE*;

for they are contained by planes equal both in multitude and in magnitude. [XI. Def. 10]

Hence the whole solid *AB* is bisected by the plane *CDEF*.

Q. E. D.

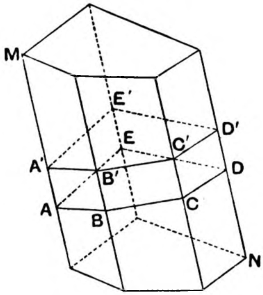
Simson properly observes that it ought to be proved that the diagonals of two opposite faces *are* in one plane, before we speak of drawing a plane through them. Clavius supplied the proof, which is of course simple enough.

Since *EF*, *CD* are both parallel to *AG* or *BH*, they are parallel to one another.

Consequently a plane can be drawn through *CD*, *EF* and the diagonals *DE*, *CF* are in that plane [XI. 7]. Moreover *CD*, *EF* are equal as well as parallel; so that *CF*, *DE* are also equal and parallel.

Simson does not, however, seem to have noticed a more serious difficulty. The two prisms are shown by Euclid to be contained by equal faces—the faces are in fact equal and similar—and Euclid then infers at once that the prisms are *equal*. But they are not equal in the only sense in which we have, at present, a right to speak of solids being equal, namely in the sense that they can be *applied*, the one to the other. They cannot be so applied because the faces, though equal respectively, are not *similarly arranged*; consequently the prisms are *symmetrical*, and it ought to be proved that they are, though not equal *and similar*, equal in content, or *equivalent*, as Legendre has it.

Legendre addressed himself to proving that the two prisms are equivalent, and his method has been adopted, though his name is not mentioned, by Schultze and Sevenoak and by Holgate. Certain preliminary propositions are necessary.



1. *The sections of a prism made by parallel planes cutting all the lateral edges are equal polygons.*

Suppose a prism *MN* cut by parallel planes which make sections *ABCDE*, *A'B'C'D'E'*.

Now *AB*, *BC*, *CD*, ... are respectively parallel to *A'B'*, *B'C'*, *C'D'*, ... [XI. 16]

Therefore the angles *ABC*, *BCD*, ... are equal to the angles *A'B'C'*, *B'C'D'*, ... respectively. [XI. 10]

Also *AB*, *BC*, *CD*, ... are respectively equal to *A'B'*, *B'C'*, *C'D'*, ... [I. 34]

Thus the polygons *ABCDE*, *A'B'C'D'E'* are equilateral and equiangular to one another.

2. *Two prisms are equal when they have a solid angle in each contained by three faces equal each to each and similarly arranged.*

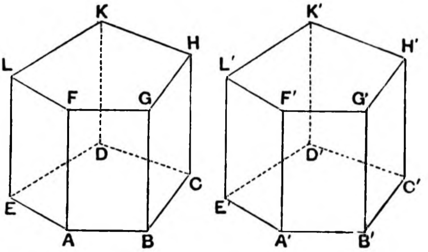
Let the faces *ABCDE*, *AG*, *AL* be equal and similarly placed to the faces *A'B'C'D'E'*, *A'G'*, *A'L'*.

Since the three plane angles at *A*, *A'* are equal respectively and are similarly placed, the trihedral angle at *A* is equal to the trihedral angle at *A'*. [(3) in note to XI. 21]

Place the trihedral angle at A on that at A'.

Then the face $ABCDE$ coincides with the face $A'B'C'D'E'$, the face AG with the face $A'G'$, and the face AL with the face $A'L'$.

The point C falls on C' and D on D'.



Since the lateral edges of a prism are parallel, CH will fall an $C'H'$, and DK on $D'K'$.

And the points F, G, L coincide respectively with F', G', L', so that

the planes GK, $G'K'$ coincide.

Hence H, K coincide with H', K' respectively.

Thus the prisms coincide throughout and are equal.

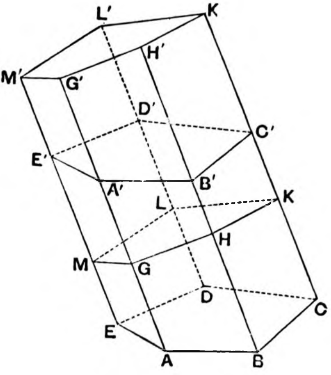
In the same way we can prove that two *truncated* prisms with three faces forming a solid angle related to one another as in the above proposition are identically equal.

In particular,

Cor. *Two right prisms having equal bases and equal heights are equal.*

3. *An oblique prism is equivalent to a right prism whose base is a right section of the oblique prism and whose height is equal to a lateral edge of the oblique prism.*

Suppose GL to be a right section of the oblique prism AD', and let GL be a right prism on GL as base and with height equal to a lateral edge of AD'.



Now the lateral edges of GL' are equal to the lateral edges of AD'.

Therefore $AG = A'G'$, $BH = B'H'$, $CK = C'K'$, etc.

Thus the faces AH, BK, CL are equal respectively to the faces $A'H'$, $B'K'$, $C'L'$.

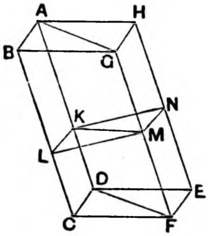
Therefore [by the proposition above]

(truncated prism AL) = (truncated prism $A'L'$).

Subtracting each from the whole solid AL', we see that

the prisms AD', GL' are equivalent.

Now suppose the parallelepiped of Euclid's proposition to be cut by the plane through *AG*, *DF*.



Let *KLMN* be a right section of the parallelepiped cutting the edges *AD*, *BC*, *GF*, *HE*.

Then *KLMN* is a parallelogram; and, if the diagonal *KM* be drawn,

$$\triangle KLM = \triangle MNK.$$

Now the prism of which the △s *ABG*, *DCF* are the bases is equal to the right prism on △*KLM* as base and of height *AD*.

Similarly the prism of which the △s *AGH*, *DFE* are the bases is equal to the right prism on △*MNK* as base and with height *AD*. [(3) above]

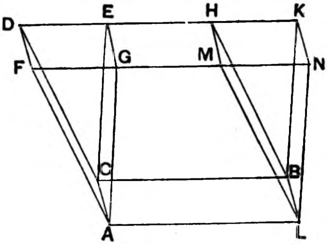
And the right prisms on △s *KLM*, *MNK* as bases and of equal height *AD* are equal. [(2), Cor. above]

Consequently the two prisms into which the parallelepiped is divided are *equivalent*.

PROPOSITION 29.

Parallelepipedal solids which are on the same base and of the same height, and in which the extremities of the sides which stand up are on the same straight lines, are equal to one another.

Let *CM*, *CN* be parallelepipedal solids on the same base *AB* and of the same height,



and let the extremities of their sides which stand up, namely *AG*, *AF*, *LM*, *LN*, *CD*, *CE*, *BH*, *BK*, be on the same straight lines *FN*, *DK*;

I say that the solid *CM* is equal to the solid *CN*.

For, since each of the figures *CH*, *CK* is a parallelogram, *CB* is equal to each of the straight lines *DH*, *EK*, [I. 34]

hence *DH* is also equal to *EK*.

Let *EH* be subtracted from each;

therefore the remainder *DE* is equal to the remainder *HK*.

Hence the triangle *DCE* is also equal to the triangle *HBK*, [I. 8, 4]

and the parallelogram *DG* to the parallelogram *HN*. [I. 36]

For the same reason
the triangle AFG is also equal to the triangle MLN.

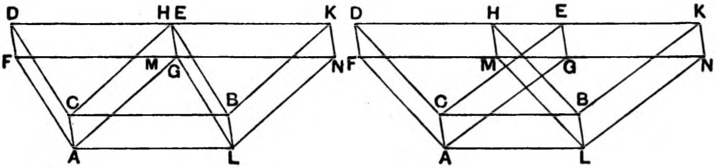
But the parallelogram CF is equal to the parallelogram BM, and CG to BN, for they are opposite;
therefore the prism contained by the two triangles AFG, DCE and the three parallelograms AD, DG, CG is equal to the prism contained by the two triangles MLN, HBK and the three parallelograms BM, HN, BN.

Let there be added to each the solid of which the parallelogram AB is the base and $GEHM$ its opposite;
therefore the whole parallelepipedal solid CM is equal to the whole parallelepipedal solid CN.

Therefore etc.

Q. E. D.

As usual, Euclid takes one case only and leaves the reader to prove for himself the two other possible cases shown in the subjoined figures. Euclid's proof holds with a very slight change in each case. With the first figure, the only difference is that the prism of which the △s GAL, ECB are the bases takes the place of "the solid of which the parallelogram AB is the base and $GEHM$ its opposite"; while with the second figure we have to subtract the prisms which are proved equal successively from the solid of which the parallelogram AB is the base and $FDKN$ its opposite.

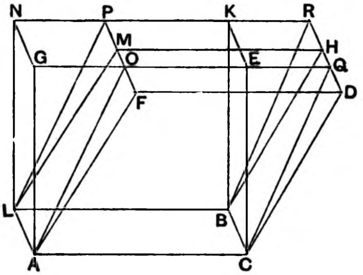


Simson, as usual, suspects mutilation by "some unskilful editor," but gives a curious reason why the case in which the two parallelograms opposite to AB have a side common ought not to have been omitted, namely that this case "is immediately deduced from the preceding 28th Prop which seems for this purpose to have been premised to the 29th." But, apart from the fact that Euclid's Prop. 28 does *not* prove the theorem which it enunciates (as we have seen), that theorem is not in the least necessary for the proof of this case of Prop. 29, as Euclid's proof applies to it perfectly well.

PROPOSITION 30.

Parallelepipedal solids which are on the same base and of the same height, and in which the extremities of the sides which stand up are not on the same straight lines, are equal to one another.

Let *CM*, *CN* be parallelepipedal solids on the same base *AB* and of the same height,



and let the extremities of their sides which stand up, namely *AF*, *AG*, *LM*, *LN*, *CD*, *CE*, *BH*, *BK*, not be on the same straight lines;

I say that the solid *CM* is equal to the solid *CN*.

For let *NK*, *DH* be produced and meet one another at *R*,

and further let *FM*, *GE* be produced to *P*, *Q*;

let *AO*, *LP*, *CQ*, *BR* be joined.

Then the solid *CM*, of which the parallelogram *ACBL* is the base, and *FDHM* its opposite, is equal to the solid *CP*, of which the parallelogram *ACBL* is the base, and *OQRP* its opposite;

for they are on the same base *ACBL* and of the same height, and the extremities of their sides which stand up, namely *AF*, *AO*, *LM*, *LP*, *CD*, *CQ*, *BH*, *BR*, are on the same straight lines *FP*, *DR*. [XI. 29]

But the solid *CP*, of which the parallelogram *ACBL* is the base, and *OQRP* its opposite, is equal to the solid *CN*, of which the parallelogram *ACBL* is the base and *GEKN* its opposite;

for they are again on the same base *ACBL* and of the same height, and the extremities of their sides which stand up, namely *AG*, *AO*, *CE*, *CQ*, *LN*, *LP*, *BK*, *BR*, are on the same straight lines *GQ*, *NR*.

Hence the solid *CM* is also equal to the solid *CN*.

Therefore etc.

Q. E. D.

This proposition completes the proof of the theorem that

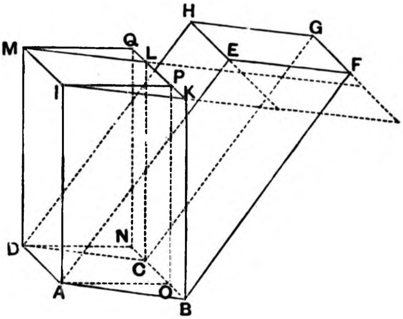
Two parallelepipeds on the same base and of the same height are equivalent.

Legendre deduced the useful theorem that

Every parallelepiped can be changed into an equivalent rectangular parallelepiped having the same height and an equivalent base.

For suppose we have a parallelepiped on the base *ABCD* with *EFGH* for the opposite face.

Draw AI, BK, CL, DM perpendicular to the plane through $EFGH$ and all equal to the height of the parallelepiped AG. Then, on joining IK, KL, LM, MI, we have a parallelepiped equivalent to the original one and having its lateral faces AK, BL, CM, DI rectangles.

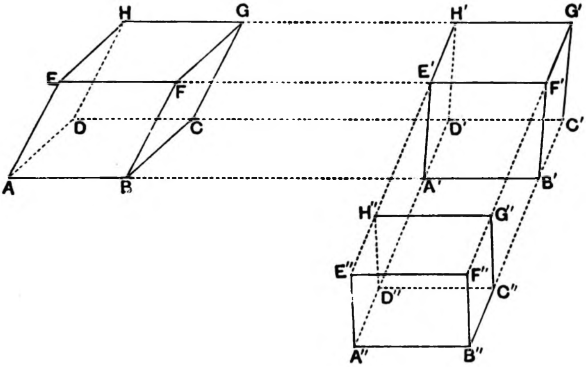


If $ABCD$ is not a rectangle, draw AO, DN in the plane AC perpendicular to BC, and IP, MQ in the plane IL perpendicular to KL.

Joining OP, NQ, we have a *rectangular* parallelepiped on $AOND$ as base which is equivalent to the parallelepiped with $ABCD$ as base and $IKLM$ as opposite face, since we may regard these parallelepipeds as being on the same base $ADMI$ and of the same height (AO).

That is, a rectangular parallelepiped has been constructed which is equivalent to the given parallelepiped and has (1) the same height, (2) an equivalent base.

The American text-books which I have quoted adopt a somewhat different construction shown in the subjoined figure.



The edges AB, DC, EF, HG of the original parallelepiped are produced and cut at right angles by two parallel planes at a distance apart $A'B'$ equal to AB.

Thus a parallelepiped is formed in which all the faces are rectangles except $A'H'$, $B'G'$.

Next produce $D'A'$, $C'B'$, $G'F'$, $H'E'$ and cut them perpendicularly by two parallel planes at a distance apart $B''C''$ equal to $B'C'$.

The points of section determine a *rectangular* parallelepiped.

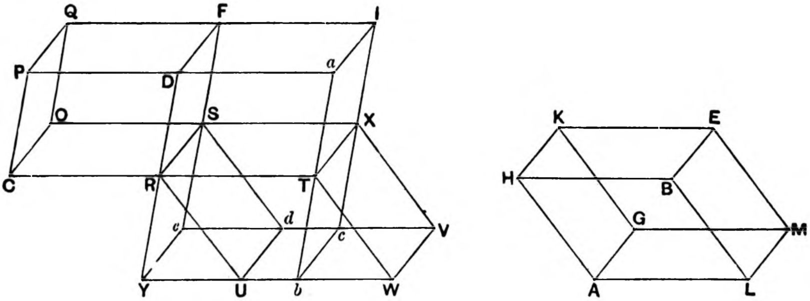
The equivalence of the three parallelepipeds is proved, not by Eucl. XI. 29, 30, but by the proposition about a right section of a prism given above in the note to XI. 28 (3 in that note).

PROPOSITION 31.

Parallelepipedal solids which are on equal bases and of the same height are equal to one another.

Let the parallelepipedal solids *AE*, *CF*, of the same height, be on equal bases *AB*, *CD*.

I say that the solid *AE* is equal to the solid *CF*.



First, let the sides which stand up, *HK*, *BE*, *AG*, *LM*, *PQ*, *DF*, *CO*, *RS*, be at right angles to the bases *AB*, *CD*;

let the straight line *RT* be produced in a straight line with *CR*;

on the straight line *RT*, and at the point *R* on it, let the angle *TRU* be constructed equal to the angle *ALB*, [I. 23]

let *RT* be made equal to *AL*, and *RU* equal to *LB*,

and let the base *RW* and the solid *XU* be completed.

Now, since the two sides *TR*, *RU* are equal to the two sides *AL*, *LB*,

and they contain equal angles,

therefore the parallelogram *RW* is equal and similar to the parallelogram *HL*.

Since again *AL* is equal to *RT*, and *LM* to *RS*,

and they contain right angles,

therefore the parallelogram *RX* is equal and similar to the parallelogram *AM*.

For the same reason
LE is also equal and similar to *SU*;
therefore three parallelograms of the solid *AE* are equal and similar to three parallelograms of the solid *XU*.

But the former three are equal and similar to the three opposite, and the latter three to the three opposite; [XI. 24]
therefore the whole parallelepipedal solid *AE* is equal to the whole parallelepipedal solid *XU*. [XI. Def. 10]

Let *DR*, *WU* be drawn through and meet one another at *Y*,
let *aTb* be drawn through *T* parallel to *DY*,
let *PD* be produced to *a*,
and let the solids *YX*, *RI* be completed.

Then the solid *XY*, of which the parallelogram *RX* is the base and *Yc* its opposite, is equal to the solid *XU* of which the parallelogram *RX* is the base and *UV* its opposite,
for they are on the same base *RX* and of the same height, and the extremities of their sides which stand up, namely *RY*, *RU*, *Tb*, *TW*, *Se*, *Sd*, *Xc*, *XV*, are on the same straight lines *YW*, *eV*. [XI. 29]

But the solid *XU* is equal to *AE*:
therefore the solid *XY* is also equal to the solid *AE*.

And, since the parallelogram *RUWT* is equal to the parallelogram *YT*
for they are on the same base *RT* and in the same parallels *RT*, *YW*, [I. 35]
while *RUWT* is equal to *CD*, since it is also equal to *AB*,
therefore the parallelogram *YT* is also equal to *CD*.

But *DT* is another parallelogram;
therefore, as the base *CD* is to *DT*, so is *YT* to *DT*. [V. 7]

And, since the parallelepipedal solid *CI* has been cut by the plane *RF* which is parallel to opposite planes,
as the base *CD* is to the base *DT*, so is the solid *CF* to the solid *RI*. [XI. 25]

For the same reason,

since the parallelepipedal solid *YI* has been cut by the plane *RX* which is parallel to opposite planes,

as the base *YT* is to the base *TD*, so is the solid *YX* to the solid *RI*. [XI. 25]

But, as the base *CD* is to *DT*, so is *YT* to *DT*;

therefore also, as the solid *CF* is to the solid *RI*, so is the solid *YX* to *RI*. [V. 11]

Therefore each of the solids *CF*, *YX* has to *RI* the same ratio;

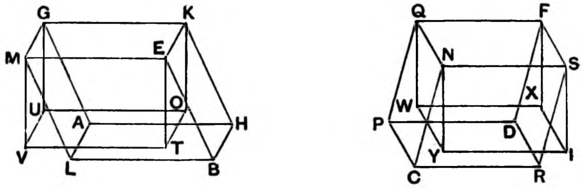
therefore the solid *CF* is equal to the solid *YX*. [V. 9]

But *YX* was proved equal to *AE*;

therefore *AE* is also equal to *CF*.

Next, let the sides standing up, *AG*, *HK*, *BE*, *LM*, *CN*, *PQ*, *DF*, *RS*, not be at right angles to the bases *AB*, *CD*;

I say again that the solid *AE* is equal to the solid *CF*.



For from the points *K*, *E*, *G*, *M*, *Q*, *F*, *N*, *S* let *KO*, *ET*, *GU*, *MV*, *QW*, *FX*, *NY*, *SI* be drawn perpendicular to the plane of reference, and let them meet the plane at the points *O*, *T*, *U*, *V*, *W*, *X*, *Y*, *I*,

and let *OT*, *OU*, *UV*, *TV*, *WX*, *WY*, *YI*, *IX* be joined.

Then the solid *KV* is equal to the solid *QI*,

for they are on the equal bases *KM*, *QS* and of the same height, and their sides which stand up are at right angles to their bases. [First part of this Prop.]

But the solid *KV* is equal to the solid *AE*,

and *QI* to *CF*;

for they are on the same base and of the same height, while the extremities of their sides which stand up are not on the same straight lines. [XI. 30]

Therefore the solid AE is also equal to the solid CF.

Therefore etc.

Q. E. D.

It is interesting to observe that, in the figure of this proposition, the bases are represented as lying "in the plane of the paper," as it were, and the third dimension as "standing up" from that plane. The figure is that of the manuscript P slightly corrected as regards the solid AE.

Nothing could well be more ingenious than the proof of this proposition, which recalls the brilliant proposition I. 44 and the proofs of VI. 14 and 23.

As the proof occupies considerable space in the text, it will no doubt be well to give a summary.

I. First, suppose that the edges terminating at the angular points of the bases are *perpendicular* to the bases.

AB, CD being the bases, Euclid constructs a solid identically equal to AE (he might simply have *moved* AE itself), placing it so that RS is the edge corresponding to HK ($RS = HK$ because the heights are equal), and the face RX corresponding to HE is in the plane of CS.

The faces CD, RW are in one plane because both are perpendicular to RS. Thus DR, WU meet, if produced, in Y say.

Complete the parallelograms YT, DT and the solids YX, FT.

Then $(\text{solid } YX) = (\text{solid } UX)$,

because they are on the same base ST and of the same height. [XI. 29]

Also, CI, YI being parallelepipeds cut by planes RF, RX parallel to pairs of opposite faces respectively,

$(\text{solid } CF) : (\text{solid } RI) = \square\, CD : \square\, DT$, [XI. 25]

and $(\text{solid } YX) : (\text{solid } RI) = \square\, YT : \square\, DT$.

But [I. 35] $\square\, YT = \square\, UT$

$= \square\, AB$

$= \square\, CD$, by hypothesis.

Therefore $(\text{solid } CF) = (\text{solid } YX)$

$= (\text{solid } UX)$

$= (\text{solid } AE)$.

II. If the edges terminating at the base are *not* perpendicular to it, turn each solid into an equivalent one on the same base with edges perpendicular to it (by drawing four perpendiculars from the angular points of the base to the plane of the opposite face). (XI. 29, 30 prove the equivalence.)

Then the equivalent solids are equal, by Part I.; so that the original solids are also equal.

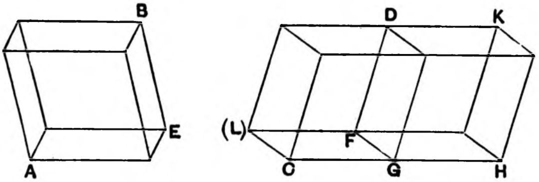
Simson observes that Euclid has made no mention of the case in which the bases of the two solids are *equiangular*, and he prefixes this case to Part I. in the text. This is surely unnecessary, as Part I. covers it well enough: the only difference in the figure is that UW would coincide with Yb and dV with ec.

Simson further remarks that in the demonstration of Part II. it is not proved that the new solids constructed in the manner described *are* parallelepipeds. The proof is, however, so simple that it scarcely needed insertion into the text. He is correct in his remark that the words "while the extremities of their sides which stand up are not on the same straight lines" just before the end of the proposition would be better absent, since they *may* be "on the same straight lines."

PROPOSITION 32.

Parallelepipedal solids which are of the same height are to one another as their bases.

Let *AB*, *CD* be parallelepipedal solids of the same height; I say that the parallelepipedal solids *AB*, *CD* are to one another as their bases, that is, that, as the base *AE* is to the base *CF*, so is the solid *AB* to the solid *CD*.



For let *FH* equal to *AE* be applied to *FG*, [I. 45]
and on *FH* as base, and with the same height as that of *CD*, let the parallelepipedal solid *GK* be completed.

Then the solid *AB* is equal to the solid *GK*;
for they are on equal bases *AE*, *FH* and of the same height. [XI. 31]

And, since the parallelepipedal solid *CK* is cut by the plane *DG* which is parallel to opposite planes,
therefore, as the base *CF* is to the base *FH*, so is the solid *CD* to the solid *DH*. [XI. 25]

But the base *FH* is equal to the base *AE*,
and the solid *GK* to the solid *AB*;
therefore also, as the base *AE* is to the base *CF*, so is the solid *AB* to the solid *CD*.

Therefore etc.

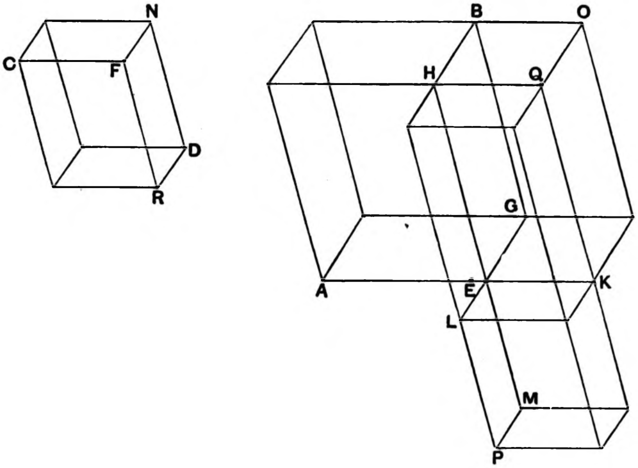
Q. E. D.

As Clavius observed, Euclid should have said, in applying the parallelogram *FH* to *FG*, that it should be applied "*in the angle* FGH *equal to the angle* LCG." Simson is however, I think, hypercritical when he states as regards the completion of the solid *GK* that it ought to be said, "complete the solid of which the base is *FH*, ***and one of its insisting straight lines is FD.***" Surely, when we have two faces *DG*, *FH* meeting in an edge, to say "***complete*** the solid" is quite sufficient, though the words "on *FH* as base" might perhaps as well be left out. The same "completion" of a parallelepipedal solid occurs in XI. 31 and 33.

PROPOSITION 33.

Similar parallelepipedal solids are to one another in the triplicate ratio of their corresponding sides.

Let *AB*, *CD* be similar parallelepipedal solids,
and let *AE* be the side corresponding to *CF*;
I say that the solid *AB* has to the solid *CD* the ratio triplicate of that which *AE* has to *CF*.



For let *EK*, *EL*, *EM* be produced in a straight line with *AE*, *GE*, *HE*,
let *EK* be made equal to *CF*, *EL* equal to *FN*, and further *EM* equal to *FR*,
and let the parallelogram *KL* and the solid *KP* be completed.

Now, since the two sides *KE*, *EL* are equal to the two sides *CF*, *FN*,
while the angle *KEL* is also equal to the angle *CFN*, inasmuch as the angle *AEG* is also equal to the angle *CFN* because of the similarity of the solids *AB*, *CD*,

therefore the parallelogram *KL* is equal < and similar > to the parallelogram *CN*.

For the same reason
the parallelogram *KM* is also equal and similar to *CR*,
and further *EP* to *DF*;
therefore three parallelograms of the solid *KP* are equal and similar to three parallelograms of the solid *CD*.

But the former three parallelograms are equal and similar to their opposites, and the latter three to their opposites; [XI. 24]
therefore the whole solid *KP* is equal and similar to the whole solid *CD*. [XI. Def. 10]

Let the parallelogram *GK* be completed,
and on the parallelograms *GK*, *KL* as bases, and with the same height as that of *AB*, let the solids *EO*, *LQ* be completed.

Then since; owing to the similarity of the solids *AB*, *CD*,
as *AE* is to *CF*, so is *EG* to *FN*, and *EH* to *FR*,
while *CF* is equal to *EK*, *FN* to *EL*, and *FR* to *EM*,
therefore, as *AE* is to *EK*, so is *GE* to *EL*, and *HE* to *EM*.

But, as *AE* is to *EK*, so is *AG* to the parallelogram *GK*,
as *GE* is to *EL*, so is *GK* to *KL*,
and, as *HE* is to *EM*, so is *QE* to *KM*; [VI. 1]
therefore also, as the parallelogram *AG* is to *GK*, so is *GK* to *KL*, and *QE* to *KM*.

But, as *AG* is to *GK*, so is the solid *AB* to the solid *EO*,
as *GK* is to *KL*, so is the solid *OE* to the solid *QL*,
and, as *QE* is to *KM*, so is the solid *QL* to the solid *KP*; [XI. 32]
therefore also, as the solid *AB* is to *EO*, so is *EO* to *QL*, and *QL* to *KP*.

But, if four magnitudes be continuously proportional, the first has to the fourth the ratio triplicate of that which it has to the second; [V. Def. 10]
therefore the solid *AB* has to *KP* the ratio triplicate of that which *AB* has to *EO*.

But, as *AB* is to *EO*, so is the parallelogram *AG* to *GK*, and the straight line *AE* to *EK* [VI. 1];

hence the solid *AB* has also to *KP* the ratio triplicate of that which *AE* has to *EK*.

But the solid *KP* is equal to the solid *CD*,
and the straight line *EK* to *CF*;
therefore the solid *AB* has also to the solid *CD* the ratio triplicate of that which the corresponding side of it, *AE*, has to the corresponding side *CF*.

Therefore etc.

Q. E. D.

PORISM. From this it is manifest that, if four straight lines be < continuously > proportional, as the first is to the fourth, so will a parallelepipedal solid on the first be to the similar and similarly described parallelepipedal solid on the second, inasmuch as the first has to the fourth the ratio triplicate of that which it has to the second.

The proof may be summarised as follows.

The three edges *AE*, *GE*, *HE* of the parallelepiped *AB* which meet at *E*, the vertex corresponding to *R* in the other parallelepiped, are produced, and lengths *EK*, *EL*, *EM* are marked off equal respectively to the edges *CF*, *FN*, *FR* of *CD*.

The parallelograms and solids are then completed as shown in the figure.

Euclid first shows that the solid *CD* and the new solid *PK* are equal and similar according to the criterion in XI. Def. 10, viz. that they are contained by the same number of equal and similar planes. (They are arranged in the same order, and it would be easy to prove equality by proving the equality of a pair of solid angles and then applying one solid to the other.)

We have now, by hypothesis,

$$AE : CF = EG : FN = EH : FR;$$

that is,

$$AE : EK = EG : EL = EH : EM.$$

But

$$AE : EK = \text{▱}\, AG : \text{▱}\, GK, \qquad \text{[VI. 1]}$$

$$EG : EL = \text{▱}\, GK : \text{▱}\, KL,$$

$$EH : EM = \text{▱}\, HK : \text{▱}\, KM.$$

Again, by XI. 25 or 32,

$$\text{▱}\, AG : \text{▱}\, GK = (\text{solid } AB) : (\text{solid } EO),$$

$$\text{▱}\, GK : \text{▱}\, KL = (\text{solid } EO) : (\text{solid } QL),$$

$$\text{▱}\, HK : \text{▱}\, KM = (\text{solid } QL) : (\text{solid } KP).$$

Therefore

$$(\text{solid } AB) : (\text{solid } EO) = (\text{solid } EO) : (\text{solid } QL) = (\text{solid } QL) : (\text{solid } KP),$$

or the solid *AB* is to the solid *KP* (that is, *CD*) in the ratio triplicate of that which the solid *AB* has to the solid *EO*, i.e. the ratio triplicate of that which *AE* has to *EK* (or *CF*).

Heiberg doubts whether the Porism appended to this proposition is genuine.

Simson adds, as Prop. D, a useful theorem which we should have expected to find here, on the analogy of VI. 23 following VI. 19, 20, viz. that *Solid parallelepipeds contained by parallelograms equiangular to one another, each to each, that is, of which the solid angles are equal, each to each, have to one another the ratio compounded of the ratios of their sides.*

The proof follows the method of the proposition XI. 33, and we can use the same figure. In order to obtain one ratio between lines to represent the ratio compounded of the ratios of the sides, after the manner of VI. 23, we take any straight line a, and then determine three other straight lines b, c, d, such that

$$\begin{aligned} AE : CF &= a : b, \\ EG : FN &= b : c, \\ EH : FR &= c : d, \end{aligned}$$

whence $a : d$ represents the ratio compounded of the ratios of the sides.

We obtain, in the same manner as above,

$$\begin{aligned} (\text{solid } AB) : (\text{solid } EO) &= \text{▱}\, AG : \text{▱}\, GK = AE : EK = AE : CF \\ &= a : b, \\ (\text{solid } EO) : (\text{solid } QL) &= \text{▱}\, GK : \text{▱}\, KL = GE : EL = GE : FN \\ &= b : c, \\ (\text{solid } QL) : (\text{solid } KP) &= \text{▱}\, HK : \text{▱}\, KM = EH : EM = EH : FR \\ &= c : d, \end{aligned}$$

whence, by composition [V. 22],

$$(\text{solid } AB) : (\text{solid } KP) = a : d,$$

or

$$(\text{solid } AB) : (\text{solid } CD) = a : d.$$

PROPOSITION 34.

In equal parallelepipedal solids the bases are reciprocally proportional to the heights; and those parallelepipedal solids in which the bases are reciprocally proportional to the heights are equal.

Let AB, CD be equal parallelepipedal solids;

I say that in the parallelepipedal solids AB, CD the bases are reciprocally proportional to the heights,

that is, as the base EH is to the base NQ, so is the height of the solid CD to the height of the solid AB.

First, let the sides which stand up, namely AG, EF, LB, HK, CM, NO, PD, QR, be at right angles to their bases;

I say that, as the base EH is to the base NQ, so is CM to AG.

If now the base EH is equal to the base NQ,

while the solid AB is also equal to the solid CD,

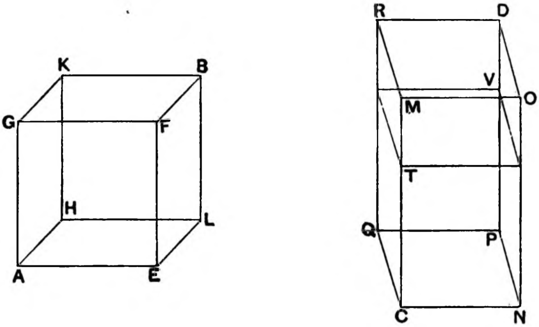
CM will also be equal to AG.

For parallelepipedal solids of the same height are to one another as the bases; [XI. 32]

and, as the base *EH* is to *NQ*, so will *CM* be to *AG*,

and it is manifest that in the parallelepipedal solids *AB*, *CD* the bases are reciprocally proportional to the heights.

Next, let the base *EH* not be equal to the base *NQ*, but let *EH* be greater.



Now the solid *AB* is equal to the solid *CD*;

therefore *CM* is also greater than *AG*.

Let then *CT* be made equal to *AG*,

and let the parallelepipedal solid *VC* be completed on *NQ* as base and with *CT* as height.

Now, since the solid *AB* is equal to the solid *CD*,

and *CV* is outside them,

while equals have to the same the same ratio, [V. 7]

therefore, as the solid *AB* is to the solid *CV*, so is the solid *CD* to the solid *CV*.

But, as the solid *AB* is to the solid *CV*, so is the base *EH* to the base *NQ*,

for the solids *AB*, *CV* are of equal height; [XI. 32]

and, as the solid *CD* is to the solid *CV*, so is the base *MQ* to the base *TQ* [XI. 25] and *CM* to *CT* [VI. 1];

therefore also, as the base *EH* is to the base *NQ*, so is *MC* to *CT*.

But *CT* is equal to *AG*;

therefore also, as the base *EH* is to the base *NQ*, so is *MC* to *AG*.

Therefore in the parallelepipedal solids AB, CD the bases are reciprocally proportional to the heights.

Again, in the parallelepipedal solids AB, CD let the bases be reciprocally proportional to the heights, that is, as the base EH is to the base NQ, so let the height of the solid CD be to the height of the solid AB;
I say that the solid AB is equal to the solid CD.

Let the sides which stand up be again at right angles to the bases.

Now, if the base EH is equal to the base NQ,
and, as the base EH is to the base NQ, so is the height of the solid CD to the height of the solid AB,
therefore the height of the solid CD is also equal to the height of the solid AB.

But parallelepipedal solids on equal bases and of the same height are equal to one another; [XI. 31]
therefore the solid AB is equal to the solid CD.

Next, let the base EH not be equal to the base NQ,
but let EH be greater;
therefore the height of the solid CD is also greater than the height of the solid AB,
that is, CM is greater than AG.

Let CT be again made equal to AG,
and let the solid CV be similarly completed.

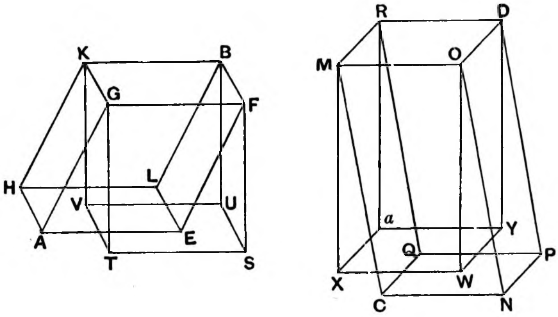
Since, as the base EH is to the base NQ, so is MC to AG,
while AG is equal to CT,
therefore, as the base EH is to the base NQ, so is CM to CT.

But, as the base EH is to the base NQ, so is the solid AB to the solid CV,
for the solids AB, CV are of equal height; [XI. 32]
and, as CM is to CT, so is the base MQ to the base QT [VI. 1]
and the solid CD to the solid CV. [XI. 25]

Therefore also, as the solid AB is to the solid CV, so is the solid CD to the solid CV;
therefore each of the solids AB, CD has to CV the same ratio.

Therefore the solid *AB* is equal to the solid *CD*. [v. 9]

Now let the sides which stand up, *FE*, *BL*, *GA*, *HK*, *ON*, *DP*, *MC*, *RQ*, not be at right angles to their bases;
let perpendiculars be drawn from the points *F*, *G*, *B*, *K*, *O*, *M*, *D*, *R* to the planes through *EH*, *NQ*, and let them meet the planes at *S*, *T*, *U*, *V*, *W*, *X*, *Y*, *a*,
and let the solids *FV*, *Oa* be completed;
I say that, in this case too, if the solids *AB*, *CD* are equal, the bases are reciprocally proportional to the heights, that is, as the base *EH* is to the base *NQ*, so is the height of the solid *CD* to the height of the solid *AB*.



Since the solid *AB* is equal to the solid *CD*,
while *AB* is equal to *BT*,
for they are on the same base *FK* and of the same height; [XI. 29, 30]
and the solid *CD* is equal to *DX*,
for they are again on the same base *RO* and of the same height; [*id.*]
therefore the solid *BT* is also equal to the solid *DX*.

Therefore, as the base *FK* is to the base *OR*, so is the height of the solid *DX* to the height of the solid *BT*. [Part I.]

But the base *FK* is equal to the base *EH*,
and the base *OR* to the base *NQ*;
therefore, as the base *EH* is to the base *NQ*, so is the height of the solid *DX* to the height of the solid *BT*.

But the solids *DX*, *BT* and the solids *DC*, *BA* have the same heights respectively;
therefore, as the base *EH* is to the base *NQ*, so is the height of the solid *DC* to the height of the solid *AB*.

Therefore in the parallelepipedal solids *AB*, *CD* the bases are reciprocally proportional to the heights.

Again, in the parallelepipedal solids *AB*, *CD* let the bases be reciprocally proportional to the heights,
that is, as the base *EH* is to the base *NQ*, so let the height of the solid *CD* be to the height of the solid *AB*;
I say that the solid *AB* is equal to the solid *CD*.

For, with the same construction,
since, as the base *EH* is to the base *NQ*, so is the height of the solid *CD* to the height of the solid *AB*,
while the base *EH* is equal to the base *FK*,
and *NQ* to *OR*,
therefore, as the base *FK* is to the base *OR*, so is the height of the solid *CD* to the height of the solid *AB*.

But the solids *AB*, *CD* and *BT*, *DX* have the same heights respectively;
therefore, as the base *FK* is to the base *OR*, so is the height of the solid *DX* to the height of the solid *BT*.

Therefore in the parallelepipedal solids *BT*, *DX* the bases are reciprocally proportional to the heights;
therefore the solid *BT* is equal to the solid *DX*. [Part I.]

But *BT* is equal to *BA*,
for they are on the same base *FK* and of the same height; [XI. 29, 30]
and the solid *DX* is equal to the solid *DC*. [*id.*]

Therefore the solid *AB* is also equal to the solid *CD*.

Q. E. D.

In this proposition Euclid makes two assumptions which require notice, (1) that, if two parallelepipeds are equal, and have equal bases, their heights are equal, and (2) that, if the bases of two equal parallelepipeds are unequal, that which has the lesser base has the greater height. In justification of the former statement Euclid says, according to what Heiberg holds to be the genuine reading, "for parallelepipedal solids of the same height are to one another as their bases" [XI. 32]. This apparently struck some very early editor as not being sufficient, and he added the explanation appearing in Simson's text, "For if, the bases *EH*, *NQ* being equal, the heights *AG*, *CM* were not equal, neither would the solid *AB* be equal to *CD*. But it is by hypothesis equal. Therefore the height *CM* is not unequal to the height *AG*; therefore it is equal." Then, it being perceived that there ought not to be two explanations, the genuine one was erased from the inferior MSS. While the interpolated explanation does not take us very far, the truth of the statement may be deduced with perhaps greater ease from XI. 31 than from XI. 32 quoted by Euclid. For, assuming one height greater than the other, while the bases are equal, we have only to cut from the higher solid so much as will make its height equal to that of the other. Then this *part* of the higher solid is equal to the whole of the other solid which is by hypothesis equal to the higher solid itself. That is, the whole is equal to its part: which is impossible.

The genuine text contains no explanation of the second assumption that, if the base *EH* be greater than the base *NQ*, while the solids are equal, the height *CM* is greater than the height *AG*; for the added words "for, if not, neither again will the solids *AB*, *CD* be equal; but they are equal by hypothesis" are no doubt interpolated. In this case the truth of the assumption is easily deduced from XI. 32 by *reductio ad absurdum*. If the height *CM* were *equal* to the height *AG*, the solid *AB* would be to the solid *CD* as the base *EH* is to the base *NQ*, i.e. as a greater to a less, so that the solids would not be equal, as they are by hypothesis. Again, if the height *CM* were *less* than the height *AG*, we could increase the height of *CD* till it was equal to that of *AB*, and it would then appear that *AB* is greater than the heightened solid and *a fortiori* greater than *CD*: which contradicts the hypothesis.

Clavius rather ingeniously puts the first assumption the other way, saying that, if the heights are equal in the equal parallelepipeds, the bases must be equal This follows *directly* from XI. 32, which proves that the parallelepipeds are to one another as their bases; though Clavius deduces it indirectly from XI. 31. The advantage of Clavius' alternative is that it makes the second assumption unnecessary. He merely says, if the *heights* be not equal, let *CM* be the greater, and then proceeds with Euclid's construction.

It is also to be observed that, when Euclid comes to the corresponding proposition for cones and cylinders [XII. 15], he begins by supposing the *heights* equal, inferring by XII. 11 (corresponding to XI. 32) that, the solids being equal, the bases are also equal, and then proceeds to the case where the heights are unequal without making any preliminary inference about the bases. The analogy then of XII. 15, and the fact that he quotes XI. 32 here (which directly proves that, if the solids are equal, and also their heights, their bases are also equal), make Clavius' form the more convenient to adopt.

The two assumptions being proved as above, the proposition can be put shortly as follows.

I. Suppose the edges terminating at the corners of the base to be *perpendicular* to it.

Then (*a*), if the base *EH* be equal to the base *NQ*, the parallelepipeds being also equal, the heights must be equal (converse of XI. 31), so that the bases are reciprocally proportional to the heights, the ratio of the bases and the ratio of the heights being both ratios of equality.

(*b*) If the base *EH* be greater than the base *NQ*, and consequently (by deduction from XI. 32) the height *CM* greater than the height *AG*, cut off *CT* from *CM* equal to *AG*, and draw the plane *TV* through *T* parallel to the base *NQ*, making the parallelepiped *CV*, with *CT* (= *AG*) for its height.

Then, since the solids *AB*, *CD* are equal,

$$(\text{solid } AB) : (\text{solid } CV) = (\text{solid } CD) : (\text{solid } CV). \qquad [\text{v. } 7]$$

But $(\text{solid } AB) : (\text{solid } CV) = \text{▱}\,HE : \text{▱}\,NQ$, [XI. 32]

and $(\text{solid } CD) : (\text{solid } CV) = \text{▱}\,MQ : \text{▱}\,TQ$ [XI. 25]

$= CM : CT.$ [VI. 1]

Therefore $\text{▱}\,HE : \text{▱}\,NQ = CM : CT$

$= CM : AG.$

Conversely (*a*), if the bases EH, NQ be equal and reciprocally proportional to the heights, the heights must be equal.

Consequently $(\text{solid } AB) = (\text{solid } CD)$. [XI. 31]

(*b*) If the bases EH, NQ be unequal, if, e.g. $\text{▱}\,EH > \text{▱}\,NQ$,

then, since $\text{▱}\,EH : \text{▱}\,NQ = CM : AG$,

$CM > AG.$

Make the same construction as before.

Then $\text{▱}\,EH : \text{▱}\,NQ = (\text{solid } AB) : (\text{solid } CV)$, [XI. 32]

and $CM : AG = CM : CT$

$= \text{▱}\,MQ : \text{▱}\,TQ$ [VI. 1]

$= (\text{solid } CD) : (\text{solid } CV).$ [XI. 25]

Therefore

$$(\text{solid } AB) : (\text{solid } CV) = (\text{solid } CD) : (\text{solid } CV),$$

whence $(\text{solid } AB) = \text{solid } CD.$ [V. 9]

II. Suppose that the edges terminating at the corners of the bases are *not* perpendicular to it.

Drop perpendiculars on the bases from the corners of the faces opposite to the bases.

We thus have two parallelepipeds equal to AB, CD respectively, since they are on the same bases FK, RO and of the same height respectively. [XI. 29, 30]

If then (1) the solid AB is equal to the solid CD,

$$(\text{solid } BT) = (\text{solid } DX),$$

and, by the first part of this proposition,

$$\text{▱}\,KF : \text{▱}\,OR = MX : GT,$$

or $\text{▱}\,HE : \text{▱}\,NQ = MX : GT.$

(2) If $\text{▱}\,HE : \text{▱}\,NQ = MX : GT$,

then $\text{▱}\,KF : \text{▱}\,OR = MX : GT$,

so that, by the first half of the proposition, the solids BT, DX are equal, and consequently

$$(\text{solid } AB) = (\text{solid } CD).$$

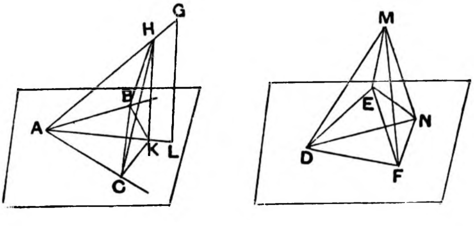
The text of the second part of the proposition four times contains, after the words "of the same height," the words "in which the sides which stand up are not on the same straight lines." As Simson observed, they are inept, as the extremities of the edges may or may not be "on the same straight lines"; cf. the similar words incorrectly inserted at the end of XI. 31.

Words purporting to quote the result of the first part of the proposition are also twice inserted; but they are rejected as unnecessary and as containing an absurd expression—"(solids) in which *the heights* are at right angles to their bases," as if the *heights* could be otherwise than perpendicular to the bases.

PROPOSITION 35.

If there be two equal plane angles, and on their vertices there be set up elevated straight lines containing equal angles with the original straight lines respectively, if on the elevated straight lines points be taken at random and perpendiculars be drawn from them to the planes in which the original angles are, and if from the points so arising in the planes straight lines be joined to the vertices of the original angles, they will contain, with the elevated straight lines, equal angles.

Let the angles *BAC*, *EDF* be two equal rectilineal angles, and from the points *A*, *D* let the elevated straight lines *AG*, *DM* be set up containing, with the original straight lines, equal angles respectively, namely, the angle *MDE* to the angle *GAB* and the angle *MDF* to the angle *GAC*,
let points *G*, *M* be taken at random on *AG*, *DM*,
let *GL*, *MN* be drawn from the points *G*, *M* perpendicular to the planes through *BA*, *AC* and *ED*, *DF*, and let them meet the planes at *L*, *N*,
and let *LA*, *ND* be joined;
I say that the angle *GAL* is equal to the angle *MDN*.



Let *AH* be made equal to *DM*,
and let *HK* be drawn through the point *H* parallel to *GL*.

But *GL* is perpendicular to the plane through *BA*, *AC*;
therefore *HK* is also perpendicular to the plane through *BA*, *AC*. [XI. 8]

From the points *K*, *N* let *KC*, *NF*, *KB*, *NE* be drawn perpendicular to the straight lines *AC*, *DF*, *AB*, *DE*,
and let *HC*, *CB*, *MF*, *FE* be joined.

Since the square on *HA* is equal to the squares on *HK*, *KA*,
and the squares on *KC*, *CA* are equal to the square on *KA*, [I. 47]
therefore the square on *HA* is also equal to the squares on *HK*, *KC*, *CA*.

But the square on *HC* is equal to the squares on *HK*, *KC*; [I. 47]
therefore the square on *HA* is equal to the squares on *HC*, *CA*.

Therefore the angle *HCA* is right. [I. 48]

For the same reason
the angle *DFM* is also right.

Therefore the angle *ACH* is equal to the angle *DFM*.

But the angle *HAC* is also equal to the angle *MDF*.

Therefore *MDF*, *HAC* are two triangles which have two angles equal to two angles respectively, and one side equal to one side, namely, that subtending one of the equal angles, that is, *HA* equal to *MD*;
therefore they will also have the remaining sides equal to the remaining sides respectively. [I. 26]

Therefore *AC* is equal to *DF*.

Similarly we can prove that *AB* is also equal to *DE*.

Since then *AC* is equal to *DF*, and *AB* to *DE*,
the two sides *CA*, *AB* are equal to the two sides *FD*, *DE*.

But the angle *CAB* is also equal to the angle *FDE*;
therefore the base *BC* is equal to the base *EF*, the triangle to the triangle, and the remaining angles to the remaining angles; [I. 4]
therefore the angle *ACB* is equal to the angle *DFE*.

But the right angle *ACK* is also equal to the right angle *DFN*;
therefore the remaining angle *BCK* is also equal to the remaining angle *EFN*.

For the same reason
the angle *CBK* is also equal to the angle *FEN*.

Therefore BCK, EFN are two triangles which have two angles equal to two angles respectively, and one side equal to one side, namely, that adjacent to the equal angles, that is, BC equal to EF;
therefore they will also have the remaining sides equal to the remaining sides. [I. 26]

Therefore CK is equal to FN.

But AC is also equal to DF;
therefore the two sides AC, CK are equal to the two sides DF, FN;
and they contain right angles.

Therefore the base AK is equal to the base DN. [I. 4]

And, since AH is equal to DM,
the square on AH is also equal to the square on DM.

But the squares on AK, KH are equal to the square on AH,
for the angle AKH is right; [I. 47]
and the squares on DN, NM are equal to the square on DM,
for the angle DNM is right; [I. 47]
therefore the squares on AK, KH are equal to the squares on DN, NM;
and of these the square on AK is equal to the square on DN;
therefore the remaining square on KH is equal to the square on NM;
therefore HK is equal to MN.

And, since the two sides HA, AK are equal to the two sides MD, DN respectively,
and the base HK was proved equal to the base MN,
therefore the angle HAK is equal to the angle MDN. [I. 8]

Therefore etc.

PORISM. From this it is manifest that, if there be two equal plane angles, and if there be set up on them elevated straight lines which are equal and contain equal angles with the original straight lines respectively, the perpendiculars drawn from their extremities to the planes in which are the original angles are equal to one another.

Q. E. D.

This proposition is required for the next, where it is necessary to know that, if in two equiangular parallelepipeds equal angles, one in each, be contained by three plane angles respectively, one of which is an angle of the parallelogram forming the *base* in one parallelepiped, while its equal is likewise in the *base* of the other, and the edges in which the two remaining angles forming the solid angles meet are *equal*, the parallelepipeds *are of the same height.*

Bearing in mind the definition of *the inclination of a straight line to a plane*, we might enunciate the proposition more shortly thus.

If there be two trihedral angles identically equal to one another, corresponding edges in each are equally inclined to the planes through the other two edges respectively.

The proof, which is necessarily somewhat long, may be summarised thus.

It is required to prove that the angles GAL, MDN in the figure are equal, G, M being any points on AG, DM, and GL, MN perpendicular to the planes BAC, EDF respectively.

If AH is made equal to DM, and HK is drawn in the plane GAL parallel to GL,

HK is also perpendicular to the plane BAC. [XI. 8]

Draw KB, KC perpendicular to AB, AC respectively and NE, NF perpendicular to DE, DF respectively, and complete the figures.

Now (1)
$$\left.\begin{aligned} HA^2 &= HK^2 + KA^2 \\ &= HK^2 + KC^2 + CA^2 \\ &= HC^2 + CA^2 \end{aligned}\right\} \qquad [\text{I. } 47]$$

Therefore $\angle HCA =$ a right angle.

Similarly $\angle MFD =$ a right angle.

(2) $\triangle$s HAC, MDF have therefore two angles equal and one side.

Therefore $\triangle HAC \equiv \triangle MDF$, and $AC = DF$. [I. 26]

(3) Similarly $\triangle HAB \equiv \triangle MDE$, and $AB = DE$.

(4) Hence $\triangle$s ABC, DEF are equal in all respects, so that $BC = EF$, and
$$\angle ABC = \angle DEF,$$
$$\angle ACB = \angle DFE.$$

(5) Therefore the complements of these angles are equal,

i.e. $\angle KBC = \angle NEF$,

and $\angle KCB = \angle NFE$.

(6) The $\triangle$s KBC, NEF have two angles equal and one side, and are therefore equal in all respects, so that
$$KB = NE,$$
$$KC = NF.$$

(7) The right-angled triangles KAC, NDF are equal in all respects, since $AC = DF$ [(2) above], $KC = NF$.

Consequently $AK = DN$.

(8) In $\triangle$s HAK, MDN,
$$\begin{aligned} HK^2 + KA^2 &= HA^2 \\ &= MD^2, \text{ by hypothesis,} \\ &= MN^2 + ND^2. \end{aligned}$$

Subtracting the equals KA^2, ND^2,
we have $$HK^2 = MN^2,$$
or $$HK = MN.$$

(9) $\triangle$s HAK, MDN are now equal in all respects, by I. 8 and I. 4, and therefore
$$\angle HAK = \angle MDN.$$

The Porism is merely a statement of the result arrived at in (8).

Legendre uses, practically, the construction and argument of this proposition to prove the theorem given under (3) of the note on XI. 21 above that *In two equal trihedral angles, corresponding pairs of face angles include equal dihedral angles.* This fact is readily deduced from the above proposition.

Since [(1)] HC, KC are both perpendicular to AC, and MF, NF both perpendicular to DF, the angles HCK, MFN are the measures of the *dihedral* angles between the planes HAC, BAC, and MDF, EDF respectively. [XI. Def. 6]

By (6), $$KC = NF,$$
and, by (8), $$HK = MN,$$
while the angles HKC, MNF, both being right, are equal.

Consequently the $\triangle$s HCK, MFN are equal in all respects, [I. 4]
so that $$\angle HCK = \angle MFN.$$

Simson substituted a different proof of (1) in the above summary, as follows.

Since HK is perpendicular to the plane BAC, the plane HBK, passing through HK, is also perpendicular to the plane BAC. [XI. 18]

And AB, being drawn in the plane BAC perpendicular to BK, the common section of the planes HBK, BAC, is perpendicular to the plane HBK [XI. Def. 4], and is therefore perpendicular to every straight line meeting it in that plane [XI. Def. 3].

Hence the angle ABH is a right angle.

I think Euclid's proof much preferable to this with its references to definitions which are more of the nature of theorems.

PROPOSITION 36.

If three straight lines be proportional, the parallelepipedal solid formed out of the three is equal to the parallelepipedal solid on the mean which is equilateral, but equiangular with the aforesaid solid.

Let A, B, C be three straight lines in proportion, so that, as A is to B, so is B to C;

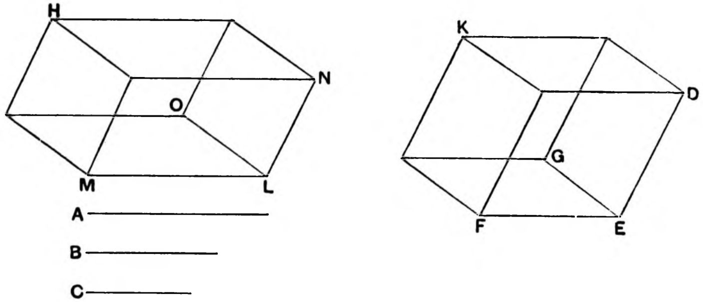
I say that the solid formed out of A, B, C is equal to the solid on B which is equilateral, but equiangular with the aforesaid solid.

Let there be set out the solid angle at E contained by the angles DEG, GEF, FED, let each of the straight lines *DE*, *GE*, *EF* be made equal to *B*, and let the parallelepipedal solid *EK* be completed,

let *LM* be made equal to *A*,

and on the straight line *LM*, and at the point *L* on it, let there be constructed a solid angle equal to the solid angle at *E*, namely that contained by *NLO*, *OLM*, *MLN*;

let *LO* be made equal to *B*, and *LN* equal to *C*.



Now, since, as *A* is to *B*, so is *B* to *C*,

while *A* is equal to *LM*, *B* to each of the straight lines *LO*, *ED*, and *C* to *LN*,

therefore, as *LM* is to *EF*, so is *DE* to *LN*.

Thus the sides about the equal angles *NLM*, *DEF* are reciprocally proportional;

therefore the parallelogram *MN* is equal to the parallelogram *DF*. [VI. 14]

And, since the angles *DEF*, *NLM* are two plane rectilineal angles, and on them the elevated straight lines *LO*, *EG* are set up which are equal to one another and contain equal angles with the original straight lines respectively,

therefore the perpendiculars drawn from the points *G*, *O* to the planes through *NL*, *LM* and *DE*, *EF* are equal to one another; [XI. 35, Por.]

hence the solids *LH*, *EK* are of the same height.

But parallelepipedal solids on equal bases and of the same height are equal to one another; [XI. 31]

therefore the solid *HL* is equal to the solid *EK*.

And *LH* is the solid formed out of *A*, *B*, *C*, and *EK* the solid on *B*;

therefore the parallelepipedal solid formed out of A, B, C is equal to the solid on B which is equilateral, but equiangular with the aforesaid solid.

Q. E. D.

The edges of the parallelepiped HL being respectively equal to A, B, C, and those of the equiangular parallelepiped KE being all equal to B, we regard MN (*not* containing the edge OL equal to B) as the base of the first parallelepiped, and consequently FD, equiangular to MN, as the base of KE.

Then the solids have the same height. [XI. 35, Por.]

Hence (solid HL) : (solid KE) = ▱ MN : ▱ FD. [XI. 32]

But, since A, B, C are in continued proportion,

$$A : B = B : C,$$

or

$$LM : EF = DE : LN.$$

Thus the sides of the equiangular ▱s MN, FD are reciprocally proportional, whence

▱ MN = ▱ FD, [VI. 14]

and therefore (solid HL) = (solid KE).

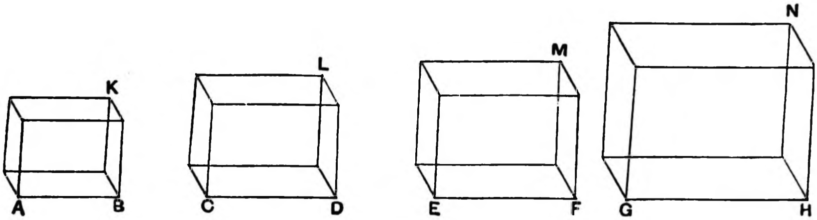
PROPOSITION 37.

If four straight lines be proportional, the parallelepipedal solids on them which are similar and similarly described will also be proportional; and, if the parallelepipedal solids on them which are similar and similarly described be proportional, the straight lines will themselves also be proportional.

Let AB, CD, EF, GH be four straight lines in proportion, so that, as AB is to CD, so is EF to GH;

and let there be described on AB, CD, EF, GH the similar and similarly situated parallelepipedal solids KA, LC, ME, NG;

I say that, as KA is to LC, so is ME to NG.



For, since the parallelepipedal solid KA is similar to LC, therefore KA has to LC the ratio triplicate of that which AB has to CD. [XI. 33]

For the same reason

ME also has to *NG* the ratio triplicate of that which *EF* has to *GH*. [*id.*]

And, as *AB* is to *CD*, so is *EF* to *GH*.

Therefore also, as *AK* is to *LC*, so is *ME* to *NG*.

Next, as the solid *AK* is to the solid *LC*, so let the solid *ME* be to the solid *NG*;

I say that, as the straight line *AB* is to *CD*, so is *EF* to *GH*.

For since, again, *KA* has to *LC* the ratio triplicate of that which *AB* has to *CD*, [XI. 33]

and *ME* also has to *NG* the ratio triplicate of that which *EF* has to *GH*, [*id.*]

and, as *KA* is to *LC*, so is *ME* to *NG*,

therefore also, as *AB* is to *CD*, so is *EF* to *GH*.

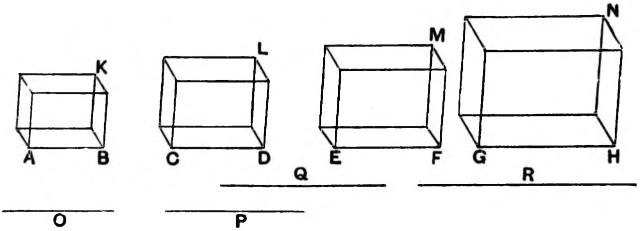
Therefore etc.

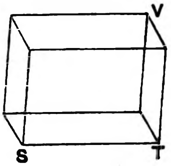
Q. E. D.

In this proposition it is assumed that, if two ratios be equal, the ratio triplicate of one is equal to the ratio triplicate of the other and, conversely, that, if ratios which are the triplicate of two other ratios are equal, those other ratios are themselves equal.

To avoid the necessity for these assumptions Simson adopts the alternative proof found in the MS. which Heiberg calls b, and also adopted by Clavius, who, however, gives Euclid's proof as well, attributing it to Theon. The alternative proof proceeds after the manner of VI. 22, thus.

Make *AB*, *CD*, *O*, *P* continuous proportionals, and also *EF*, *GH*, *Q*, *R*.





I. Then, since

$$AB : CD = EF : GH,$$

we have, *ex aequali*,

$$AB : P = EF : R. \qquad [\text{v. } 22]$$

But (solid AK) : (solid CL) = $AB : P$,

[XI. 33 and Por.]

and (solid EM) : (solid GN) = $EF : R$.

Therefore

$$(\text{solid } AK) : (\text{solid } CL) = (\text{solid } EM) : (\text{solid } GN)$$

II. If the *solids* are proportional, take ST such that

$$AB : CD = EF : ST,$$

and on ST describe the parallelepiped SV similar and similarly situated to either of the parallelepipeds EM, GN.

Then, by the first part,

$$(\text{solid } AK) : (\text{solid } CL) = (\text{solid } EM) : (\text{solid } SV),$$

whence it follows that

$$(\text{solid } GN) = (\text{solid } SV).$$

But these solids are similar and similarly situated;

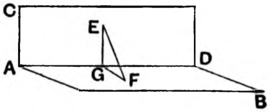
therefore their faces are similar and equal; [xi. Def. 10]

therefore the corresponding sides GH, ST are equal.

[For this inference cf. note on vi. 22. The equality of GH, ST may readily be proved by application of the two parallelepipeds to one another, since, being similar, they are equiangular.]

Hence $AB : CD = EF : GH$.

The text of the MSS. has here a proposition which is as badly placed as it is unnecessary. *If a plane be at right angles to a plane, and from any one of the points in one of the planes a perpendicular be drawn to the other plane, the perpendicular so drawn will fall on the common section of the planes.* It is of the nature of a lemma to xii. 17, where alone the fact is made use of. Heiberg observes that it is omitted in b and that the copyist of P knew other texts which did not contain it. From these facts it is fairly concluded that the proposition was interpolated. The truth of it is of course immediately obvious by *reductio ad absurdum*. Let the plane CAD be perpendicular to the plane AB, and let a perpendicular be drawn to the latter from any point E in the former.



If it does not fall on AD, the common section, let it meet the plane AB in F.

Draw FG in AB perpendicular to AD, and join EG.

Then FG is perpendicular to the plane CAD [xi. Def. 4], and therefore to GE [xi. Def. 3]. Therefore $\angle EGF$ is right.

Also, since EF is perpendicular to AB,

the angle EFG is right.

That is, the triangle EGF has two right angles:

which is impossible.

PROPOSITION 38.

If the sides of the opposite planes of a cube be bisected, and planes be carried through the points of section, the common section of the planes and the diameter of the cube bisect one another.

For let the sides of the opposite planes CF, AH of the cube AF be bisected at the points K, L, M, N, O, Q, P, R,

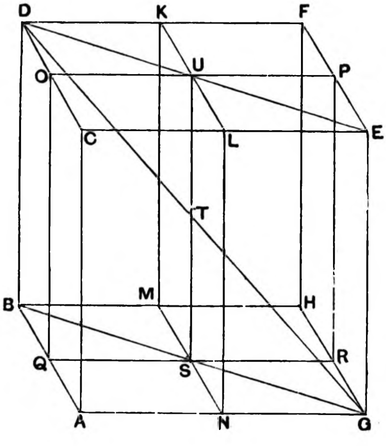
and through the points of section let the planes *KN*, *OR* be carried;
let *US* be the common section of the planes, and *DG* the diameter of the cube *AF*.

I say that *UT* is equal to *TS*, and *DT* to *TG*.

For let *DU*, *UE*, *BS*, *SG* be joined.

Then, since *DO* is parallel to *PE*,
the alternate angles *DOU*, *UPE* are equal to one another. [I. 29]

And, since *DO* is equal to *PE*, and *OU* to *UP*,
and they contain equal angles,
therefore the base *DU* is equal to the base *UE*,
the triangle *DOU* is equal to the triangle *PUE*,
and the remaining angles are equal to the remaining angles; [I. 4]
therefore the angle *OUD* is equal to the angle *PUE*.



For this reason *DUE* is a straight line. [I. 14]

For the same reason, *BSG* is also a straight line,
and *BS* is equal to *SG*.

Now, since *CA* is equal and parallel to *DB*,
while *CA* is also equal and parallel to *EG*,
therefore *DB* is also equal and parallel to *EG*. [XI. 9]

And the straight lines DE, BG join their extremities; therefore DE is parallel to BG. [I. 33]

Therefore the angle EDT is equal to the angle BGT, for they are alternate; [I. 29]
and the angle DTU is equal to the angle GTS. [I. 15]

Therefore DTU, GTS are two triangles which have two angles equal to two angles, and one side equal to one side, namely that subtending one of the equal angles, that is, DU equal to GS,
for they are the halves of DE, BG;
therefore they will also have the remaining sides equal to the remaining sides. [I. 26]

Therefore DT is equal to TG, and UT to TS.

Therefore etc.

Q. E. D.

Euclid enunciates this proposition of a *cube* only, though it is true of any parallelepiped, no doubt because its truth for a cube is all that was wanted for the only proposition where it is needed, viz. XIII. 17.

Simson remarks that it should be proved that the straight lines bisecting the corresponding opposite sides of opposite planes *are* in one plane. This is, however, clear because e.g. since DK, CL are equal and parallel, KL is equal and parallel to CD. And, since KL, AB are both parallel to DC, KL is parallel to AB. And lastly, since KL, MN are both parallel to AB, KL is parallel to MN and therefore in one plane with it.

The essential thing to be proved is that the plane passing through the opposite edges DB, EG passes through the straight line US, since, only if this be the case, can US, DG intersect one another.

To prove this we have only to prove that, if DU, UE and BS, SG be joined, DUE and BSG are both straight lines.

Now, since DO is parallel to PE,

$$\angle DOU = \angle EPU.$$

Thus, in the $\triangle$s DUO, EUP, two sides DO, OU are equal to two sides EP, PU, and the included angles are equal.

Therefore $$\triangle DUO \equiv \triangle EUP,$$
$$DU = UE,$$
and $$\angle DUO = \angle EUP,$$
so that DUE is a straight line, bisected at U. Similarly BSG is a straight line, bisected at S.

Thus the plane through DB, EG (DB, EG being equal and parallel) contains the straight lines DUE, BSG (which are therefore equal and parallel also) and also [XI. 7] the straight lines US, DG (which accordingly intersect).

In $\triangle$s DTU, GTS, the angles UDT, SGT are equal (being alternate), and the angles UTD, STG are also equal (being vertically opposite), while DU (half of DE) is equal to GS (half of BG).

Therefore [I. 26] the triangles DTU, GTS are equal in all respects, so that

$$DT = TG,$$
$$UT = TS.$$

PROPOSITION 39.

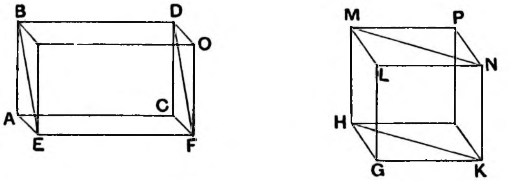
If there be two prisms of equal height, and one have a parallelogram as base and the other a triangle, and if the parallelogram be double of the triangle, the prisms will be equal.

Let $ABCDEF$, $GHKLMN$ be two prisms of equal height,

let one have the parallelogram AF as base, and the other the triangle GHK,

and let the parallelogram AF be double of the triangle GHK;

I say that the prism $ABCDEF$ is equal to the prism $GHKLMN$.



For let the solids AO, GP be completed.

Since the parallelogram AF is double of the triangle GHK, while the parallelogram HK is also double of the triangle GHK, [I. 34]

therefore the parallelogram AF is equal to the parallelogram HK.

But parallelepipedal solids which are on equal bases and of the same height are equal to one another; [XI. 31]

therefore the solid AO is equal to the solid GP.

And the prism $ABCDEF$ is half of the solid AO,

and the prism $GHKLMN$ is half of the solid GP; [XI. 28]

therefore the prism $ABCDEF$ is equal to the prism $GHKLMN$.

Therefore etc.

Q. E. D.

This proposition is made use of in XII. 3, 4. The phraseology is interesting because we find one of the *parallelogrammic* faces of one of the triangular prisms called its *base*, and the perpendicular on this plane from that vertex of either *triangular* face which is not in this plane the *height*.

The proof is simple because we have only to complete parallelepipeds which are double the prisms respectively and then use XI. 31. It has to be borne in mind, however, that, if the parallelepipeds are not rectangular, the proof in XI. 28 is not sufficient to establish the fact that the parallelepipeds are double of the prisms, but has to be supplemented as shown in the note on that proposition. XII. 4 does, however, require the theorem in its general form.

BOOK XII.

HISTORICAL NOTE.

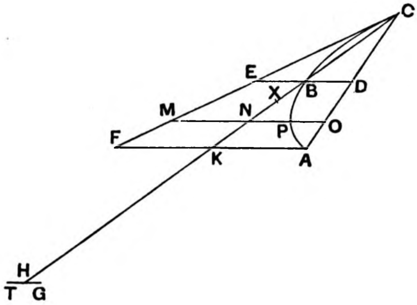
The predominant feature of Book xii. is the use of the *method of exhaustion*, which is applied in Propositions 2, 3—5, 10, 11, 12, and (in a slightly different form) in Propositions 16—18. We conclude therefore that for the content of this Book Euclid was greatly indebted to Eudoxus, to whom the discovery of the method of exhaustion is attributed. The evidence for this attribution comes mainly from Archimedes. (1) In the preface to *On the Sphere and Cylinder* i., after stating the main results obtained by himself regarding the surface of a sphere or a segment thereof, and the volume and surface of a right cylinder with height equal to its diameter as compared with those of a sphere with the same diameter, Archimedes adds: "Having now discovered that the properties mentioned are true of these figures, I cannot feel any hesitation in setting them side by side both with my former investigations and *with those of the theorems of Eudoxus on solids* which are held to be most irrefragably established, namely that *any pyramid is one third part of the prism which has the same base with the pyramid and equal height* [i.e. Eucl. xii. 7], and that *any cone is one third part of the cylinder which has the same base with the cone and equal height* [i.e. Eucl. xii. 10]. For, though these properties also were naturally inherent in the figures all along, yet they were in fact unknown to all the many able geometers who lived before Eudoxus and had not been observed by any one." (2) In the preface to the treatise known as the *Quadrature of the Parabola* Archimedes states the "lemma" assumed by him and known as the "Axiom of Archimedes" (see note on x. 1 above) and proceeds: "Earlier geometers (οἱ πρότερον γεωμέτραι) have also used this lemma; for it is by the use of this same lemma that they have shown that *circles are to one another in the duplicate ratio of their diameters* [Eucl. xii. 2], and that *spheres are to one another in the triplicate ratio of their diameters* [Eucl. xii. 18], and further that *every pyramid is one third part of the prism which has the same base with the pyramid and equal height* [Eucl. xii. 7]; also, that *every cone is one third part of the cylinder which has the same base with the cone and equal height* [Eucl. xii. 10] they proved by assuming *a certain lemma similar to that aforesaid.*" Thus in the first passage two theorems of Eucl. xii. are definitely attributed to Eudoxus; and, when Archimedes says, in the second passage, that "earlier geometers" proved these two theorems by means of the lemma known as the "Axiom of Archimedes" and of a lemma similar to it respectively, we can hardly suppose him to be alluding to any other proof than that given by Eudoxus. As a matter of fact, the lemma used by Euclid to prove both propositions (XII. 3—5 and 7, and XII. 10) is the theorem of Eucl. X. 1. As regards the connexion between the two "lemmas" see note on X. 1.

We are not, however, to suppose that none of the *results* obtained by the method of exhaustion had been discovered before the time of Eudoxus (fl. about 368—5 B.C.). Two at least are of earlier date, those of Eucl. XII. 2 and XII. 7.

(*a*) Simplicius (*Comment. in Aristot. Phys.* p. 61, ed. Diels) quotes Eudemus as saying, in his *History of Geometry*, that Hippocrates of Chios (fl. say 430 B.C.) first laid it down (ἔθετο) that similar segments of circles are in the ratio of the squares on their bases and that he proved this (ἐδείκνυεν) by proving (ἐκ τοῦ δεῖξαι) that the squares on the diameters have the same ratio as the (whole) circles. We know nothing of the method by which Hippocrates proved this proposition; but, having regard to the evidence from Archimedes quoted above, it is not permissible to suppose that the method was the fully developed *method of exhaustion* as we know it.

(*b*) As regards the two theorems about the volume of a pyramid and of a cone respectively, which Eudoxus was the first to prove, we now have authentic evidence in the short treatise by Archimedes discovered by Heiberg in a MS. at Constantinople in 1906 and published in *Hermes* the following year (see now *Archimedis opera omnia*, ed. Heiberg, 2. ed., Vol. II., 1913, pp. 425—507; T. L. Heath, *The* Method *of Archimedes*, Cambridge, 1912). The said treatise, complete in all essentials, bears the title Ἀρχιμήδους περὶ τῶν μηχανικῶν θεωρημάτων πρὸς Ἐρατοσθένην ἔφοδος. This "Method" (or "Plan of attack"), addressed to Eratosthenes, is none other than the ἐφόδιον on which, according to Suidas, Theodosius wrote a commentary, and which is several times cited by Heron in his *Metrica*; its discovery adds a new and important chapter to the history of the integral calculus. In the preface to this work Archimedes alludes to the theorems which he first discovered by means of mechanical considerations, but proved afterwards by geometry, because the investigation by means of mechanics did not constitute a rigid proof; he observes, however, that the mechanical method is of great use for the *discovery* of theorems, and it is much easier to provide the rigid proof when the fact to be proved has once been discovered than it would be if nothing were known to begin with. He goes on: "Hence too, in the case of those theorems the proof of which was first discovered by Eudoxus, namely those relating to the cone and the pyramid, that the cone is one third part of the cylinder, and the pyramid one third part of the prism, having the same base and equal height, no small part of the credit will naturally be assigned to Democritus, who was the first to make the statement (of the fact) regarding the said figure [i.e. property], though without proving it." Hence the *discovery* of the two theorems must now be attributed to Democritus (fl. towards the end of 5th cent. B.C.). The words "without proving it" (χωρὶς ἀποδείξεως) do not mean that Democritus gave no sort of proof, but only that he did not give a proof on the rigorous lines required later; for the same words are used by Archimedes of his own investigations by means of mechanics, which, however, do constitute a reasoned argument. The character of Archimedes' mechanical arguments combined with a passage of Plutarch about a particular question in infinitesimals said to have been raised by Democritus may perhaps give a clue to the line of Democritus' argument as regards the pyramid. The essential feature of Archimedes' mechanical arguments in this tract is that he regards an area as the sum of an infinite number of *straight lines* parallel to one another and terminated by the boundary or boundaries of the closed figure the area of which is to be found, and a volume as the sum of an infinite number of *plane sections* parallel to one another: which is of course the same thing as taking (as we do in the integral calculus) the sum of an infinite number of strips of breadth dx (say), when dx becomes indefinitely small, or the sum of an infinite number of parallel laminae of depth dz (say), when dz becomes indefinitely small. To give only one instance, we may take the case of the area of a segment of a parabola cut off by a chord.

Let CBA be the parabolic segment, CE the tangent at C meeting the diameter EBD through the middle point of the chord CA in E, so that



$$EB = BD.$$

Draw AF parallel to ED meeting CE produced in F. Produce CB to H so that $CK = KH$, where K is the point in which CH meets AF; and suppose CH to be a lever.

Let any diameter $MNPO$ be drawn meeting the curve in P and CF, CK, CA in M, N, O respectively.

Archimedes then observes that

$$CA \, . \, AO = MO : OP$$

("for this is proved in a lemma"),

whence $$HK : KN = MO : OP,$$

so that, if a straight line TG equal to PO be placed with its middle point at H, the straight line MO with centre of gravity at N, and the straight line TG with centre of gravity at H, will balance about K.

Taking all other parts of diameters like PO intercepted between the curve and CA, and placing equal straight lines with their centres of gravity at H, these straight lines collected at H will balance (about K) all the lines like MO parallel to FA intercepted within the triangle CFA in the positions in which they severally lie in the figure.

Hence Archimedes infers that an area equal to that of the parabolic segment hung at H will balance (about K) the triangle CFA hung at its centre of gravity, the point X (a point on CK such that $CK = 3XK$), and therefore that

$$\begin{aligned}(\text{area of triangle } CFA) : (\text{area of segment}) &= HK : KX \\ &= 3 : 1,\end{aligned}$$

from which it follows that

$$\text{area of parabolic segment} = \tfrac{4}{3}\triangle ABC.$$

The same sort of argument is used for solids, *plane sections* taking the place of *straight lines*.

Archimedes is careful to state once more that this method of argument does not constitute a *proof*. Thus, at the end of the above proposition about the parabolic segment, he adds: "This property is of course not proved by what has just been said; but it has furnished a sort of *indication* (ἔμφασίν τινα) that the conclusion is true."

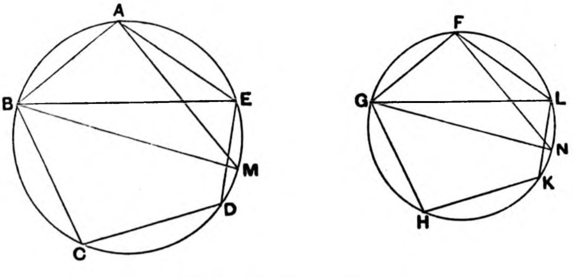
Let us now turn to the passage of Plutarch (*De Comm. Not. adv. Stoicos* XXXIX 3) about Democritus above referred to. Plutarch speaks of Democritus as having raised the question in natural philosophy (φυσικῶς): "if a cone were cut by a plane parallel to the base [by which is clearly meant a plane indefinitely near to the base], what must we think of the surfaces of the sections, that they are equal or unequal? For, if they are unequal, they will make the cone irregular, as having many indentations, like steps, and unevennesses; but, if they are equal, the sections will be equal, and the cone will appear to have the property of the cylinder and to be made up of equal, not unequal circles, which is very absurd." The phrase "*made up* of equal...circles" (ἐξ ἴσων συγκείμενος...κύκλων) shows that Democritus already had the idea of a solid being the sum of an infinite number of parallel planes, or indefinitely thin laminae, indefinitely near together: a most important anticipation of the same thought which led to such fruitful results in Archimedes. If then one may hazard a conjecture as to Democritus' argument with regard to a pyramid, it seems probable that he would notice that, if two pyramids of the same height and equal triangular bases are respectively cut by planes parallel to the base and dividing the heights in the same ratio, the corresponding sections of the two pyramids are equal, whence he would infer that the pyramids are equal as being the sum of the same infinite number of equal plane sections or indefinitely thin laminae. (This would be a particular anticipation of Cavalieri's proposition that the areal or solid contents of two figures are equal if two sections of them taken at the same height, whatever the height may be, always give equal straight lines or equal surfaces respectively.) And Democritus would of course see that the three pyramids into which a prism on the same base and of equal height with the original pyramid is divided (as in Eucl. XII. 7) satisfy this test of equality, so that the pyramid would be one third part of the prism. The extension to a pyramid with a polygonal base would be easy. And Democritus may have stated the proposition for the cone (of course without an absolute proof) as a natural inference from the result of increasing indefinitely the number of sides in a regular polygon forming the base of a pyramid.

BOOK XII. PROPOSITIONS.

PROPOSITION 1.

Similar polygons inscribed in circles are to one another as the squares on the diameters.

Let *ABC*, *FGH* be circles,
let *ABCDE*, *FGHKL* be similar polygons inscribed in them,
and let *BM*, *GN* be diameters of the circles;
I say that, as the square on *BM* is to the square on *GN*, so is the polygon *ABCDE* to the polygon *FGHKL*.



For let *BE*, *AM*, *GL*, *FN* be joined.

Now, since the polygon *ABCDE* is similar to the polygon *FGHKL*,
the angle *BAE* is equal to the angle *GFL*,
and, as *BA* is to *AE*, so is *GF* to *FL*. [vi. Def. 1]

Thus *BAE*, *GFL* are two triangles which have one angle equal to one angle, namely the angle *BAE* to the angle *GFL*, and the sides about the equal angles proportional;
therefore the triangle *ABE* is equiangular with the triangle *FGL*. [vi. 6]

Therefore the angle *AEB* is equal to the angle *FLG*.

But the angle *AEB* is equal to the angle *AMB*, for they stand on the same circumference; [III. 27]
and the angle *FLG* to the angle *FNG*;
therefore the angle *AMB* is also equal to the angle *FNG*.

But the right angle *BAM* is also equal to the right angle *GFN*; [III. 31]
therefore the remaining angle is equal to the remaining angle. [I. 32]

Therefore the triangle *ABM* is equiangular with the triangle *FGN*.

Therefore, proportionally, as *BM* is to *GN*, so is *BA* to *GF*. [VI. 4]

But the ratio of the square on *BM* to the square on *GN* is duplicate of the ratio of *BM* to *GN*,
and the ratio of the polygon *ABCDE* to the polygon *FGHKL* is duplicate of the ratio of *BA* to *GF*; [VI. 20]
therefore also, as the square on *BM* is to the square on *GN*, so is the polygon *ABCDE* to the polygon *FGHKL*.

Therefore etc.

Q. E. D.

As, from this point onward, the text of each proposition usually occupies considerable space, I shall generally give in the notes a summary of the argument, to enable it to be followed more easily.

Here we have to prove that a pair of corresponding sides are in the ratio of the corresponding diameters.

Since $\angle$s *BAE*, *GFL* are equal, and the sides about those angles proportional,

$\triangle$s *ABE*, *FGL* are equiangular,

so that $\angle AEB = \angle FLG$

Hence their equals in the same segments, $\angle$s *AMB*, *FNG*, are equal.
And the right angles *BAM*, *GFN* are equal.
Therefore $\triangle$s *ABM*, *FGN* are equiangular, so that

$$BM : GN = BA : GF.$$

The duplicates of these ratios are therefore equal,

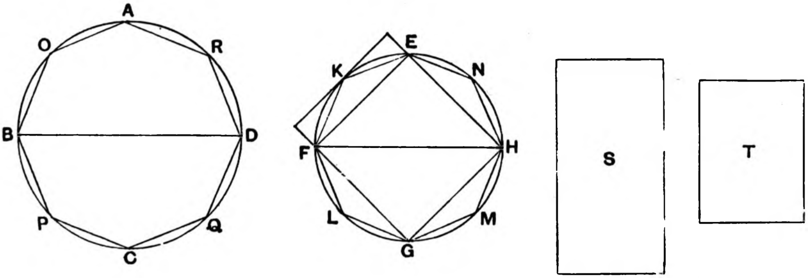
whence (polygon *ABCDE*) : (polygon *FGHKL*)
= duplicate ratio of *BA* to *GF*
= duplicate ratio of *BM* to *GN*
$= BM^2 : GN^2$.

PROPOSITION 2.

Circles are to one another as the squares on the diameters.

Let *ABCD*, *EFGH* be circles, and *BD*, *FH* their diameters;

I say that, as the circle *ABCD* is to the circle *EFGH*, so is the square on *BD* to the square on *FH*.



For, if the square on *BD* is not to the square on *FH* as the circle *ABCD* is to the circle *EFGH*,

then, as the square on *BD* is to the square on *FH*, so will the circle *ABCD* be either to some less area than the circle *EFGH*, or to a greater.

First, let it be in that ratio to a less area *S*.

Let the square *EFGH* be inscribed in the circle *EFGH*; then the inscribed square is greater than the half of the circle *EFGH*, inasmuch as, if through the points *E*, *F*, *G*, *H* we draw tangents to the circle, the square *EFGH* is half the square circumscribed about the circle, and the circle is less than the circumscribed square;

hence the inscribed square *EFGH* is greater than the half of the circle *EFGH*.

Let the circumferences *EF*, *FG*, *GH*, *HE* be bisected at the points *K*, *L*, *M*, *N*,

and let *EK*, *KF*, *FL*, *LG*, *GM*, *MH*, *HN*, *NE* be joined;

therefore each of the triangles *EKF*, *FLG*, *GMH*, *HNE* is also greater than the half of the segment of the circle about it, inasmuch as, if through the points *K*, *L*, *M*, *N* we draw tangents to the circle and complete the parallelograms on the straight lines *EF*, *FG*, *GH*, *HE*, each of the triangles *EKF*,

FLG, *GMH*, *HNE* will be half of the parallelogram about it,
while the segment about it is less than the parallelogram;
hence each of the triangles *EKF*, *FLG*, *GMH*, *HNE* is greater than the half of the segment of the circle about it.

Thus, by bisecting the remaining circumferences and joining straight lines, and by doing this continually, we shall leave some segments of the circle which will be less than the excess by which the circle *EFGH* exceeds the area *S*.

For it was proved in the first theorem of the tenth book that, if two unequal magnitudes be set out, and if from the greater there be subtracted a magnitude greater than the half, and from that which is left a greater than the half, and if this be done continually, there will be left some magnitude which will be less than the lesser magnitude set out.

Let segments be left such as described, and let the segments of the circle *EFGH* on *EK*, *KF*, *FL*, *LG*, *GM*, *MH*, *HN*, *NE* be less than the excess by which the circle *EFGH* exceeds the area *S*.

Therefore the remainder, the polygon *EKFLGMHN*, is greater than the area *S*.

Let there be inscribed, also, in the circle *ABCD* the polygon *AOBPCQDR* similar to the polygon *EKFLGMHN*;
therefore, as the square on *BD* is to the square on *FH*, so is the polygon *AOBPCQDR* to the polygon *EKFLGMHN*. [XII. 1]

But, as the square on *BD* is to the square on *FH*, so also is the circle *ABCD* to the area *S*;
therefore also, as the circle *ABCD* is to the area *S*, so is the polygon *AOBPCQDR* to the polygon *EKFLGMHN*; [V. 11]
therefore, alternately, as the circle *ABCD* is to the polygon inscribed in it, so is the area *S* to the polygon *EKFLGMHN*. [V. 16]

But the circle *ABCD* is greater than the polygon inscribed in it;
therefore the area *S* is also greater than the polygon *EKFLGMHN*.

But it is also less:
which is impossible.

Therefore, as the square on BD is to the square on FH, so is not the circle $ABCD$ to any area less than the circle $EFGH$.

Similarly we can prove that neither is the circle $EFGH$ to any area less than the circle $ABCD$ as the square on FH is to the square on BD.

I say next that neither is the circle $ABCD$ to any area greater than the circle $EFGH$ as the square on BD is to the square on FH.

For, if possible, let it be in that ratio to a greater area S.

Therefore, inversely, as the square on FH is to the square on DB, so is the area S to the circle $ABCD$.

But, as the area S is to the circle $ABCD$, so is the circle $EFGH$ to some area less than the circle $ABCD$;
therefore also, as the square on FH is to the square on BD, so is the circle $EFGH$ to some area less than the circle $ABCD$: [v. 11]
which was proved impossible.

Therefore, as the square on BD is to the square on FH, so is not the circle $ABCD$ to any area greater than the circle $EFGH$.

And it was proved that neither is it in that ratio to any area less than the circle $EFGH$;
therefore, as the square on BD is to the square on FH, so is the circle $ABCD$ to the circle $EFGH$.

Therefore etc.

Q. E. D.

LEMMA.

I say that, the area S being greater than the circle $EFGH$, as the area S is to the circle $ABCD$, so is the circle $EFGH$ to some area less than the circle $ABCD$.

For let it be contrived that, as the area S is to the circle $ABCD$, so is the circle $EFGH$ to the area T.

I say that the area T is less than the circle $ABCD$.

For since, as the area S is to the circle $ABCD$, so is the circle $EFGH$ to the area T, therefore, alternately, as the area S is to the circle $EFGH$, so is the circle $ABCD$ to the area T. [V. 16]

But the area S is greater than the circle $EFGH$;
therefore the circle $ABCD$ is also greater than the area T.

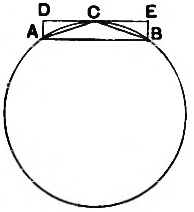
Hence, as the area S is to the circle $ABCD$, so is the circle $EFGH$ to some area less than the circle $ABCD$.

Q. E. D.

Though this theorem is said to have been proved by Hippocrates, we may with tolerable certainty attribute the proof of it given by Euclid to Eudoxus, to whom XII. 7 Por. and XII. 10 (which Euclid proves in exactly the same manner) are specifically attributed by Archimedes. As regards the lemma used herein (Eucl. X. 1) and the somewhat different lemma by means of which Archimedes says that the theorems of XII. 2, XII. 7 Por. and XII. 18 were proved, see my note on X. 1 above.

The first essential in this proposition is to prove that we can *exhaust* a circle, in the sense of X. 1, by successively inscribing in it regular polygons, each of which has twice as many sides as the preceding one. We take first an inscribed square, then bisect the arcs subtended by the sides and so form an equilateral polygon of eight sides, then do the same with the latter, forming a polygon of 16 sides, and so on. And we have to prove that what is left over when any one of these polygons is taken away from the circle is *more than half exhausted* when the next polygon is made and subtracted from the circle.

Euclid proves that the inscribed square is greater than half the circle and that the regular octagon when subtracted takes away more than half of what was left by the square. He then infers that the same thing will happen whenever the number of sides is doubled.



This can be seen generally by taking *any* arc of a circle cut off by a chord AB. Bisect the arc in C. Draw a tangent to the circle at C, and let AD, BE be drawn perpendicular to the tangent. Join AC, CB.

Then DE is parallel to AB, since

$\angle ECB = \angle CAB$, in alternate segment, [III. 32]

$= \angle CBA$. [III. 29, I. 5]

Thus $ABED$ is a ▱;

and it is greater than the segment ACB.

Therefore its half, the $\triangle ACB$, is greater than half the segment.

Thus, by X. 1, Euclid's construction of successive regular polygons in a circle, if continued far enough, will at length leave segments which are together less than any given area.

Now let X, X' be the areas of the circles, d, d' their diameters, respectively.

Then, if
$$X : X' \neq d^2 : d'^2,$$
$$d^2 : d'^2 = X : S,$$
where S is some area either greater or less than X'.

I. Suppose $S < X'$.

Continue the construction of polygons in X' until we arrive at one which leaves over segments together less than the excess of X' over S, i.e. a polygon such that

$$X' > (\text{polygon in } X') > S.$$

Inscribe in the circle X a polygon similar to that in X.

Then $\quad (\text{polygon in } X) : (\text{polygon in } X') = d^2 : d'^2 \qquad$ [XII. 1]

$$= X : S, \text{ by hypothesis;}$$

and, alternately,

$$(\text{polygon in } X) : X = (\text{polygon in } X') : S.$$

But $\quad (\text{polygon in } X) < X$;

therefore $\quad (\text{polygon in } X') < S$.

But, by construction. $\quad (\text{polygon in } X') > S$:

which is impossible.

Hence S cannot be *less* than X' as supposed.

II. Suppose $S > X'$.

Since $\quad d^2 : d'^2 = X : S$,

we have, inversely, $\quad d'^2 : d^2 = S : X$.

Suppose that $\quad S : X = X' : T$,

whence, since $S > X'$, $\quad X > T$. $\qquad$ [V. 14]

Consequently $\quad d'^2 : d^2 = X' : T$,

where $T < X$.

This can be proved impossible in exactly the same way as shown in Part I. Hence S cannot be *greater* than X' as supposed.

Since then S is neither greater nor less than X',

$$S = X',$$

and therefore $\quad d^2 : d'^2 = X : X'$.

With reference to the assumption that there *is* some space S such that

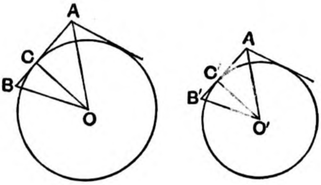
$$d^2 : d'^2 = X : S,$$

i.e. that there is a fourth proportional to the areas d^2, d'^2, X, Simson observes that it is sufficient, in this and the like cases, that a thing made use of in the reasoning can possibly *exist*, though it cannot be exhibited by a geometrical construction. As regards the assumption see note on V. 18 above.

There is grave reason for suspecting the genuineness of the Lemma at the end of the proposition; though, if it be rejected, it will be necessary to delete the words "as was before proved" in corresponding places in XII. 5, 18.

It will be observed that Euclid proves the impossibility in the second case by reducing it to the first. If it is desired to prove the second case independently, we must *circumscribe* successive polygons to the circles instead of inscribing them, in the way shown by Archimedes in his first proposition on the *Measurement of a circle.* Of course we require, as a preliminary, the proposition corresponding to XII. 1, that *Similar polygons circumscribed about circles are to one another as the squares on the diameters.*

Let AB, $A'B'$ be corresponding sides of the two similar polygons. Then $\angle$s OAB, $O'A'B'$ are equal, since AO, $A'O'$ bisect equal angles.



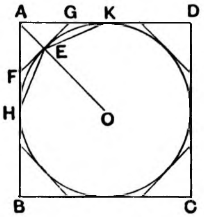
Similarly $\angle ABO = \angle A'B'O'$.

Therefore $\triangle$s AOB, $A'O'B'$ are similar, so that their areas are in the duplicate ratio of AB to $A'B'$.

The radii OC, $O'C'$ drawn to the points of contact are perpendicular to AB, $A'B'$, and it follows that

$$AB : A'B' = CO : C'O'.$$

Thus the polygons are to one another in the duplicate ratio of the radii, and therefore of the diameters.



Now suppose a square $ABCD$ described about a circle.

Make an octagon described about the circle by drawing tangents at the points E etc., where OA etc. meet the circle.

Then shall the tangent at E cut off more than half of the area between AK, AH and the arc HEK.

For the angle AEG is right, and is therefore $> \angle EAG$.

Therefore $AG > EG$

$> GK$.

Therefore $\triangle AGE > \triangle EGK$.

Similarly $\triangle AFE > \triangle EFH$.

Hence $\triangle AFG > \frac{1}{2}$ (re-entrant quadrilateral $AHEK$),

and *a fortiori*, $\triangle AFG > \frac{1}{2}$ (area between AH, AK and the arc).

Thus the octagon takes from the square more than half the space between the square and the circle.

Similarly, if a figure of 16 equal sides be circumscribed by cutting off symmetrically the corners of the octagon, it will take away more than half of the space between the octagon and circle.

Suppose now, with the original notation, that

$$d^2 : d'^2 = X : S,$$

where S is greater than X'.

Continue the construction of circumscribed polygons about X' until the total area between the polygon and the circle is less than the difference between S and X', i.e. till

$$S > (\text{polygon about } X') > X'.$$

Circumscribe a similar polygon about X.

Then (polygon about X) : (polygon about X') $= d^2 : d'^2$

$= X : S$, by hypothesis,

and, alternately,

$$(\text{polygon about } X) : X = (\text{polygon about } X') : S.$$

But (polygon about X) $> X$.

Therefore (polygon about X') $> S$.

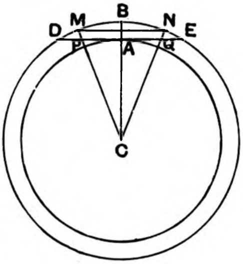
But $S >$ (polygon about X') : [above]

which is impossible.

Hence S cannot be *greater* than X'.

Legendre proves this proposition by a method equally rigorous but not, I think, possessing any advantages over Euclid's. It depends on a lemma corresponding to Eucl. XII. 16, but with another part added to it.

Two concentric circles being given, we can always inscribe in the greater a regular polygon such that its sides do not meet the circumference of the lesser, and we can also circumscribe about the lesser a regular polygon such that its sides do not meet the circumference of the greater.



Let CA, CB be the radii of the circles.

I. At A on the inner circle draw the tangent DE meeting the outer circle in D, E.

Inscribe in the outer circle any of the regular polygons which we can inscribe, e.g. a square.

Bisect the arc subtended by a side, bisect the half, bisect that again, and so on, until we arrive at an arc less than the arc DBE.

Let this arc be MN, and suppose it so placed that B is its middle point.

Then the chord MN is clearly more distant from the centre C than DE is; and the regular polygon, of which MN is a side, does not anywhere meet the circumference of the inner circle.

II. Join CM, CN, meeting DE in P, Q.

Then PQ will be the side of a polygon circumscribed about the inner circle and similar to the polygon inscribed in the outer;

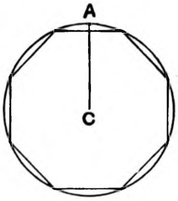
and the circumscribed polygon of which PQ is a side will not anywhere meet the outer circle.

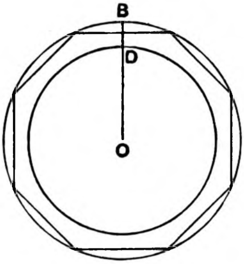
Legendre now proves XII. 2 after the following manner.

For brevity, let us denote the area of the circle with radius CA by (circ. CA).

Then it is required to prove that, if OB be the radius of a second circle,

$$(\text{circ. } CA) : (\text{circ. } OB) = CA^2 : OB^2.$$





Suppose, if possible, that this relation is not true. Then CA^2 will be to OB^2 as (circ. CA) is to an area greater or less than (circ. OB).

I. Suppose, first, that

$$CA^2 : OB^2 = (\text{circ. } CA) : (\text{circ. } OD),$$

where OD is less than OB.

Inscribe in the circle with radius OB a regular polygon such that its sides do not anywhere meet the circumference of the circle with centre OD; [Lemma]

and inscribe a similar polygon in the other circle.

The areas of the polygons will then be in the duplicate ratio of CA to OB, [XII. 1]

or

$$\begin{aligned}(\text{polygon in circ. } CA) : (\text{polygon in circ. } OB) &= CA^2 : OB^2 \\ &= (\text{circ. } CA) : (\text{circ. } OD), \text{ by hypothesis.}\end{aligned}$$

But this is impossible, because the polygon in (circ. CA) is *less* than (circ. CA), but the polygon in (circ. OB) is *greater* than (circ. OD).

Therefore CA^2 cannot be to OB^2 as (circ. CA) is to a *less* circle than (circ. OB).

II. Suppose, if possible, that

$$CA^2 : OB^2 = (\text{circ. } CA) : (\text{some circle} > \text{circ. } OB).$$

Then inversely

$$OB^2 : CA^2 = (\text{circ. } OB) : (\text{some circle} < \text{circ. } CA),$$

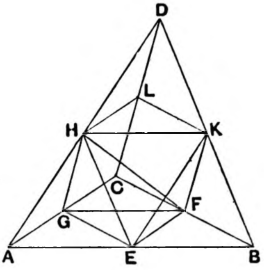
and this is proved impossible exactly as in Part I.

Therefore $CA^2 : OB^2 = (\text{circ. } CA) : (\text{circ. } OB)$.

PROPOSITION 3.

Any pyramid which has a triangular base is divided into two pyramids equal and similar to one another, similar to the whole and having triangular bases, and into two equal prisms; and the two prisms are greater than the half of the whole pyramid.

Let there be a pyramid of which the triangle ABC is the base and the point D the vertex;



I say that the pyramid $ABCD$ is divided into two pyramids equal to one another, having triangular bases and similar to the whole pyramid, and into two equal prisms; and the two prisms are greater than the half of the whole pyramid.

For let AB, BC, CA, AD, DB, DC be bisected at the points E, F, G, H, K, L, and let HE, EG, GH, HK, KL, LH, KF, FG be joined.

Since AE is equal to EB, and AH to DH, therefore EH is parallel to DB. [VI. 2]

For the same reason
HK is also parallel to *AB*.

Therefore *HEBK* is a parallelogram;
therefore *HK* is equal to *EB*. [I. 34]

But *EB* is equal to *EA*;
therefore *AE* is also equal to *HK*.

But *AH* is also equal to *HD*;
therefore the two sides *EA*, *AH* are equal to the two sides *KH*, *HD* respectively,
and the angle *EAH* is equal to the angle *KHD*;
therefore the base *EH* is equal to the base *KD*. [I. 4]

Therefore the triangle *AEH* is equal and similar to the triangle *HKD*.

For the same reason
the triangle *AHG* is also equal and similar to the triangle *HLD*.

Now, since two straight lines *EH*, *HG* meeting one another are parallel to two straight lines *KD*, *DL* meeting one another, and are not in the same plane, they will contain equal angles. [XI. 10]

Therefore the angle *EHG* is equal to the angle *KDL*.

And, since the two straight lines *EH*, *HG* are equal to the two *KD*, *DL* respectively,
and the angle *EHG* is equal to the angle *KDL*,
therefore the base *EG* is equal to the base *KL*; [I. 4]
therefore the triangle *EHG* is equal and similar to the triangle *KDL*.

For the same reason
the triangle *AEG* is also equal and similar to the triangle *HKL*.

Therefore the pyramid of which the triangle *AEG* is the base and the point *H* the vertex is equal and similar to the pyramid of which the triangle *HKL* is the base and the point *D* the vertex. [XI. Def. 10]

And, since *HK* has been drawn parallel to *AB*, one of the sides of the triangle *ADB*,

the triangle *ADB* is equiangular to the triangle *DHK*, [I. 29] and they have their sides proportional;
therefore the triangle *ADB* is similar to the triangle *DHK*. [VI. Def. 1]

For the same reason
the triangle *DBC* is also similar to the triangle *DKL*, and the triangle *ADC* to the triangle *DLH*.

Now, since the two straight lines *BA*, *AC* meeting one another are parallel to the two straight lines *KH*, *HL* meeting one another, not in the same plane, they will contain equal angles. [XI. 10]

Therefore the angle *BAC* is equal to the angle *KHL*.

And, as *BA* is to *AC*, so is *KH* to *HL*;
therefore the triangle *ABC* is similar to the triangle *HKL*.

Therefore also the pyramid of which the triangle *ABC* is the base and the point *D* the vertex is similar to the pyramid of which the triangle *HKL* is the base and the point *D* the vertex.

But the pyramid of which the triangle *HKL* is the base and the point *D* the vertex was proved similar to the pyramid of which the triangle *AEG* is the base and the point *H* the vertex.

Therefore each of the pyramids *AEGH*, *HKLD* is similar to the whole pyramid *ABCD*.

Next, since *BF* is equal to *FC*,
the parallelogram *EBFG* is double of the triangle *GFC*.

And since, if there be two prisms of equal height, and one have a parallelogram as base, and the other a triangle, and if the parallelogram be double of the triangle, the prisms are equal, [XI. 39]
therefore the prism contained by the two triangles *BKF*, *EHG*, and the three parallelograms *EBFG*, *EBKH*, *HKFG* is equal to the prism contained by the two triangles *GFC*, *HKL* and the three parallelograms *KFCL*, *LCGH*, *HKFG*.

And it is manifest that each of the prisms, namely that in which the parallelogram *EBFG* is the base and the straight line *HK* is its opposite, and that in which the triangle *GFC* is the base and the triangle *HKL* its opposite, is greater than each of the pyramids of which the triangles *AEG*, *HKL* are the bases and the points *H*, *D* the vertices, inasmuch as, if we join the straight lines *EF*, *EK*, the prism in which the parallelogram *EBFG* is the base and the straight line *HK* its opposite is greater than the pyramid of which the triangle *EBF* is the base and the point *K* the vertex.

But the pyramid of which the triangle *EBF* is the base and the point *K* the vertex is equal to the pyramid of which the triangle *AEG* is the base and the point *H* the vertex; for they are contained by equal and similar planes.

Hence also the prism in which the parallelogram *EBFG* is the base and the straight line *HK* its opposite is greater than the pyramid of which the triangle *AEG* is the base and the point *H* the vertex.

But the prism in which the parallelogram *EBFG* is the base and the straight line *HK* its opposite is equal to the prism in which the triangle *GFC* is the base and the triangle *HKL* its opposite,
and the pyramid of which the triangle *AEG* is the base and the point *H* the vertex is equal to the pyramid of which the triangle *HKL* is the base and the point *D* the vertex.

Therefore the said two prisms are greater than the said two pyramids of which the triangles *AEG*, *HKL* are the bases and the points *H*, *D* the vertices.

Therefore the whole pyramid, of which the triangle *ABC* is the base and the point *D* the vertex, has been divided into two pyramids equal to one another and into two equal prisms, and the two prisms are greater than the half of the whole pyramid.

Q. E. D.

We will denote a pyramid with vertex *D* and base *ABC* by *D* (*ABC*) or *D-ABC* and the triangular prism with triangles *GCF*, *HLK* for bases by (*GCF*, *HLK*).

The following are the steps of the proof.

I. To prove pyramid $H(AEG)$ equal and similar to pyramid $D(HKL)$.
Since sides of $\triangle DAB$ are bisected at H, E, K,

$$HE \parallel DB, \text{ and } HK \parallel AB.$$

Hence

$$HK = EB = EA,$$
$$HE = KB = DK.$$

Therefore (1) $\triangle$s HAE, DHK are equal and similar.
Similarly (2) $\triangle$s HAG, DHL are equal and similar.
Again, LH, HK are respectively $\parallel$ to GA, AE in a different plane;
therefore

$$\angle GAE = \angle LHK.$$

And LH, HK are respectively equal to GA, AE.

Therefore (3) $\triangle$s GAE, LHK are equal and similar.

Similarly (4) $\triangle$s HGE, DLK are equal and similar.

Therefore [XI. Def. 10] the pyramids $H(AEG)$ and $D(HKL)$ are equal and similar.

II. To prove the pyramid $D(HKL)$ similar to the pyramid $D(ABC)$.

(1) The $\triangle$s DHK, DAB are equiangular and therefore similar.

Similarly (2) $\triangle$s DLH, DCA are similar, as also (3) the $\triangle$s DLK, DCB.

Again, BA, AC are respectively parallel to KH, HL in a different plane; therefore $\angle BAC = \angle KHL$.

And $BA : AC = KH : HL$.

Therefore (4) $\triangle$s BAC, KHL are similar.

Consequently the pyramid $D(ABC)$ is similar to the pyramid $D(HKL)$, and therefore also to the pyramid $H(AEG)$.

III. To prove prism (GCF, HLK) equal to prism (HGE, KFB).

The prisms may be regarded as having the same *height* (the distance between the planes HKL, ABC) and having for bases (1) the $\triangle\, CGF$ and (2) the ▱ $EBFG$, which is the double of the $\triangle\, CGF$.

Therefore, by XI. 39, the prisms are equal.

IV. To prove the prisms greater than the small pyramids.

Prism (HGE, KFB) is clearly greater than pyramid $K(EFB)$ and therefore greater than pyramid $H(AEG)$.

Therefore each of the prisms is greater than each of the small pyramids; and the sum of the two prisms is greater than the sum of the two small pyramids, which, with the two prisms, make up the whole pyramid.

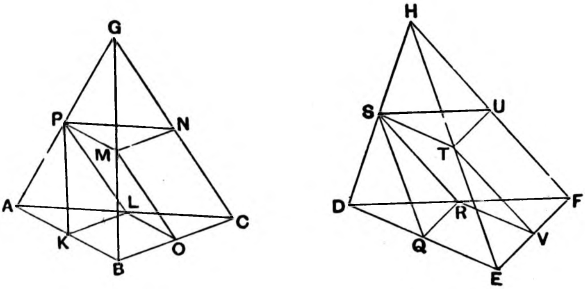
PROPOSITION 4.

If there be two pyramids of the same height which have triangular bases, and each of them be divided into two pyramids equal to one another and similar to the whole, and into two equal prisms, then, as the base of the one pyramid is to the base of the other pyramid, so will all the prisms in the one pyramid be to all the prisms, being equal in multitude, in the other pyramid.

Let there be two pyramids of the same height which have the triangular bases ABC, DEF, and vertices the points G, H,

and let each of them be divided into two pyramids equal to one another and similar to the whole and into two equal prisms; [XII. 3]

I say that, as the base ABC is to the base DEF, so are all the prisms in the pyramid $ABCG$ to all the prisms, being equal in multitude, in the pyramid $DEFH$, For, since *BO* is equal to *OC*, and *AL* to *LC*, therefore *LO* is parallel to *AB*, and the triangle *ABC* is similar to the triangle *LOC*.



For the same reason the triangle *DEF* is also similar to the triangle *RVF*.

And, since *BC* is double of *CO*, and *EF* of *FV*, therefore, as *BC* is to *CO*, so is *EF* to *FV*.

And on *BC*, *CO* are described the similar and similarly situated rectilineal figures *ABC*, *LOC*, and on *EF*, *FV* the similar and similarly situated figures *DEF*, *RVF*; therefore, as the triangle *ABC* is to the triangle *LOC*, so is the triangle *DEF* to the triangle *RVF*; [VI. 22]

therefore, alternately, as the triangle *ABC* is to the triangle *DEF*, so is the triangle *LOC* to the triangle *RVF*. [V. 16]

But, as the triangle *LOC* is to the triangle *RVF*, so is the prism in which the triangle *LOC* is the base and *PMN* its opposite to the prism in which the triangle *RVF* is the base and *STU* its opposite; [Lemma following]

therefore also, as the triangle *ABC* is to the triangle *DEF*, so is the prism in which the triangle *LOC* is the base and *PMN* its opposite to the prism in which the triangle *RVF* is the base and *STU* its opposite.

But, as the said prisms are to one another, so is the prism in which the parallelogram *KBOL* is the base and the straight line *PM* its opposite to the prism in which the parallelogram *QEVR* is the base and the straight line *ST* its opposite. [XI. 39; cf. XII. 3]

Therefore also the two prisms, that in which the parallelogram *KBOL* is the base and *PM* its opposite, and that in which the triangle *LOC* is the base and *PMN* its opposite, are to the prisms in which *QEVR* is the base and the straight line *ST* its opposite and in which the triangle *RVF* is the base and *STU* its opposite in the same ratio [v. 12]

Therefore also, as the base *ABC* is to the base *DEF*, so are the said two prisms to the said two prisms.

And similarly, if the pyramids *PMNG*, *STUH* be divided into two prisms and two pyramids,

as the base *PMN* is to the base *STU*, so will the two prisms in the pyramid *PMNG* be to the two prisms in the pyramid *STUH*.

But, as the base *PMN* is to the base *STU*, so is the base *ABC* to the base *DEF*;

for the triangles *PMN*, *STU* are equal to the triangles *LOC*, *RVF* respectively.

Therefore also, as the base *ABC* is to the base *DEF*, so are the four prisms to the four prisms.

And similarly also, if we divide the remaining pyramids into two pyramids and into two prisms, then, as the base *ABC* is to the base *DEF*, so will all the prisms in the pyramid *ABCG* be to all the prisms, being equal in multitude, in the pyramid *DEFH*.

Q. E. D.

LEMMA.

But that, as the triangle *LOC* is to the triangle *RVF*, so is the prism in which the triangle *LOC* is the base and *PMN* its opposite to the prism in which the triangle *RVF* is the base and *STU* its opposite, we must prove as follows.

For in the same figure let perpendiculars be conceived drawn from *G*, *H* to the planes *ABC*, *DEF*; these are of course equal because, by hypothesis, the pyramids are of equal height.

Now, since the two straight lines *GC* and the perpendicular from *G* are cut by the parallel planes *ABC*, *PMN*,

they will be cut in the same ratios. [XI. 17]

And GC is bisected by the plane PMN at N;
therefore the perpendicular from G to the plane ABC will also be bisected by the plane PMN.

For the same reason
the perpendicular from H to the plane DEF will also be bisected by the plane STU.

And the perpendiculars from G, H to the planes ABC, DEF are equal;
therefore the perpendiculars from the triangles PMN, STU to the planes ABC, DEF are also equal.

Therefore the prisms in which the triangles LOC, RVF are bases, and PMN, STU their opposites, are of equal height.

Hence also the parallelepipedal solids described from the said prisms are of equal height and are to one another as their bases; [XI. 32]
therefore their halves, namely the said prisms, are to one another as the base LOC is to the base RVF.

Q. E. D.

We can incorporate the lemma at the end of the proposition and summarise the proof thus.

Since LO is parallel to AB,

$$\triangle\text{s } ABC,\ LOC \text{ are similar.}$$

In like manner $\triangle$s DEF, RVF are similar.

And, since $BC : CO = EF : FV$,

$$\triangle ABC : \triangle LOC = \triangle DEF : \triangle RVF, \qquad \text{[VI. 22]}$$

and, alternately,

$$\triangle ABC : \triangle DEF = \triangle LOC : \triangle RVF.$$

Now the prisms (LOC, PMN) and (RVF, STU) are equal in height: for the perpendiculars from G, H on the bases ABC, DEF are divided by the planes PMN, STU (parallel to the bases) in the same proportion as GC, HF are divided by those planes [XI. 17], i.e. they are bisected;
hence the heights of the prisms, being half the equal heights of the pyramids, are equal.

And the prisms are the halves respectively of parallelepipeds of the same height on parallelogrammic bases double of the $\triangle$s LOC, RVF respectively; [XI. 28 and note]
hence they are in the same ratio as those parallelepipeds, and therefore as their bases [XI. 32].

Therefore

$$(\text{prism } LOC,\ PMN) : (\text{prism } RVF,\ STU) = \triangle LOC : \triangle RVF$$
$$= \triangle ABC : \triangle DEF.$$

And since the other prisms in the pyramids are equal to these prisms respectively,

(sum of prisms in $GABC$) : (sum of prisms in $HDEF$) $= \triangle ABC : \triangle DEF$.

Similarly, if the pyramids $GPMN$, $HSTU$ be divided in like manner, and also the pyramids $PAKL$, $SDQR$, we shall have e.g.

(sum of prisms in $GPMN$) : (sum of prisms in $HSTU$) $= \triangle PMN : \triangle STU$
$= \triangle ABC : \triangle DEF$,

and similarly for the second pair of pyramids.

The process may be continued indefinitely, and we shall always have

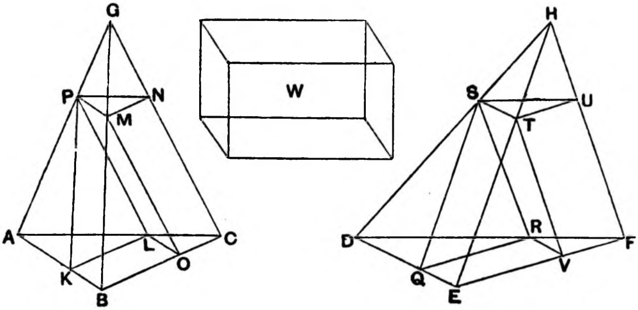
(sum of prisms in $GABC$) : (sum of prisms in $HDEF$) $= \triangle ABC : \triangle DEF$.

PROPOSITION 5.

Pyramids which are of the same height and have triangular bases are to one another as the bases.

Let there be pyramids of the same height, of which the triangles ABC, DEF are the bases and the points G, H the vertices;

I say that, as the base ABC is to the base DEF, so is the pyramid $ABCG$ to the pyramid $DEFH$.



For, if the pyramid $ABCG$ is not to the pyramid $DEFH$ as the base ABC is to the base DEF,

then, as the base ABC is to the base DEF, so will the pyramid $ABCG$ be either to some solid less than the pyramid $DEFH$ or to a greater.

Let it, first, be in that ratio to a less solid W, and let the pyramid $DEFH$ be divided into two pyramids equal to one another and similar to the whole and into two equal prisms;

then the two prisms are greater than the half of the whole pyramid. [XII. 3]

Again, let the pyramids arising from the division be similarly divided,
and let this be done continually until there are left over from the pyramid *DEFH* some pyramids which are less than the excess by which the pyramid *DEFH* exceeds the solid *W*. [X. 1]

Let such be left, and let them be, for the sake of argument, *DQRS*, *STUH*;
therefore the remainders, the prisms in the pyramid *DEFH*, are greater than the solid *W*.

Let the pyramid *ABCG* also be divided similarly, and a similar number of times, with the pyramid *DEFH*;
therefore, as the base *ABC* is to the base *DEF*, so are the prisms in the pyramid *ABCG* to the prisms in the pyramid *DEFH*. [XII. 4]

But, as the base *ABC* is to the base *DEF*, so also is the pyramid *ABCG* to the solid *W*;
therefore also, as the pyramid *ABCG* is to the solid *W*, so are the prisms in the pyramid *ABCG* to the prisms in the pyramid *DEFH*; [V. 11]
therefore, alternately, as the pyramid *ABCG* is to the prisms in it, so is the solid *W* to the prisms in the pyramid *DEFH*. [V. 16]

But the pyramid *ABCG* is greater than the prisms in it;
therefore the solid *W* is also greater than the prisms in the pyramid *DEFH*.

But it is also less:
which is impossible.

Therefore the prism *ABCG* is not to any solid less than the pyramid *DEFH* as the base *ABC* is to the base *DEF*.

Similarly it can be proved that neither is the pyramid *DEFH* to any solid less than the pyramid *ABCG* as the base *DEF* is to the base *ABC*.

I say next that neither is the pyramid *ABCG* to any solid greater than the pyramid *DEFH* as the base *ABC* is to the base *DEF*.

For, if possible, let it be in that ratio to a greater solid *W*;
therefore, inversely, as the base *DEF* is to the base *ABC*, so is the solid *W* to the pyramid *ABCG*.

But, as the solid W is to the solid $ABCG$, so is the pyramid $DEFH$ to some solid less than the pyramid $ABCG$, as was before proved; [XII. 2, Lemma]

therefore also, as the base DEF is to the base ABC, so is the pyramid $DEFH$ to some solid less than the pyramid $ABCG$: [V. 11]

which was proved absurd.

Therefore the pyramid $ABCG$ is not to any solid greater than the pyramid $DEFH$ as the base ABC is to the base DEF.

But it was proved that neither is it in that ratio to a less solid.

Therefore, as the base ABC is to the base DEF, so is the pyramid $ABCG$ to the pyramid $DEFH$.

Q. E. D.

In the two preceding propositions it has been shown how we can divide a pyramid with a triangular base into (1) two equal prisms which are together greater than half the pyramid and (2) two equal pyramids similar to the original one, and that, if this process be continued with the two pyramids, then with the four resulting pyramids, and so on, and if, further, another pyramid of the same height as the original one be similarly divided, the subdivision being made the same number of times, the sum of all the prisms in one pyramid is to the sum of all the prisms in the other as the base of the first is to the base of the second.

We can now prove in the manner of XII. 2 that the volumes of the pyramids themselves are as the bases.

Let us call the pyramids P, P' and their respective bases B, B'.

If $P : P' \neq B : B'$,

suppose that $B : B' = P : W$.

I. Let W be $< P'$.

Divide P' into two prisms and two pyramids, subdivide the latter similarly, and so on, until the sum of the *pyramids* remaining is less than the difference between P' and W [X. 1], so that

$$P' > (\text{prisms in } P') > W.$$

Then divide P similarly, the same number of times.

Now $(\text{prisms in } P) : (\text{prisms in } P') = B : B'$ [XII. 4]

$= P : W$, by hypothesis,

and, alternately,

$$(\text{prisms in } P) : P = (\text{prisms in } P') : W.$$

But $(\text{prisms in } P) < P$;

therefore $(\text{prisms in } P') < W$.

But, by construction, $(\text{prisms in } P') > W$.

Hence W cannot be less than P'.

II. Suppose, if possible, that $W > P'$.

Then, inversely, $$B' : B = W : P.$$
$$= P' : V,$$
where V is *some solid less than P.* [Cf. XII. 2, Lemma, and note.]

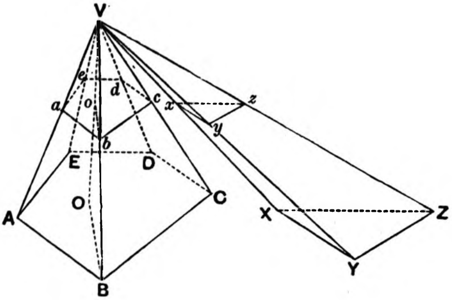
But this can be proved impossible exactly as in Part I.

Therefore W is neither less nor greater than P',

so that $$B : B' = P : P'.$$

Legendre, followed by the American editors already mentioned, and by others, approaches the subject by a different route, proving the following propositions.

1. *If a pyramid be cut by a plane parallel to the base, (a) the lateral edges and the height will be cut in the same proportion, (b) the section by the plane will be a polygon similar to the base.*



(*a*) Since a lateral face VAB of the pyramid $V(ABCDE)$ is cut by two parallel planes in AB, ab,
$$AB \parallel ab;$$
Similarly $BC \parallel bc$, and so on.

Therefore $VA : Va = VB : Vb = VC : Vc = \ldots.$

And, if VO the height be cut in O, o,

$BO \parallel bo$; and each of the above ratios is equal to $VO : Vo$.

(*b*) Since $BA \parallel ba$, and $BC \parallel bc$,
$$\angle ABC = \angle abc. \qquad \text{[XI. 10]}$$
Similarly for all the other angles of the polygons, which are therefore equiangular.

Also, by similar triangles,
$$VA : Va = AB : ab,$$
and so on.

Therefore, by the ratios above,
$$AB : ab = BC : bc = \ldots.$$
Therefore the polygons are similar.

2. *If two pyramids of the same height be cut by planes which are at the same perpendicular distance from the vertices, the sections are as the respective bases.*

For, if we place the pyramids so that the vertices coincide and the bases are in one plane, the planes of the sections will coincide.

If, e.g., the base of the second pyramid be XYZ and the section xyz, we shall have, by the argument of the last proposition,

$$VX : Vx = VY : Vy = VZ : Vz = VO : Vo = VA : Va = \dots,$$

and XYZ, xyz will be similar.

Now
$$(\text{polygon } ABCDE) : (\text{polygon } abcde) = AB^2 : ab^2$$
$$= VA^2 : Va^2,$$

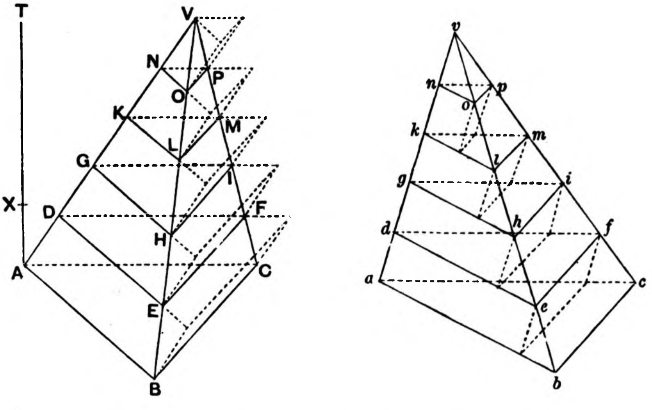
and
$$\triangle XYZ : \triangle xyz = XY^2 : xy^2$$
$$= VX^2 : Vx^2$$
$$= VA^2 : Va^2.$$

Therefore
$$(\text{polygon } ABCDE) : (\text{polygon } abcde) = \triangle XYZ : \triangle xyz.$$

As a particular case, *if the bases of the two pyramids are equivalent, the sections are also equivalent.*

3. *Two triangular pyramids which have equivalent bases and equal heights are equivalent.*

Let $VABC$, $vabc$ be pyramids with equivalent bases ABC, abc, which for convenience we will suppose placed in one plane, and let TA be the common height.



Then, if the pyramids are not equivalent, one must be greater than the other.

Let $VABC$ be the greater; and let AX be the height of a prism on ABC as base which is equal in volume to the difference of the pyramids.

Divide the height AT into equal parts such that each is less than AX, and let each part be equal to z.

Through the points of division draw planes parallel to the bases cutting both pyramids in the sections DEF, GHI,... and def, ghi, ...

The sections DEF, def will then be equivalent; so will the sections GHI, ghi, and so on. [(2) above]

On the triangles ABC, DEF, GHI, ... as bases draw *exterior* prisms having for edges the parts AD, DG, GK, ... of the edge AV;

and on the triangles *def*, *ghi*, ... as bases draw *interior* prisms having for edges the parts *ad*, *dg*, ... of *av*.

All the partial prisms will then have the same height z.

Now the sum of the exterior prisms of the pyramid $VABC$ is *greater* than that pyramid;

and the sum of the interior prisms in the pyramid *vabc* is *less* than that pyramid.

Consequently the difference between the sum of the first set of prisms and the sum of the second set of prisms is greater than the difference between the two pyramids.

Again, if we start from the bases ABC, *abc*, the *second* exterior prism $DEFG$ is equivalent to the *first* interior prism *defa*, since their bases are equivalent and they have the same height z. [XI. 28 and note; XI. 32]

Similarly the third exterior prism is equivalent to the second interior prism, and so on, until we arrive at the last of each.

Therefore the prism $ABCD$, the *first* exterior prism, is the difference between the sums of the exterior and interior prisms respectively.

Therefore the difference between the two pyramids is *less* than the prism $ABCD$, which should therefore be greater than the prism with base ABC and height AX.

But the prism $ABCD$ is, by hypothesis, less than the latter prism:

which is impossible.

Consequently the pyramid $VABC$ cannot be greater than the pyramid *vabc*.

Similarly it may be proved that *vabc* cannot be greater than $VABC$.

Therefore the pyramids are *equivalent*.

Legendre next establishes a proposition corresponding to Eucl. XII. 7, viz.

4. *Any triangular pyramid is one third of the triangular prism on the same base and of the same height*,

and from this he deduces that

COR. *The volume of a triangular pyramid is equal to a third of the product of its base by its height.*

He has previously proved that the volume of a triangular prism is equal to the product of its base and height, since (1) the prism is half of a parallelepiped of the same height and with a parallelogram for base which is double of the base of the prism, and (2) this parallelepiped can be transformed into an equivalent *rectangular* parallelepiped with the same height and an equivalent base.

The theorem (4) is then extended to *any* pyramid in the proposition

5. *Any pyramid has for its measure the third part of the product of its base and its height*, from which follow

COR. I. *Any pyramid is the third part of the prism on the same base and of the same height.*

COR. II. *Two pyramids of the same height are to one another as their bases, and two pyramids on the same base are to one another as their heights.*

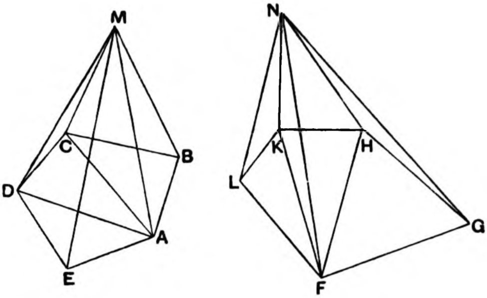
The first part of the second corollary corresponds to the present proposition as extended by the next, XII. 6.

PROPOSITION 6.

Pyramids which are of the same height and have polygonal bases are to one another as the bases.

Let there be pyramids of the same height of which the polygons *ABCDE*, *FGHKL* are the bases and the points *M*, *N* the vertices;

I say that, as the base *ABCDE* is to the base *FGHKL*, so is the pyramid *ABCDEM* to the pyramid *FGHKLN*.



For let *AC*, *AD*, *FH*, *FK* be joined.

Since then *ABCM*, *ACDM* are two pyramids which have triangular bases and equal height,

they are to one another as the bases; [XII. 5]

therefore, as the base *ABC* is to the base *ACD*, so is the pyramid *ABCM* to the pyramid *ACDM*.

And, *componendo*, as the base *ABCD* is to the base *ACD*, so is the pyramid *ABCDM* to the pyramid *ACDM*. [V. 18]

But also, as the base *ACD* is to the base *ADE*, so is the pyramid *ACDM* to the pyramid *ADEM*. [XII. 5]

Therefore, *ex aequali*, as the base *ABCD* is to the base *ADE*, so is the pyramid *ABCDM* to the pyramid *ADEM*. [V. 22]

And again *componendo*, as the base *ABCDE* is to the base *ADE*, so is the pyramid *ABCDEM* to the pyramid *ADEM*. [V. 18]

Similarly also it can be proved that, as the base *FGHKL* is to the base *FGH*, so is the pyramid *FGHKLN* to the pyramid *FGHN*.

And, since $ADEM$, $FGHN$ are two pyramids which have triangular bases and equal height,

therefore, as the base ADE is to the base FGH, so is the pyramid $ADEM$ to the pyramid $FGHN$. [XII. 5]

But, as the base ADE is to the base $ABCDE$, so was the pyramid $ADEM$ to the pyramid $ABCDEM$.

Therefore also, *ex aequali*, as the base $ABCDE$ is to the base FGH, so is the pyramid $ABCDEM$ to the pyramid $FGHN$. [V. 22]

But further, as the base FGH is to the base $FGHKL$, so also was the pyramid $FGHN$ to the pyramid $FGHKLN$.

Therefore also, *ex aequali*, as the base $ABCDE$ is to the base $FGHKL$, so is the pyramid $ABCDEM$ to the pyramid $FGHKLN$. [V. 22]

Q. E. D.

It will be seen that, in order to obtain the proportion

$$(\text{base } ABCDE) : \triangle ADE = (\text{pyramid } MABCDE) : (\text{pyramid } MADE),$$

Euclid employs V. 18 (*componendo*) twice over, with an *ex aequali* step [V. 22] intervening.

We might arrive at it more concisely by using V. 24 extended to any number of antecedents.

Thus

$$\triangle ABC : \triangle ADE = (\text{pyramid } MABC) : (\text{pyramid } MADE),$$

$$\triangle ACD : \triangle ADE = (\text{pyramid } MACD) : (\text{pyramid } MADE),$$

and lastly

$$\triangle ADE : \triangle ADE = (\text{pyramid } MADE) : (\text{pyramid } MADE).$$

Therefore, adding the antecedents [V. 24], we have

$$(\text{polygon } ABCDE) : \triangle ADE = (\text{pyramid } MABCDE) : (\text{pyramid } MADE).$$

Again, since the pyramids $MADE$, $NFGH$ are of the same height,

$$\triangle ADE : \triangle FGH = (\text{pyramid } MADE) : (\text{pyramid } NFGH).$$

Lastly, using the same argument for the pyramid $NFGHKL$ as for $MABCDE$, and inverting, we have

$$\triangle FGH : (\text{polygon } FGHKL) = (\text{pyramid } NFGH) : (\text{pyramid } NFGHKL).$$

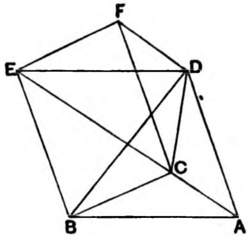
Thus from the three proportions, *ex aequali*,

$$(\text{polygon } ABCDE) : (\text{polygon } FGHKL)$$
$$= (\text{pyramid } MABCDE) : (\text{pyramid } NFGHKL).$$

PROPOSITION 7.

Any prism which has a triangular base is divided into three pyramids equal to one another which have triangular bases.

Let there be a prism in which the triangle *ABC* is the base and *DEF* its opposite;



I say that the prism *ABCDEF* is divided into three pyramids equal to one another, which have triangular bases.

For let *BD*, *EC*, *CD* be joined.

Since *ABED* is a parallelogram, and *BD* is its diameter, therefore the triangle *ABD* is equal to the triangle *EBD*; [I. 34]

therefore also the pyramid of which the triangle *ABD* is the base and the point *C* the vertex is equal to the pyramid of which the triangle *DEB* is the base and the point *C* the vertex. [XII. 5]

But the pyramid of which the triangle *DEB* is the base and the point *C* the vertex is the same with the pyramid of which the triangle *EBC* is the base and the point *D* the vertex;

for they are contained by the same planes.

Therefore the pyramid of which the triangle *ABD* is the base and the point *C* the vertex is also equal to the pyramid of which the triangle *EBC* is the base and the point *D* the vertex.

Again, since *FCBE* is a parallelogram, and *CE* is its diameter, the triangle *CEF* is equal to the triangle *CBE*. [I. 34]

Therefore also the pyramid of which the triangle *BCE* is the base and the point *D* the vertex is equal to the pyramid of which the triangle *ECF* is the base and the point *D* the vertex. [XII. 5]

But the pyramid of which the triangle *BCE* is the base and the point *D* the vertex was proved equal to the pyramid of which the triangle *ABD* is the base and the point *C* the vertex;

therefore also the pyramid of which the triangle CEF is the base and the point D the vertex is equal to the pyramid of which the triangle ABD is the base and the point C the vertex;
therefore the prism $ABCDEF$ has been divided into three pyramids equal to one another which have triangular bases.

And, since the pyramid of which the triangle ABD is the base and the point C the vertex is the same with the pyramid of which the triangle CAB is the base and the point D the vertex,
for they are contained by the same planes,
while the pyramid of which the triangle ABD is the base and the point C the vertex was proved to be a third of the prism in which the triangle ABC is the base and DEF its opposite,
therefore also the pyramid of which the triangle ABC is the base and the point D the vertex is a third of the prism which has the same base, the triangle ABC, and DEF as its opposite.

PORISM. From this it is manifest that any pyramid is a third part of the prism which has the same base with it and equal height.

Q. E. D.

If we denote by C-ABD a pyramid with vertex C and base ABD, Euclid's argument is easily followed thus.

The ▱ $ABED$ being bisected by BD,

(pyramid C-ABD) = (pyramid C-DEB) [XII. 5]
≡ (pyramid D-EBC).

And, the ▱ $EBCF$ being bisected by EC,

(pyramid D-EBC) = (pyramid D-ECF).

Thus (pyramid C-ABD) = (pyramid D-EBC) = (pyramid D-ECF), and these three pyramids make up the whole prism, so that each is one-third of the prism.

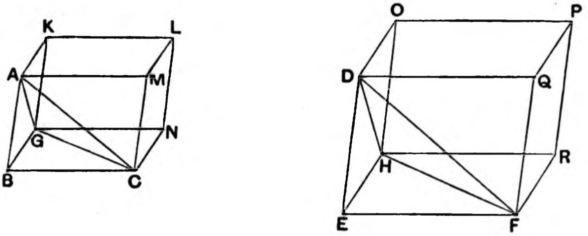
And, since (pyramid C-ABD) ≡ (pyramid D-ABC),
(pyramid D-ABC) = $\frac{1}{3}$ (prism ABC, DEF).

PROPOSITION 8.

Similar pyramids which have triangular bases are in the triplicate ratio of their corresponding sides.

Let there be similar and similarly situated pyramids of which the triangles *ABC*, *DEF* are the bases and the points *G*, *H* the vertices;

I say that the pyramid *ABCG* has to the pyramid *DEFH* the ratio triplicate of that which *BC* has to *EF*.



For let the parallelepipedal solids *BGML*, *EHQP* be completed.

Now, since the pyramid *ABCG* is similar to the pyramid *DEFH*,

therefore the angle *ABC* is equal to the angle *DEF*,

the angle *GBC* to the angle *HEF*,

and the angle *ABG* to the angle *DEH*;

and, as *AB* is to *DE*, so is *BC* to *EF*, and *BG* to *EH*.

And since, as *AB* is to *DE*, so is *BC* to *EF*,

and the sides are proportional about equal angles,

therefore the parallelogram *BM* is similar to the parallelogram *EQ*.

For the same reason

BN is also similar to *ER*, and *BK* to *EO*;

therefore the three parallelograms *MB*, *BK*, *BN* are similar to the three *EQ*, *EO*, *ER*.

But the three parallelograms *MB*, *BK*, *BN* are equal and similar to their three opposites,

and the three *EQ*, *EO*, *ER* are equal and similar to their three opposites. [XI. 24]

Therefore the solids *BGML*, *EHQP* are contained by similar planes equal in multitude.

Therefore the solid *BGML* is similar to the solid *EHQP*.

But similar parallelepipedal solids are in the triplicate ratio of their corresponding sides. [XI. 33]

Therefore the solid *BGML* has to the solid *EHQP* the ratio triplicate of that which the corresponding side *BC* has to the corresponding side *EF*.

But, as the solid *BGML* is to the solid *EHQP*, so is the pyramid *ABCG* to the pyramid *DEFH*,

inasmuch as the pyramid is a sixth part of the solid, because the prism which is half of the parallelepipedal solid [XI. 28] is also triple of the pyramid. [XII. 7]

Therefore the pyramid *ABCG* also has to the pyramid *DEFH* the ratio triplicate of that which *BC* has to *EF*.

Q. E. D.

PORISM. From this it is manifest that similar pyramids which have polygonal bases are also to one another in the triplicate ratio of their corresponding sides.

For, if they are divided into the pyramids contained in them which have triangular bases, by virtue of the fact that the similar polygons forming their bases are also divided into similar triangles equal in multitude and corresponding to the wholes [VI. 20],

then, as the one pyramid which has a triangular base in the one complete pyramid is to the one pyramid which has a triangular base in the other complete pyramid, so also will all the pyramids which have triangular bases contained in the one pyramid be to all the pyramids which have triangular bases contained in the other pyramid [V. 12], that is, the pyramid itself which has a polygonal base to the pyramid which has a polygonal base.

But the pyramid which has a triangular base is to the pyramid which has a triangular base in the triplicate ratio of the corresponding sides;

therefore also the pyramid which has a polygonal base has to the pyramid which has a similar base the ratio triplicate of that which the side has to the side.

It is at once proved that, the pyramids being similar, the parallelepipeds constructed as shown in the figure are also similar.

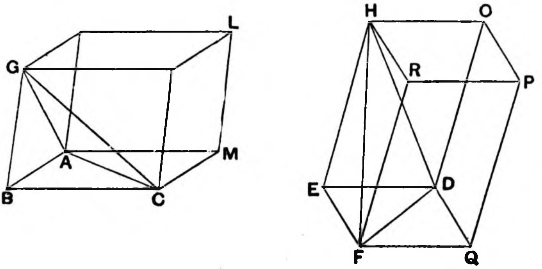
Consequently, as these latter are in the triplicate ratio of their corresponding sides [XI. 33], so are the pyramids which are their sixth parts respectively (being one third of the respective *prisms* on the same bases, i.e. of the halves of the respective parallelepipeds, XI. 28).

As the Porism is not used where Euclid might have been expected to use it (see note on XII. 12, p. 416), there is some reason to doubt its genuineness. P only has it in the margin, though in the first hand.

PROPOSITION 9.

In equal pyramids which have triangular bases the bases are reciprocally proportional to the heights; and those pyramids in which the bases are reciprocally proportional to the heights are equal.

For let there be equal pyramids which have the triangular bases *ABC*, *DEF* and vertices the points *G*, *H*;
I say that in the pyramids *ABCG*, *DEFH* the bases are reciprocally proportional to the heights, that is, as the base *ABC* is to the base *DEF*, so is the height of the pyramid *DEFH* to the height of the pyramid *ABCG*.



For let the parallelepipedal solids *BGML*, *EHQP* be completed.

Now, since the pyramid *ABCG* is equal to the pyramid *DEFH*,
and the solid *BGML* is six times the pyramid *ABCG*,
and the solid *EHQP* six times the pyramid *DEFH*,
therefore the solid *BGML* is equal to the solid *EHQP*.

But in equal parallelepipedal solids the bases are reciprocally proportional to the heights; [XI. 34]
therefore, as the base *BM* is to the base *EQ*, so is the height of the solid *EHQP* to the height of the solid *BGML*.

But, as the base *BM* is to *EQ*, so is the triangle *ABC* to the triangle *DEF*. [I. 34]

Therefore also, as the triangle *ABC* is to the triangle *DEF*, so is the height of the solid *EHQP* to the height of the solid *BGML*. [V. 11]

But the height of the solid *EHQP* is the same with the height of the pyramid *DEFH*,

and the height of the solid *BGML* is the same with the height of the pyramid *ABCG*,

therefore, as the base *ABC* is to the base *DEF*, so is the height of the pyramid *DEFH* to the height of the pyramid *ABCG*.

Therefore in the pyramids *ABCG*, *DEFH* the bases are reciprocally proportional to the heights.

Next, in the pyramids *ABCG*, *DEFH* let the bases be reciprocally proportional to the heights;

that is, as the base *ABC* is to the base *DEF*, so let the height of the pyramid *DEFH* be to the height of the pyramid *ABCG*;

I say that the pyramid *ABCG* is equal to the pyramid *DEFH*.

For, with the same construction,

since, as the base *ABC* is to the base *DEF*, so is the height of the pyramid *DEFH* to the height of the pyramid *ABCG*,

while, as the base *ABC* is to the base *DEF*, so is the parallelogram *BM* to the parallelogram *EQ*,

therefore also, as the parallelogram *BM* is to the parallelogram *EQ*, so is the height of the pyramid *DEFH* to the height of the pyramid *ABCG*. [v. 11]

But the height of the pyramid *DEFH* is the same with the height of the parallelepiped *EHQP*,

and the height of the pyramid *ABCG* is the same with the height of the parallelepiped *BGML*;

therefore, as the base *BM* is to the base *EQ*, so is the height of the parallelepiped *EHQP* to the height of the parallelepiped *BGML*.

But those parallelepipedal solids in which the bases are reciprocally proportional to the heights are equal; [xi. 34]

therefore the parallelepipedal solid *BGML* is equal to the parallelepipedal solid *EHQP*.

And the pyramid *ABCG* is a sixth part of *BGML*, and the pyramid *DEFH* a sixth part of the parallelepiped *EHQP*;

therefore the pyramid *ABCG* is equal to the pyramid *DEFH*.

Therefore etc

Q. E. D.

The volumes of the pyramids are respectively one sixth part of the volumes of the parallelepipeds described, as in the figure, on double the bases and with the same heights as the pyramids.

I. Thus the parallelepipeds are equal if the pyramids are equal.

And, the parallelepipeds being equal, their bases are reciprocally proportional to their heights; [XI. 34]

hence the bases of the equal pyramids (which are the halves of the bases of the parallelepipeds) are proportional to their heights.

II. If the bases of the pyramids are reciprocally proportional to their heights, so are the bases of the parallelepipeds to their heights (since the bases of the parallelepipeds are double of the bases of the pyramids respectively).

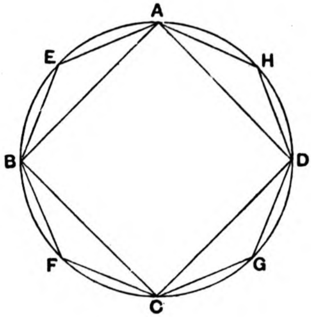
Consequently the parallelepipeds are equal. [XI. 34]

Therefore their sixth parts, the pyramids, are also equal.

PROPOSITION 10.

Any cone is a third part of the cylinder which has the same base with it and equal height.

For let a cone have the same base, namely the circle *ABCD*, with a cylinder and equal height;



I say that the cone is a third part of the cylinder, that is, that the cylinder is triple of the cone.

For if the cylinder is not triple of the cone, the cylinder will be either greater than triple or less than triple of the cone.

First let it be greater than triple,

and let the square *ABCD* be inscribed in the circle *ABCD*; [IV. 6]

then the square *ABCD* is greater than the half of the circle *ABCD*.

From the square *ABCD* let there be set up a prism of equal height with the cylinder.

Then the prism so set up is greater than the half of the cylinder, inasmuch as, if we also circumscribe a square about the circle *ABCD* [IV. 7], the square inscribed in the circle *ABCD* is half of that circumscribed about it,

and the solids set up from them are parallelepipedal prisms of equal height,

while parallelepipedal solids which are of the same height are to one another as their bases; [XI. 32]

therefore also the prism set up on the square *ABCD* is half of the prism set up from the square circumscribed about the circle *ABCD*; [cf. XI. 28, or XII. 6 and 7, Por.]

and the cylinder is less than the prism set up from the square circumscribed about the circle *ABCD*;

therefore the prism set up from the square *ABCD* and of equal height with the cylinder is greater than the half of the cylinder.

Let the circumferences *AB*, *BC*, *CD*, *DA* be bisected at the points *E*, *F*, *G*, *H*,

and let *AE*, *EB*, *BF*, *FC*, *CG*, *GD*, *DH*, *HA* be joined;

then each of the triangles *AEB*, *BFC*, *CGD*, *DHA* is greater than the half of that segment of the circle *ABCD* which is about it, as we proved before. [XII. 2]

On each of the triangles *AEB*, *BFC*, *CGD*, *DHA* let prisms be set up of equal height with the cylinder;

then each of the prisms so set up is greater than the half part of that segment of the cylinder which is about it,

inasmuch as, if we draw through the points *E*, *F*, *G*, *H* parallels to *AB*, *BC*, *CD*, *DA*, complete the parallelograms on *AB*, *BC*, *CD*, *DA*, and set up from them parallelepipedal solids of equal height with the cylinder, the prisms on the triangles *AEB*, *BFC*, *CGD*, *DHA* are halves of the several solids set up;

and the segments of the cylinder are less than the parallelepipedal solids set up;

hence also the prisms on the triangles *AEB*, *BFC*, *CGD*, *DHA* are greater than the half of the segments of the cylinder about them.

Thus, bisecting the circumferences that are left, joining straight lines, setting up on each of the triangles prisms of equal height with the cylinder,

and doing this continually,

we shall leave some segments of the cylinder which will be less than the excess by which the cylinder exceeds the triple of the cone. [X. 1]

Let such segments be left, and let them be *AE*, *EB*, *BF*, *FC*, *CG*, *GD*, *DH*, *HA*;

therefore the remainder, the prism of which the polygon *AEBFCGDH* is the base and the height is the same as that of the cylinder, is greater than triple of the cone.

But the prism of which the polygon *AEBFCGDH* is the base and the height the same as that of the cylinder is triple of the pyramid of which the polygon *AEBFCGDH* is the base and the vertex is the same as that of the cone; [XII. 7, Por.]

therefore also the pyramid of which the polygon *AEBFCGDH* is the base and the vertex is the same as that of the cone is greater than the cone which has the circle *ABCD* as base.

But it is also less, for it is enclosed by it:

which is impossible.

Therefore the cylinder is not greater than triple of the cone.

I say next that neither is the cylinder less than triple of the cone,

For, if possible, let the cylinder be less than triple of the cone,

therefore, inversely, the cone is greater than a third part of the cylinder.

Let the square *ABCD* be inscribed in the circle *ABCD*;

therefore the square *ABCD* is greater than the half of the circle *ABCD*.

Now let there be set up from the square *ABCD* a pyramid having the same vertex with the cone;

therefore the pyramid so set up is greater than the half part of the cone,

seeing that, as we proved before, if we circumscribe a square about the circle, the square *ABCD* will be half of the square circumscribed about the circle,

and if we set up from the squares parallelepipedal solids of equal height with the cone, which are also called prisms, the solid set up from the square *ABCD* will be half of that set up from the square circumscribed about the circle;

for they are to one another as their bases. [XI. 32]

Hence also the thirds of them are in that ratio;

therefore also the pyramid of which the square *ABCD* is the base is half of the pyramid set up from the square circumscribed about the circle.

And the pyramid set up from the square about the circle is greater than the cone,

for it encloses it.

Therefore the pyramid of which the square *ABCD* is the base and the vertex is the same with that of the cone is greater than the half of the cone.

Let the circumferences *AB*, *BC*, *CD*, *DA* be bisected at the points *E*, *F*, *G*, *H*,

and let *AE*, *EB*, *BF*, *FC*, *CG*, *GD*, *DH*, *HA* be joined;

therefore also each of the triangles *AEB*, *BFC*, *CGD*, *DHA* is greater than the half part of that segment of the circle *ABCD* which is about it.

Now, on each of the triangles *AEB*, *BFC*, *CGD*, *DHA* let pyramids be set up which have the same vertex as the cone;

therefore also each of the pyramids so set up is, in the same manner, greater than the half part of that segment of the cone which is about it.

Thus, by bisecting the circumferences that are left, joining straight lines, setting up on each of the triangles a pyramid which has the same vertex as the cone,

and doing this continually,

we shall leave some segments of the cone which will be less than the excess by which the cone exceeds the third part of the cylinder. [X. 1]

Let such be left, and let them be the segments on *AE*, *EB*, *BF*, *FC*, *CG*, *GD*, *DH*, *HA*;

therefore the remainder, the pyramid of which the polygon *AEBFCGDH* is the base and the vertex the same with that of the cone, is greater than a third part of the cylinder.

But the pyramid of which the polygon *AEBFCGDH* is the base and the vertex the same with that of the cone is a third part of the prism of which the polygon *AEBFCGDH* is the base and the height is the same with that of the cylinder;
therefore the prism of which the polygon *AEBFCGDH* is the base and the height is the same with that of the cylinder is greater than the cylinder of which the circle *ABCD* is the base.

But it is also less, for it is enclosed by it:
which is impossible.

Therefore the cylinder is not less than triple of the cone.

But it was proved that neither is it greater than triple;
therefore the cylinder is triple of the cone;
hence the cone is a third part of the cylinder.

Therefore etc.

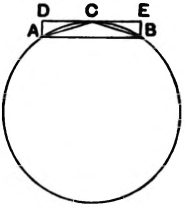
Q. E. D.

We observe the use in this proposition of the term "parallelepipedal prism," which recalls Heron's "parallelogrammic" or "parallel-sided prism."

The course of the proof is exactly the same as in XII. 2, except that an arithmetical fraction takes the place of a ratio which, being incommensurable, could only be expressed as a ratio. Consequently we do not need *proportions* in this proposition, as we did in XII. 2, and shall again in XII. 11, etc.

Euclid *exhausts* the cylinder and cone respectively by setting up prisms and pyramids of the same height on the successive regular polygons inscribed in the circle which is the common base, viz. the square, the regular polygon of 8 sides, that of 16 sides, etc.

If *AB* be the side of one polygon, we obtain two sides of the next by bisecting the arc *ACB* and joining *AC*, *CB*. Draw the tangent *DE* at *C* and complete the parallelogram *ABED*.



Now suppose a prism erected on the polygon of which *AB* is a side, and of the same height as that of the cylinder.

To obtain the prism of the same height on the next polygon we add all the triangular prisms of the same height on the bases *ACB* and the rest.

Now the prism on *ACB* is half the prism of the same height on the ▱ *ABED* as base.

[cf. XI. 28]

And the prism on $\square\, ABED$ includes, and is greater than, the portion of the cylinder standing on the segment ACB of the circle.

The same thing is true in regard to the other sides of the polygon of which AB is one side.

Thus the process begins with a prism on the square inscribed in the circle, which is more than half the cylinder, the next prism (with eight lateral faces) takes away more than half the remainder, and so on;

hence [x. 1], if we proceed far enough, we shall ultimately arrive at a prism leaving over portions of the cylinder together less than any assigned volume.

The construction of pyramids on the successive polygons exhausts the cone in exactly the same way.

Now, if the cone is not equal to one-third of the cylinder, it must be either greater or less.

I. Suppose, if possible, that, V, O being their volumes respectively,

$$O > 3V.$$

Construct successive inscribed polygons in the bases and prisms on them until we arrive at a prism P leaving over portions of the cylinder together less than $(O - 3V)$, i.e. such that

$$O > P > 3V.$$

But P is triple of the pyramid on the same base and of the same height; and this pyramid is included by, and is therefore less than, V;

therefore $P < 3V$.

But, by construction, $P > 3V$:

which is impossible.

Therefore $O \not> 3V$.

II. Suppose, if possible, that $O < 3V$.

Therefore $V > \frac{1}{3}O$.

Construct successive pyramids in the cone in the manner described until we arrive at a pyramid Π leaving over portions of the cone together less than $(V - \frac{1}{3}O)$, i.e. such that

$$V > \Pi > \tfrac{1}{3}O.$$

Now Π is one-third of the prism on the same base and of the same height; and this prism is included by, and is therefore less than, the cylinder;

therefore $\Pi < \frac{1}{3}O$.

But, by construction, $\Pi > \frac{1}{3}O$:

which is impossible.

Therefore O is neither greater nor less than $3V$, so that

$$O = 3V.$$

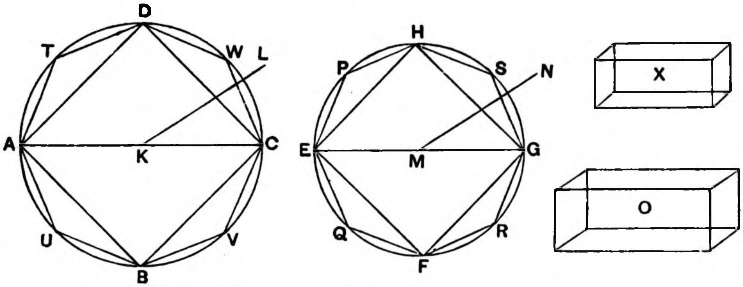
It will be observed that here, as in XII. 2, Euclid always *exhausts* the solid by (as it were) building up to it from inside. Hence the solid to be exhausted must, with him, be supposed *greater* than the solid to which it is to be proved equal; and this is the reason why, in the second part, the initial supposition is turned round.

In this case too Euclid might have approximated to the cone and cylinder by *circumscribing* successive pyramids and prisms in the way shown, after Archimedes, in the note on XII. 2.

PROPOSITION 11.

Cones and cylinders which are of the same height are to one another as their bases.

Let there be cones and cylinders of the same height,
let the circles *ABCD*, *EFGH* be their bases, *KL*, *MN* their axes and *AC*, *EG* the diameters of their bases;
I say that, as the circle *ABCD* is to the circle *EFGH*, so is the cone *AL* to the cone *EN*.



For, if not, then, as the circle *ABCD* is to the circle *EFGH*, so will the cone *AL* be either to some solid less than the cone *EN* or to a greater.

First, let it be in that ratio to a less solid *O*, and let the solid *X* be equal to that by which the solid *O* is less than the cone *EN*;
therefore the cone *EN* is equal to the solids *O*, *X*.

Let the square *EFGH* be inscribed in the circle *EFGH*;
therefore the square is greater than the half of the circle.

Let there be set up from the square *EFGH* a pyramid of equal height with the cone;
therefore the pyramid so set up is greater than the half of the cone,
inasmuch as, if we circumscribe a square about the circle, and set up from it a pyramid of equal height with the cone, the inscribed pyramid is half of the circumscribed pyramid,
for they are to one another as their bases, [xii. 6]
while the cone is less than the circumscribed pyramid.

Let the circumferences *EF*, *FG*, *GH*, *HE* be bisected at the points *P*, *Q*, *R*, *S*,
and let *HP*, *PE*, *EQ*, *QF*, *FR*, *RG*, *GS*, *SH* be joined.

Therefore each of the triangles *HPE*, *EQF*, *FRG*, *GSH* is greater than the half of that segment of the circle which is about it.

On each of the triangles *HPE*, *EQF*, *FRG*, *GSH* let there be set up a pyramid of equal height with the cone;
therefore, also, each of the pyramids so set up is greater than the half of that segment of the cone which is about it.

Thus, bisecting the circumferences which are left, joining straight lines, setting up on each of the triangles pyramids of equal height with the cone,
and doing this continually,
we shall leave some segments of the cone which will be less than the solid *X*. [x. 1]

Let such be left, and let them be the segments on *HP*, *PE*. *EQ*, *QF*, *FR*, *RG*, *GS*, *SH*;
therefore the remainder, the pyramid of which the polygon *HPEQFRGS* is the base and the height the same with that of the cone, is greater than the solid *O*.

Let there also be inscribed in the circle *ABCD* the polygon *DTAUBVCW* similar and similarly situated to the polygon *HPEQFRGS*,
and on it let a pyramid be set up of equal height with the cone *AL*.

Since then, as the square on *AC* is to the square on *EG*, so is the polygon *DTAUBVCW* to the polygon *HPEQFRGS*, [XII. 1]
while, as the square on *AC* is to the square on *EG*, so is the circle *ABCD* to the circle *EFGH*, [XII. 2]
therefore also, as the circle *ABCD* is to the circle *EFGH*, so is the polygon *DTAUBVCW* to the polygon *HPEQFRGS*.

But, as the circle *ABCD* is to the circle *EFGH*, so is the cone *AL* to the solid *O*,
and, as the polygon *DTAUBVCW* is to the polygon *HPEQFRGS*, so is the pyramid of which the polygon *DTAUBVCW* is the base and the point *L* the vertex to the pyramid of which the polygon *HPEQFRGS* is the base and the point *N* the vertex. [XII. 6]

Therefore also, as the cone *AL* is to the solid *O*, so is the pyramid of which the polygon *DTAUBVCW* is the base and the point *L* the vertex to the pyramid of which the polygon *HPEQFRGS* is the base and the point *N* the vertex; [v. 11]
therefore, alternately, as the cone *AL* is to the pyramid in it, so is the solid *O* to the pyramid in the cone *EN*. [v. 16]

But the cone *AL* is greater than the pyramid in it;
therefore the solid *O* is also greater than the pyramid in the cone *EN*.

But it is also less:
which is absurd.

Therefore the cone *AL* is not to any solid less than the cone *EN* as the circle *ABCD* is to the circle *EFGH*.

Similarly we can prove that neither is the cone *EN* to any solid less than the cone *AL* as the circle *EFGH* is to the circle *ABCD*.

I say next that neither is the cone *AL* to any solid greater than the cone *EN* as the circle *ABCD* is to the circle *EFGH*.

For, if possible, let it be in that ratio to a greater solid *O*;
therefore, inversely, as the circle *EFGH* is to the circle *ABCD*, so is the solid *O* to the cone *AL*.

But, as the solid *O* is to the cone *AL*, so is the cone *EN* to some solid less than the cone *AL*;
therefore also, as the circle *EFGH* is to the circle *ABCD*, so is the cone *EN* to some solid less than the cone *AL*:
which was proved impossible.

Therefore the cone *AL* is not to any solid greater than the cone *EN* as the circle *ABCD* is to the circle *EFGH*.

But it was proved that neither is it in this ratio to a less solid;
therefore, as the circle *ABCD* is to the circle *EFGH*, so is the cone *AL* to the cone *EN*.

But, as the cone is to the cone, so is the cylinder to the cylinder,
for each is triple of each; [XII. 10]

Therefore also, as the circle *ABCD* is to the circle *EFGH*, so are the cylinders on them which are of equal height.

Therefore etc.

Q. E. D.

We need not again repeat the preliminary construction of successive pyramids and prisms exhausting the cones and cylinders.

Let Z, Z' be the volumes of the two cones, β, β' their respective bases.

If $$\beta : \beta' \neq Z : Z',$$
then must $$\beta : \beta' = Z : O,$$
where O is either less or greater than Z'.

I. Suppose, if possible, that O is *less* than Z'.

Inscribe in Z' a pyramid (Π') leaving over portions of it together less than $(Z' - O)$, i.e. such that
$$Z' > \Pi' > O.$$

Inscribe in Z a pyramid Π on a polygon inscribed in the circular base of Z similar to the polygon which is the base of Π'.

Now, if d, d' be the diameters of the bases,
$$\beta : \beta' = d^2 : d'^2 \qquad [\text{XII. } 2]$$
$$= (\text{polygon in } \beta) : (\text{polygon in } \beta') \qquad [\text{XII. } 1]$$
$$= \Pi : \Pi'. \qquad [\text{XII. } 6]$$

Therefore $$Z : O = \Pi : \Pi',$$
and, alternately, $$Z : \Pi = O : \Pi'.$$

But $Z > \Pi$, since it includes it;

therefore $$O > \Pi'.$$

But, by construction, $$O < \Pi':$$
which is impossible.

Therefore $$O \nless Z.$$

II. Suppose, if possible, that
$$\beta : \beta' = Z : O,$$
where O is *greater* than Z'.

Therefore $$\beta : \beta' = O' : Z',$$
where O' is some solid less than Z

That is, $$\beta' : \beta = Z' : O',$$
where $O' < Z$.

This is proved impossible exactly in the same way as the assumption in Part I. was proved impossible.

Therefore Z has not either to a less solid than Z' or to a greater solid than Z' the ratio of β to β';

therefore $$\beta : \beta' = Z : Z'.$$

The same is true of the cylinders which are equal to $3Z$, $3Z'$ respectively.

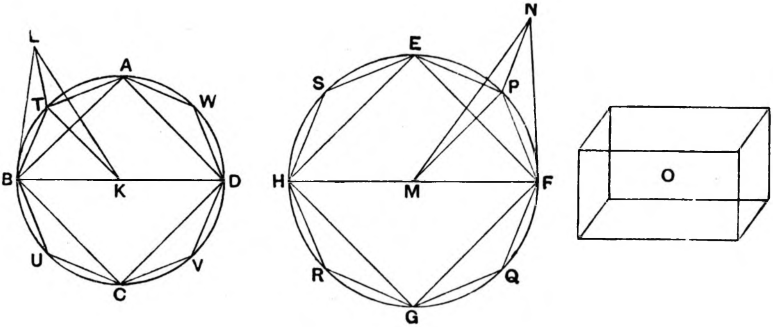
PROPOSITION 12.

Similar cones and cylinders are to one another in the triplicate ratio of the diameters in their bases.

Let there be similar cones and cylinders,

let the circles *ABCD, EFGH* be their bases, *BD, FH* the diameters of the bases, and *KL, MN* the axes of the cones and cylinders;

I say that the cone of which the circle *ABCD* is the base and the point *L* the vertex has to the cone of which the circle *EFGH* is the base and the point *N* the vertex the ratio triplicate of that which *BD* has to *FH*.



For, if the cone *ABCDL* has not to the cone *EFGHN* the ratio triplicate of that which *BD* has to *FH*,

the cone *ABCDL* will have that triplicate ratio either to some solid less than the cone *EFGHN* or to a greater.

First, let it have that triplicate ratio to a less solid *O*.

Let the square *EFGH* be inscribed in the circle *EFGH*; [IV. 6]

therefore the square *EFGH* is greater than the half of the circle *EFGH*.

Now let there be set up on the square *EFGH* a pyramid having the same vertex with the cone;

therefore the pyramid so set up is greater than the half part of the cone.

Let the circumferences *EF*, *FG*, *GH*, *HE* be bisected at the points *P*, *Q*, *R*, *S*,

and let *EP*, *PF*, *FQ*, *QG*, *GR*, *RH*, *HS*, *SE* be joined.

Therefore each of the triangles *EPF*, *FQG*, *GRH*, *HSE* is also greater than the half part of that segment of the circle *EFGH* which is about it.

Now on each of the triangles *EPF*, *FQG*, *GRH*, *HSE* let a pyramid be set up having the same vertex with the cone;

therefore each of the pyramids so set up is also greater than the half part of that segment of the cone which is about it.

Thus, bisecting the circumferences so left, joining straight lines, setting up on each of the triangles pyramids having the same vertex with the cone,

and doing this continually,

we shall leave some segments of the cone which will be less than the excess by which the cone *EFGHN* exceeds the solid *O*. [x. 1]

Let such be left, and let them be the segments on *EP*, *PF*, *FQ*, *QG*, *GR*, *RH*, *HS*, *SE*;

therefore the remainder, the pyramid of which the polygon *EPFQGRHS* is the base and the point *N* the vertex, is greater than the solid *O*.

Let there be also inscribed in the circle *ABCD* the polygon *ATBUCVDW* similar and similarly situated to the polygon *EPFQGRHS*,

and let there be set up on the polygon *ATBUCVDW* a pyramid having the same vertex with the cone;

of the triangles containing the pyramid of which the polygon *ATBUCVDW* is the base and the point *L* the vertex let *LBT* be one,

and of the triangles containing the pyramid of which the polygon *EPFQGRHS* is the base and the point *N* the vertex let *NFP* be one;

and let *KT*, *MP* be joined.

Now, since the cone *ABCDL* is similar to the cone *EFGHN*,

therefore, as *BD* is to *FH*, so is the axis *KL* to the axis *MN*. [XI. Def. 24]

But, as BD is to FH, so is BK to FM;
therefore also, as BK is to FM, so is KL to MN.

And, alternately, as BK is to KL, so is FM to MN. [V. 16]

And the sides are proportional about equal angles, namely the angles BKL, FMN;
therefore the triangle BKL is similar to the triangle FMN. [VI. 6]

Again, since, as BK is to KT, so is FM to MP,
and they are about equal angles, namely the angles BKT, FMP,
inasmuch as, whatever part the angle BKT is of the four right angles at the centre K, the same part also is the angle FMP of the four right angles at the centre M;
since then the sides are proportional about equal angles,
therefore the triangle BKT is similar to the triangle FMP. [VI. 6]

Again, since it was proved that, as BK is to KL, so is FM to MN,
while BK is equal to KT, and FM to PM,
therefore, as TK is to KL, so is PM to MN;
and the sides are proportional about equal angles, namely the angles TKL, PMN, for they are right;
therefore the triangle LKT is similar to the triangle NMP. [VI. 6]

And since, owing to the similarity of the triangles LKB, NMF,
as LB is to BK, so is NF to FM,
and, owing to the similarity of the triangles BKT, FMP,
as KB is to BT, so is MF to FP,
therefore, *ex aequali*, as LB is to BT, so is NF to FP. [V. 22]

Again since, owing to the similarity of the triangles LTK, NPM,
as LT is to TK, so is NP to PM,
and, owing to the similarity of the triangles TKB, PMF,
as KT is to TB, so is MP to PF;
therefore, *ex aequali*, as LT is to TB, so is NP to PF. [V. 22]

But it was also proved that, as *TB* is to *BL*, so is *PF* to *FN*.

Therefore, *ex aequali*, as *TL* is to *LB*, so is *PN* to *NF*. [V. 22]

Therefore in the triangles *LTB*, *NPF* the sides are proportional;

therefore the triangles *LTB*, *NPF* are equiangular; [VI. 5]

hence they are also similar. [VI. Def. 1]

Therefore the pyramid of which the triangle *BKT* is the base and the point *L* the vertex is also similar to the pyramid of which the triangle *FMP* is the base and the point *N* the vertex,

for they are contained by similar planes equal in multitude. [XI. Def. 9]

But similar pyramids which have triangular bases are to one another in the triplicate ratio of their corresponding sides. [XII. 8]

Therefore the pyramid *BKTL* has to the pyramid *FMPN* the ratio triplicate of that which *BK* has to *FM*.

Similarly, by joining straight lines from *A*, *W*, *D*, *V*, *C*, *U* to *K*, and from *E*, *S*, *H*, *R*, *G*, *Q* to *M*, and setting up on each of the triangles pyramids which have the same vertex with the cones,

we can prove that each of the similarly arranged pyramids will also have to each similarly arranged pyramid the ratio triplicate of that which the corresponding side *BK* has to the corresponding side *FM*, that is, which *BD* has to *FH*.

And, as one of the antecedents is to one of the consequents, so are all the antecedents to all the consequents; [V. 12]

therefore also, as the pyramid *BKTL* is to the pyramid *FMPN*, so is the whole pyramid of which the polygon *ATBUCVDW* is the base and the point *L* the vertex to the whole pyramid of which the polygon *EPFQGRHS* is the base and the point *N* the vertex;

hence also the pyramid of which *ATBUCVDW* is the base and the point *L* the vertex has to the pyramid of which the polygon *EPFQGRHS* is the base and the point *N* the vertex the ratio triplicate of that which *BD* has to *FH*.

But, by hypothesis, the cone of which the circle *ABCD* is the base and the point *L* the vertex has also to the solid *O* the ratio triplicate of that which *BD* has to *FH*;

therefore, as the cone of which the circle *ABCD* is the base and the point *L* the vertex is to the solid *O*, so is the pyramid of which the polygon *ATBUCVDW* is the base and *L* the vertex to the pyramid of which the polygon *EPFQGRHS* is the base and the point *N* the vertex;

therefore, alternately, as the cone of which the circle *ABCD* is the base and *L* the vertex is to the pyramid contained in it of which the polygon *ATBUCVDW* is the base and *L* the vertex, so is the solid *O* to the pyramid of which the polygon *EPFQGRHS* is the base and *N* the vertex. [v. 16]

But the said cone is greater than the pyramid in it;

for it encloses it.

Therefore the solid *O* is also greater than the pyramid of which the polygon *EPFQGRHS* is the base and *N* the vertex.

But it is also less:

which is impossible.

Therefore the cone of which the circle *ABCD* is the base and *L* the vertex has not to any solid less than the cone of which the circle *EFGH* is the base and the point *N* the vertex the ratio triplicate of that which *BD* has to *FH*.

Similarly we can prove that neither has the cone *EFGHN* to any solid less than the cone *ABCDL* the ratio triplicate of that which *FH* has to *BD*.

I say next that neither has the cone *ABCDL* to any solid greater than the cone *EFGHN* the ratio triplicate of that which *BD* has to *FH*.

For, if possible, let it have that ratio to a greater solid *O*.

Therefore, inversely, the solid *O* has to the cone *ABCDL* the ratio triplicate of that which *FH* has to *BD*.

But, as the solid *O* is to the cone *ABCDL*, so is the cone *EFGHN* to some solid less than the cone *ABCDL*.

Therefore the cone *EFGHN* also has to some solid less than the cone *ABCDL* the ratio triplicate of that which *FH* has to *BD*:

which was proved impossible.

Therefore the cone *ABCDL* has not to any solid greater than the cone *EFGHN* the ratio triplicate of that which *BD* has to *FH*.

But it was proved that neither has it this ratio to a less solid than the cone *EFGHN*.

Therefore the cone *ABCDL* has to the cone *EFGHN* the ratio triplicate of that which *BD* has to *FH*.

But, as the cone is to the cone, so is the cylinder to the cylinder,

for the cylinder which is on the same base as the cone and of equal height with it is triple of the cone; [XII. 10]

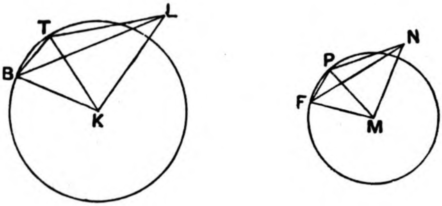
therefore the cylinder also has to the cylinder the ratio triplicate of that which *BD* has to *FH*.

Therefore etc.

Q. E. D.

The method of proof is precisely that of the previous proposition. The only addition is caused by the necessity of proving that, if similar equilateral polygons be inscribed in the bases of two similar cones, and pyramids be erected on them with the same vertices as those of the cones, the pyramids (are similar and) are to one another in the triplicate ratio of corresponding edges.

Let *KL*, *MN* be the axes of the cones, *L*, *N* the vertices, and let *BT*, *FP* be sides of similar polygons inscribed in the bases. Join *BK*, *TK*, *BL*, *TL*, *PM*, *FM*, *PN*, *FN*.



Now *BKL*, *FMN* are right-angled triangles, and, since the cones are similar,

$$BK : KL = FM : MN. \qquad \text{[XI. Def. 24]}$$

Therefore (1) $\triangle$s *BKL*, *FMN* are similar. [VI. 6]

Similarly (2) $\triangle$s *TKL*, *PMN* are similar.

Next, in $\triangle$s *BKT*, *FMP*, the angles *BKT*, *FMP* are equal, since each is the same fraction of four right angles; and the sides about the equal angles are proportional;

therefore (3) $\triangle$s *BKT*, *FMP* are similar.

Again, since from the similar △s *BKL*, *FMN*, and the similar △s *BKT*, *FMP* respectively,

$$LB : BK = NF : FM,$$

$$BK : BT = MF : FP,$$

ex aequali, $LB : BT = NF : FP.$

Similarly $LT : TB = NP : PF.$

Inverting the latter ratio and compounding it with the preceding one, we have, *ex aequali*,

$$LB : LT = NF : NP.$$

Thus in △s *LTB*, *NPF* the sides are proportional in pairs;

therefore (4) △s *LTB*, *NPF* are similar.

Thus the partial pyramids *L-BKT*, *N-FMP* are similar.

In exactly the same way it is proved that all the other partial pyramids are similar.

Now

(pyramid *L-BKT*) : (pyramid *N-FMP*) = ratio triplicate of ($BK : FM$).

The other partial pyramids are to one another in the same triplicate ratio.

The sum of the antecedents is therefore to the sum of the consequents in the same triplicate ratio,

i.e. (pyramid *L-ATBU*...) : (pyramid *N-EPFQ*...)
= ratio triplicate of ratio ($BK : FM$)
= ratio triplicate of ratio ($BD : FH$).

[The fact that Euclid makes this transition from the partial pyramids to the whole pyramids in the body of this proposition seems to me to suggest grave doubts as to the genuineness of the Porism to XII. 8, which contains a similar but rather more general extension from the case of triangular pyramids to pyramids with polygonal bases. Were that Porism genuine, Euclid would have been more likely to refer to it than to repeat here the same arguments which it contains.]

Now we are in a position to apply the method of exhaustion.

If X, X' be the volumes of the cones, d, d' the diameters of their bases, and if

(ratio triplicate of $d : d'$) $\neq X : X'$,

then must (ratio triplicate of $d : d'$) $= X : O$,

where O is either less or greater than X'.

I. Suppose that O is *less* than X'.

Construct in the way described a pyramid (Π') in X' leaving over portions of X' together less than $(X' - O)$, so that $X' > \Pi' > O$,

and construct in X a pyramid (Π), with the same vertex as X has, on a polygon inscribed in its base similar to the base of Π'.

Then, by what has just been proved,

$\Pi : \Pi' =$ (ratio triplicate of $d : d'$)
$= X : O$, by hypothesis,

and, alternately, $\Pi : X = \Pi' : O.$

But X includes, and is therefore greater than, Π;

therefore $O > \Pi'.$

But, by construction, $O < \Pi'$:

which is impossible.

Therefore O cannot be less than X'.

II. Suppose, if possible, that

(ratio triplicate of $d : d'$) $= X : O$,

where O is *greater* than X';

then (ratio triplicate of $d : d'$) $= Z : X'$,

or, inversely, (ratio triplicate of $d' : d$) $= X' : Z$,

where Z is *some solid less than* X.

This is proved impossible by the exact method of Part I.

Hence O cannot be either greater or less than X',

and $X : X' =$ (ratio triplicate of ratio $d : d'$).

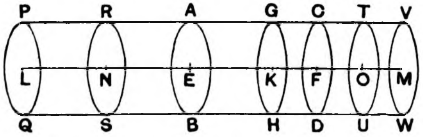
PROPOSITION 13.

If a cylinder be cut by a plane which is parallel to its opposite planes, then, as the cylinder is to the cylinder, so will the axis be to the axis.

For let the cylinder AD be cut by the plane GH which is parallel to the opposite planes AB, CD,

and let the plane GH meet the axis at the point K;

I say that, as the cylinder BG is to the cylinder GD, so is the axis EK to the axis KF.



For let the axis EF be produced in both directions to the points L, M,

and let there be set out any number whatever of axes EN, NL equal to the axis EK,

and any number whatever FO, OM equal to FK;

and let the cylinder PW on the axis LM be conceived of which the circles PQ, VW are the bases.

Let planes be carried through the points N, O parallel to AB, CD and to the bases of the cylinder PW,

and let them produce the circles RS, TU about the centres N, O.

Then, since the axes LN, NE, EK are equal to one another, therefore the cylinders *QR*, *RB*, *BG* are to one another as their bases. [XII. 11]

But the bases are equal;

therefore the cylinders *QR*, *RB*, *BG* are also equal to one another.

Since then the axes *LN*, *NE*, *EK* are equal to one another,

and the cylinders *QR*, *RB*, *BG* are also equal to one another,

and the multitude of the former is equal to the multitude of the latter,

therefore, whatever multiple the axis *KL* is of the axis *EK*, the same multiple also will the cylinder *QG* be of the cylinder *GB*.

For the same reason, whatever multiple the axis *MK* is of the axis *KF*, the same multiple also is the cylinder *WG* of the cylinder *GD*.

And, if the axis *KL* is equal to the axis *KM*, the cylinder *QG* will also be equal to the cylinder *GW*,

if the axis is greater than the axis, the cylinder will also be greater than the cylinder,

and if less, less.

Thus, there being four magnitudes, the axes *EK*, *KF* and the cylinders *BG*, *GD*,

there have been taken equimultiples of the axis *EK* and of the cylinder *BG*, namely the axis *LK* and the cylinder *QG*,

and equimultiples of the axis *KF* and of the cylinder *GD*, namely the axis *KM* and the cylinder *GW*;

and it has been proved that,

if the axis *KL* is in excess of the axis *KM*, the cylinder *QG* is also in excess of the cylinder *GW*,

if equal, equal,

and if less, less.

Therefore, as the axis *EK* is to the axis *KF*, so is the cylinder *BG* to the cylinder *GD*. [V. Def. 5]

Q. E. D.

It is not necessary to reproduce the proof, as it follows exactly the method of VI. 1 and XI. 25.

The fact that cylinders described about axes of equal length and having equal bases are equal is inferred from XII. 11 to the effect that cylinders of equal height are to one another as their bases.

That, of two cylinders with unequal axes but equal bases, the greater is that which has the longer axis is of course obvious either by application or by cutting off from the cylinder with the longer axis a cylinder with an axis of the same length as that of the other given cylinder.

PROPOSITION 14.

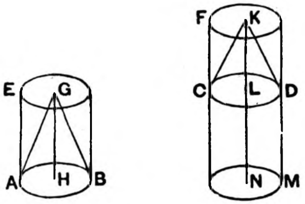
Cones and cylinders which are on equal bases are to one another as their heights.

For let *EB*, *FD* be cylinders on equal bases, the circles *AB*, *CD*;

I say that, as the cylinder *EB* is to the cylinder *FD*, so is the axis *GH* to the axis *KL*.

For let the axis *KL* be produced to the point *N*,

let *LN* be made equal to the axis *GH*,



and let the cylinder *CM* be conceived about *LN* as axis.

Since then the cylinders *EB*, *CM* are of the same height, they are to one another as their bases. [XII. 11]

But the bases are equal to one another:

therefore the cylinders *EB*, *CM* are also equal.

And, since the cylinder *FM* has been cut by the plane *CD* which is parallel to its opposite planes,

therefore, as the cylinder *CM* is to the cylinder *FD*, so is the axis *LN* to the axis *KL*. [XII. 13]

But the cylinder *CM* is equal to the cylinder *EB*,

and the axis *LN* to the axis *GH*;

therefore, as the cylinder *EB* is to the cylinder *FD*, so is the axis *GH* to the axis *KL*.

But, as the cylinder *EB* is to the cylinder *FD*, so is the cone *ABG* to the cone *CDK*. [XII. 10]

Therefore also, as the axis *GH* is to the axis *KL*, so is the cone *ABG* to the cone *CDK* and the cylinder *EB* to the cylinder *FD*. Q. E. D.

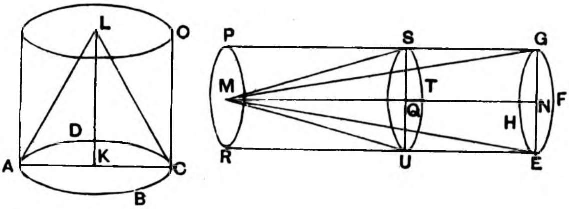
No separate proposition corresponding to this is necessary in the case of parallelepipeds, for XI. 25 really contains the property corresponding to that in this proposition as well as the property corresponding to that in XII. 13.

PROPOSITION 15.

In equal cones and cylinders the bases are reciprocally proportional to the heights; and those cones and cylinders in which the bases are reciprocally proportional to the heights are equal.

Let there be equal cones and cylinders of which the circles *ABCD*, *EFGH* are the bases;
let *AC*, *EG* be the diameters of the bases,
and *KL*, *MN* the axes, which are also the heights of the cones or cylinders;
let the cylinders *AO*, *EP* be completed.

I say that in the cylinders *AO*, *EP* the bases are reciprocally proportional to the heights,
that is, as the base *ABCD* is to the base *EFGH*, so is the height *MN* to the height *KL*.



For the height *LK* is either equal to the height *MN* or not equal.

First, let it be equal.

Now the cylinder *AO* is also equal to the cylinder *EP*.

But cones and cylinders which are of the same height are to one another as their bases; [XII. 11]
therefore the base *ABCD* is also equal to the base *EFGH*.

Hence also, reciprocally, as the base *ABCD* is to the base *EFGH*, so is the height *MN* to the height *KL*.

Next, let the height *LK* not be equal to *MN*,
but let *MN* be greater;
from the height *MN* let *QN* be cut off equal to *KL*,
through the point *Q* let the cylinder *EP* be cut by the plane *TUS* parallel to the planes of the circles *EFGH*, *RP*,

and let the cylinder *ES* be conceived erected from the circle *EFGH* as base and with height *NQ*.

Now, since the cylinder *AO* is equal to the cylinder *EP*, therefore, as the cylinder *AO* is to the cylinder *ES*, so is the cylinder *EP* to the cylinder *ES*. [v. 7]

But, as the cylinder *AO* is to the cylinder *ES*, so is the base *ABCD* to the base *EFGH*,
for the cylinders *AO*, *ES* are of the same height; [XII. 11]
and, as the cylinder *EP* is to the cylinder *ES*, so is the height *MN* to the height *QN*,
for the cylinder *EP* has been cut by a plane which is parallel to its opposite planes. [XII. 13]

Therefore also, as the base *ABCD* is to the base *EFGH*, so is the height *MN* to the height *QN*. [v. 11]

But the height *QN* is equal to the height *KL*;
therefore, as the base *ABCD* is to the base *EFGH*, so is the height *MN* to the height *KL*.

Therefore in the cylinders *AO*, *EP* the bases are reciprocally proportional to the heights.

Next, in the cylinders *AO*, *EP* let the bases be reciprocally proportional to the heights,
that is, as the base *ABCD* is to the base *EFGH*, so let the height *MN* be to the height *KL*;
I say that the cylinder *AO* is equal to the cylinder *EP*.

For, with the same construction,
since, as the base *ABCD* is to the base *EFGH*, so is the height *MN* to the height *KL*,
while the height *KL* is equal to the height *QN*,
therefore, as the base *ABCD* is to the base *EFGH*, so is the height *MN* to the height *QN*

But, as the base *ABCD* is to the base *EFGH*, so is the cylinder *AO* to the cylinder *ES*,
for they are of the same height; [XII. 11]
and, as the height *MN* is to *QN*, so is the cylinder *EP* to the cylinder *ES*; [XII. 13]
therefore, as the cylinder *AO* is to the cylinder *ES*, so is the cylinder *EP* to the cylinder *ES*. [v. 11]

Therefore the cylinder AO is equal to the cylinder EP. [V. 9]

And the same is true for the cones also.

Q. E. D.

I. If the heights of the two cylinders are equal, and their volumes are equal, the bases are equal, since the latter are proportional to the volumes. [XII. 11]

If the heights are *not* equal, cut off from the higher cylinder a cylinder of the same height as the lower.

Then, if LK, QN be the equal heights,
we have, by XII. 11,

$$\begin{aligned}(\text{base } ABCD) : (\text{base } EFGH) &= (\text{cylinder } AO) : (\text{cylinder } ES)\\ &= (\text{cylinder } EP) : (\text{cylinder } ES),\\ &\qquad\qquad \text{by hypothesis,}\\ &= MN : QN \qquad\qquad [\text{XII. } 13]\\ &= MN : KL.\end{aligned}$$

II. In the converse part of the proposition, Euclid omits the case where the cylinders have equal heights. In this case of course the reciprocal ratios are both ratios of equality; the bases are therefore equal, and consequently the cylinders.

If the heights are *not* equal, we have, with the same construction as before,

$$(\text{base } ABCD) : (\text{base } EFGH) = MN : KL.$$

But [XII. 11]

$$(\text{base } ABCD) : (\text{base } EFGH) = (\text{cylinder } AO) : (\text{cylinder } ES),$$

and

$$\begin{aligned}MN : KL &= MN : QN\\ &= (\text{cylinder } EP) : (\text{cylinder } ES). \qquad [\text{XII. } 13]\end{aligned}$$

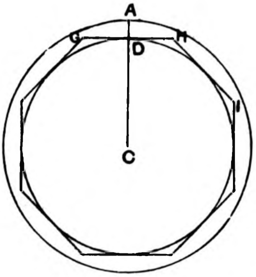
Therefore

$$(\text{cylinder } AO) : (\text{cylinder } ES) = (\text{cylinder } EP) : (\text{cylinder } ES),$$

and consequently $(\text{cylinder } AO) = (\text{cylinder } EP)$.

Similarly for the cones, which are equal to one-third of the cylinders respectively.

Legendre deduces these propositions about cones and cylinders from two others which he establishes by a method similar to that adopted by him for the theorem of XII. 2 (see note on that proposition).



The first (for the cylinder) is as follows.

The volume of a cylinder is equal to the product of its base by its height.

Suppose CA to be the radius of the base of the given cylinder, h its height.

For brevity let us denote by (surf. CA) the area of the circle of which CA is the radius.

If (surf. CA) $\times h$ is not the measure of the given cylinder, it will be the measure of a cylinder greater or less than it.

I. First let it be the measure of a less cylinder, that, for example, of which the circle with radius CD is the base, and h is the height.

Circumscribe about the circle with radius CD a regular polygon GHI... such that its sides do not anywhere meet the circle with radius CA. [See note on XII. 2, p. 393 above, for Legendre's lemma relating to this construction.]

Imagine a prism erected on the polygon as base and with height h.

Then (volume of prism) = (polygon GHI...) × h.

[Legendre has previously proved this proposition, first for a parallelepiped (by transforming it into a rectangular one), then for a triangular prism (half of a parallelepiped of the same height), and lastly for a prism with a polygonal base.]

But (polygon GHI...) < (surf. CA).

Therefore (volume of prism) < (surf. CA) × h

< (cylinder on circle of rad. CD),

by hypothesis.

But the prism is *greater* than the latter cylinder, since it includes it: which is impossible.

II. In order not to multiply figures let us, in this second case, suppose that CD is the radius of the base of the given cylinder, and that (surf. CD) × h is the measure of a cylinder greater than it, e.g. a cylinder on the circle with radius CA as base and of height h.

Then, with the same construction,

(volume of prism) = (polygon GHI...) × h.

And (polygon GHI...) > (surf. CD).

Therefore (volume of prism) > (surf. CD) × h

> (cylinder on surf. CA), by hypothesis.

But the volume of the prism is also *less* than that cylinder, being included by it:

which is impossible.

Therefore (volume of cylinder) = (its base) × (its height).

It follows as a corollary that

Cylinders of the same height are to one another as their bases [XII. 13], *and cylinders on the same base are to one another as their heights* [XII. 14].

Also

Similar cylinders are as the cubes of their heights, or as the cubes of the diameters of their bases [Eucl. XII. 12].

For the bases are as the squares on their diameters; and, since the cylinders are similar, the diameters of the bases are as their heights.

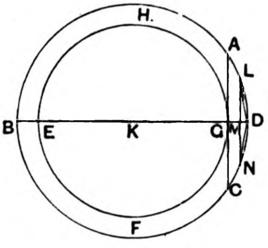
Therefore the bases are as the squares on the heights, and the bases multiplied by the heights, or the cylinders themselves, are as the cubes of the heights.

I need not reproduce Legendre's proofs of the corresponding propositions for the cone.

PROPOSITION 16.

Given two circles about the same centre, to inscribe in the greater circle an equilateral polygon with an even number of sides which does not touch the lesser circle.

Let *ABCD*, *EFGH* be the two given circles about the same centre *K*;



thus it is required to inscribe in the greater circle *ABCD* an equilateral polygon with an even number of sides which does not touch the circle *EFGH*.

For let the straight line *BKD* be drawn through the centre *K*,

and from the point *G* let *GA* be drawn at right angles to the straight line *BD* and carried through to *C*;

therefore *AC* touches the circle *EFGH*. [III. 16, Por.]

Then, bisecting the circumference *BAD*, bisecting the half of it, and doing this continually, we shall leave a circumference less than *AD*. [X. 1]

Let such be left, and let it be *LD*;

from *L* let *LM* be drawn perpendicular to *BD* and carried through to *N*,

and let *LD*, *DN* be joined;

therefore *LD* is equal to *DN*. [III. 3, I. 4]

Now, since *LN* is parallel to *AC*,

and *AC* touches the circle *EFGH*,

therefore *LN* does not touch the circle *EFGH*;

therefore *LD*, *DN* are far from touching the circle *EFGH*.

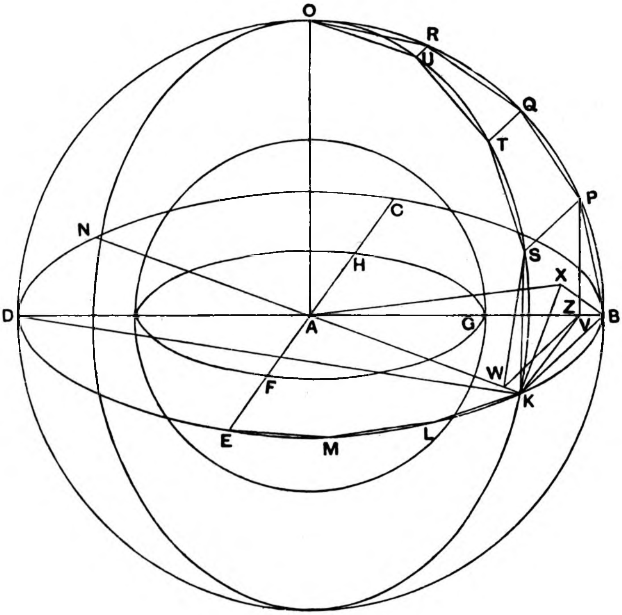
If then we fit into the circle *ABCD* straight lines equal to the straight line *LD* and placed continuously, there will be inscribed in the circle *ABCD* an equilateral polygon with an even number of sides which does not touch the lesser circle *EFGH*. Q. E. F.

It must be carefully observed that the polygon inscribed in the outer circle in this proposition is such that not only do its own sides not touch the inner circle, but also *the chords*, as *LN*, *joining angular points next but one to each other do not touch the inner circle either*. In other words, the polygon is the second in order, not the first, which satisfies the condition of the enunciation. This is important, because such a polygon is wanted in the next proposition; hence in that proposition the *exact* construction here given must be followed.

PROPOSITION 17.

Given two spheres about the same centre, to inscribe in the greater sphere a polyhedral solid which does not touch the lesser sphere at its surface.

Let two spheres be conceived about the same centre *A*; thus it is required to inscribe in the greater sphere a polyhedral solid which does not touch the lesser sphere at its surface.



Let the spheres be cut by any plane through the centre; then the sections will be circles,

inasmuch as the sphere was produced by the diameter remaining fixed and the semicircle being carried round it;

[XI. Def. 14]

hence, in whatever position we conceive the semicircle to be, the plane carried through it will produce a circle on the circumference of the sphere.

And it is manifest that this circle is the greatest possible, inasmuch as the diameter of the sphere, which is of course the diameter both of the semicircle and of the circle, is greater than all the straight lines drawn across in the circle or the sphere.

Let then *BCDE* be the circle in the greater sphere,
and *FGH* the circle in the lesser sphere;
let two diameters in them, *BD*, *CE*, be drawn at right angles to one another;
then, given the two circles *BCDE*, *FGH* about the same centre, let there be inscribed in the greater circle *BCDE* an equilateral polygon with an even number of sides which does not touch the lesser circle *FGH*,
let *BK*, *KL*, *LM*, *ME* be its sides in the quadrant *BE*.
let *KA* be joined and carried through to *N*,
let *AO* be set up from the point *A* at right angles to the plane of the circle *BCDE*, and let it meet the surface of the sphere at *O*,
and through *AO* and each of the straight lines *BD*, *KN* let planes be carried;
they will then make greatest circles on the surface of the sphere, for the reason stated.

Let them make such,
and in them let *BOD*, *KON* be the semicircles on *BD*, *KN*.

Now, since *OA* is at right angles to the plane of the circle *BCDE*,
therefore all the planes through *OA* are also at right angles to the plane of the circle *BCDE*; [XI. 18]
hence the semicircles *BOD*, *KON* are also at right angles to the plane of the circle *BCDE*.

And, since the semicircles *BED*, *BOD*, *KON* are equal,
for they are on the equal diameters *BD*, *KN*,
therefore the quadrants *BE*, *BO*, *KO* are also equal to one another.

Therefore there are as many straight lines in the quadrants *BO*, *KO* equal to the straight lines *BK*, *KL*, *LM*, *ME* as there are sides of the polygon in the quadrant *BE*.

Let them be inscribed, and let them be *BP*, *PQ*, *QR*, *RO* and *KS*, *ST*, *TU*, *UO*,
let *SP*, *TQ*, *UR* be joined,

and from *P*, *S* let perpendiculars be drawn to the plane of the circle *BCDE*; [XI. 11]

these will fall on *BD*, *KN*, the common sections of the planes, inasmuch as the planes of *BOD*, *KON* are also at right angles to the plane of the circle *BCDE*. [cf. XI. Def. 4]

Let them so fall, and let them be *PV*, *SW*,
and let *WV* be joined.

Now since, in the equal semicircles *BOD*, *KON*, equal straight lines *BP*, *KS* have been cut off,
and the perpendiculars *PV*, *SW* have been drawn,
therefore *PV* is equal to *SW*, and *BV* to *KW*. [III. 27, I. 26]

But the whole *BA* is also equal to the whole *KA*;
therefore the remainder *VA* is also equal to the remainder *WA*;
therefore, as *BV* is to *VA*, so is *KW* to *WA*;
therefore *WV* is parallel to *KB*. [VI. 2]

And, since each of the straight lines *PV*, *SW* is at right angles to the plane of the circle *BCDE*,
therefore *PV* is parallel to *SW*. [XI. 6]

But it was also proved equal to it;
therefore *WV*, *SP* are also equal and parallel. [I. 33]

And, since *WV* is parallel to *SP*,
while *WV* is parallel to *KB*,
therefore *SP* is also parallel to *KB*. [XI. 9]

And *BP*, *KS* join their extremities;
therefore the quadrilateral *KBPS* is in one plane,
inasmuch as, if two straight lines be parallel, and points be taken at random on each of them, the straight line joining the points is in the same plane with the parallels. [XI. 7]

For the same reason
each of the quadrilaterals *SPQT*, *TQRU* is also in one plane.

But the triangle *URO* is also in one plane. [XI. 2]

If then we conceive straight lines joined from the points *P*, *S*, *Q*, *T*, *R*, *U* to *A*, there will be constructed a certain polyhedral solid figure between the circumferences *BO*, *KO*, consisting of pyramids of which the quadrilaterals *KBPS*, *SPQT*, *TQRU* and the triangle *URO* are the bases and the point *A* the vertex.

And, if we make the same construction in the case of each of the sides KL, LM, ME as in the case of BK, and further in the case of the remaining three quadrants,

there will be constructed a certain polyhedral figure inscribed in the sphere and contained by pyramids, of which the said quadrilaterals and the triangle URO, and the others corresponding to them, are the bases and the point A the vertex.

I say that the said polyhedron will not touch the lesser sphere at the surface on which the circle FGH is.

Let AX be drawn from the point A perpendicular to the plane of the quadrilateral $KBPS$, and let it meet the plane at the point X; [XI. 11]

let XB, XK be joined.

Then, since AX is at right angles to the plane of the quadrilateral $KBPS$,

therefore it is also at right angles to all the straight lines which meet it and are in the plane of the quadrilateral. [XI. Def. 3]

Therefore AX is at right angles to each of the straight lines BX, XK.

And, since AB is equal to AK,

the square on AB is also equal to the square on AK.

And the squares on AX, XB are equal to the square on AB,

for the angle at X is right; [I. 47]

and the squares on AX, XK are equal to the square on AK. [*id.*]

Therefore the squares on AX, XB are equal to the squares on AX, XK.

Let the square on AX be subtracted from each;

therefore the remainder, the square on BX, is equal to the remainder, the square on XK;

therefore BX is equal to XK.

Similarly we can prove that the straight lines joined from X to P, S are equal to each of the straight lines BX, XK.

Therefore the circle described with centre X and distance one of the straight lines XB, XK will pass through P, S also, and $KBPS$ will be a quadrilateral in a circle.

Now, since KB is greater than WV,
while WV is equal to SP,
therefore KB is greater than SP.

But KB is equal to each of the straight lines KS, BP;
therefore each of the straight lines KS, BP is greater than SP.

And, since $KBPS$ is a quadrilateral in a circle,
and KB, BP, KS are equal, and PS less,
and BX is the radius of the circle,
therefore the square on KB is greater than double of the square on BX.

Let KZ be drawn from K perpendicular to BV.

Then, since BD is less than double of DZ,
and, as BD is to DZ, so is the rectangle DB, BZ to the rectangle DZ, ZB,
if a square be described upon BZ and the parallelogram on ZD be completed,
then the rectangle DB, BZ is also less than double of the rectangle DZ, ZB.

And, if KD be joined,
the rectangle DB, BZ is equal to the square on BK,
and the rectangle DZ, ZB equal to the square on KZ;
[III. 31, VI. 8 and Por.]
therefore the square on KB is less than double of the square on KZ.

But the square on KB is greater than double of the square on BX;
therefore the square on KZ is greater than the square on BX.

And, since BA is equal to KA,
the square on BA is equal to the square on AK.

And the squares on BX, XA are equal to the square on BA,
and the squares on KZ, ZA equal to the square on KA;
[I. 47]
therefore the squares on BX, XA are equal to the squares on KZ, ZA,

and of these the square on KZ is greater than the square on BX;

therefore the remainder, the square on ZA, is less than the square on XA.

Therefore AX is greater than AZ;

therefore AX is much greater than AG.

And AX is the perpendicular on one base of the polyhedron,

and AG on the surface of the lesser sphere;

hence the polyhedron will not touch the lesser sphere on its surface.

Therefore, given two spheres about the same centre, a polyhedral solid has been inscribed in the greater sphere which does not touch the lesser sphere at its surface.

Q. E. F.

PORISM. But if in another sphere also a polyhedral solid be inscribed similar to the solid in the sphere $BCDE$,

the polyhedral solid in the sphere $BCDE$ has to the polyhedral solid in the other sphere the ratio triplicate of that which the diameter of the sphere $BCDE$ has to the diameter of the other sphere.

For, the solids being divided into their pyramids similar in multitude and arrangement, the pyramids will be similar.

But similar pyramids are to one another in the triplicate ratio of their corresponding sides; [XII. 8, Por.]

therefore the pyramid of which the quadrilateral $KBPS$ is the base, and the point A the vertex, has to the similarly arranged pyramid in the other sphere the ratio triplicate of that which the corresponding side has to the corresponding side, that is, of that which the radius AB of the sphere about A as centre has to the radius of the other sphere.

Similarly also each pyramid of those in the sphere about A as centre has to each similarly arranged pyramid of those in the other sphere the ratio triplicate of that which AB has to the radius of the other sphere.

And, as one of the antecedents is to one of the consequents, so are all the antecedents to all the consequents;

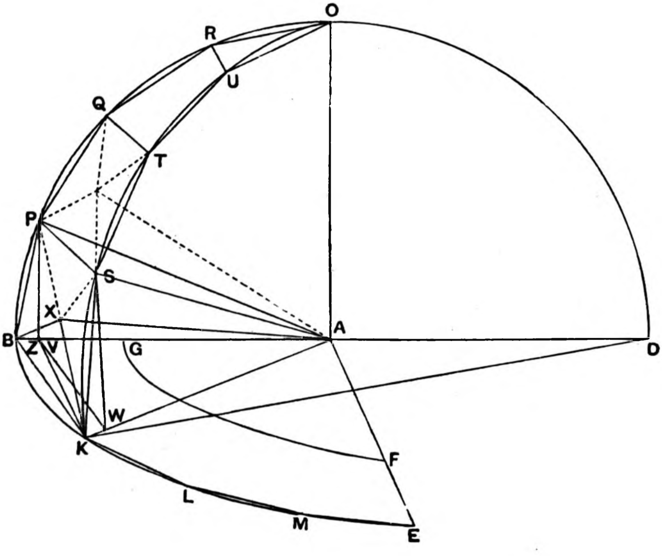
[V. 12]

hence the whole polyhedral solid in the sphere about *A* as centre has to the whole polyhedral solid in the other sphere the ratio triplicate of that which *AB* has to the radius of the other sphere, that is, of that which the diameter *BD* has to the diameter of the other sphere.

Q. E. D.

This proposition is of great length and therefore requires summarising in order to make it easier to grasp. Moreover there are some assumptions in it which require to be proved, and some omissions to be supplied. The figure also is one of some complexity, and, in addition, the text and the figure treat two points *Z* and *V*, which are really one and the same, as different.

The first thing needed is to know that all sections of a sphere by planes through the centre are circles and equal to one another (great circles or "greatest circles" as Euclid calls them, more appropriately). Euclid uses his definition of a sphere as the figure described by a semicircle revolving about its diameter. This of course establishes that all planes through the particular diameter make equal circular sections; but it is also assumed that the same sphere is generated by *any other* semicircle of the same size and with its centre at the same point.



The construction and argument of the proposition may be shortly given as follows.

A plane through the centre of two concentric spheres cuts them in great circles of which *BE*, *GF* are quadrants.

A regular polygon with an even number of sides is inscribed (*exactly* as in Prop. 16) to the outer circle such that its sides do not touch the inner circle. *BK*, *KL*, *LM*, *ME* are the sides in the quadrant *BE*.

AO is drawn at right angles to the plane ABE, and through AO are drawn planes passing through B, K, L, M, E, etc., cutting the sphere in great circles.

OB, OK are quadrants of two of these great circles.

As these quadrants are equal to the quadrant BE, they will be divisible into arcs equal in number and magnitude to the arcs BK, KL, LM, ME.

Dividing the other quadrants of these circles, and also all the quadrants of the other circles through OA, in this way we shall have in all the circles a polygon equal to that in the circle of which BE is a quadrant.

BP, PQ, QR, RO and KS, ST, TU, UO are the sides of these polygons in the quadrants BO, KO.

Joining PS, QT, RU, and making the same construction all round the circles through AO, we have a certain polyhedron inscribed in the outer sphere.

Draw PV perpendicular to AB and therefore (since the planes OAB, BAE are at right angles) perpendicular to the plane BAE; [XI. Def. 4] draw SW perpendicular to AK and therefore (for a like reason) perpendicular to the plane BAE.

Draw KZ perpendicular to BA. (Since $BK = BP$, and $DB \cdot BV = BP^2$, $DB \cdot BZ = BK^2$, it follows that $BV = BZ$, and Z, V coincide.)

Now, since $\angle$s PAV, SAW, being angles subtended at the centre by equal arcs of equal circles, are equal,

and since $\angle$s PVA, SWA are right,

while $AS = AP$,

$\triangle$s PAV, SAW are equal in all respects, [I. 26]

and $AV = AW$.

Consequently $AB : AV = AK : AW$;

and VW, BK are parallel.

But PV, SW are parallel (being both perpendicular to one plane) and equal (by the equal $\triangle$s PAV, SAW),

therefore VW, PS are equal and parallel.

Therefore BK (being parallel to VW) is parallel to PS.

Consequently (1) $BPSK$ is a quadrilateral in *one plane*.

Similarly the other quadrilaterals $PQTS$, $QRUT$ are in one plane; and the triangle ORU is in one plane.

In order now to prove that the plane BPSK *does not anywhere touch the inner sphere we have to prove that the shortest distance from* A *to the plane is greater than* AZ, *which by the construction in* XII. 16 *is greater than* AG.

Draw AX perpendicular to the plane $BPSK$.

Then $AX^2 + XB^2 = AX^2 + XK^2 = AX^2 + XS^2 = AX^2 + XP^2 = AB^2$,

whence $XB = XK = XS = XP$,

or (2) the quadrilateral $BPSK$ is inscribable in a circle with X as centre and radius XB.

Now
$$BK > VW$$
$$> PS;$$

therefore in the quadrilateral $BPSK$ three sides BK, BP, KS are equal, but PS is less.

Consequently the angles about X are three equal angles and one smaller angle;

therefore any one of the equal angles is greater than a right angle, i.e. $\angle BXK$ is obtuse.

Therefore (3) $BK^2 > 2BX^2$. [II. 12]

Next, consider the semicircle BKD with KZ drawn perpendicular to BD.

We have $BD < 2DZ$,

so that $DB \cdot BZ < 2DZ \cdot ZB$,

or $BK^2 < 2KZ^2$;

therefore, *a fortiori*, [by (3) above]

(4) $BX^2 < KZ^2$.

Now $AK^2 = AB^2$;

therefore $AZ^2 + ZK^2 = AX^2 + XB^2$.

And $BX^2 < KZ^2$;

therefore $AX^2 > AZ^2$,

or (5) $AX > AZ$.

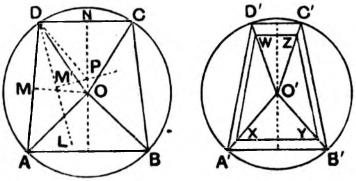
But, by the construction in XII. 16, $AZ > AG$; therefore, *a fortiori*, $AX > AG$.

And, since the perpendicular AX is the shortest distance from A to the plane $BPSK$,

(6) the plane $BPSK$ does not anywhere meet the inner sphere.

Euclid omits to prove that, *a fortiori*, the other quadrilaterals $PQTS$, $QRUT$, and the triangle ROU, do not anywhere meet the inner sphere.

For this purpose it is only necessary to show that the radii of the circles circumscribing $BPSK$, $PQTS$, $QRUT$ and ROU are in descending order of magnitude.



We have therefore to prove that, if $ABCD$, $A'B'C'D'$ are two quadrilaterals inscribable in circles, and

$$AD = BC = A'D' = B'C',$$

while AB is not greater than AD, $A'B' = CD$, and $AB > CD > C'D'$,

then the radius OA of the circle circumscribing the first quadrilateral is greater than the radius $O'A'$ of the circle circumscribing the second.

Clavius, and Simson after him, prove this by *reductio ad absurdum*.

(1) If $OA = O'A'$,

it follows that $\angle$s AOD, BOC, $A'O'D'$, $B'O'C'$ are all equal.

Also $\angle AOB > \angle A'O'B'$,

$\angle COD > \angle C'O'D'$,

whence the four angles about O are together greater than the four angles about O', i.e. greater than four right angles;

which is impossible.

(2) If $O'A' > OA$,

cut off from $O'A'$, $O'B'$, $O'C'$, $O'D'$ lengths equal to OA, and draw the inner quadrilateral as shown in the figure ($XYZW$).

Then

$$\begin{aligned} AB &> A'B' > XY, \\ CD &> C'D' > ZW, \\ AD &= A'D' > WX, \\ BC &= B'C' > YZ. \end{aligned}$$

Consequently the same absurdity as in (1) follows *a fortiori*.

Therefore, since OA is neither equal to nor less than $O'A'$,

$$OA > O'A'.$$

The fact is also sufficiently clear if we draw MO, NO bisecting DA, DC perpendicularly and therefore meeting in O, the centre of the circumscribed circle, and then suppose the side DA with the perpendicular MO to turn inwards about D as centre. Then the intersection of MO and NO, as P, will gradually move towards N.

Simson gives his proof as "Lemma II." immediately before XII. 17. He adds to the Porism some words explaining how we may construct a similar polyhedron in another sphere and how we may prove that the polyhedra are similar.

The Porism is of course of the essence of the matter because it is the porism which as much as the construction is wanted in the next proposition. It would therefore not have been amiss to include the Porism in the enunciation of XII. 17 so as to call attention to it.

PROPOSITION 18.

Spheres are to one another in the triplicate ratio of their respective diameters.

Let the spheres ABC, DEF be conceived,

and let BC, EF be their diameters;

I say that the sphere ABC has to the sphere DEF the ratio triplicate of that which BC has to EF.

For, if the sphere ABC has not to the sphere DEF the ratio triplicate of that which BC has to EF,

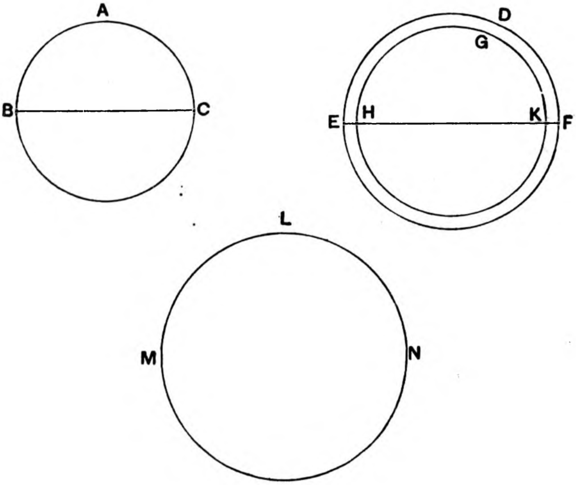
then the sphere ABC will have either to some less sphere than the sphere DEF, or to a greater, the ratio triplicate of that which BC has to EF.

First, let it have that ratio to a less sphere GHK,

let DEF be conceived about the same centre with GHK,

let there be inscribed in the greater sphere DEF a polyhedral solid which does not touch the lesser sphere GHK at its surface, [XII. 17] and let there also be inscribed in the sphere *ABC* a polyhedral solid similar to the polyhedral solid in the sphere *DEF*;

therefore the polyhedral solid in *ABC* has to the polyhedral solid in *DEF* the ratio triplicate of that which *BC* has to *EF*. [XII. 17, Por.]



But the sphere *ABC* also has to the sphere *GHK* the ratio triplicate of that which *BC* has to *EF*;

therefore, as the sphere *ABC* is to the sphere *GHK*, so is the polyhedral solid in the sphere *ABC* to the polyhedral solid in the sphere *DEF*;

and, alternately, as the sphere *ABC* is to the polyhedron in it, so is the sphere *GHK* to the polyhedral solid in the sphere *DEF*. [V. 16]

But the sphere *ABC* is greater than the polyhedron in it;

therefore the sphere *GHK* is also greater than the polyhedron in the sphere *DEF*.

But it is also less,

for it is enclosed by it.

Therefore the sphere *ABC* has not to a less sphere than the sphere *DEF* the ratio triplicate of that which the diameter *BC* has to *EF*.

Similarly we can prove that neither has the sphere *DEF* to a less sphere than the sphere *ABC* the ratio triplicate of that which *EF* has to *BC*.

I say next that neither has the sphere *ABC* to any greater sphere than the sphere *DEF* the ratio triplicate of that which *BC* has to *EF*.

For, if possible, let it have that ratio to a greater, *LMN*;

therefore, inversely, the sphere *LMN* has to the sphere *ABC* the ratio triplicate of that which the diameter *EF* has to the diameter *BC*.

But, inasmuch as *LMN* is greater than *DEF*,

therefore, as the sphere *LMN* is to the sphere *ABC*, so is the sphere *DEF* to some less sphere than the sphere *ABC*, as was before proved. [XII. 2, Lemma]

Therefore the sphere *DEF* also has to some less sphere than the sphere *ABC* the ratio triplicate of that which *EF* has to *BC*:

which was proved impossible.

Therefore the sphere *ABC* has not to any sphere greater than the sphere *DEF* the ratio triplicate of that which *BC* has to *EF*.

But it was proved that neither has it that ratio to a less sphere.

Therefore the sphere *ABC* has to the sphere *DEF* the ratio triplicate of that which *BC* has to *EF*.

Q. E. D.

It is the method of this proposition which Legendre adopted for his proof of XII. 2 (see note on that proposition).

The argument can be put very shortly. We will suppose S, S' to be the volumes of the spheres, and d, d' to be their diameters; and we will for brevity express the triplicate ratio of d to d' by $d^3 : d'^3$.

If $$d^3 : d'^3 \neq S : S',$$

then $$d^3 : d'^3 = S : T,$$

where T is the volume of some sphere either greater or less than S'.

I. Suppose, if possible, that $T < S'$.

Let T be supposed concentric with S'.

As in XII. 17, inscribe a polyhedron in S' such that its faces do not anywhere touch T;

and inscribe in S a polyhedron similar to that in S'.

Then $S : T = d^3 : d'^3$

$= (\text{polyhedron in } S) : (\text{polyhedron in } S')$;

or, alternately,

$S : (\text{polyhedron in } S) = T : (\text{polyhedron in } S')$.

And $S > (\text{polyhedron in } S)$;

therefore $T > (\text{polyhedron in } S')$.

But, by construction, $T < (\text{polyhedron in } S')$:

which is impossible.

Therefore $T \nless S'$.

II. Suppose, if possible, that $T > S'$.

Now $d^3 : d'^3 = S : T$

$= X : S'$,

where X is the volume of some sphere less than S, [XII. 2, Lemma]

or, inversely, $d'^3 : d^3 = S' : X$,

where $X < S$.

This is proved impossible exactly as in Part I.

Therefore $T \ngtr S'$.

Hence T, not being greater or less than S', is equal to it, and

$$d^3 : d'^3 = S : S'.$$

BOOK XIII.

HISTORICAL NOTE.

I have already given, in the note to IV. 10, the evidence upon which the construction of the five regular solids is attributed to the Pythagoreans. Some of them, the cube, the tetrahedron (which is nothing but a pyramid), and the octahedron (which is only a double pyramid with a square base), cannot but have been known to the Egyptians. And it appears that dodecahedra have been found, of bronze or other material, which may belong to periods earlier than Pythagoras' time by some centuries (for references see Cantor's *Geschichte der Mathematik* I_3, pp. 175—6).

It is true that the author of the scholium No. 1 to Eucl. XIII. says that the Book is about "the five so-called Platonic figures, which however do not belong to Plato, three of the aforesaid five figures being due to the Pythagoreans, namely the cube, the pyramid and the dodecahedron, while the octahedron and the icosahedron are due to Theaetetus." This statement (taken probably from Geminus) may perhaps rest on the fact that Theaetetus was the first to write at any length about the two last-mentioned solids. We are told indeed by Suidas (s. v. Θεαίτητος) that Theaetetus "first wrote on the 'five solids' as they are called." This no doubt means that Theaetetus was the first to write a complete and systematic treatise on all the regular solids; it does not exclude the possibility that Hippasus or others had already written on the dodecahedron. The fact that Theaetetus wrote upon the regular solids agrees very well with the evidence which we possess of his contributions to the theory of irrationals, the connexion between which and the investigation of the regular solids is seen in Euclid's Book XIII.

Theaetetus flourished about 380 B.C., and his work on the regular solids was soon followed by another, that of Aristaeus, an elder contemporary of Euclid, who also wrote an important book on *Solid Loci*, i.e. on conics treated as loci. This Aristaeus (known as "the elder") wrote in the period about 320 B.C. We hear of his *Comparison of the five regular solids* from Hypsicles (2nd cent. B.C.), the writer of the short book commonly included in the editions of the *Elements* as Book XIV. Hypsicles gives in this Book some six propositions supplementing Eucl. XIII.; and he introduces the second of the propositions (Heiberg's Euclid, Vol. V. p. 6) as follows:

"*The same circle circumscribes both the pentagon of the dodecahedron and the triangle of the icosahedron when both are inscribed in the same sphere.* This is proved by Aristaeus in the book entitled *Comparison of the five figures.*"

Hypsicles proceeds (pp. 7 sqq.) to give a proof of this theorem. Allman pointed out (*Greek Geometry from Thales to Euclid*, 1889, pp. 201—2) that this proof depends on eight theorems, six of which appear in Euclid's Book XIII. (in Propositions 8, 10, 12, 15, 16 with Por., 17); two other propositions not mentioned by Allman are also used, namely XIII. 4 and 9. This seems, as Allman says, to confirm the inference of Bretschneider (p. 171) that, as Aristaeus' work was the newest and latest in which, before Euclid's time, this subject was treated, we have in Eucl. XIII. at least a partial recapitulation of the contents of the treatise of Aristaeus.

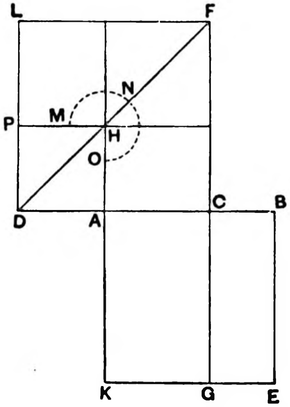
After Euclid, Apollonius wrote on the comparison of the dodecahedron and the icosahedron inscribed in one and the same sphere. This we also learn from Hypsicles, who says in the next words following those about Aristaeus above quoted: "But it is proved by Apollonius in the second edition of his *Comparison of the dodecahedron with the icosahedron* that, as the surface of the dodecahedron is to the surface of the icosahedron [inscribed in the same sphere], so is the dodecahedron itself [i.e. its volume] to the icosahedron, because the perpendicular is the same from the centre of the sphere to the pentagon of the dodecahedron and to the triangle of the icosahedron."

BOOK XIII. PROPOSITIONS.

PROPOSITION 1.

If a straight line be cut in extreme and mean ratio, the square on the greater segment added to the half of the whole is five times the square on the half.

For let the straight line *AB* be cut in extreme and mean ratio at the point *C*,



and let *AC* be the greater segment;
let the straight line *AD* be produced in a straight line with *CA*,
and let *AD* be made half of *AB*;
I say that the square on *CD* is five times the square on *AD*.

For let the squares *AE*, *DF* be described on *AB*, *DC*,
and let the figure in *DF* be drawn;
let *FC* be carried through to *G*.

Now, since *AB* has been cut in extreme and mean ratio at *C*,
therefore the rectangle *AB*, *BC* is equal to the square on *AC*.
[VI. Def. 3, VI. 17]

And *CE* is the rectangle *AB*, *BC*, and *FH* the square on *AC*;
therefore *CE* is equal to *FH*.

And, since *BA* is double of *AD*,
while *BA* is equal to *KA*, and *AD* to *AH*,
therefore *KA* is also double of *AH*.

But, as *KA* is to *AH*, so is *CK* to *CH*; [VI. 1]
therefore *CK* is double of *CH*.

But *LH*, *HC* are also double of *CH*.

Therefore *KC* is equal to *LH*, *HC*.

But CE was also proved equal to HF;
therefore the whole square AE is equal to the gnomon MNO.

And, since BA is double of AD,
the square on BA is quadruple of the square on AD,
that is, AE is quadruple of DH.

But AE is equal to the gnomon MNO;
therefore the gnomon MNO is also quadruple of AP;
therefore the whole DF is five times AP.

And DF is the square on DC, and AP the square on DA;
therefore the square on CD is five times the square on DA.

Therefore etc.

Q. E. D.

The first five propositions are in the nature of lemmas, which are required for later propositions but are not in themselves of much importance.

It will be observed that, while the method of the propositions is that of Book II., being strictly geometrical and not algebraical, none of the results of that Book are made use of (except indeed in the Lemma to XIII. 2, which is probably not genuine). It would therefore appear as though these propositions were taken from an earlier treatise without being revised or rewritten in the light of Book II. It will be remembered that, according to Proclus (p. 67, 6), Eudoxus "greatly added to the number of the theorems which originated with Plato regarding *the section*" (i.e. presumably the "*golden* section"); and it is therefore probable that the five theorems are due to Eudoxus.

That, if AB is divided at C in extreme and mean ratio, the rectangle AB, BC is equal to the square on AC is inferred from VI. 17.

AD is made equal to half AB, and we have to prove that

$$(\text{sq. on } CD) = 5\,(\text{sq. on } AD).$$

The figure shows at once that

$$\square\, CH = \square\, HL,$$

so that

$$\square\, CH + \square\, HL = 2\,(\square\, CH)$$
$$= \square\, AG.$$

Also

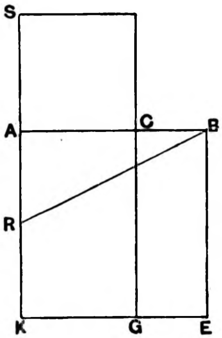
$$\text{sq. } HF = (\text{sq. on } AC)$$
$$= \text{rect. } AB,\, BC$$
$$= CE.$$

By addition,

$$(\text{gnomon } MNO) = \text{sq. on } AB$$
$$= 4\,(\text{sq. on } AD);$$

whence, adding the sq. on AD to each, we have

$$(\text{sq. on } CD) = 5\,(\text{sq. on } AD).$$



The result here, and in the next propositions, is really seen more readily by means of the figure of II. 11.

In this figure $SR = AC + \frac{1}{2}AB$, by construction; and we have therefore to prove that

$$(\text{sq. on } SR) = 5\,(\text{sq. on } AR).$$

This is obvious, for

$$\begin{aligned}(\text{sq. on } SR) &= (\text{sq. on } RB)\\ &= \text{sum of sqs. on } AB,\ AR\\ &= 5\,(\text{sq. on } AR).\end{aligned}$$

The mss. contain a curious addition to xiii. 1—5 in the shape of analyses and syntheses for each proposition prefaced by the heading:

"What is analysis and what is synthesis.

"Analysis is the assumption of that which is sought as if it were admitted <and the arrival> by means of its consequences at something admitted to be true.

"Synthesis is an assumption of that which is admitted <and the arrival> by means of its consequences at something admitted to be true."

There must apparently be some corruption in the text; it does not, in the case of synthesis, give what is wanted. B and V have, instead of "something admitted to be true," the words "the end or attainment of what is sought."

The whole of this addition is evidently interpolated. To begin with, the analyses and syntheses of the five propositions are placed all together in four mss.; in P, q they come after an alternative proof of xiii. 5 (which alternative proof P gives after xiii. 6, while q gives it instead of xiii. 6), in B (which has not the alternative proof of xiii. 5) after xiii. 6, and in b (in which xiii. 6 is wanting, and the alternative proof of xiii. 5 is in the margin, in the first hand) after xiii. 5, while V has the analyses of 1—3 in the text after xiii. 6 and those of 4—5 in the same place in the margin, by the second hand. Further, the addition is altogether alien from the plan and manner of the *Elements*. The interpolation took place before Theon's time, and the probability is that it was originally in the margin, whence it crept into the text of P after xiii. 5. Heiberg (after Bretschneider) suggested in his edition (Vol. v. p. lxxxiv.) that it might be a relic of analytical investigations by Theaetetus or Eudoxus, and he cited the remark of Pappus (v. p. 410) at the beginning of his "comparisons of the five [regular solid] figures which have an equal surface," to the effect that he will not use "the so-called analytical investigation by means of which some of the ancients effected their demonstrations." More recently (*Paralipomena zu Euklid* in *Hermes* xxxviii., 1903) Heiberg conjectures that the author is Heron, on the ground that the sort of analysis and synthesis recalls Heron's remarks on analysis and synthesis in his commentary on the beginning of Book ii. (quoted by an-Nairīzī, ed. Curtze, p. 89) and his quasi-algebraical alternative proofs of propositions in that Book.

To show the character of the interpolated matter I need only give the analysis and synthesis of one proposition. In the case of xiii. 1 it is in substance as follows. The figure is a mere straight line.

Let AB be divided in extreme and mean ratio at C, AC being the greater segment; and let

$$AD = \tfrac{1}{2}AB.$$

I say that $(\text{sq. on } CD) = 5\,(\text{sq. on } AD)$.

(Analysis.)

"For, since $(\text{sq. on } CD) = 5\,(\text{sq. on } AD)$,"

and $(\text{sq. on } CD) = (\text{sq. on } CA) + (\text{sq. on } AD) + 2\,(\text{rect. } CA,\ AD)$,

therefore $(\text{sq. on } CA) + 2\,(\text{rect. } CA,\ AD) = 4\,(\text{sq. on } AD)$.

But $\text{rect. } BA\,.\,AC = 2\,(\text{rect. } CA\,.\,AD)$,

and $(\text{sq. on } CA) = (\text{rect. } AB,\ BC)$.

Therefore

(rect. BA, AC) + (rect. AB, BC) = 4 (sq. on AD),

or (sq. on AB) = 4 (sq. on AD):

and this is true, since $AD = \frac{1}{2}AB$.

(Synthesis.)

Since (sq. on AB) = 4 (sq. on AD),

and (sq. on AB) = (rect. BA, AC) + (rect. AB, BC),

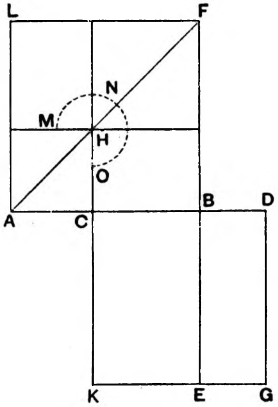
therefore 4 (sq. on AD) = 2 (rect. DA, AC) + sq. on AC.

Adding to each the square on AD, we have

(sq. on CD) = 5 (sq. on AD).

PROPOSITION 2.

If the square on a straight line be five times the square on a segment of it, then, when the double of the said segment is cut in extreme and mean ratio, the greater segment is the remaining part of the original straight line.



For let the square on the straight line AB be five times the square on the segment AC of it,

and let CD be double of AC;

I say that, when CD is cut in extreme and mean ratio, the greater segment is CB.

Let the squares AF, CG be described on AB, CD respectively, let the figure in AF be drawn, and let BE be drawn through.

Now, since the square on BA is five times the square on AC,

AF is five times AH.

Therefore the gnomon MNO is quadruple of AH.

And, since DC is double of CA,

therefore the square on DC is quadruple of the square on CA, that is, CG is quadruple of AH.

But the gnomon MNO was also proved quadruple of AH; therefore the gnomon MNO is equal to CG.

And, since DC is double of CA,

while DC is equal to CK, and AC to CH,

therefore KB is also double of BH. [VI. 1]

But LH, HB are also double of HB;
therefore KB is equal to LH, HB.

But the whole gnomon MNO was also proved equal to the whole CG;
therefore the remainder HF is equal to BG.

And BG is the rectangle CD, DB,
for CD is equal to DG;
and HF is the square on CB;
therefore the rectangle CD, DB is equal to the square on CB.

Therefore, as DC is to CB, so is CB to BD.

But DC is greater than CB;
therefore CB is also greater than BD.

Therefore, when the straight line CD is cut in extreme and mean ratio, CB is the greater segment.

Therefore etc.

Q. E. D.

LEMMA.

That the double of AC is greater than BC is to be proved thus.

If not, let BC be, if possible, double of CA.

Therefore the square on BC is quadruple of the square on CA;
therefore the squares on BC, CA are five times the square on CA.

But, by hypothesis, the square on BA is also five times the square on CA;
therefore the square on BA is equal to the squares on BC, CA:
which is impossible. [II. 4]

Therefore CB is not double of AC.

Similarly we can prove that neither is a straight line less than CB double of CA;
for the absurdity is much greater.

Therefore the double of AC is greater than CB.

Q. E. D.

This proposition is the converse of Prop. 1. We have to prove that, if AB be so divided at C that

$$(\text{sq. on } AB) = 5\,(\text{sq. on } AC),$$

and if $CD = 2AC$,

then $$(\text{rect. } CD, DB) = (\text{sq. on } CB).$$

Subtract from each side the sq. on AC;

then $$(\text{gnomon } MNO) = 4\,(\text{sq. on } AC)$$
$$= (\text{sq. on } CD).$$

Now, as in the last proposition,
$$\text{▱}\,CE = 2\,(\text{▱}\,BH)$$
$$= \text{▱}\,BH + \text{▱}\,HL.$$

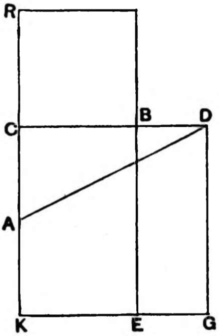
Subtracting these equals from the equals, the square on CD and the gnomon MNO respectively, we have
$$\text{▱}\,BG = (\text{square } HF),$$
i.e. $$(\text{rect. } CD, DB) = (\text{sq. on } CB).$$

Here again the proposition can readily be proved by means of a figure similar to that of II. 11.

Draw CA through C *at right angles* to CB and of length equal to CA in the original figure; make CD double of CA; produce AC to R so that $CR = CB$.

Complete the squares on CB and CD, and join AD.

Now we are given the fact that
$$(\text{sq. on } AR) = 5\,(\text{sq. on } CA).$$
But
$$5\,(\text{sq. on } AC) = (\text{sq. on } AC) + (\text{sq. on } CD)$$
$$= (\text{sq. on } AD).$$
Therefore
$$(\text{sq. on } AR) = (\text{sq. on } AD),$$
or $$AR = AD.$$



Now
$$(\text{rect. } KR, RC) + (\text{sq. on } AC) = (\text{sq. on } AR)$$
$$= (\text{sq. on } AD)$$
$$= (\text{sq. on } AC) + (\text{sq. on } CD).$$
Therefore $$(\text{rect. } KR\,.\,RC) = (\text{sq. on } CD).$$
That is, $$(\text{rectangle } RE) = (\text{square } CG).$$
Subtract the common part CE,

and $$(\text{rect. } BG) = (\text{sq. } RB),$$
or $$\text{rect. } CD, DB = (\text{sq. on } CB).$$

Heiberg, with reason, doubts the genuineness of the Lemma following this proposition.

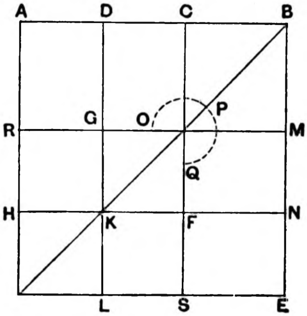
PROPOSITION 3.

If a straight line be cut in extreme and mean ratio, the square on the lesser segment added to the half of the greater segment is five times the square on the half of the greater segment.

For let any straight line *AB* be cut in extreme and mean ratio at the point *C*,

let *AC* be the greater segment, and let *AC* be bisected at *D*;

I say that the square on *BD* is five times the square on *DC*.



For let the square *AE* be described on *AB*,

and let the figure be drawn double.

Since *AC* is double of *DC*,

therefore the square on *AC* is quadruple of the square on *DC*,

that is, *RS* is quadruple of *FG*.

And, since the rectangle *AB*, *BC* is equal to the square on *AC*,

and *CE* is the rectangle *AB*, *BC*,

therefore *CE* is equal to *RS*.

But *RS* is quadruple of *FG*;

therefore *CE* is also quadruple of *FG*.

Again, since *AD* is equal to *DC*,

HK is also equal to *KF*.

Hence the square *GF* is also equal to the square *HL*.

Therefore *GK* is equal to *KL*, that is, *MN* to *NE*;

hence *MF* is also equal to *FE*.

But *MF* is equal to *CG*;

therefore *CG* is also equal to *FE*.

Let *CN* be added to each;

therefore the gnomon *OPQ* is equal to *CE*.

But *CE* was proved quadruple of *GF*;

therefore the gnomon *OPQ* is also quadruple of the square *FG*.

Therefore the gnomon *OPQ* and the square *FG* are five times *FG*.

But the gnomon *OPQ* and the square *FG* are the square *DN*.

And *DN* is the square on *DB*, and *GF* the square on *DC*.

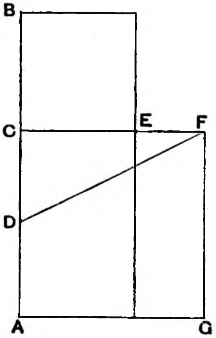
Therefore the square on *DB* is five times the square on *DC*.

Q. E. D.

In this case we have

$$\begin{aligned}(\text{sq. on } BD) &= (\text{sq. } FG) + (\text{rect. } CG) + (\text{rect. } CN)\\ &= (\text{sq. } FG) + (\text{rect. } FE) + (\text{rect. } CN)\\ &= (\text{sq. } FG) + (\text{rect. } CE)\\ &= (\text{sq. } FG) + (\text{rect. } AB, BC)\\ &= (\text{sq. } FG) + (\text{sq. on } AC), \text{ by hypothesis,}\\ &= 5\,(\text{sq. on } DC).\end{aligned}$$

The theorem is still more obvious if the figure of II. 11 be used. Let CF be divided in extreme and mean ratio at E, by the method of II. 11.



Then, since

$$\begin{aligned}(\text{rect. } AB, BC) + (\text{sq. on } CD) &= \text{sq. on } BD\\ &= \text{sqs. on } CD, CF,\\ (\text{rect. } AB, BC) &= (\text{sq. on } CF)\\ &= (\text{sq. on } CA),\end{aligned}$$

and AB is divided at C in extreme and mean ratio.

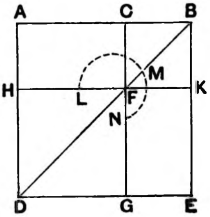
And $$\begin{aligned}(\text{sq. on } BD) &= (\text{sq. on } DF)\\ &= 5\,(\text{sq. on } CD).\end{aligned}$$

PROPOSITION 4.

If a straight line be cut in extreme and mean ratio, the square on the whole and the square on the lesser segment together are triple of the square on the greater segment.

Let AB be a straight line, let it be cut in extreme and mean ratio at C, and let AC be the greater segment;

I say that the squares on AB, BC are triple of the square on CA.



For let the square $ADEB$ be described on AB, and let the figure be drawn.

Since then AB has been cut in extreme and mean ratio at C, and AC is the greater segment, therefore the rectangle AB, BC is equal to the square on AC. [VI. Def. 3, VI. 17]

And AK is the rectangle AB, BC, and HG the square on AC; therefore AK is equal to HG.

And, since *AF* is equal to *FE*,
let *CK* be added to each;
therefore the whole *AK* is equal to the whole *CE*;
therefore *AK*, *CE* are double of *AK*.

But *AK*, *CE* are the gnomon *LMN* and the square *CK*; therefore the gnomon *LMN* and the square *CK* are double of *AK*.

But, further, *AK* was also proved equal to *HG*; therefore the gnomon *LMN* and the squares *CK*, *HG* are triple of the square *HG*.

And the gnomon *LMN* and the squares *CK*, *HG* are the whole square *AE* and *CK*, which are the squares on *AB*, *BC*,
while *HG* is the square on *AC*.

Therefore the squares on *AB*, *BC* are triple of the square on *AC*.

Q. E. D.

Here, as in the preceding propositions, the results are proved *de novo* by the method of Book II., without reference to that Book. Otherwise the proof might have been shorter.

For, by II. 7,

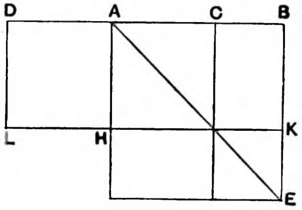
$$(\text{sq. on } AB) + (\text{sq. on } BC) = 2\,(\text{rect. } AB,\ BC) + (\text{sq. on } AC)$$
$$= 3\,(\text{sq. on } AC).$$

PROPOSITION 5.

If a straight line be cut in extreme and mean ratio, and there be added to it a straight line equal to the greater segment, the whole straight line has been cut in extreme and mean ratio, and the original straight line is the greater segment.

For let the straight line *AB* be cut in extreme and mean ratio at the point *C*,
let *AC* be the greater segment,
and let *AD* be equal to *AC*.

I say that the straight line *DB* has been cut in extreme and mean ratio at *A*, and the original straight line *AB* is the greater segment.



For let the square *AE* be described on *AB*, and let the figure be drawn.

Since AB has been cut in extreme and mean ratio at C, therefore the rectangle AB, BC is equal to the square on AC. [VI. Def. 3, VI. 17]

And CE is the rectangle AB, BC, and CH the square on AC;
therefore CE is equal to HC.

But HE is equal to CE,
and DH is equal to HC;
therefore DH is also equal to HE.

Therefore the whole DK is equal to the whole AE.

And DK is the rectangle BD, DA,
for AD is equal to DL;
and AE is the square on AB;
therefore the rectangle BD, DA is equal to the square on AB.

Therefore, as DB is to BA, so is BA to AD. [VI. 17]

And DB is greater than BA;
therefore BA is also greater than AD. [V. 14]

Therefore DB has been cut in extreme and mean ratio at A, and AB is the greater segment.

Q. E. D.

We have

$$\begin{aligned}(\text{sq. } DH) &= (\text{sq. } HC)\\ &= (\text{rect. } CE), \text{ by hypothesis,}\\ &= (\text{rect. } HE).\end{aligned}$$

Add to each side the rectangle AK, and

$$(\text{rect. } DK) = (\text{sq. } AE),$$

or

$$(\text{rect. } BD, DA) = (\text{sq. on } AB).$$

The result is of course obvious from II. 11.

There is an alternative proof given in P after XIII. 6, which depends on Book V.

By hypothesis, $BA : AC = AC : CB$,

or, inversely, $AC : AB = CB : AC$.

Componendo, $(AB + AC) : AB = AB : AC$,

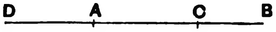
or $DB : BA = BA : AD$.

PROPOSITION 6.

If a rational straight line be cut in extreme and mean ratio, each of the segments is the irrational straight line called apotome.

Let AB be a rational straight line,

let it be cut in extreme and mean ratio at C,



and let AC be the greater segment;

I say that each of the straight lines AC, CB is the irrational straight line called apotome.

For let BA be produced, and let AD be made half of BA.

Since then the straight line AB has been cut in extreme and mean ratio,

and to the greater segment AC is added AD which is half of AB,

therefore the square on CD is five times the square on DA. [XIII. 1]

Therefore the square on CD has to the square on DA the ratio which a number has to a number;

therefore the square on CD is commensurable with the square on DA. [X. 6]

But the square on DA is rational,

for DA is rational, being half of AB which is rational;

therefore the square on CD is also rational; [X. Def. 4]

therefore CD is also rational.

And, since the square on CD has not to the square on DA the ratio which a square number has to a square number,

therefore CD is incommensurable in length with DA; [X. 9]

therefore CD, DA are rational straight lines commensurable in square only;

therefore AC is an apotome. [X. 73]

Again, since AB has been cut in extreme and mean ratio, and AC is the greater segment,

therefore the rectangle AB, BC is equal to the square on AC. [VI. Def. 3, VI. 17]

Therefore the square on the apotome AC, if applied to the rational straight line AB, produces BC as breadth.

But the square on an apotome, if applied to a rational straight line, produces as breadth a first apotome; [X. 97]

therefore CB is a first apotome.

And CA was also proved to be an apotome.

Therefore etc.

Q. E. D

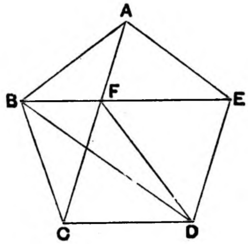
It seems certain that this proposition is an interpolation. P has it, but the copyist (or rather the copyist of its archetype) says that "this theorem is not found in most copies of the new recension, but is found in those of the old." In the first place, there is a scholium to XIII. 17 in P itself which proves the same thing as XIII. 6, and which would therefore have been useless if XIII. 6 had preceded. Hence, when the scholium was written, this proposition had not yet been interpolated. Secondly, P has it before the alternative proof of XIII. 5; this proof is considered, on general grounds, to be interpolated, and it would appear that it must have been a *later* interpolation (XIII. 6) which divorced it from the proposition to which it belonged. Thirdly, there is cause for suspicion in the proposition itself, for, while the enunciation states that each segment of the straight line is an *apotome*, the proposition adds that the lesser segment is a *first* apotome. The scholium in P referred to has not this blot. What is actually wanted in XIII. 17 is the fact that the *greater* segment is an apotome. It is probable that Euclid *assumed* this fact as evident enough from XIII. 1 without further proof, and that he neither wrote XIII. 6 nor the quotation of its enunciation in XIII. 17.

PROPOSITION 7.

If three angles of an equilateral pentagon, taken either in order or not in order, be equal, the pentagon will be equiangular.

For in the equilateral pentagon $ABCDE$ let, first, three angles taken in order, those at A, B, C, be equal to one another;

I say that the pentagon $ABCDE$ is equiangular.



For let AC, BE, FD be joined.

Now, since the two sides CB, BA are equal to the two sides BA, AE respectively,

and the angle CBA is equal to the angle BAE,

therefore the base AC is equal to the base BE,

the triangle ABC is equal to the triangle ABE,

and the remaining angles will be equal to the remaining angles, namely those which the equal sides subtend, [I. 4]

that is, the angle BCA to the angle BEA, and the angle ABE to the angle CAB;

hence the side AF is also equal to the side BF. [I. 6]

But the whole *AC* was also proved equal to the whole *BE*; therefore the remainder *FC* is also equal to the remainder *FE*.

But *CD* is also equal to *DE*.

Therefore the two sides *FC*, *CD* are equal to the two sides *FE*, *ED*;
and the base *FD* is common to them;
therefore the angle *FCD* is equal to the angle *FED*. [I. 8]

But the angle *BCA* was also proved equal to the angle *AEB*;
therefore the whole angle *BCD* is also equal to the whole angle *AED*.

But, by hypothesis, the angle *BCD* is equal to the angles at *A*, *B*;
therefore the angle *AED* is also equal to the angles at *A*, *B*.

Similarly we can prove that the angle *CDE* is also equal to the angles at *A*, *B*, *C*;
therefore the pentagon *ABCDE* is equiangular.

Next, let the given equal angles not be angles taken in order, but let the angles at the points *A*, *C*, *D* be equal;
I say that in this case too the pentagon *ABCDE* is equiangular.

For let *BD* be joined.

Then, since the two sides *BA*, *AE* are equal to the two sides *BC*, *CD*,
and they contain equal angles,
therefore the base *BE* is equal to the base *BD*,
the triangle *ABE* is equal to the triangle *BCD*,
and the remaining angles will be equal to the remaining angles, namely those which the equal sides subtend; [I. 4]
therefore the angle *AEB* is equal to the angle *CDB*.

But the angle *BED* is also equal to the angle *BDE*,
since the side *BE* is also equal to the side *BD*. [I. 5]

Therefore the whole angle *AED* is equal to the whole angle *CDE*.

But the angle *CDE* is, by hypothesis, equal to the angles at *A*, *C*;
therefore the angle *AED* is also equal to the angles at *A*, *C*.

For the same reason
the angle ABC is also equal to the angles at A, C, D.

Therefore the pentagon $ABCDE$ is equiangular.

Q. E. D.

This proposition is required in XIII. 17.

The steps of the proof may be shown thus.

I. Suppose that the angles at A, B, C are all equal.

Then the isosceles triangles BAE, ABC are equal in all respects;
thus $BE = AC$, $\angle BCA = \angle BEA$, $\angle CAB = \angle EBA$.

By the last equality, $FA = FB$,
so that, since $BE = AC$, $FC = FE$.

The $\triangle$s FED, FCD are now equal in all respects, [I. 8, 4]
and $\angle FCD = \angle FED$.

But $\angle ACB = \angle AEB$, from above,
whence, by addition, $\angle BCD = \angle AED$.

Similarly it may be proved that $\angle CDE$ is also equal to any one of the angles at A, B, C.

II. Suppose the angles at A, C, D to be equal.

Then the isosceles triangles ABE, CBD are equal in all respects, and hence $BE = BD$ (so that $\angle BDE = \angle BED$),
and $\angle CDB = \angle AEB$.

By addition of the equal angles,

$$\angle CDE = \angle DEA.$$

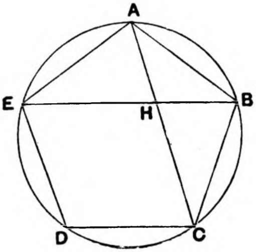
Similarly it may be proved that $\angle ABC$ is also equal to each of the angles at A, C, D.

PROPOSITION 8.

If in an equilateral and equiangular pentagon straight lines subtend two angles taken in order, they cut one another in extreme and mean ratio, and their greater segments are equal to the side of the pentagon.

For in the equilateral and equiangular pentagon $ABCDE$ let the straight lines AC, BE, cutting one another at the point H, subtend two angles taken in order, the angles at A, B;

I say that each of them has been cut in extreme and mean ratio at the point H, and their greater segments are equal to the side of the pentagon.



For let the circle $ABCDE$ be circumscribed about the pentagon $ABCDE$. [IV. 14]

Then, since the two straight lines *EA*, *AB* are equal to the two *AB*, *BC*,
and they contain equal angles,
therefore the base *BE* is equal to the base *AC*,
the triangle *ABE* is equal to the triangle *ABC*,
and the remaining angles will be equal to the remaining angles respectively, namely those which the equal sides subtend. [I. 4]

Therefore the angle *BAC* is equal to the angle *ABE*;
therefore the angle *AHE* is double of the angle *BAH*. [I. 32]

But the angle *EAC* is also double of the angle *BAC*,
inasmuch as the circumference *EDC* is also double of the circumference *CB*; [III. 28, VI. 33]
therefore the angle *HAE* is equal to the angle *AHE*;
hence the straight line *HE* is also equal to *EA*, that is, to *AB*. [I. 6]

And, since the straight line *BA* is equal to *AE*,
the angle *ABE* is also equal to the angle *AEB*. [I. 5]

But the angle *ABE* was proved equal to the angle *BAH*;
therefore the angle *BEA* is also equal to the angle *BAH*.

And the angle *ABE* is common to the two triangles *ABE* and *ABH*;
therefore the remaining angle *BAE* is equal to the remaining angle *AHB*; [I. 32]
therefore the triangle *ABE* is equiangular with the triangle *ABH*;
therefore, proportionally, as *EB* is to *BA*, so is *AB* to *BH*. [VI. 4]

But *BA* is equal to *EH*;
therefore, as *BE* is to *EH*, so is *EH* to *HB*.

And *BE* is greater than *EH*;
therefore *EH* is also greater than *HB*. [V. 14]

Therefore *BE* has been cut in extreme and mean ratio at *H*, and the greater segment *HE* is equal to the side of the pentagon.

Similarly we can prove that *AC* has also been cut in extreme and mean ratio at *H*, and its greater segment *CH* is equal to the side of the pentagon.

Q. E. D.

In order to prove this theorem we have to show (1) that the $\triangle$s AEB, HAB are similar, and (2) that $EH = EA$ $(= AB)$.

To prove (2) we have

$$\triangle\text{s } AEB, BAC \text{ equal in all respects,}$$

whence $$EB = AC,$$

and $$\angle BAC = \angle ABE.$$

Therefore $$\angle AHE = 2 \angle BAC$$
$$= \angle EAC,$$

so that $$EH = EA$$
$$= AB.$$

To prove (1) we have, in the $\triangle$s AEB, HAB,

$$\angle BAH = \angle EBA$$
$$= \angle AEB,$$

and $\angle ABE$ is common,

therefore the third $\angle$s AHB, EAB are equal,

and $\triangle$s AEB, HAB are similar.

Now, since these triangles are similar,

$$EB : BA = BA : BH,$$

or $$(\text{rect. } EB, BH) = (\text{sq. on } BA)$$
$$= (\text{sq. on } EH),$$

so that EB is divided in extreme and mean ratio at H.

Similarly its equal, CA, is divided in extreme and mean ratio at H.

PROPOSITION 9.

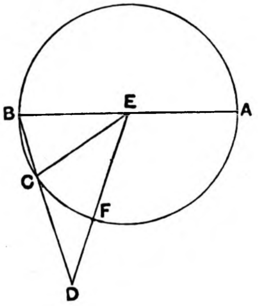
If the side of the hexagon and that of the decagon inscribed in the same circle be added together, the whole straight line has been cut in extreme and mean ratio, and its greater segment is the side of the hexagon.

Let ABC be a circle;

of the figures inscribed in the circle ABC let BC be the side of a decagon, CD that of a hexagon,

and let them be in a straight line;

I say that the whole straight line BD has been cut in extreme and mean ratio, and CD is its greater segment.



For let the centre of the circle, the point E, be taken,

let EB, EC, ED be joined,

and let BE be carried through to A.

Since BC is the side of an equilateral decagon,

therefore the circumference *ACB* is five times the circumference *BC*;
therefore the circumference *AC* is quadruple of *CB*.

But, as the circumference *AC* is to *CB*, so is the angle *AEC* to the angle *CEB*; [VI. 33]
therefore the angle *AEC* is quadruple of the angle *CEB*.

And, since the angle *EBC* is equal to the angle *ECB*, [I. 5]
therefore the angle *AEC* is double of the angle *ECB*. [I. 32]

And, since the straight line *EC* is equal to *CD*,
for each of them is equal to the side of the hexagon inscribed in the circle *ABC*, [IV. 15, Por.]
the angle *CED* is also equal to the angle *CDE*; [I. 5]
therefore the angle *ECB* is double of the angle *EDC*. [I. 32]

But the angle *AEC* was proved double of the angle *ECB*;
therefore the angle *AEC* is quadruple of the angle *EDC*.

But the angle *AEC* was also proved quadruple of the angle *BEC*;
therefore the angle *EDC* is equal to the angle *BEC*.

But the angle *EBD* is common to the two triangles *BEC* and *BED*;
therefore the remaining angle *BED* is also equal to the remaining angle *ECB*; [I. 32]
therefore the triangle *EBD* is equiangular with the triangle *EBC*.

Therefore, proportionally, as *DB* is to *BE*, so is *EB* to *BC*. [VI. 4]

But *EB* is equal to *CD*.

Therefore, as *BD* is to *DC*, so is *DC* to *CB*.

And *BD* is greater than *DC*;
therefore *DC* is also greater than *CB*.

Therefore the straight line *BD* has been cut in extreme and mean ratio, and *DC* is its greater segment.

Q. E. D.

BC is the side of a regular decagon inscribed in the circle; *CD* is the side of the inscribed regular hexagon, and is therefore equal to the radius *BE* or *EC*.

Therefore, in order to prove our theorem, we have only to show that *the triangles* EBC, DBE *are similar.*

Since BC is the side of a regular decagon,

$$(\text{arc } BCA) = 5\,(\text{arc } BC),$$

so that $$(\text{arc } CFA) = 4\,(\text{arc } BC),$$

whence $$\angle CEA = 4\,\angle BEC.$$

But $$\angle CEA = 2\,\angle ECB.$$

Therefore $$\angle ECB = 2\,\angle BEC \quad \text{........................(1)}.$$

But, since $CD = CE$,

$$\angle CDE = \angle CED,$$

so that $$\angle ECB = 2\,\angle CDE.$$

It follows from (1) that $\angle BEC = \angle CDE$.

Now, in the $\triangle$s EBC, DBE,

$$\angle BEC = \angle BDE,$$

and $$\angle EBC \text{ is common},$$

so that $$\angle ECB = \angle DEB,$$

and $$\triangle\text{s } EBC, DBE \text{ are similar.}$$

Hence $$DB : BE = EB : BC,$$

or $$(\text{rect. } DB, BC) = (\text{sq. on } EB)$$
$$= (\text{sq. on } CD),$$

and DB is divided at C in extreme and mean ratio.

To find the side of the decagon algebraically in terms of the radius we have, if x be the side required,

$$(r + x)x = r^2,$$

whence $$x = \frac{r}{2}(\sqrt{5} - 1).$$

PROPOSITION 10.

If an equilateral pentagon be inscribed in a circle, the square on the side of the pentagon is equal to the squares on the side of the hexagon and on that of the decagon inscribed in the same circle.

Let $ABCDE$ be a circle,
and let the equilateral pentagon $ABCDE$ be inscribed in the circle $ABCDE$.

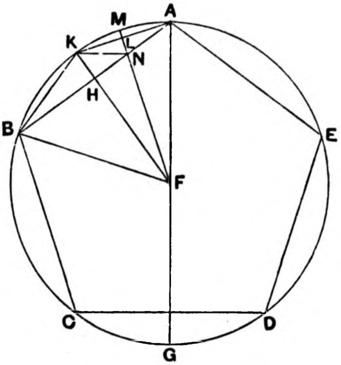
I say that the square on the side of the pentagon $ABCDE$ is equal to the squares on the side of the hexagon and on that of the decagon inscribed in the circle $ABCDE$.

For let the centre of the circle, the point F, be taken,
let AF be joined and carried through to the point G,
let FB be joined,
let FH be drawn from F perpendicular to AB and be carried through to K,

let AK, KB be joined,

let FL be again drawn from F perpendicular to AK, and be carried through to M,

and let KN be joined.



Since the circumference $ABCG$ is equal to the circumference $AEDG$,

and in them ABC is equal to AED,

therefore the remainder, the circumference CG, is equal to the remainder GD.

But CD belongs to a pentagon;

therefore CG belongs to a decagon.

And, since FA is equal to FB,

and FH is perpendicular,

therefore the angle AFK is also equal to the angle KFB. [I. 5, I. 26]

Hence the circumference AK is also equal to KB; [III. 26]

therefore the circumference AB is double of the circumference BK;

therefore the straight line AK is a side of a decagon.

For the same reason

AK is also double of KM.

Now, since the circumference AB is double of the circumference BK,

while the circumference CD is equal to the circumference AB,

therefore the circumference CD is also double of the circumference BK.

But the circumference CD is also double of CG;

therefore the circumference CG is equal to the circumference BK.

But BK is double of KM, since KA is so also;

therefore CG is also double of KM.

But, further, the circumference *CB* is also double of the circumference *BK*,
for the circumference *CB* is equal to *BA*.

Therefore the whole circumference *GB* is also double of *BM*;
hence the angle *GFB* is also double of the angle *BFM*. [VI. 33]

But the angle *GFB* is also double of the angle *FAB*,
for the angle *FAB* is equal to the angle *ABF*.

Therefore the angle *BFN* is also equal to the angle *FAB*.

But the angle *ABF* is common to the two triangles *ABF* and *BFN*;
therefore the remaining angle *AFB* is equal to the remaining angle *BNF*; [I. 32]
therefore the triangle *ABF* is equiangular with the triangle *BFN*.

Therefore, proportionally, as the straight line *AB* is to *BF*, so is *FB* to *BN*; [VI. 4]
therefore the rectangle *AB*, *BN* is equal to the square on *BF*. [VI. 17]

Again, since *AL* is equal to *LK*,
while *LN* is common and at right angles,
therefore the base *KN* is equal to the base *AN*; [I. 4]
therefore the angle *LKN* is also equal to the angle *LAN*.

But the angle *LAN* is equal to the angle *KBN*;
therefore the angle *LKN* is also equal to the angle *KBN*.

And the angle at *A* is common to the two triangles *AKB* and *AKN*.

Therefore the remaining angle *AKB* is equal to the remaining angle *KNA*; [I. 32]
therefore the triangle *KBA* is equiangular with the triangle *KNA*.

Therefore, proportionally, as the straight line *BA* is to *AK*, so is *KA* to *AN*; [VI. 4]
therefore the rectangle *BA*, *AN* is equal to the square on *AK*. [VI. 17]

But the rectangle *AB*, *BN* was also proved equal to the square on *BF*;

therefore the rectangle AB, BN together with the rectangle BA, AN, that is, the square on BA [II. 2], is equal to the square on BF together with the square on AK.

And BA is a side of the pentagon, BF of the hexagon [IV. 15, Por.], and AK of the decagon.

Therefore etc.

Q. E. D.

$ABCDE$ being a regular pentagon inscribed in a circle, and AG the diameter through A, it follows that

$$(\text{arc } CG) = (\text{arc } GD),$$

and CG, GD are sides of an inscribed regular decagon.

FHK being drawn perpendicular to AB, it follows, by I. 26, that $\angle$s AFK, BFK are equal, and BK, KA are sides of the regular decagon.

Similarly it may be proved that, FLM being perpendicular to AK,

$$AL = LK,$$

and $$(\text{arc } AM) = (\text{arc } MK).$$

The main facts to prove are that

(1) the triangles ABF, FBN are similar, and (2) the triangles ABK, AKN are similar.

(1) $$2\,(\text{arc } CG) = (\text{arc } CD)$$
$$= (\text{arc } AB)$$
$$= 2\,(\text{arc } BK),$$

or $$(\text{arc } CG) = (\text{arc } BK) = (\text{arc } AK)$$
$$= 2\,(\text{arc } KM).$$

And $$(\text{arc } CB) = 2\,(\text{arc } BK).$$

Therefore, by addition,

$$(\text{arc } BCG) = 2\,(\text{arc } BKM).$$

Therefore $$\angle BFG = 2\angle BFN.$$

But $$\angle BFG = 2\angle FAB,$$

so that $$\angle FAB = \angle BFN.$$

Hence, in the $\triangle$s ABF, FBN,

$$\angle FAB = \angle BFN,$$

and $\angle ABF$ is common;

therefore $$\angle AFB = \angle BNF,$$

and $\triangle$s ABF, FBN are similar.

(2) Since $AL = LK$, and the angles at L are right,

$$AN = NK,$$

and $$\angle NKA = \angle NAK$$
$$= \angle KBA.$$

Hence, in the $\triangle$s ABK, AKN,

$$\angle ABK = \angle AKN,$$

and $\angle KAN$ is common,

whence the third angles are equal;

therefore the triangles ABK, AKN are similar.

Now from the similarity of △s ABF, FBN it follows that

$$AB : BF = BF : BN,$$

or $$(\text{rect. } AB, BN) = (\text{sq. on } BF).$$

And, from the similarity of ABK, AKN,

$$BA : AK = AK : AN,$$

or $$(\text{rect. } BA, AN) = (\text{sq. on } AK).$$

Therefore, by addition,

$$(\text{rect. } AB, BN) + (\text{rect. } BA, AN) = (\text{sq. on } BF) + (\text{sq. on } AK),$$

that is, $$(\text{sq. on } AB) = (\text{sq. on } BF) + (\text{sq. on } AK).$$

If r be the radius of the circle, we have seen (XIII. 9, note) that

$$AK = \frac{r}{2}(\sqrt{5} - 1).$$

Therefore $$(\text{side of pentagon})^2 = r^2 + \frac{r^2}{4}(6 - 2\sqrt{5})$$
$$= \frac{r^2}{4}(10 - 2\sqrt{5}),$$

so that $$(\text{side of pentagon}) = \frac{r}{2}\sqrt{10 - 2\sqrt{5}}.$$

PROPOSITION 11.

If in a circle which has its diameter rational an equilateral pentagon be inscribed, the side of the pentagon is the irrational straight line called minor.

For in the circle *ABCDE* which has its diameter rational let the equilateral pentagon *ABCDE* be inscribed;

I say that the side of the pentagon is the irrational straight line called minor.

For let the centre of the circle, the point *F*, be taken,

let *AF*, *FB* be joined and carried through to the points, *G*, *H*,

let *AC* be joined,

and let *FK* be made a fourth part of *AF*.

Now *AF* is rational;

therefore *FK* is also rational.

But *BF* is also rational;

therefore the whole *BK* is rational.

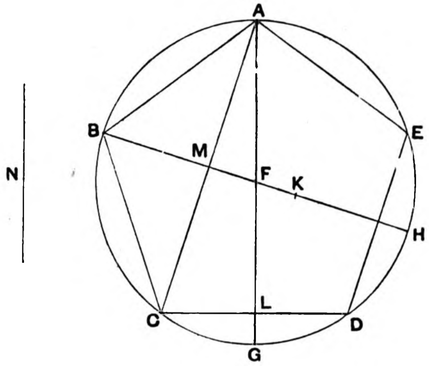
And, since the circumference *ACG* is equal to the circumference *ADG*,

and in them *ABC* is equal to *AED*,

therefore the remainder *CG* is equal to the remainder *GD*.

And, if we join AD, we conclude that the angles at L are right,
and CD is double of CL.

For the same reason
the angles at M are also right,
and AC is double of CM.



Since then the angle ALC is equal to the angle AMF,
and the angle LAC is common to the two triangles ACL and AMF,
therefore the remaining angle ACL is equal to the remaining angle MFA; [I. 32]
therefore the triangle ACL is equiangular with the triangle AMF;
therefore, proportionally, as LC is to CA, so is MF to FA.

And the doubles of the antecedents may be taken;
therefore, as the double of LC is to CA, so is the double of MF to FA.

But, as the double of MF is to FA, so is MF to the half of FA;
therefore also, as the double of LC is to CA, so is MF to the half of FA.

And the halves of the consequents may be taken;
therefore, as the double of LC is to the half of CA, so is MF to the fourth of FA.

And *DC* is double of *LC*, *CM* is half of *CA*, and *FK* a fourth part of *FA*;
therefore, as *DC* is to *CM*, so is *MF* to *FK*.

Componendo also, as the sum of *DC*, *CM* is to *CM*, so is *MK* to *KF*; [v. 18]
therefore also, as the square on the sum of *DC*, *CM* is to the square on *CM*, so is the square on *MK* to the square on *KF*.

And since, when the straight line subtending two sides of the pentagon, as *AC*, is cut in extreme and mean ratio, the greater segment is equal to the side of the pentagon, that is, to *DC*, [XIII. 8]
while the square on the greater segment added to the half of the whole is five times the square on the half of the whole, [XIII. 1]
and *CM* is half of the whole *AC*,
therefore the square on *DC*, *CM* taken as one straight line is five times the square on *CM*.

But it was proved that, as the square on *DC*, *CM* taken as one straight line is to the square on *CM*, so is the square on *MK* to the square on *KF*;
therefore the square on *MK* is five times the square on *KF*.

But the square on *KF* is rational,
for the diameter is rational;
therefore the square on *MK* is also rational;
therefore *MK* is rational

And, since *BF* is quadruple of *FK*,
therefore *BK* is five times *KF*;
therefore the square on *BK* is twenty-five times the square on *KF*.

But the square on *MK* is five times the square on *KF*;
therefore the square on *BK* is five times the square on *KM*;
therefore the square on *BK* has not to the square on *KM* the ratio which a square number has to a square number;
therefore *BK* is incommensurable in length with *KM*. [x. 9]

And each of them is rational.

Therefore *BK*, *KM* are rational straight lines commensurable in square only.

But, if from a rational straight line there be subtracted a rational straight line which is commensurable with the whole in square only, the remainder is irrational, namely an apotome; therefore MB is an apotome and MK the annex to it. [x. 73]

I say next that MB is also a fourth apotome.

Let the square on N be equal to that by which the square on BK is greater than the square on KM;

therefore the square on BK is greater than the square on KM by the square on N.

And, since KF is commensurable with FB,

componendo also, KB is commensurable with FB. [x. 15]

But BF is commensurable with BH;

therefore BK is also commensurable with BH. [x. 12]

And, since the square on BK is five times the square on KM,

therefore the square on BK has to the square on KM the ratio which 5 has to 1.

Therefore, *convertendo*, the square on BK has to the square on N the ratio which 5 has to 4 [v. 19, Por.], and this is not the ratio which a square number has to a square number;

therefore BK is incommensurable with N; [x. 9]

therefore the square on BK is greater than the square on KM by the square on a straight line incommensurable with BK.

Since then the square on the whole BK is greater than the square on the annex KM by the square on a straight line incommensurable with BK,

and the whole BK is commensurable with the rational straight line, BH, set out,

therefore MB is a fourth apotome. [x. Deff. III. 4]

But the rectangle contained by a rational straight line and a fourth apotome is irrational,

and its square root is irrational, and is called minor. [x. 94]

But the square on AB is equal to the rectangle HB, BM,

because, when AH is joined, the triangle ABH is equiangular with the triangle ABM, and, as HB is to BA, so is AB to BM.

Therefore the side AB of the pentagon is the irrational straight line called minor.

Q. E. D.

Here we require certain definitions and propositions of Book X.

First we require the definition of an *apotome* [see X. 73], which is a straight line of the form $(\rho \sim \sqrt{k}\,.\,\rho)$, where ρ is a "rational" straight line and k is any integer or numerical fraction, the square root of which is not integral or expressible in integers. The lesser of the straight lines ρ, $\sqrt{k}\,.\,\rho$ is the *annex*.

Next we require the definition of the *fourth apotome* [X. Deff. III. (after X. 84)], which is a straight line of the form $(x - y)$, where x, y (being both rational and commensurable in square only) are also such that $\sqrt{x^2 - y^2}$ is incommensurable with x, while x is commensurable with a given rational straight line ρ. As shown on X. 88 (note), the *fourth apotome* is of the form

$$\left(k\rho - \frac{k\rho}{\sqrt{1+\lambda}}\right).$$

Lastly the *minor* (straight line) is the irrational straight line defined in X. 76. It is of the form $(x - y)$, where x, y are incommensurable in square, and $(x^2 + y^2)$ is 'rational,' while xy is 'medial.' As shown in the note on X. 76, the *minor* irrational straight line is of the form

$$\frac{\rho}{\sqrt{2}}\sqrt{1 + \frac{k}{\sqrt{1+k^2}}} - \frac{\rho}{\sqrt{2}}\sqrt{1 - \frac{k}{\sqrt{1+k^2}}}.$$

The proposition may be put as follows. $ABCDE$ being a regular pentagon inscribed in a circle, AG, BH the diameters through A, B meeting CD in L and AC in M respectively, FK is made equal to $\frac{1}{4}AF$.

Now, the radius $AF\,(r)$ being rational, so are FK, BK.

The arcs CG, GD are equal;
hence $\angle$ s at L are right, and $CD = 2CL$

Similarly $\angle$ s at M are right, and $AC = 2CM$.

We have to prove

(1) that BM is an apotome,
(2) that BM is a fourth apotome,
(3) that BA is a minor irrational straight line.

Remembering that, if CA is divided in extreme and mean ratio, the greater segment is equal to the side of the pentagon [XIII. 8], and that accordingly [XIII. 1] $(CD + \frac{1}{2}CA)^2 = 5\,(\frac{1}{2}CA)^2$, we work towards a proportion containing the ratio $(CD + CM)^2 : CM^2$, thus.

The $\triangle$s ACL, AFM are equiangular and therefore similar.

Therefore $\quad LC : CA = MF : FA$,
and accordingly $\quad 2LC : CA = MF : \frac{1}{2}FA$;
thus $\quad 2LC : \frac{1}{2}CA = MF : \frac{1}{4}FA$,
or $\quad DC : CM = MF : FK$;

whence, *componendo*, and squaring,

$$(DC + CM)^2 : CM^2 = MK^2 : KF^2.$$

But $\quad (DC + CM)^2 = 5\,CM^2$;
therefore $\quad MK^2 = 5KF^2$.

[This means that $MK^2 = \frac{5}{16} r^2$,

or $MK = \frac{\sqrt{5}}{4} r$.]

It follows that, KF being rational, MK^2, and therefore MK, is rational.

(1) To prove that BM is an *apotome* and MK its *annex*.

We have $BF = 4FK$;

therefore $BK = 5FK$,

$$BK^2 = 25FK^2$$

$$= 5MK^2, \text{ from above;}$$

therefore BK^2 has not to MK^2 the ratio of a square number to a square number;

therefore BK, MK are incommensurable in length.

They are therefore rational and commensurable in square only; accordingly BM is an *apotome*.

[$BK^2 = 5MK^2 = \frac{25}{16} r^2$, and $BK = \frac{5}{4} r$.

Consequently $BK - MK = \left(\frac{5}{4} r - \frac{\sqrt{5}}{4} r\right)$.]

(2) To prove that BM is a *fourth apotome*.

First, since KF, FB are commensurable, BK, BF are commensurable, i.e. BK is commensurable with BH, a given rational straight line.

Secondly, if $N^2 = BK^2 - KM^2$,

since $BK^2 : KM^2 = 5 : 1$,

it follows that $BK^2 : N^2 = 5 : 4$,

whence BK, N are incommensurable.

Therefore BM is a *fourth apotome*.

(3) To prove that BA is a *minor* irrational straight line.

If a fourth apotome form a rectangle with a rational straight line, the side of the square equivalent to the rectangle is *minor* [x. 94].

Now $BA^2 = HB \,.\, BM$,

HB is rational, and BM is a fourth apotome;

therefore BA is a *minor* irrational straight line.

[$BA = r\sqrt{2} \,.\, \sqrt{\frac{5}{4} - \frac{\sqrt{5}}{4}} = \frac{r}{2}\sqrt{10 - 2\sqrt{5}}$.

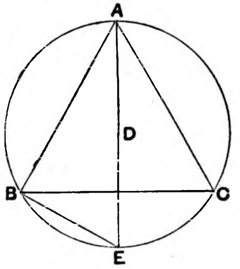
If this is separated into the difference between two straight lines, we have

$$BA = \frac{r}{2}\sqrt{5 + 2\sqrt{5}} - \frac{r}{2}\sqrt{5 - 2\sqrt{5}}.]$$

PROPOSITION 12.

If an equilateral triangle be inscribed in a circle, the square on the side of the triangle is triple of the square on the radius of the circle.

Let ABC be a circle,
and let the equilateral triangle ABC be inscribed in it;
I say that the square on one side of the triangle ABC is triple of the square on the radius of the circle.



For let the centre D of the circle ABC be taken,
let AD be joined and carried through to E,
and let BE be joined.

Then, since the triangle ABC is equilateral,
therefore the circumference BEC is a third part of the circumference of the circle ABC.

Therefore the circumference BE is a sixth part of the circumference of the circle;
therefore the straight line BE belongs to a hexagon;
therefore it is equal to the radius DE. [IV. 15, Por.]

And, since AE is double of DE,
the square on AE is quadruple of the square on ED, that is, of the square on BE.

But the square on AE is equal to the squares on AB, BE; [III. 31, I. 47]
therefore the squares on AB, BE are quadruple of the square on BE.

Therefore, *separando*, the square on AB is triple of the square on BE.

But BE is equal to DE;
therefore the square on AB is triple of the square on DE.

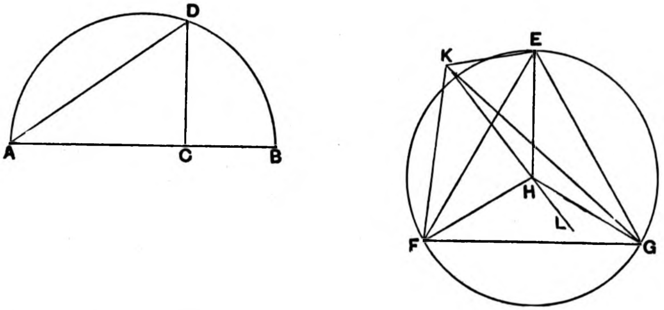
Therefore the square on the side of the triangle is triple of the square on the radius.

Q. E. D.

PROPOSITION 13.

To construct a pyramid, to comprehend it in a given sphere, and to prove that the square on the diameter of the sphere is one and a half times the square on the side of the pyramid.

Let the diameter *AB* of the given sphere be set out,
and let it be cut at the point *C* so that *AC* is double of *CB*;
let the semicircle *ADB* be described on *AB*,
let *CD* be drawn from the point *C* at right angles to *AB*,
and let *DA* be joined;
let the circle *EFG* which has its radius equal to *DC* be set out,
let the equilateral triangle *EFG* be inscribed in the circle *EFG*, [IV. 2]
let the centre of the circle, the point *H*, be taken, [III. 1]
let *EH*, *HF*, *HG* be joined;
from the point *H* let *HK* be set up at right angles to the plane of the circle *EFG*, [XI. 12]
let *HK* equal to the straight line *AC* be cut off from *HK*,
and let *KE*, *KF*, *KG* be joined.



Now, since *KH* is at right angles to the plane of the circle *EFG*,
therefore it will also make right angles with all the straight lines which meet it and are in the plane of the circle *EFG*. [XI. Def. 3]

But each of the straight lines *HE*, *HF*, *HG* meets it:
therefore *HK* is at right angles to each of the straight lines *HE*, *HF*, *HG*.

And, since *AC* is equal to *HK*, and *CD* to *HE*,
and they contain right angles,
therefore the base *DA* is equal to the base *KE*. [I. 4]

For the same reason
each of the straight lines *KF*, *KG* is also equal to *DA*;
therefore the three straight lines *KE*, *KF*, *KG* are equal to one another.

And, since *AC* is double of *CB*,
therefore *AB* is triple of *BC*.

But, as *AB* is to *BC*, so is the square on *AD* to the square on *DC*, as will be proved afterwards.

Therefore the square on *AD* is triple of the square on *DC*.

But the square on *FE* is also triple of the square on *EH*, [XIII. 12]
and *DC* is equal to *EH*;
therefore *DA* is also equal to *EF*.

But *DA* was proved equal to each of the straight lines *KE*, *KF*, *KG*;
therefore each of the straight lines *EF*, *FG*, *GE* is also equal to each of the straight lines *KE*, *KF*, *KG*;
therefore the four triangles *EFG*, *KEF*, *KFG*, *KEG* are equilateral.

Therefore a pyramid has been constructed out of four equilateral triangles, the triangle *EFG* being its base and the point *K* its vertex.

It is next required to comprehend it in the given sphere and to prove that the square on the diameter of the sphere is one and a half times the square on the side of the pyramid.

For let the straight line *HL* be produced in a straight line with *KH*,
and let *HL* be made equal to *CB*.

Now, since, as *AC* is to *CD*, so is *CD* to *CB*, [VI. 8, Por.]
while *AC* is equal to *KH*, *CD* to *HE*, and *CB* to *HL*,
therefore, as *KH* is to *HE*, so is *EH* to *HL*;
therefore the rectangle *KH*, *HL* is equal to the square on *EH*. [VI. 17]

And each of the angles *KHE*. *EHL* is right;
therefore the semicircle described on *KL* will pass through *E* also. [cf. VI. 8, III. 31]

If then, KL remaining fixed, the semicircle be carried round and restored to the same position from which it began to be moved, it will also pass through the points F, G,

since, if FL, LG be joined, the angles at F, G similarly become right angles;

and the pyramid will be comprehended in the given sphere.

For KL, the diameter of the sphere, is equal to the diameter AB of the given sphere, inasmuch as KH was made equal to AC, and HL to CB.

I say next that the square on the diameter of the sphere is one and a half times the square on the side of the pyramid

For, since AC is double of CB,

therefore AB is triple of BC;

and, *convertendo*, BA is one and a half times AC.

But, as BA is to AC, so is the square on BA to the square on AD.

Therefore the square on BA is also one and a half times the square on AD.

And BA is the diameter of the given sphere, and AD is equal to the side of the pyramid.

Therefore the square on the diameter of the sphere is one and a half times the square on the side of the pyramid.

Q. E. D.

LEMMA.

It is to be proved that, as AB is to BC, so is the square on AD to the square on DC.

For let the figure of the semicircle be set out,

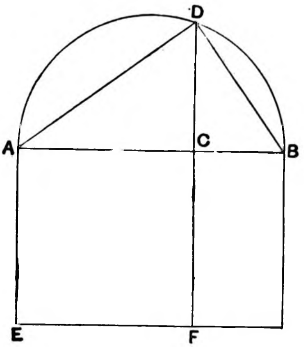
let DB be joined,

let the square EC be described on AC,

and let the parallelogram FB be completed.

Since then, because the triangle DAB is equiangular with the triangle DAC,

as BA is to AD, so is DA to AC, [VI. 8, VI. 4]



therefore the rectangle BA, AC is equal to the square on AD. [VI. 17]

And since, as AB is to BC, so is EB to BF, [VI. 1]
and EB is the rectangle BA, AC, for EA is equal to AC,
and BF is the rectangle AC, CB,
therefore, as AB is to BC, so is the rectangle BA, AC to the rectangle AC, CB.

And the rectangle BA, AC is equal to the square on AD,
and the rectangle AC, CB to the square on DC,
for the perpendicular DC is a mean proportional between the segments AC, CB of the base, because the angle ADB is right. [VI. 8, Por.]

Therefore, as AB is to BC, so is the square on AD to the square on DC.

Q. E. D.

The Lemma is with reason suspected. Euclid commonly takes more difficult theorems for granted in the stereometrical Books. It is also clumsy in itself, while, from a gloss in the proposition rejected as an interpolation, it is clear that the interpolator of the gloss had not the Lemma. With the Lemma should disappear the words "as will be proved afterwards" (p. 469).

In the figure of the proposition, the semicircle really represents half of a section of the sphere through its centre and one edge of the inscribed tetrahedron (AD being the length of that edge).

The proof is in three parts, the object of which is to prove

(1) that $KEFG$ is a tetrahedron with all its edges equal to AD,

(2) that it is inscribable in a sphere of diameter equal to AB,

(3) that $AB^2 = \frac{3}{2} AD^2$.

To prove (1) we have to show

(*a*) that $KE = KF = KG = AD$,

(*b*) that $AD = EF$.

(*a*) Since $HE = HF = HG = CD$,

$$KH = AC,$$

and $\angle$s ACD, KHE, KHF, KHG are right,

$\triangle$s ACD, KHE, KHF, KHG are equal in all respects;

therefore $KE = KF = KG = AD$.

(*b*) Since $AB = 3BC$,

and $AB : BC = AB . AC : AC . CB$

$$= AD^2 : CD^2,$$

it follows that $AD^2 = 3CD^2$.

But [XIII. 12] $EF^2 = 3EH^2$;

and $EH = CD$, by construction

Therefore $AD = EF$.

Thus $EFGK$ is a *regular* tetrahedron.

(2) We now observe the usefulness of Euclid's description of a sphere [in XI. Def. 14].

Producing $KH (= AC)$ to L so that $HL = CB$,
we have KL equal to AB;
thus KL is a diameter of the sphere which should circumscribe our tetrahedron,
and we have only to prove that E, F, G lie on semicircles described on KL as diameter.

E.g. for the point E,
since $AC : CD = CD : CB$,
while $AC = KH$, $CD = HE$, $CB = HL$,
we have $KH : HE = HE : HL$,
or $KH . HL = HE^2$,
whence, the angles KHE, EHL being right,
EKL is a triangle right-angled at E [cf. VI. 8].

Hence E lies on a semicircle on KL as diameter.

Similarly for F, G.

Thus a semicircle on KL as diameter revolving round KL passes successively through E, F, G.

(3) $AB = 3BC$;
therefore $BA = \frac{3}{2} AC$.

And $BA : AC = BA^2 : BA . AC$
$= BA^2 : AD^2$.

Therefore $BA^2 = \frac{3}{2} AD^2$.

If r be the radius of the circumscribed sphere,

$$\text{(edge of tetrahedron)} = \frac{2\sqrt{2}}{\sqrt{3}} . r = \tfrac{2}{3}\sqrt{6} . r.$$

It will be observed that, although in these cases Euclid's construction is equivalent to inscribing the particular regular solid in a given sphere, he does not actually construct the solid *in* the sphere but constructs a solid which a sphere *equal* to the given sphere will circumscribe. Pappus, on the other hand, in dealing with the same problems, actually constructs the respective solids in the given spheres. His method is to find circular sections in the given spheres containing a certain number of the angular points of the given solids. His solutions are interesting, although they require a knowledge of some properties of a sphere which are of course not found in the *Elements* but belonged to treatises such as the *Sphaerica* of Theodosius.

Pappus' solution of the problem of Eucl. XIII. 13.

In order to inscribe a regular pyramid or tetrahedron in a given sphere, Pappus (III. pp. 142—144) finds two circular sections equal and parallel to one another, each of which contains one of two opposite edges as its diameter. In this and the other similar problems he proceeds in the orthodox manner by analysis and synthesis. The following is a reproduction of his solution of this case.

Analysis.

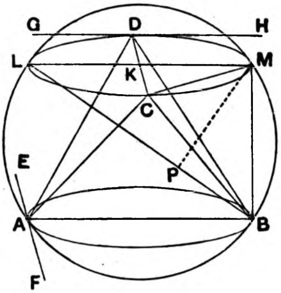
Suppose the problem solved, A, B, C, D being the angular points of the required pyramid.

Through A draw EF parallel to CD; this will make equal angles with AC, AD; and, since AB does so too, EF is perpendicular to AB [Pappus has a lemma for this, p. 140, 12—24], and is therefore a tangent to the sphere (for EF is parallel to CD, the base of the triangle ACD, and therefore touches the circle circumscribing it, while it also touches the circular section AB made by the plane passing through AB and EF perpendicular to it).

Similarly GH drawn through D parallel to AB touches the sphere.

And the plane through GH, CD makes a circular section equal and parallel to AB.

Through the centre K of that circular section, and in the plane of the section, draw LM perpendicular to CD and therefore parallel to AB. Join BL, BM.



BM is then perpendicular to AB, LM, and LB is a diameter of the sphere.

Join MC.

Then $LM^2 = 2MC^2$,

and $BC = AB = LM$,

so that $BC^2 = 2MC^2$.

And BM, being perpendicular to the plane of the circle LM, is perpendicular to CM,

whence $BC^2 = BM^2 + MC^2$,

so that $BM = MC$.

But $BC = LM$;

therefore $LM^2 = 2BM^2$.

And, since the angle LMB is right,

$$BL^2 = LM^2 + MB^2 = \tfrac{3}{2} LM^2.$$

Synthesis.

Draw two parallel circular sections of the sphere with diameter d', such that

$$d'^2 = \tfrac{2}{3} d^2,$$

where d is the diameter of the sphere.

[This is easily done by dividing BL, any diameter of the sphere, at P, so that $LP = 2PB$, and then drawing PM at right angles to LB meeting the great circle LMB of the sphere in M. Then $LM^2 : LB^2 = LP : LB = 2 : 3$.]

Draw sections through M, B perpendicular to MB, and in these sections respectively draw the parallel diameters LM, AB.

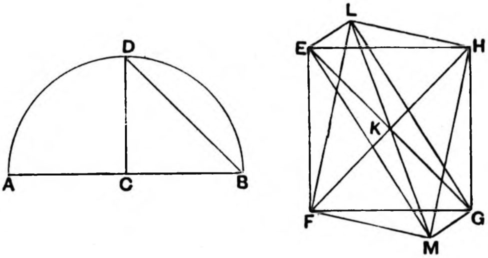
Lastly, in the section LM draw CD through the centre K perpendicular to LM.

$ABCD$ is then the required regular pyramid or tetrahedron.

PROPOSITION 14.

To construct an octahedron and comprehend it in a sphere, as in the preceding case; and to prove that the square on the diameter of the sphere is double of the square on the side of the octahedron.

Let the diameter *AB* of the given sphere be set out,
and let it be bisected at *C*;
let the semicircle *ADB* be described on *AB*,
let *CD* be drawn from *C* at right angles to *AB*,
let *DB* be joined;
let the square *EFGH*, having each of its sides equal to *DB*, be set out,
let *HF*, *EG* be joined,
from the point *K* let the straight line *KL* be set up at right angles to the plane of the square *EFGH* [XI. 12], and let it be carried through to the other side of the plane, as *KM*;
from the straight lines *KL*, *KM* let *KL*, *KM* be respectively cut off equal to one of the straight lines *EK*, *FK*, *GK*, *HK*,
and let *LE*, *LF*, *LG*, *LH*, *ME*, *MF*, *MG*, *MH* be joined.



Then, since *KE* is equal to *KH*,
and the angle *EKH* is right,
therefore the square on *HE* is double of the square on *EK*. [I. 47]

Again, since *LK* is equal to *KE*,
and the angle *LKE* is right,
therefore the square on *EL* is double of the square on *EK*. [*id.*]

But the square on *HE* was also proved double of the square on *EK*;
therefore the square on *LE* is equal to the square on *EH*;
therefore *LE* is equal to *EH*.

For the same reason
LH is also equal to *HE*;
therefore the triangle *LEH* is equilateral.

Similarly we can prove that each of the remaining triangles of which the sides of the square *EFGH* are the bases, and the points *L*, *M* the vertices, is equilateral;
therefore an octahedron has been constructed which is contained by eight equilateral triangles.

It is next required to comprehend it in the given sphere, and to prove that the square on the diameter of the sphere is double of the square on the side of the octahedron.

For, since the three straight lines *LK*, *KM*, *KE* are equal to one another,
therefore the semicircle described on *LM* will also pass through *E*.

And for the same reason,
if, *LM* remaining fixed, the semicircle be carried round and restored to the same position from which it began to be moved,
it will also pass through the points *F*, *G*, *H*,
and the octahedron will have been comprehended in a sphere.

I say next that it is also comprehended in the given sphere.

For, since *LK* is equal to *KM*,
while *KE* is common,
and they contain right angles,
therefore the base *LE* is equal to the base *EM*. [I. 4]

And, since the angle *LEM* is right, for it is in a semicircle, [III. 31]
therefore the square on *LM* is double of the square on *LE*. [I. 47]

Again, since *AC* is equal to *CB*,
AB is double of *BC*.

But, as *AB* is to *BC*, so is the square on *AB* to the square on *BD*;
therefore the square on *AB* is double of the square on *BD*.

But the square on *LM* was also proved double of the square on *LE*.

And the square on *DB* is equal to the square on *LE*, for *EH* was made equal to *DB*.

Therefore the square on *AB* is also equal to the square on *LM*;
therefore *AB* is equal to *LM*.

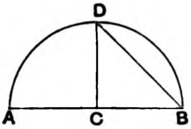
And *AB* is the diameter of the given sphere;
therefore *LM* is equal to the diameter of the given sphere.

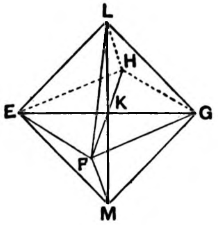
Therefore the octahedron has been comprehended in the given sphere, and it has been demonstrated at the same time that the square on the diameter of the sphere is double of the square on the side of the octahedron.

Q. E. D.

I think the accompanying figure will perhaps be clearer than that in Euclid's text.

EFGH being a square with side equal to *BD*, it follows that *KE*, *KF*, *KG*, *KH* are all equal to *CB*.





So are *KL*, *KM*, by construction;
hence *LE*, *LF*, *LG*, *LH* and *ME*, *MF*, *MG*, *MH* are all equal to *EF* or *BD*.

Thus (1) the figure is made up of eight equilateral triangles and is therefore a regular octahedron.

(2) Since $KE = KL = KM$,
the semicircle on *LM* in the plane *LKE* passes through *E*.

Similarly *F*, *G*, *H* lie on semicircles on *LM* as diameter.

Thus all the vertices of the tetrahedron lie on the sphere of which *LM* is a diameter.

(3) $LE = EM = BD$;

therefore $LM^2 = 2EL^2 = 2BD^2$

$= AB^2$,

or $LM = AB$.

(4)
$$AB^2 = 2BD^2$$
$$= 2EF^2.$$

If r be the radius of the circumscribed sphere,

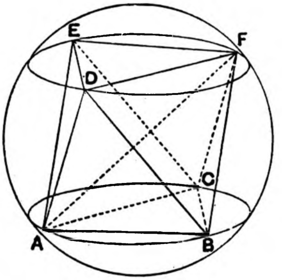
$$(\text{edge of octahedron}) = \sqrt{2} \cdot r.$$

Pappus' method.

Pappus (III. pp. 148—150) finds the two equal and parallel sections of the sphere which circumscribe two opposite faces of the octahedron thus.

Analysis.

Suppose the octahedron inscribed, A, B, C; D, E, F being the vertices.



Through ABC, DEF describe planes making the circular sections ABC, DEF.

Since the straight lines DA, DB, DE, DF are equal, the points A, E, F, B lie on a circle of which D is the pole.

Again, since AB, BF, FE, EA are equal, $ABFE$ is a square inscribed in the said circle, and AB, EF are parallel.

Similarly DE is parallel to BC, and DF to AC.

Therefore the circles through D, E, F and A, B, C are parallel; and they are also equal because the equilateral triangles inscribed in them are equal.

Now, ABC, DEF being equal and parallel circular sections, and AB, EF equal and parallel chords "not on the same side of the centres,"

AF is a diameter of the sphere.

[Pappus has a lemma for this, pp. 136—138].

And $AE = EF$, so that $AF^2 = 2FE^2$.

But, if d' be the diameter of the circle DEF,

$$d'^2 = \tfrac{4}{3}EF^2. \qquad \text{[cf. XIII. 12]}$$

Therefore, if d be the diameter of the sphere,

$$d^2 : d'^2 = 3 : 2.$$

Now d is given, and therefore d' is given; hence the circles DEF, ABC are given.

Synthesis.

Draw two equal and parallel circular sections with diameter d', such that

$$d^2 = \tfrac{3}{2}d'^2,$$

where d is the diameter of the sphere.

Inscribe an equilateral triangle ABC in either circle (ABC).

In the other circle draw EF equal and parallel to AB but on the opposite side of the centre, and complete the inscribed equilateral triangle DEF.

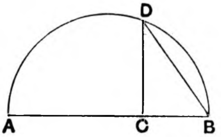
$ABCDEF$ is the octahedron required.

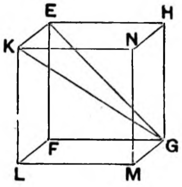
It will be observed that, whereas in the problem of XIII. 13 Euclid first finds the circle circumscribing a face and Pappus first finds an edge, in this problem Euclid finds the edge first and Pappus the circle circumscribing a face.

PROPOSITION 15.

To construct a cube and comprehend it in a sphere, like the pyramid; and to prove that the square on the diameter of the sphere is triple of the square on the side of the cube.

Let the diameter AB of the given sphere be set out,
and let it be cut at C so that AC is double of CB;
let the semicircle ADB be described on AB,
let CD be drawn from C at right angles to AB,
and let DB be joined;
let the square $EFGH$ having its side equal to DB be set out,
from E, F, G, H let EK, FL, GM, HN be drawn at right angles to the plane of the square $EFGH$,
from EK, FL, GM, HN let EK, FL, GM, HN respectively be cut off equal to one of the straight lines EF, FG, GH, HE,
and let KL, LM, MN, NK be joined;
therefore the cube FN has been constructed which is contained by six equal squares.





It is then required to comprehend it in the given sphere, and to prove that the square on the diameter of the sphere is triple of the square on the side of the cube.

For let KG, EG be joined.

Then, since the angle KEG is right, because KE is also at right angles to the plane EG and of course to the straight line EG also, [XI. Def. 3]
therefore the semicircle described on KG will also pass through the point E.

Again, since GF is at right angles to each of the straight lines FL, FE,
GF is also at right angles to the plane FK;
hence also, if we join FK, GF will be at right angles to FK;

and for this reason again the semicircle described on *GK* will also pass through *F*.

Similarly it will also pass through the remaining angular points of the cube.

If then, *KG* remaining fixed, the semicircle be carried round and restored to the same position from which it began to be moved,

the cube will be comprehended in a sphere.

I say next that it is also comprehended in the given sphere.

For, since *GF* is equal to *FE*,

and the angle at *F* is right,

therefore the square on *EG* is double of the square on *EF*.

But *EF* is equal to *EK*;

therefore the square on *EG* is double of the square on *EK*;

hence the squares on *GE*, *EK*, that is the square on *GK* [I. 47], is triple of the square on *EK*.

And, since *AB* is triple of *BC*,

while, as *AB* is to *BC*, so is the square on *AB* to the square on *BD*,

therefore the square on *AB* is triple of the square on *BD*.

But the square on *GK* was also proved triple of the square on *KE*.

And *KE* was made equal to *DB*;

therefore *KG* is also equal to *AB*.

And *AB* is the diameter of the given sphere;

therefore *KG* is also equal to the diameter of the given sphere.

Therefore the cube has been comprehended in the given sphere; and it has been demonstrated at the same time that the square on the diameter of the sphere is triple of the square on the side of the cube.

Q. E. D.

AB is divided so that $AC = 2CB$; *CD* is drawn at right angles to *AB*, and *BD* is joined.

KG is, by construction, a cube of side equal to *BD*.

To prove (1) that it is inscribable in a sphere.

Since *KE* is perpendicular to *EH*, *EF*,

KE is perpendicular to *EG*.

Thus, KEG being a right angle, E lies on a semicircle with diameter KG.

The same thing is proved in the same way of the other vertices F, H, L, M, N.

Thus the cube is inscribed in the sphere of which KG is a diameter.

(2)
$$\begin{aligned} KG^2 &= KE^2 + EG^2 \\ &= KE^2 + 2EF^2 \\ &= 3EK^2. \end{aligned}$$

Also $$AB = 3BC,$$

while $$\begin{aligned} AB : BC &= AB^2 : AB \,.\, BC \\ &= AB^2 : BD^2 ; \end{aligned}$$

therefore $$AB^2 = 3BD^2.$$

But $$BD = EK;$$

therefore $$KG = AB.$$

(3)
$$\begin{aligned} AB^2 &= 3BD^2 \\ &= 3KE^2. \end{aligned}$$

If r be the radius of the circumscribed sphere,

$$(\text{edge of cube}) = \frac{2}{\sqrt{3}} \cdot r = \tfrac{2}{3}\sqrt{3} \cdot r.$$

Pappus' solution.

In this case too Pappus (III. pp. 144—148) gives the full analysis and synthesis.

Analysis.

Suppose the problem solved, and let the vertices of the cube be A, B, C, D, E, F, G, H.

Draw planes through A, B, C, D and E, F, G, H respectively; these will produce parallel circular sections, which are also equal since the inscribed squares are equal.

And CE will be a diameter of the sphere.

Join EG.

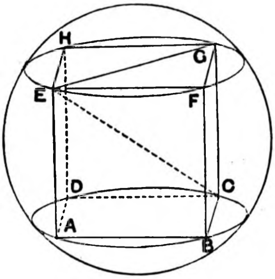
Now, since $EG^2 = 2EH^2 = 2GC^2$,

and the angle CGE is right,

$$CE^2 = GC^2 + EG^2 = \tfrac{3}{2}EG^2.$$

But CE^2 is given;

therefore EG^2 is given, so that the circles $EFGH$, $ABCD$, and the squares inscribed in them, are given.



Synthesis.

Draw two parallel circular sections with equal diameters d', such that

$$d^2 = \tfrac{3}{2}d'^2,$$

where d is the diameter of the given sphere.

Inscribe a square in one of the circles, as $ABCD$.

In the other circle draw FG equal and parallel to BC, and complete the square on FG inscribed in the circle $EFGH$.

The eight vertices of the required cube are thus determined.

PROPOSITION 16.

To construct an icosahedron and comprehend it in a sphere, like the aforesaid figures; and to prove that the side of the icosahedron is the irrational straight line called minor.

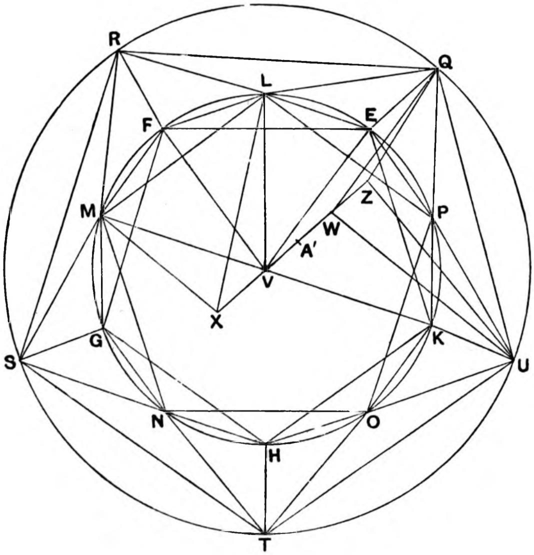
Let the diameter AB of the given sphere be set out,

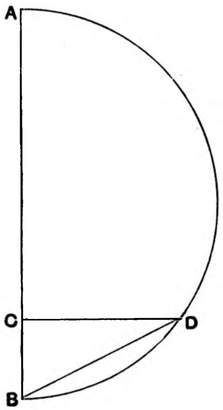
and let it be cut at C so that AC is quadruple of CB,

let the semicircle ADB be described on AB,

let the straight line CD be drawn from C at right angles to AB,

and let DB be joined;





let the circle $EFGHK$ be set out and let its radius be equal to DB,

let the equilateral and equiangular pentagon $EFGHK$ be inscribed in the circle $EFGHK$,

let the circumferences EF, FG, GH, HK, KE be bisected at the points L, M, N, O, P,

and let LM, MN, NO, OP, PL, EP be joined.

Therefore the pentagon *LMNOP* is also equilateral,
and the straight line *EP* belongs to a decagon.

Now from the points *E*, *F*, *G*, *H*, *K* let the straight lines *EQ*, *FR*, *GS*, *HT*, *KU* be set up at right angles to the plane of the circle, and let them be equal to the radius of the circle *EFGHK*,
let *QR*, *RS*, *ST*, *TU*, *UQ*, *QL*, *LR*, *RM*, *MS*, *SN*, *NT*, *TO*, *OU*, *UP*, *PQ* be joined.

Now, since each of the straight lines *EQ*, *KU* is at right angles to the same plane,
therefore *EQ* is parallel to *KU*. [XI. 6]

But it is also equal to it;
and the straight lines joining those extremities of equal and parallel straight lines which are in the same direction are equal and parallel. [I. 33]

Therefore *QU* is equal and parallel to *EK*.

But *EK* belongs to an equilateral pentagon;
therefore *QU* also belongs to the equilateral pentagon inscribed in the circle *EFGHK*.

For the same reason
each of the straight lines *QR*, *RS*, *ST*, *TU* also belongs to the equilateral pentagon inscribed in the circle *EFGHK*;
therefore the pentagon *QRSTU* is equilateral.

And, since *QE* belongs to a hexagon,
and *EP* to a decagon,
and the angle *QEP* is right,
therefore *QP* belongs to a pentagon;
for the square on the side of the pentagon is equal to the square on the side of the hexagon and the square on the side of the decagon inscribed in the same circle. [XIII. 10]

For the same reason
PU is also a side of a pentagon.

But *QU* also belongs to a pentagon;
therefore the triangle *QPU* is equilateral.

For the same reason
each of the triangles *QLR*, *RMS*, *SNT*, *TOU* is also equilateral.

And, since each of the straight lines *QL*, *QP* was proved to belong to a pentagon,
and *LP* also belongs to a pentagon,
therefore the triangle *QLP* is equilateral.

For the same reason
each of the triangles *LRM*, *MSN*, *NTO*, *OUP* is also equilateral.

Let the centre of the circle *EFGHK*. the point *V*, be taken;
from *V* let *VZ* be set up at right angles to the plane of the circle,
let it be produced in the other direction, as *VX*,
let there be cut off *VW*, the side of a hexagon, and each of the straight lines *VX*, *WZ*, being sides of a decagon,
and let *QZ*, *QW*, *UZ*, *EV*, *LV*, *LX*, *XM* be joined.

Now, since each of the straight lines *VW*, *QE* is at right angles to the plane of the circle,
therefore *VW* is parallel to *QE*. [XI. 6]
But they are also equal;
therefore *EV*, *QW* are also equal and parallel. [I. 33]
But *EV* belongs to a hexagon;
therefore *QW* also belongs to a hexagon.
And, since *QW* belongs to a hexagon,
and *WZ* to a decagon,
and the angle *QWZ* is right,
therefore *QZ* belongs to a pentagon. [XIII. 10]

For the same reason
UZ also belongs to a pentagon,
inasmuch as, if we join *VK*, *WU*, they will be equal and opposite, and *VK*, being a radius, belongs to a hexagon; [IV. 15, Por.]
therefore *WU* also belongs to a hexagon.
But *WZ* belongs to a decagon,
and the angle *UWZ* is right;
therefore *UZ* belongs to a pentagon. [XIII. 10]
But *QU* also belongs to a pentagon;
therefore the triangle *QUZ* is equilateral.

For the same reason
each of the remaining triangles of which the straight lines QR, RS, ST, TU are the bases, and the point Z the vertex, is also equilateral.

Again, since VL belongs to a hexagon,
and VX to a decagon,
and the angle LVX is right,
therefore LX belongs to a pentagon. [XIII. 10]

For the same reason,
if we join MV, which belongs to a hexagon,
MX is also inferred to belong to a pentagon.

But LM also belongs to a pentagon;
therefore the triangle LMX is equilateral.

Similarly it can be proved that each of the remaining triangles of which MN, NO, OP, PL are the bases, and the point X the vertex, is also equilateral.

Therefore an icosahedron has been constructed which is contained by twenty equilateral triangles.

It is next required to comprehend it in the given sphere, and to prove that the side of the icosahedron is the irrational straight line called minor.

For, since VW belongs to a hexagon,
and WZ to a decagon,
therefore VZ has been cut in extreme and mean ratio at W,
and VW is its greater segment; [XIII. 9]
therefore, as ZV is to VW, so is VW to WZ.

But VW is equal to VE, and WZ to VX;
therefore, as ZV is to VE, so is EV to VX.

And the angles ZVE, EVX are right;
therefore, if we join the straight line EZ, the angle XEZ will be right because of the similarity of the triangles XEZ, VEZ.

For the same reason,
since, as ZV is to VW, so is VW to WZ,
and ZV is equal to XW, and VW to WQ,
therefore, as XW is to WQ, so is QW to WZ.

And for this reason again,
if we join *QX*, the angle at *Q* will be right; [VI. 8]
therefore the semicircle described on *XZ* will also pass through *Q*. [III. 31]

And if, *XZ* remaining fixed, the semicircle be carried round and restored to the same position from which it began to be moved, it will also pass through *Q* and the remaining angular points of the icosahedron,
and the icosahedron will have been comprehended in a sphere.

I say next that it is also comprehended in the given sphere.

For let *VW* be bisected at *A'*.

Then, since the straight line *VZ* has been cut in extreme and mean ratio at *W*,
and *ZW* is its lesser segment,
therefore the square on *ZW* added to the half of the greater segment, that is *WA'*, is five times the square on the half of the greater segment; [XIII. 3]
therefore the square on *ZA'* is five times the square on *A'W*.

And *ZX* is double of *ZA'*, and *VW* double of *A'W*;
therefore the square on *ZX* is five times the square on *WV*.

And, since *AC* is quadruple of *CB*,
therefore *AB* is five times *BC*.

But, as *AB* is to *BC*, so is the square on *AB* to the square on *BD*; [VI. 8, V. Def. 9]
therefore the square on *AB* is five times the square on *BD*.

But the square on *ZX* was also proved to be five times the square on *VW*.

And *DB* is equal to *VW*,
for each of them is equal to the radius of the circle *EFGHK*;
therefore *AB* is also equal to *XZ*.

And *AB* is the diameter of the given sphere;
therefore *XZ* is also equal to the diameter of the given sphere.

Therefore the icosahedron has been comprehended in the given sphere

I say next that the side of the icosahedron is the irrational straight line called minor.

For, since the diameter of the sphere is rational,

and the square on it is five times the square on the radius of the circle *EFGHK*,

therefore the radius of the circle *EFGHK* is also rational;

hence its diameter is also rational.

But, if an equilateral pentagon be inscribed in a circle which has its diameter rational, the side of the pentagon is the irrational straight line called minor. [XIII. 11]

And the side of the pentagon *EFGHK* is the side of the icosahedron.

Therefore the side of the icosahedron is the irrational straight line called minor.

PORISM. From this it is manifest that the square on the diameter of the sphere is five times the square on the radius of the circle from which the icosahedron has been described, and that the diameter of the sphere is composed of the side of the hexagon and two of the sides of the decagon inscribed in the same circle.

Q. E. D.

Euclid's method is

(1) to find the pentagons in the two parallel circular sections of the sphere, the sides of which form ten (five in each circle) of the edges of the icosahedron,

(2) to find the two points which are the poles of the two circular sections,

(3) to prove that the triangles formed by joining the angular points of the pentagons which are nearest to one another two and two are equilateral,

(4) to prove that the triangles of which the poles are the vertices and the sides of the pentagons the bases are also equilateral,

(5) that all the angular points other than the poles lie on a sphere the diameter of which is the straight line joining the poles,

(6) that this sphere is of the same size as the given sphere,

(7) that, if the diameter of the sphere is rational, the edge of the icosahedron is the *minor* irrational straight line.

I have drawn another figure which will perhaps show the pentagons, and the position of the poles with regard to them, more clearly than does Euclid's figure.

(1) If AB is the diameter of the given sphere, divide AB at C so that

$$AC = 4CB;$$

draw CD at right angles to AB meeting the semicircle on AB in D.

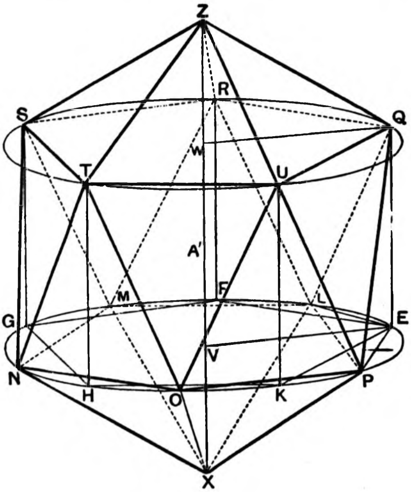
Join BD.

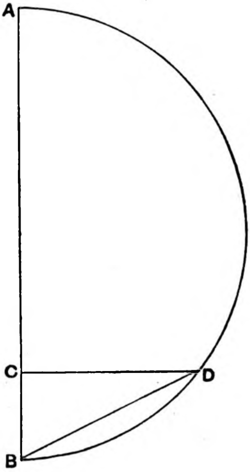
BD *is the radius of the circular sections containing the pentagons.*

[If r is the radius of the sphere,

since $$AB : BC = AB^2 : AB \,.\, BC$$
$$= AB^2 : BD^2,$$
while $$AB = 5BC,$$
it follows that $$AB^2 = 5BD^2,$$
or $$(\text{radius of section})^2 = \tfrac{4}{5}r^2.$$

Thus [XIII. 10, note] $$(\text{side of pentagon})^2 = \frac{r^2}{5}(10 - 2\sqrt{5}).]$$





Inscribe the regular pentagon $EFGHK$ in the circle $EFGHK$ of radius equal to BD.

Bisect the arcs EF, FG, ..., so forming a decagon in the circle.

Joining successive points of bisection, we obtain another regular pentagon $LMNOP$.

LMNOP *is one of the pentagons containing five edges of the icosahedron.*

The other circle and inscribed pentagon are obtained by drawing perpendiculars from E, F, G, H, K to the plane of the circle, as EQ, FR, GS, HT, KU, and making each of these perpendiculars equal to the radius of the circle, or, as Euclid says, the side of the regular hexagon in it.

QRSTU *is the second pentagon* (of course equal to the first) *containing five edges of the icosahedron.*

Joining each angular point of one of the two pentagons to the two nearest angular points in the other pentagon, we complete ten triangles each of which has for one side a side of one or other of the two pentagons.

V, W are the centres of the two circles, and VW is of course perpendicular to the planes of both.

(2) Produce VW in both directions, making VX and WZ both equal to *a side of the regular decagon in the circle* (as EL).

Joining X, Z to the angular points of the corresponding pentagons, we have five more triangles formed with the sides of each pentagon as bases, ten more triangles in all.

Now we come to the proof.

(3) Taking two adjacent perpendiculars, EQ, KU, to the plane of the circle $EFGHK$, we see that they are parallel as well as equal;
therefore QU, EK are equal and parallel.

Similarly for QR, EF etc.

Thus the pentagons have their sides equal.

To prove that the triangles QPL etc., are equilateral, we have, e.g.

$$QL^2 = LE^2 + EQ^2$$
$$= (\text{side of decagon})^2 + (\text{side of hexagon})^2$$
$$= (\text{side of pentagon})^2, \qquad [\text{XIII. 10}]$$

i.e.
$$QL = (\text{side of pentagon in circle})$$
$$= LP.$$

Similarly $$QP = LP,$$

and $$\triangle\, QPL \text{ is equilateral.}$$

So for the other triangles between the two pentagons.

(4) Since VW, EQ are equal and parallel,

VE, WQ are equal and parallel.

Thus WQ is equal to the side of a regular hexagon in the circles.

Now the angle ZWQ is right;

therefore $$ZQ^2 = ZW^2 + WQ^2$$
$$= (\text{side of decagon})^2 + (\text{side of hexagon})^2$$
$$= (\text{side of pentagon})^2. \qquad [\text{XIII. 10}]$$

Thus ZQ, ZR, ZS, ZT, ZU are all equal to QR, RS etc.; and the triangles with Z as vertex and bases QR, RS etc. are equilateral.

Similarly for the triangles with X as vertex and LM, MN etc. as bases.

Hence the figure is an icosahedron, being contained by twenty equal equilateral triangles.

(5) To prove that all the vertices of the icosahedron lie on the sphere which has XZ for diameter.

VW being equal to the side of a regular hexagon, and WZ to the side of a regular decagon inscribed in the same circle,

VZ is divided at W in extreme and mean ratio. [XIII. 9]

Therefore $$ZV : VW = VW : WZ,$$

or, since $$VW = VE,\quad WZ = VX,$$
$$ZV : VE = VE : VX.$$

Thus E lies on the semicircle on ZX as diameter. [VI. 8]

Similarly for all the other vertices of the icosahedron.

Hence the sphere with diameter XZ circumscribes it.

(6) To prove $XZ = AB$.

Since VZ is divided in extreme and mean ratio at W, and VW is bisected at A',

$$A'Z^2 = 5A'W^2. \qquad [\text{XIII. 3}]$$

Taking the doubles of $A'Z$, $A'W$, we have

$$XZ^2 = 5\,VW^2$$
$$= 5BD^2$$
$$= AB^2. \qquad [\text{see under (1) above}]$$

That is, $XZ = AB$.

[If r is the radius of the sphere,

$$VW = BD = \frac{2}{\sqrt{5}}r,$$

$$VX = (\text{side of decagon in circle of radius } BD)$$

$$= \frac{BD}{2}(\sqrt{5} - 1) \qquad [\text{XIII. 9, note}]$$

$$= \frac{r}{\sqrt{5}}(\sqrt{5} - 1).$$

Consequently

$$XZ = VW + 2\,VX$$

$$= \frac{2}{\sqrt{5}}r + \frac{2}{\sqrt{5}}r(\sqrt{5} - 1)$$

$$= 2r.]$$

(6) The radius of the circle $EFGHK$ is equal to $\frac{2}{\sqrt{5}}r$, and is therefore "rational" in Euclid's sense.

Hence the side of the inscribed pentagon is the irrational straight line called *minor*. [XIII. 11]

[The side of this pentagon is the edge of the icosahedron, and its value is (note on XIII. 10)

$$\frac{BD}{2}\sqrt{10 - 2\sqrt{5}}$$

$$= \frac{r}{\sqrt{5}}\sqrt{10 - 2\sqrt{5}}$$

$$= \frac{r}{5}\sqrt{10(5 - \sqrt{5})}.]$$

Pappus' solution.

This solution (Pappus, III. pp. 150—6) differs considerably from that of Euclid. Whereas Euclid uses *two* circular sections of the sphere (those circumscribing the pentagons of his construction), Pappus finds *four* parallel circular sections each passing through *three* of the vertices of the icosahedron; two of the circles are small circles circumscribing two opposite triangular faces respectively, and the other two circles are between these two circles, parallel to them and equal to one another.

Analysis.

Suppose the problem solved, the vertices of the icosahedron being A, B, C; D, E, F; G, H, K; L, M, N.

Since the straight lines BA, BC, BF, BG, BE drawn from B to the surface of the sphere are equal,

A, C, F, G, E are in one plane.

And AC, CF, FG, GE, EA are equal;

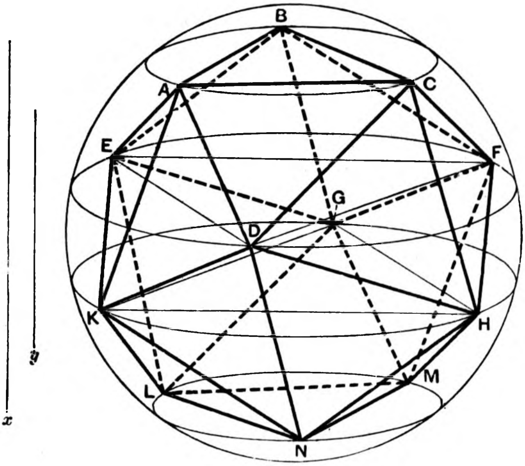
therefore $ACFGE$ is an equilateral and equiangular pentagon.

So are the figures $KEBCD$, $DHFBA$, $AKLGB$, $AKNHC$, and $CHMGB$.

Join EF, KH.

Now AC will be parallel to EF (in the pentagon $ACFGE$) and to KH (in the pentagon $AKNHC$), so that EF, KH are also parallel;

and further KH is parallel to LM (in the pentagon $LKDHM$).



Similarly BC, ED, GH, LN are all parallel;

and likewise BA, FD, GK, MN are all parallel.

Since BC is equal and parallel to LN, and BA to MN, the circles ABC, LMN are equal and parallel.

Similarly the circles DEF, KGH are equal and parallel; for the triangles inscribed in them are equal (since each of the sides in both is the chord subtending an angle of equal pentagons), and their sides are parallel respectively.

Now in the equal and parallel circles DEF, KGH the chords EF, KH are equal and parallel, and on opposite sides of the centres;

therefore FK is a diameter of the sphere [Pappus' lemma, pp. 136—8], and the angle FEK is right [Pappus' lemma, p. 138, 20—26].

[The diameter FK is not actually drawn in the figure.]

In the pentagon $GEACF$, if EF be divided in extreme and mean ratio, the greater segment is equal to AC. [Eucl. XIII. 8]

Therefore $EF : AC =$ (side of hexagon) : (side of decagon in same circle). [XIII. 9]

And $EF^2 + AC^2 = EF^2 + EK^2 = d^2$,

where d is the diameter of the sphere.

Thus FK, EF, AC are as the sides of the pentagon, hexagon and decagon respectively inscribed in the same circle. [XIII. 10]

But FK, the diameter of the sphere, is given;

therefore EF, AC are given respectively;

thus the radii of the circles EFD, ACB are given (if r, r' are their radii, $r^2 = \frac{1}{3}EF^2$, $r'^2 = \frac{1}{3}AC^2$).

Hence the circles are given·
and so are the circles KHG, LMN which are equal and parallel to them respectively.

Synthesis.

If d be the diameter of the sphere, set out two straight lines x, y, such that d, x, y are in the ratio of the sides of the pentagon, hexagon and decagon respectively inscribed in one and the same circle.

Draw (1) two equal and parallel circular sections in the sphere, with radii equal to r, where $r^2 = \frac{1}{3}x^2$, as DEF, KGH,
and (2) two equal and parallel circular sections as ABC, LMN, with radius r' such that $r'^2 = \frac{1}{3}y^2$.

In the circles (1) draw EF, KH as sides of inscribed equilateral triangles, parallel to one another, and on opposite sides of the centres;
and in the circles (2) draw AC, LM as sides of inscribed equilateral triangles parallel to one another and to EF, KH, and so that AC, EF are on opposite sides of the centres, and likewise KH, LM.

Complete the figure.

The correctness of the construction is proved as in the analysis.

It follows also (says Pappus) that

$$(\text{diam. of sphere})^2 = 3\,(\text{side of pentagon in } DEF)^2.$$

For, by construction, $\qquad KF : FE = p : h,$

where p, h are the sides of the pentagon and hexagon inscribed in the same circle DEF.

And $FE : h$ = the ratio of the side of an equilateral triangle to that of a hexagon inscribed in the same circle;

that is, $\qquad FE : h = \sqrt{3} : 1,$

whence $\qquad KF : p = \sqrt{3} : 1,$

or $\qquad KF^2 = 3p^2.$

Another construction.

Mr H. M. Taylor has a neat construction for an icosahedron of edge a.

Let l be the length of the diagonal of a regular pentagon with side equal to a.

Then (figure of XIII. 8), by Ptolemy's theorem,

$$l^2 = la + a^2.$$

Construct a cube with edge equal to l.

Let O be the centre of the cube.

From O draw OL, OM, ON perpendicular to three adjacent faces, and in these draw PP', QQ', RR' parallel to AB, AD, AE respectively.

Make LP, LP', MQ, MQ', NR, NR' all equal to $\frac{1}{2}a$.

Let p, p', q, q', r, r' be the reflexes of P, P', Q, Q', R, R' respectively.

Then will P, P', Q, Q', R, R', p, p', q, q', r, r' be the vertices of a regular icosahedron.

The projections of PQ on AB, AD, AE are equal to $\frac{1}{2}(l - a)$, $\frac{1}{2}a$, $\frac{1}{2}l$ respectively.

Therefore

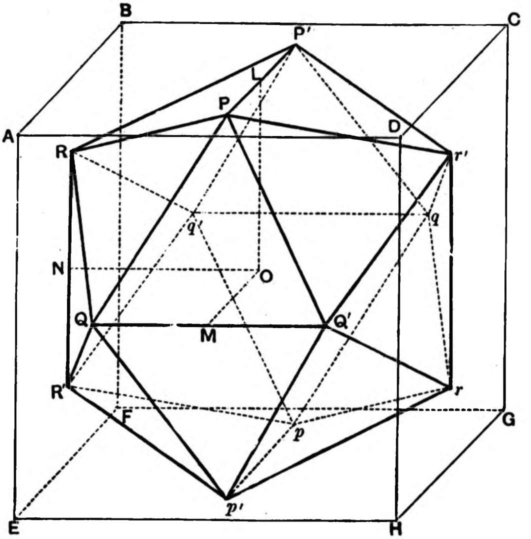
$$\begin{aligned} PQ^2 &= \tfrac{1}{4}(l-a)^2 + \tfrac{1}{4}a^2 + \tfrac{1}{4}l^2 \\ &= \tfrac{1}{2}(l^2 - al + a^2) \\ &= a^2. \end{aligned}$$

Therefore $$PQ = a.$$

Similarly it may be proved that every other edge is equal to a.

All the angular points lie on a sphere with radius OP, and

$$OP^2 = \tfrac{1}{4}(a^2 + l^2).$$



Each solid pentahedral angle is composed of five equal plane angles, each of which is the angle of an equilateral triangle.

Therefore the icosahedron is regular.

$$[a^2 = 4OP^2 - l^2.$$

And, from the equation $l^2 = la + a^2$, we derive

$$l = \frac{a}{2}(\sqrt{5} + 1).$$

Therefore, if r be the radius of the sphere,

$$a^2 \left\{1 + \frac{(\sqrt{5}+1)^2}{4}\right\} = 4r^2,$$

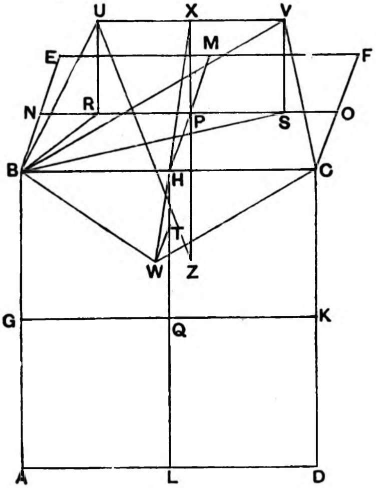
whence
$$\begin{aligned} a &= 4r/\sqrt{10 + 2\sqrt{5}} \\ &= 4r\sqrt{10 - 2\sqrt{5}}/\sqrt{80} \\ &= \frac{r}{\sqrt{5}}\sqrt{10 - 2\sqrt{5}} \\ &= \frac{r}{5}\sqrt{10(5 - \sqrt{5})}, \end{aligned}$$

as above.]

PROPOSITION 17.

To construct a dodecahedron and comprehend it in a sphere, like the aforesaid figures, and to prove that the side of the dodecahedron is the irrational straight line called apotome.

Let *ABCD*, *CBEF*, two planes of the aforesaid cube at right angles to one another, be set out,
let the sides *AB*, *BC*, *CD*, *DA*, *EF*, *EB*, *FC* be bisected at *G*, *H*, *K*, *L*, *M*, *N*, *O* respectively,
let *GK*, *HL*, *MH*, *NO* be joined,
let the straight lines *NP*, *PO*, *HQ* be cut in extreme and mean ratio at the points *R*, *S*, *T* respectively,
and let *RP*, *PS*, *TQ* be their greater segments;
from the points *R*, *S*, *T* let *RU*, *SV*, *TW* be set up at right angles to the planes of the cube towards the outside of the cube,
let them be made equal to *RP*, *PS*, *TQ*,
and let *UB*, *BW*, *WC*, *CV*, *VU* be joined.



I say that the pentagon *UBWCV* is equilateral, and in one plane, and is further equiangular.

For let *RB*, *SB*, *VB* be joined.

Then, since the straight line *NP* has been cut in extreme and mean ratio at *R*,
and *RP* is the greater segment,
therefore the squares on *PN*, *NR* are triple of the square on *RP*. [XIII. 4]

But *PN* is equal to *NB*, and *PR* to *RU*;
therefore the squares on *BN*, *NR* are triple of the square on *RU*.

But the square on *BR* is equal to the squares on *BN*, *NR*; [I. 47]
therefore the square on *BR* is triple of the square on *RU*;
hence the squares on *BR*, *RU* are quadruple of the square on *RU*.

But the square on *BU* is equal to the squares on *BR*, *RU*;
therefore the square on *BU* is quadruple of the square on *RU*;
therefore *BU* is double of *RU*.

But *VU* is also double of *UR*,
inasmuch as *SR* is also double of *PR*, that is, of *RU*;
therefore *BU* is equal to *UV*.

Similarly it can be proved that each of the straight lines *BW*, *WC*, *CV* is also equal to each of the straight lines *BU*, *UV*.

Therefore the pentagon *BUVCW* is equilateral.

I say next that it is also in one plane.

For let *PX* be drawn from *P* parallel to each of the straight lines *RU*, *SV* and towards the outside of the cube,
and let *XH*, *HW* be joined;
I say that *XHW* is a straight line.

For, since *HQ* has been cut in extreme and mean ratio at *T*, and *QT* is its greater segment,
therefore, as *HQ* is to *QT*, so is *QT* to *TH*.

But *HQ* is equal to *HP*, and *QT* to each of the straight lines *TW*, *PX*;
therefore, as *HP* is to *PX*, so is *WT* to *TH*.

And *HP* is parallel to *TW*,
for each of them is at right angles to the plane *BD*; [XI. 6]
and *TH* is parallel to *PX*,
for each of them is at right angles to the plane *BF*. [*id.*]

But if two triangles, as *XPH*, *HTW*, which have two sides proportional to two sides be placed together at one angle so that their corresponding sides are also parallel,

the remaining straight lines will be in a straight line; [VI. 32]

therefore *XH* is in a straight line with *HW*.

But every straight line is in one plane; [XI. 1]

therefore the pentagon *UBWCV* is in one plane.

I say next that it is also equiangular.

For, since the straight line *NP* has been cut in extreme and mean ratio at *R*, and *PR* is the greater segment,

while *PR* is equal to *PS*,

therefore *NS* has also been cut in extreme and mean ratio at *P*,

and *NP* is the greater segment; [XIII. 5]

therefore the squares on *NS*, *SP* are triple of the square on *NP*. [XIII. 4]

But *NP* is equal to *NB*, and *PS* to *SV*;

therefore the squares on *NS*, *SV* are triple of the square on *NB*;

hence the squares on *VS*, *SN*, *NB* are quadruple of the square on *NB*.

But the square on *SB* is equal to the squares on *SN*, *NB*;

therefore the squares on *BS*, *SV*, that is, the square on *BV* —for the angle *VSB* is right—is quadruple of the square on *NB*;

therefore *VB* is double of *BN*.

But *BC* is also double of *BN*;

therefore *BV* is equal to *BC*.

And, since the two sides *BU*, *UV* are equal to the two sides *BW*, *WC*,

and the base *BV* is equal to the base *BC*,

therefore the angle *BUV* is equal to the angle *BWC*. [I. 8]

Similarly we can prove that the angle *UVC* is also equal to the angle *BWC*;

therefore the three angles *BWC*, *BUV*, *UVC* are equal to one another.

But if in an equilateral pentagon three angles are equal to one another, the pentagon will be equiangular, [XIII. 7]
therefore the pentagon $BUVCW$ is equiangular.

And it was also proved equilateral;
therefore the pentagon $BUVCW$ is equilateral and equiangular, and it is on one side BC of the cube.

Therefore, if we make the same construction in the case of each of the twelve sides of the cube,
a solid figure will have been constructed which is contained by twelve equilateral and equiangular pentagons, and which is called a dodecahedron.

It is then required to comprehend it in the given sphere, and to prove that the side of the dodecahedron is the irrational straight line called apotome.

For let XP be produced, and let the produced straight line be XZ;
therefore PZ meets the diameter of the cube, and they bisect one another,
for this has been proved in the last theorem but one of the eleventh book. [XI. 38]

Let them cut at Z;
therefore Z is the centre of the sphere which comprehends the cube,
and ZP is half of the side of the cube.

Let UZ be joined.

Now, since the straight line NS has been cut in extreme and mean ratio at P,
and NP is its greater segment,
therefore the squares on NS, SP are triple of the square on NP. [XIII. 4]

But NS is equal to XZ,
inasmuch as NP is also equal to PZ, and XP to PS.

But further PS is also equal to XU,
since it is also equal to RP;
therefore the squares on ZX, XU are triple of the square on NP.

But the square on UZ is equal to the squares on ZX, XU;
therefore the square on UZ is triple of the square on NP.

But the square on the radius of the sphere which comprehends the cube is also triple of the square on the half of the side of the cube,
for it has previously been shown how to construct a cube and comprehend it in a sphere, and to prove that the square on the diameter of the sphere is triple of the square on the side of the cube. [XIII. 15]

But, if whole is so related to whole, so is half to half also;
and *NP* is half of the side of the cube;
therefore *UZ* is equal to the radius of the sphere which comprehends the cube.

And *Z* is the centre of the sphere which comprehends the cube;
therefore the point *U* is on the surface of the sphere.

Similarly we can prove that each of the remaining angles of the dodecahedron is also on the surface of the sphere;
therefore the dodecahedron has been comprehended in the given sphere.

I say next that the side of the dodecahedron is the irrational straight line called apotome.

For since, when *NP* has been cut in extreme and mean ratio, *RP* is the greater segment,
and, when *PO* has been cut in extreme and mean ratio, *PS* is the greater segment,
therefore, when the whole *NO* is cut in extreme and mean ratio, *RS* is the greater segment.

[Thus, since, as *NP* is to *PR*, so is *PR* to *RN*,
the same is true of the doubles also,
for parts have the same ratio as their equimultiples; [V. 15]
therefore as *NO* is to *RS*, so is *RS* to the sum of *NR*, *SO*.

But *NO* is greater than *RS*;
therefore *RS* is also greater than the sum of *NR*, *SO*;
therefore *NO* has been cut in extreme and mean ratio,
and *RS* is its greater segment.]

But *RS* is equal to *UV*;
therefore, when *NO* is cut in extreme and mean ratio, *UV* is the greater segment.

And, since the diameter of the sphere is rational,
and the square on it is triple of the square on the side of the cube,
therefore *NO*, being a side of the cube, is rational.

[But if a rational line be cut in extreme and mean ratio, each of the segments is an irrational apotome.]

Therefore *UV*, being a side of the dodecahedron, is an irrational apotome. [XIII. 6]

PORISM. From this it is manifest that, when the side of the cube is cut in extreme and mean ratio, the greater segment is the side of the dodecahedron.

Q. E. D.

In this proposition we find Euclid using two propositions which precede but are used nowhere else, notably VI. 32, which some authors, in consequence of their having overlooked its use here, have been hard put to it to explain.

Euclid's construction in this case is really identical with that given by Mr H. M. Taylor, and also referred to by Henrici and Treutlein under "crystal-formation."

Euclid starts from the cube inscribed in a sphere, as in XIII. 15, and then finds the side of the regular pentagon in which the side of the cube is a diagonal.

Mr Taylor takes l to be the diagonal of a regular pentagon of side a, so that, by Ptolemy's theorem,

$$l^2 = al + a^2,$$

constructs a cube of which l is the edge, and gets the side of the pentagon by drawing ZX from Z, the centre of the cube, perpendicular to the face BF and equal to $\frac{1}{2}(l+a)$, then drawing UV through X parallel to BC, and making UX, XV both equal to $\frac{1}{2}a$.

Euclid finds UV thus.

Draw NO, MH bisecting pairs of opposite sides in the square BF and meeting in P.

Draw GK, HL bisecting pairs of opposite sides in the square BD and meeting in Q.

Divide PN, PO, QH respectively in extreme and mean ratio at R, S, T (PR, PS, QT being the greater segments); draw RU, SV, TW outwards perpendicular to the respective faces of the cube, and all equal in length to PR, PS, TQ.

Join BU, UV, VC, CW, WB.

Then *$BUVCW$ is one of the pentagonal faces of the dodecahedron*; and the others can be constructed in the same way.

Euclid now proves

(1) that the pentagon $BUVCW$ is equilateral,

(2) that it is in one plane,

(3) that it is equiangular,

(4) that the vertex U is on the sphere which circumscribes the cube, and hence

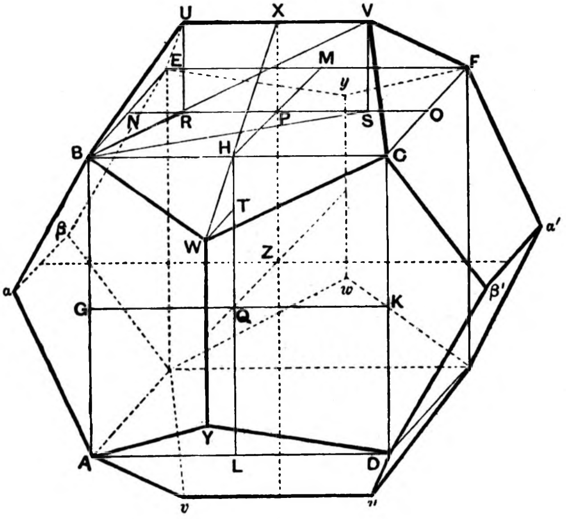
(5) that all the other vertices lie on the same sphere,

and (6) that the side of the dodecahedron is an *apotome*.

(1) To prove that the pentagon $BUVCW$ is *equilateral*.

We have
$$\begin{aligned} BU^2 &= BR^2 + RU^2 \\ &= (BN^2 + NR^2) + RP^2 \\ &= (PN^2 + NR^2) + RP^2 \\ &= 3RP^2 + RP^2 \qquad \text{[XIII. 4]} \\ &= 4RP^2 \\ &= UV^2. \end{aligned}$$

Therefore $BU = UV$.



Similarly it may be proved that BW, WC, CV are all equal to UV or BU.

[Mr Taylor proceeds in this way. With his notation, the projections of BU on BA, BC, BE are respectively $\frac{1}{2}a$, $\frac{1}{2}(l-a)$, $\frac{1}{2}l$.

Therefore
$$\begin{aligned} BU^2 &= \tfrac{1}{4}a^2 + \tfrac{1}{4}(l-a)^2 + \tfrac{1}{4}l^2 \\ &= \tfrac{1}{2}(l^2 - al + a^2) \\ &= a^2. \end{aligned}$$

Similarly for BW, WC etc.]

(2) To prove that the pentagon $BUVCW$ is *in one plane*.

Draw PX parallel to RU or SV meeting UV in X.

Join XH, HW.

Then we have to prove that XH, HW are *in one straight line.*

Now HP, WT, being both perpendicular to the face BD, are parallel.

For the same reason XP, HT are parallel.

Also, since QH is divided at T in extreme and mean ratio,

$$QH : QT = QT : TH.$$

And $$QH = HP,\ QT = WT = PX.$$

Therefore $$HP : PX = WT : TH.$$

Consequently the triangles HPX, WTH *satisfy the conditions of* VI. 32; hence XHW is a straight line.

[Mr Taylor proves this as follows:

The projections of WH, WX on BE are $\frac{1}{2}a$ and $\frac{1}{2}(a+l)$, and the projections of WH, WX on BA are $\frac{1}{2}(l-a)$ and $\frac{1}{2}l$;

and $$a : (a+l) = (l-a) : l,$$

since $$al = l^2 - a^2.$$

Therefore WHX is a straight line.]

(3) To prove that the pentagon $BUVCW$ is *equiangular.*

We have
$$\begin{aligned} BV^2 &= BS^2 + SV^2 \\ &= (BN^2 + NS^2) + SP^2 \\ &= PN^2 + (NS^2 + SP^2) \\ &= PN^2 + 3\,PN^2, \end{aligned}$$

since NS is divided in extreme and mean ratio at P [XIII. 5], so that

$$NS^2 + SP^2 = 3PN^2. \qquad \text{[XIII. 4]}$$

Consequently
$$\begin{aligned} BV^2 &= 4PN^2 \\ &= BC^2, \end{aligned}$$

or $$BV = BC.$$

The $\triangle$s UBV, WBC are therefore equal in all respects,

and $$\angle BUV = \angle BWC.$$

Similarly $$\angle CVU = \angle BWC.$$

Therefore the pentagon is *equiangular.* [XIII. 7]

(4) To prove that the sphere which circumscribes the cube also circumscribes the dodecahedron we have only to prove that, if Z be the centre of the sphere, $ZU = ZB$, for example.

Now, by XI. 38, XP produced meets the diagonal of the cube, and the portion of XP produced which is within the cube and the diagonal bisect one another.

And
$$\begin{aligned} ZU^2 &= ZX^2 + XU^2 \\ &= NS^2 + PS^2 \\ &= 3\,PN^2, \end{aligned}$$

as before.

Also (cf. XIII. 15)
$$\begin{aligned} ZB^2 &= ZP^2 + PB^2 \\ &= ZP^2 + PN^2 + NB^2 \\ &= 3\,PN^2. \end{aligned}$$

Hence $$ZU = ZB.$$

(5) Similarly for ZV, ZW etc.

(6) Since PN is divided in extreme and mean ratio at R,

$$NP : PR = PR : RN.$$

Doubling the terms, we have

$$NO : RS = RS : (NR + SO),$$

so that, if NO is divided in extreme and mean ratio, the greater segment is equal to RS.

Now, since the diameter of the sphere is rational,

and $$(\text{diam. of sphere})^2 = 3\ (\text{edge of cube})^2,$$

the edge of the cube (i.e. NO) is rational.

Consequently RS is an *apotome*.

[This is proved in the spurious XIII. 6 above; Euclid assumes it, and the words purporting to quote the theorem are probably interpolated, like XIII. 6 itself.]

As a matter of fact, with Mr Taylor's notation,

$$l^2 = la + a^2,$$

and $$a = \frac{\sqrt{5} - 1}{2}\, l.$$

Since, if r is the radius of the circumscribing sphere, $r = \sqrt{3} \cdot \frac{l}{2}$,

$$a = \frac{r}{\sqrt{3}}(\sqrt{5} - 1) = \frac{r}{3}(\sqrt{15} - \sqrt{3}).$$

Pappus' solution.

Here too Pappus (III. pp. 156—162) finds four circular sections of the sphere all parallel to one another and all passing through five of the vertices of the dodecahedron.

Analysis.

Suppose (he says) the problem solved, and let the vertices of the dodecahedron be A, B, C, D, E; F, G, H, K, L; M, N, O, P, Q; R, S, T, U, V.

Then, as before, ED is parallel to FL, and AE to FG; therefore the planes $ABCDE$, $FGHKL$ are parallel.

But, since PA is parallel to BH, and BH to OC, PA is parallel to OC; and they are equal; therefore PO, AC are parallel, so that ST, ED are also parallel.

Similarly RS, DC are parallel, and likewise the pairs (TU, EA), (UV, AB), (VR, BC).

Therefore the planes $ABCDE$, $RSTUV$ are parallel; and the circles $ABCDE$, $RSTUV$ are equal, since the inscribed pentagons are equal.

Similarly the circles $FGHKL$, $MNOPQ$ are equal, since the pentagons inscribed in them are equal.

Now CL, OU are parallel because each is parallel to KN;

therefore L, C, O, U are in one plane.

And LC, CO, OU, UL are all equal, since they subtend angles of equal pentagons.

Also L, C, O, U are on a plane section, i.e. a circle;

therefore $LCOU$ is a square.

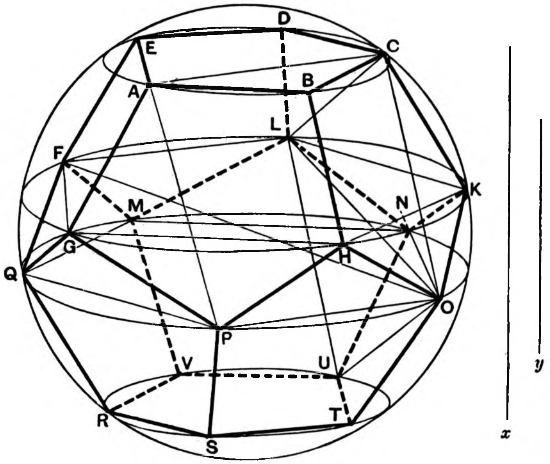
Therefore $$OL^2 = 2LC^2 = 2LF^2$$

(for LC, LF subtend angles of equal pentagons).

And the angle OLF is right; for PO, LF are equal and parallel chords in two equal and parallel circular sections of a sphere [Pappus' lemma, p. 138, 20—26].

Therefore $$OF^2 = OL^2 + FL^2 = 3FL^2.$$ [from above]

And OF is a diameter of the sphere; for PO, FL are on opposite sides of the centres of the circles in which they are [Pappus' lemma, pp. 136—8].



Now suppose p, t, h to be the sides of an equilateral pentagon, triangle and hexagon in the circle $FGHKL$, d the diameter of the sphere.

Then $$d : FL = \sqrt{3} : 1$$ [from above]
$$= t : h;$$ [Eucl. XIII. 12]

and it follows *alternando* (since $FL = p$) that
$$d : t = p : h.$$

Now let d', p', h' be the sides of a regular decagon, pentagon and hexagon respectively inscribed in *any* one circle.

Since, if FL be divided in extreme and mean ratio, the greater segment is equal to ED, [XIII. 8]
$$FL : ED = h' : d'.$$ [VI. Def. 3, XIII. 9]

And $FL : ED$ is the ratio of the sides of the regular pentagons inscribed in the circles $FGHKL$, $ABCDE$, and is therefore equal to the ratio of the sides of the equilateral triangles inscribed in the same circles.

Therefore $$t : (\text{side of } \triangle \text{ in } ABCDE) = h' : d'.$$

But $$d : t = p : h$$
$$= p' : h';$$

therefore, *ex aequali*, $d : (\text{side of } \triangle \text{ in } ABCDE) = p' : d'$.

Now d is given;

therefore the sides of the equilateral triangles inscribed in the circles $ABCDE$, $FGHKL$ respectively are given, whence the radii of those circles are also given.

Thus the two circles are given, and so accordingly are the equal and parallel circular sections.

Synthesis.

Set out two straight lines x, y such that d, x, y are in the ratio of the sides of a regular pentagon, hexagon and decagon respectively inscribed in one and the same circle.

Find two circular sections of the sphere with radii r, r', where

$$r^2 = \tfrac{1}{3}x^2,\ r'^2 = \tfrac{1}{3}y^2.$$

Let these be the circles *FGHKL*, *ABCDE* respectively, and draw the equal and parallel circles on the other side of the centre, namely *MNOPQ*, *RSTUV*.

In the first two circles inscribe regular pentagons with their sides respectively parallel, *ED* being parallel to *FL*.

Draw equal and parallel chords (on the other sides of the centres) in the other two circles, namely *ST* equal and parallel to *ED*, and *PO* equal and parallel to *FL*; and complete the regular pentagons on *ST*, *PO* inscribed in the circles.

Thus all the vertices of the dodecahedron are determined.

The proof of the correctness of the construction is clear from the analysis.

Pappus adds that the construction shows that the circles containing five vertices of the dodecahedron are the same respectively as those containing three vertices of the icosahedron, and that the same circle circumscribes the triangle of the icosahedron and the pentagonal face of the dodecahedron in the same sphere.

PROPOSITION 18.

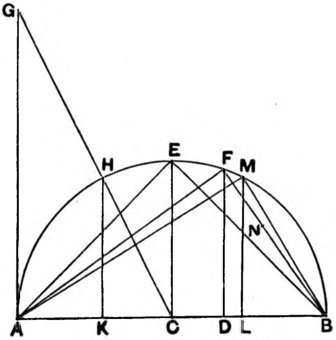
To set out the sides of the five figures and to compare them with one another.

Let *AB*, the diameter of the given sphere, be set out, and let it be cut at *C* so that *AC* is equal to *CB*, and at *D* so that *AD* is double of *DB*;

let the semicircle *AEB* be described on *AB*,

from *C*, *D* let *CE*, *DF* be drawn at right angles to *AB*,

and let *AF*, *FB*, *EB* be joined.



Then, since *AD* is double of *DB*, therefore *AB* is triple of *BD*.

Convertendo, therefore, *BA* is one and a half times *AD*.

But, as BA is to AD, so is the square on BA to the square on AF, [V. Def. 9, VI. 8]
for the triangle AFB is equiangular with the triangle AFD;
therefore the square on BA is one and a half times the square on AF.

But the square on the diameter of the sphere is also one and a half times the square on the side of the pyramid. [XIII. 13]

And AB is the diameter of the sphere;
therefore AF is equal to the side of the pyramid.

Again, since AD is double of DB,
therefore AB is triple of BD.

But, as AB is to BD, so is the square on AB to the square on BF; [VI. 8, V. Def. 9]
therefore the square on AB is triple of the square on BF.

But the square on the diameter of the sphere is also triple of the square on the side of the cube. [XIII. 15]

And AB is the diameter of the sphere;
therefore BF is the side of the cube.

And, since AC is equal to CB,
therefore AB is double of BC.

But, as AB is to BC, so is the square on AB to the square on BE;
therefore the square on AB is double of the square on BE.

But the square on the diameter of the sphere is also double of the square on the side of the octahedron. [XIII. 14]

And AB is the diameter of the given sphere;
therefore BE is the side of the octahedron.

Next, let AG be drawn from the point A at right angles to the straight line AB,
let AG be made equal to AB,
let GC be joined,
and from H let HK be drawn perpendicular to AB.

Then, since GA is double of AC,
for GA is equal to AB,
and, as GA is to AC, so is HK to KC,
therefore HK is also double of KC.

Therefore the square on *HK* is quadruple of the square on *KC*;
therefore the squares on *HK*, *KC*, that is, the square on *HC*, is five times the square on *KC*.

But *HC* is equal to *CB*;
therefore the square on *BC* is five times the square on *CK*.

And, since *AB* is double of *CB*,
and, in them, *AD* is double of *DB*,
therefore the remainder *BD* is double of the remainder *DC*.

Therefore *BC* is triple of *CD*;
therefore the square on *BC* is nine times the square on *CD*.

But the square on *BC* is five times the square on *CK*;
therefore the square on *CK* is greater than the square on *CD*;
therefore *CK* is greater than *CD*.

Let *CL* be made equal to *CK*,
from *L* let *LM* be drawn at right angles to *AB*,
and let *MB* be joined.

Now, since the square on *BC* is five times the square on *CK*,
and *AB* is double of *BC*, and *KL* double of *CK*,
therefore the square on *AB* is five times the square on *KL*.

But the square on the diameter of the sphere is also five times the square on the radius of the circle from which the icosahedron has been described. [XIII. 16, Por.]

And *AB* is the diameter of the sphere;
therefore *KL* is the radius of the circle from which the icosahedron has been described;
therefore *KL* is a side of the hexagon in the said circle. [IV. 15, Por.]

And, since the diameter of the sphere is made up of the side of the hexagon and two of the sides of the decagon inscribed in the same circle, [XIII. 16, Por.]
and *AB* is the diameter of the sphere,
while *KL* is a side of the hexagon,
and *AK* is equal to *LB*,
therefore each of the straight lines *AK*, *LB* is a side of the decagon inscribed in the circle from which the icosahedron has been described.

And, since *LB* belongs to a decagon, and *ML* to a hexagon,

for *ML* is equal to *KL*, since it is also equal to *HK*, being the same distance from the centre, and each of the straight lines *HK*, *KL* is double of *KC*,

therefore *MB* belongs to a pentagon. [XIII. 10]

But the side of the pentagon is the side of the icosahedron; [XIII. 16]

therefore *MB* belongs to the icosahedron.

Now, since *FB* is a side of the cube,

let it be cut in extreme and mean ratio at *N*,

and let *NB* be the greater segment;

therefore *NB* is a side of the dodecahedron. [XIII. 17, Por.]

And, since the square on the diameter of the sphere was proved to be one and a half times the square on the side *AF* of the pyramid, double of the square on the side *BE* of the octahedron and triple of the side *FB* of the cube,

therefore, of parts of which the square on the diameter of the sphere contains six, the square on the side of the pyramid contains four, the square on the side of the octahedron three, and the square on the side of the cube two.

Therefore the square on the side of the pyramid is four-thirds of the square on the side of the octahedron, and double of the square on the side of the cube;

and the square on the side of the octahedron is one and a half times the square on the side of the cube.

The said sides, therefore, of the three figures, I mean the pyramid, the octahedron and the cube, are to one another in rational ratios.

But the remaining two, I mean the side of the icosahedron and the side of the dodecahedron, are not in rational ratios either to one another or to the aforesaid sides;

for they are irrational, the one being minor [XIII. 16] and the other an apotome [XIII. 17].

That the side *MB* of the icosahedron is greater than the side *NB* of the dodecahedron we can prove thus.

For, since the triangle *FDB* is equiangular with the triangle *FAB*, [VI. 8]

proportionally, as *DB* is to *BF*, so is *BF* to *BA*. [VI. 4]

And, since the three straight lines are proportional,
as the first is to the third, so is the square on the first to the square on the second; [V. Def. 9, VI. 20, Por.]
therefore, as *DB* is to *BA*, so is the square on *DB* to the square on *BF*;
therefore, inversely, as *AB* is to *BD*, so is the square on *FB* to the square on *BD*.

But *AB* is triple of *BD*;
therefore the square on *FB* is triple of the square on *BD*.

But the square on *AD* is also quadruple of the square on *DB*,
for *AD* is double of *DB*;
therefore the square on *AD* is greater than the square on *FB*;
therefore *AD* is greater than *FB*;
therefore *AL* is by far greater than *FB*.

And, when *AL* is cut in extreme and mean ratio,
KL is the greater segment,
inasmuch as *LK* belongs to a hexagon, and *KA* to a decagon; [XIII. 9]
and, when *FB* is cut in extreme and mean ratio, *NB* is the greater segment;
therefore *KL* is greater than *NB*.

But *KL* is equal to *LM*;
therefore *LM* is greater than *NB*.

Therefore *MB*, which is a side of the icosahedron, is by far greater than *NB* which is a side of the dodecahedron.

Q. E. D

I say next that *no other figure, besides the said five figures, can be constructed which is contained by equilateral and equiangular figures equal to one another.*

For a solid angle cannot be constructed with two triangles, or indeed planes.

With three triangles the angle of the pyramid is constructed, with four the angle of the octahedron, and with five the angle of the icosahedron;
but a solid angle cannot be formed by six equilateral and equiangular triangles placed together at one point,

for, the angle of the equilateral triangle being two-thirds of a right angle, the six will be equal to four right angles:
which is impossible, for any solid angle is contained by angles less than four right angles. [XI. 21]

For the same reason, neither can a solid angle be constructed by more than six plane angles.

By three squares the angle of the cube is contained, but by four it is impossible for a solid angle to be contained,
for they will again be four right angles.

By three equilateral and equiangular pentagons the angle of the dodecahedron is contained;
but by four such it is impossible for any solid angle to be contained,
for, the angle of the equilateral pentagon being a right angle and a fifth, the four angles will be greater than four right angles:
which is impossible.

Neither again will a solid angle be contained by other polygonal figures by reason of the same absurdity.

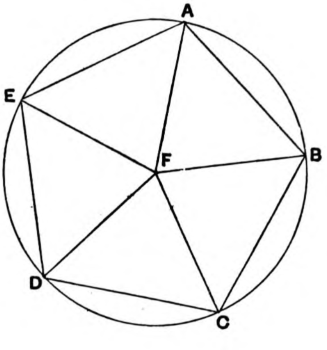
Therefore etc.

Q. E. D.

LEMMA.

But that *the angle of the equilateral and equiangular pentagon is a right angle and a fifth* we must prove thus.

Let *ABCDE* be an equilateral and equiangular pentagon,
let the circle *ABCDE* be circumscribed about it,
let its centre *F* be taken,
and let *FA*, *FB*, *FC*, *FD*, *FE* be joined.



Therefore they bisect the angles of the pentagon at *A*, *B*, *C*, *D*, *E*.

And, since the angles at *F* are equal to four right angles and are equal,

therefore one of them, as the angle *AFB*, is one right angle less a fifth;
therefore the remaining angles *FAB*, *ABF* consist of one right angle and a fifth.

But the angle *FAB* is equal to the angle *FBC*;
therefore the whole angle *ABC* of the pentagon consists of one right angle and a fifth.

Q. E. D.

We have seen in the preceding notes that, if r be the radius of the sphere circumscribing the five solid figures,

$$(\text{edge of tetrahedron}) = \tfrac{2}{3}\sqrt{6}\,.\,r,$$
$$(\text{edge of octahedron}) = \sqrt{2}\,.\,r,$$
$$(\text{edge of cube}) = \tfrac{2}{3}\sqrt{3}\,.\,r,$$
$$(\text{edge of icosahedron}) = \frac{r}{5}\sqrt{10\,(5-\sqrt{5})},$$
$$(\text{edge of dodecahedron}) = \frac{r}{3}\,(\sqrt{15}-\sqrt{3}).$$

Euclid here exhibits the edges of all the five regular solids in one figure.

(1) Make AD equal to $2DB$.

Thus $$BA = \tfrac{3}{2}AD,$$
and $$BA : AD = BA^2 : AF^2;$$
therefore $$BA^2 = \tfrac{3}{2}AF^2.$$

Thus $$AF = \sqrt{\tfrac{2}{3}}\,.\,2r = \tfrac{2}{3}\sqrt{6}\,.\,r = (\textit{edge of tetrahedron}).$$

(2) $$AB^2 : BF^2 = AB : BD$$
$$= 3 : 1.$$

Therefore $$BF^2 = \tfrac{1}{3}AB^2,$$
or $$BF = \frac{2}{\sqrt{3}}\,.\,r = \frac{2}{3}\sqrt{3}\,.\,r = (\textit{edge of cube}).$$

(3) $$AB^2 = 2BE^2.$$
Therefore $$BE = \sqrt{2}\,.\,r = (\textit{edge of octahedron}).$$

(4) Draw AG perpendicular and equal to AB. Join GC, meeting the semicircle in H, and draw HK perpendicular to AB.

Then $$GA = 2AC;$$
therefore, by similar triangles, $$HK = 2KC.$$
Hence $$HK^2 = 4KC^2,$$
and therefore $$5KC^2 = HK^2 + KC^2$$
$$= HC^2$$
$$= CB^2.$$

Again, since $AB = 2CB$, and $AD = 2DB$,
by subtraction, $$BD = 2DC,$$
or $$BC = 3DC.$$

Therefore $$9DC^2 = BC^2$$
$$= 5KC^2.$$

Hence $KC > CD$.

Make CL equal to KC, draw LM at right angles to AB, and join AM, MB.

Since $$CB^2 = 5KC^2,$$
$$AB^2 = 5KL^2.$$

It follows that KL $(= \sqrt{\frac{4}{5}} . r)$ is the radius of, or the side of the regular hexagon in, the circle containing the pentagonal sections of the icosahedron. [XIII. 16]

And, since

$$2r = (\text{side of hexagon}) + 2\ (\text{side of decagon in same circle})$$ [XIII. 16, Por.]

$$AK = LB = (\text{side of decagon in the said circle}).$$

But $$LM = HK = KL = (\text{side of hexagon in circle}).$$

Therefore $$LM^2 + LB^2\ (= BM^2) = (\text{side of pentagon in circle})^2$$ [XIII. 10]
$$= (\text{edge of icosahedron})^2,$$

and $$BM = (\textit{edge of icosahedron}).$$

[More shortly, $$HK = 2KC,$$

whence $$HK^2 = 4KC^2,$$

and $$5KC^2 = HC^2 = r^2.$$

Also $$AK = r - CK = r\left(1 - \frac{1}{\sqrt{5}}\right).$$

Thus $$BM^2 = HK^2 + AK^2$$
$$= \frac{4}{5}r^2 + r^2\left(1 - \frac{1}{\sqrt{5}}\right)^2$$
$$= r^2\left(\frac{10}{5} - \frac{2}{\sqrt{5}}\right)$$
$$= \frac{r^2}{5}(10 - 2\sqrt{5}),$$

and $$BM = \frac{r}{5}\sqrt{10(5 - \sqrt{5})} = (\textit{edge of icosahedron}).]$$

(5) Cut BF (the edge of the cube) in extreme and mean ratio at N. Then, if BN be the greater segment,

$$BN = (\textit{edge of dodecahedron}).$$ [XIII. 17]

[Solving, we obtain

$$BN = \frac{\sqrt{5} - 1}{2} . BF$$
$$= \frac{\sqrt{5} - 1}{2} . \frac{2}{\sqrt{3}} . r$$
$$= \frac{r}{3}(\sqrt{15} - \sqrt{3})$$
$$= (\textit{edge of dodecahedron}).]$$

(6) If t, o, c are the edges of the tetrahedron, octahedron and cube respectively,

$$4r^2 = \tfrac{3}{2}t^2 = 2o^2 = 3c^2.$$

If each of these equals is put equal to X,

$$4r^2 = X,$$
$$t^2 = \tfrac{2}{3} \cdot X,$$
$$o^2 = \tfrac{1}{2} \cdot X,$$
$$c^3 = \tfrac{1}{3} \cdot X,$$

whence $$4r^2 : t^2 : o^2 : c^2 = 6 : 4 : 3 : 2,$$

and the ratios between $2r$, t, o, c are all *rational* (in Euclid's sense).

The ratios between these and the edges of the icosahedron and the dodecahedron are *irrational*.

(7) To prove that

(edge of icosahedron) > (edge of dodecahedron),

i.e. that $$MB > NB.$$

By similar △s FDB, AFB,

$$DB : BF = BF : BA,$$

or $$DB : BA = DB^2 : BF^2.$$

But $$3DB = BA;$$

therefore $$BF^2 = 3DB^2.$$

By hypothesis, $$AD^2 = 4DB^2;$$

therefore $$AD > BF,$$

and, *a fortiori*, $$AL > BF.$$

Now LK is the side of a hexagon, and AK the side of a decagon in the same circle;

therefore, when AL is divided in extreme and mean ratio, KL is the greater segment.

And, when BF is divided in extreme and mean ratio, BN is the greater segment.

Therefore, since $$AL > BF,$$
$$KL > BN,$$

or $$LM > BN.$$

And therefore, *a fortiori*, $$MB > BN.$$

APPENDIX.

I. THE CONTENTS OF THE SO-CALLED BOOK XIV. BY HYPSICLES.

This supplement to Euclid's Book XIII. is worth reproducing for the sake not only of the additional theorems proved in it but of the historical notices contained in the preface and in one or two later passages. Where I translate literally from the Greek text, I shall use inverted commas; except in such passages I reproduce the contents in briefer form.

I have already quoted from the Preface (Vol. I. pp. 5—6), but I will repeat it here.

"Basilides of Tyre, O Protarchus, when he came to Alexandria and met my father, spent the greater part of his sojourn with him on account of the bond between them due to their common interest in mathematics. And on one occasion, when looking into the tract written by Apollonius about the comparison of the dodecahedron and icosahedron inscribed in one and the same sphere, that is to say, on the question what ratio they bear to one another, they came to the conclusion that Apollonius' treatment of it in this book was not correct; accordingly, as I understood from my father, they proceeded to amend and rewrite it. But I myself afterwards came across another book published by Apollonius, containing a demonstration of the matter in question, and I was greatly attracted by his investigation of the problem. Now the book published by Apollonius is accessible to all; for it has a large circulation in a form which seems to have been the result of later careful elaboration.

"For my part, I determined to dedicate to you what I deem to be necessary by way of commentary, partly because you will be able, by reason of your proficiency in all mathematics and particularly in geometry, to pass an expert judgment upon what I am about to write, and partly because, on account of your intimacy with my father and your friendly feeling towards myself, you will lend a kindly ear to my disquisition. But it is time to have done with the preamble and to begin my treatise itself.

[Prop. 1.] "*The perpendicular drawn from the centre of any circle to the side of the pentagon inscribed in the same circle is half the sum of the side of the hexagon and of the side of the decagon inscribed in the same circle.*"

Let ABC be a circle, and BC the side of the inscribed regular pentagon.

Take D the centre of the circle, draw DE from D perpendicular to BC, and produce DE both ways to meet the circle in F, A.

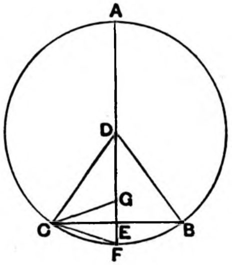
I say that DE is half the sum of the side of the hexagon and of the side of the decagon inscribed in the same circle.

Let DC, CF be joined; make GE equal to EF, and join GC.

Since the circumference of the circle is five times the arc BFC,

and half the circumference of the circle is the arc ACF,

while the arc FC is half the arc BFC,



therefore (arc ACF) = 5 (arc FC)

or (arc AC) = 4 (arc CF).

Hence $\angle ADC = 4 \angle CDF$,

and therefore $\angle AFC = 2 \angle CDF$.

Thus $\angle CGF = \angle AFC = 2 \angle CDF$;

therefore [I. 32] $\angle CDG = \angle DCG$,

so that $DG = GC = CF$.

And $GE = EF$;

therefore $$DE = EF + FC.$$

Add DE to each;

therefore $$2DE = DF + FC.$$

And DF is the side of the regular hexagon, and FC the side of the regular decagon, inscribed in the same circle.

Therefore etc.

"Next it is manifest from the theorem [12] in Book XIII. that *the perpendicular drawn from the centre of the circle to the side of the equilateral triangle* [inscribed in it] *is half of the radius of the circle.*

[**Prop. 2.**] "*The same circle circumscribes both the pentagon of the dodecahedron and the triangle of the icosahedron inscribed in the same sphere.*

"This is proved by Aristaeus in his work entitled *Comparison of the five figures.* But Apollonius proves in the second edition of his comparison of the dodecahedron with the icosahedron that, as the surface of the dodecahedron is to the surface of the icosahedron, so also is the dodecahedron itself to the icosahedron, because the perpendicular from the centre of the sphere to the pentagon of the dodecahedron and to the triangle of the icosahedron is the same.

"But it is right that I too should prove that

[**Prop. 2**] *The same circle circumscribes both the pentagon of the dodecahedron and the triangle of the icosahedron inscribed in the same sphere.*

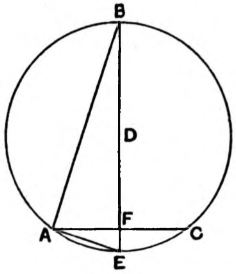
"For this I need the following

Lemma.

"*If an equilateral and equiangular pentagon be inscribed in a circle, the sum of the squares on the straight line subtending two sides and on the side of the pentagon is five times the square on the radius.*"

Let ABC be a circle, AC the side of the pentagon, D the centre; draw DF perpendicular to AC and produce it to B, E;

join AB, AE.



I say that

$$BA^2 + AC^2 = 5DE^2.$$

For, since $BE = 2ED$,

$$BE^2 = 4ED^2.$$

And $$BE^2 = BA^2 + AE^2;$$

therefore $$BA^2 + AE^2 + ED^2 = 5ED^2.$$

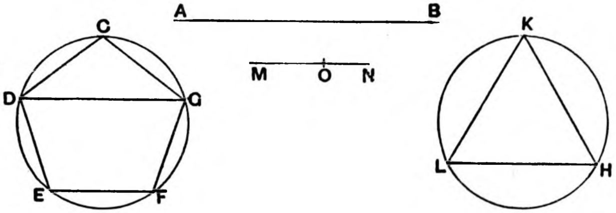
But $$AC^2 = DE^2 + EA^2;$$

[Eucl. XIII. 10]

therefore $$BA^2 + AC^2 = 5DE^2.$$

"This being proved, it is required to prove that the same circle circumscribes both the pentagon of the dodecahedron and the triangle of the icosahedron inscribed in the same sphere."

Let AB be the diameter of the sphere, and let a dodecahedron and an icosahedron be inscribed.



Let $CDEFG$ be one pentagon of the dodecahedron, and KLH one triangle of the icosahedron.

I say that the radii of the circles circumscribing them are equal.

Join DG; then DG is the side of a cube inscribed in the sphere.

[Eucl. XIII. 17]

Take a straight line MN such that $AB^2 = 5MN^2$.

Now the square on the diameter of the sphere is five times the square on the radius of the circle from which the icosahedron is described.

[XIII. 16, Por.]

Therefore MN is equal to the radius of the circle passing through the five vertices of the icosahedron which form a pentagon.

Cut MN in extreme and mean ratio at O, MO being the greater segment.

Therefore MO is the side of the decagon in the circle with radius MN.

[XIII. 9 and 5, converse]

Now $$5MN^2 = AB^2 = 3DG^2.$$ [XIII. 15]

But $$3DG^2 : 3CG^2 = 5MN^2 : 5MO^2$$

(since, if DG is cut in extreme and mean ratio, the greater segment is equal to CG, and, if two straight lines are cut in extreme and mean ratio, their segments are in the same ratio: see lemma later, pp. 518—9).

And $$5MO^2 + 5MN^2 = 5KL^2.$$

[This follows from XIII. 10, since KL is, by the construction of XIII. 16, the side of the regular pentagon in the circle with radius equal to MN, that is, the circle in which MN is the side of the inscribed hexagon and MO the side of the inscribed decagon.]

Therefore $$5KL^2 = 3CG^2 + 3DG^2.$$

But $$5KL^2 = 15\,(\text{radius of circle about } KLH)^2, \qquad [\text{XIII. 12}]$$

and $$3DG^2 + 3CG^2 = 15\,(\text{radius of circle about } CDEFG)^2.$$
[Lemma above]

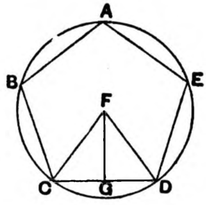
Therefore the radii of the two circles are equal.

Q. E. D.

[**Prop.** 3.] "*If there be an equilateral and equiangular pentagon and a circle circumscribed about it, and if a perpendicular be drawn from the centre to one side, then*

30 *times the rectangle contained by the side and the perpendicular is equal to the surface of the dodecahedron.*"

Let $ABCDE$ be the pentagon, F the centre of the circle, FG the perpendicular on a side CD.

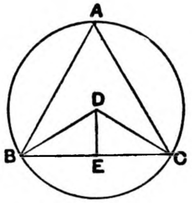


I say that
$$30\,CD \,.\, FG = 12\,(\text{area of pentagon}).$$

Let CF, FD be joined.

Then, since
$$CD \,.\, FG = 2\,(\triangle\, CDF),$$
$$5\,CD \,.\, FG = 10\,(\triangle\, CDF),$$
whence $$30\,CD \,.\, FG = 12\,(\text{area of pentagon}).$$

Similarly we can prove that,



[**Prop.** 4] *If* ABC *be an equilateral triangle in a circle,* D *the centre, and* DE *perpendicular to* BC,
$$30\text{BC} \,.\, \text{DE} = (\textit{surface of icosahedron}).$$

For $$DE \,.\, BC = 2\,(\triangle\, DBC);$$
therefore $$3\,DE \,.\, BC = 6\,(\triangle\, DBC)$$
$$= 2\,(\triangle\, ABC),$$
whence $$30\,DE \,.\, BC = 20\,(\triangle\, ABC).$$

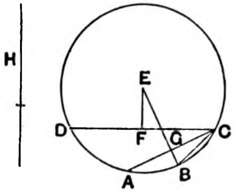
It follows that [**Prop.** 5]

(*surface of dodecahedron*) : (*surface of icosahedron*)
= (*side of pentagon*) . (*its perpendicular*) : (*side of triangle*) . (*its perp.*).

"This being clear, we have next to prove that,

[**Prop.** 6] *As the surface of the dodecahedron is to the surface of the icosahedron, so is the side of the cube to the side of the icosahedron.*"

Let ABC be the circle circumscribing the pentagon of the dodecahedron and the triangle of the icosahedron, and let CD be the side of the triangle, AC that of the pentagon.



Let E be the centre, and EF, EG perpendiculars to CD, AC.

Produce EG to meet the circle in B and join BC.

Set out H equal to the side of the cube inscribed in the same sphere.

I say that

$$(\text{surface of dodecahedron}) : (\text{surface of icosahedron}) = H : CD.$$

For, since the sum of EB, BC is divided at B in extreme and mean ratio, and BE is the greater segment, [XIII. 9]
and $EG = \frac{1}{2}(EB + BC)$, [Prop. 1]
while $EF = \frac{1}{2}BE$, [see p. 513 above]
therefore, if EG is divided in extreme and mean ratio, the greater segment is equal to EF [that is to say, since EB is the greater segment of $EB + BC$ divided in extreme and mean ratio, $\frac{1}{2}EB$ is the greater segment of $\frac{1}{2}(EB + BC)$ similarly divided].

But, if H is also divided in extreme and mean ratio, the greater segment is equal to CA. [XIII. 17, Por.]

Therefore $H : CA = EG : EF$,
or $FE . H = CA . EG$.

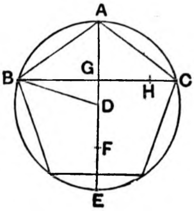
And, since $H : CD = FE . H : FE . CD$,
and $FE . H = CA . EG$,
therefore $H : CD = CA . EG : FE . CD$
$= (\text{surface of dodecahedron}) : (\text{surf. of icos.})$. [Prop. 5]

Another proof of the same theorem.

Preliminary.

Let ABC be a circle and AB, AC sides of an inscribed regular pentagon.

Join BC; take D the centre of the circle, join AD and produce it to meet the circle at E. Join BD.



Let DF be made equal to $\frac{1}{2}AD$, and CH equal to $\frac{1}{3}CG$.

I say that

rect. $AF . BH = (\text{area of pentagon})$.

For, since $AD = 2DF$,

$$AF = \tfrac{3}{2}AD.$$

And, since $GC = 3HC$,

$$GC = \tfrac{3}{2}GH.$$

Therefore $FA : AD = CG : GH$,
so that $AF . GH = AD . CG$
$= AD . BG$
$= 2(\triangle ABD)$.

Therefore

$$5AF \,.\, GH = 10\,(\triangle ABD) = 2 \text{ (area of pentagon)}.$$

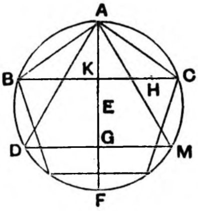
And $GH = 2HC$;

therefore $5AF \,.\, HC =$ (area of pentagon),

or $AF \,.\, BH =$ (area of pentagon).

Proof of theorem.

This being clear, let the circle be set out which circumscribes the pentagon of the dodecahedron and the triangle of the icosahedron inscribed in the same sphere.



Let ABC be the circle, and AB, AC two sides of the pentagon; join BC.

Take E the centre of the circle, join AE and produce it to F.

Let $AE = 2EG$, $KC = 3CH$.

Through G draw DM at right angles to AF meeting the circle at D, M;

DM is then the side of the inscribed equilateral triangle.

Join AD, AM, which are equal to DM.

Now, since $AG \,.\, BH =$ (area of pentagon),

and $AG \,.\, GD =$ (area of triangle),

therefore $BH : GD =$ (area of pentagon) : (area of triangle),

and $12BH : 20GD =$ (surface of dod.) : (surface of icos.).

But $12BH = 10BC$, since $BH = 5HC$, and $BC = 6HC$;

and $20GD = 10DM$;

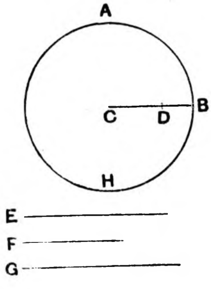
therefore (surface of dodecahedron) : (surface of icosahedron)

= (side of cube) : (side of icosahedron).

"Next we have to prove that,

[**Prop.** 7] *If any straight line whatever be cut in extreme and mean ratio, then, as is* (1) *the straight line the square on which is equal to the sum of the squares on the whole line and on the greater segment to* (2) *the straight line the square on which is equal to the sum of the squares on the whole and on the lesser segment, so is* (3) *the side of the cube to* (4) *the side of the icosahedron.*"

Let AHB be the circle circumscribing both the pentagon of the dodecahedron and the triangle of the icosahedron inscribed in the same sphere, C the centre of the circle, and CB any radius divided at D in extreme and mean ratio, CD being the greater segment.



CD is then the side of the decagon inscribed in the circle. [XIII. 9 and 5, converse]

Let E be the side of the icosahedron, F that of the dodecahedron, and G that of the cube, inscribed in the sphere.

Then E, F are the sides of the equilateral triangle and pentagon inscribed in the circle, and, if G is divided in extreme and mean ratio, the greater segment is equal to F. [XIII. 17, Por.]

Thus $E^2 = 3BC^2$, [XIII. 12]
and $CB^2 + BD^2 = 3CD^2$. [XIII. 4]

Therefore $E^2 : CB^2 = (CB^2 + BD^2) : CD^2$,
or $E^2 : (CB^2 + BD^2) = CB^2 : CD^2$
$= G^2 : F^2$.

Therefore, alternately and inversely,

$$G^2 : E^2 = F^2 : (CB^2 + BD^2).$$

But $F^2 = BC^2 + CD^2$; for the square on the side of the pentagon is equal to the sum of the squares on the sides of the hexagon and decagon inscribed in the same circle. [XIII. 10]

Therefore $G^2 : E^2 = (BC^2 + CD^2) : (CB^2 + BD^2)$,
which is the result required.

It has now to be proved that

[**Prop. 8**] (*Side of cube*) : (*side of icosahedron*)
= (*content of dodecahedron*) : (*content of icosahedron*).

Since equal circles circumscribe the pentagon of the dodecahedron and the triangle of the icosahedron inscribed in the same sphere,

and in a sphere equal circular sections are equally distant from the centre,

the perpendiculars from the centre of the sphere to the faces of the two solids are equal;

in other words, the pyramids with the centre as vertex and the pentagons of the dodecahedron and the triangles of the icosahedron respectively as bases are of equal height.

Therefore the pyramids are to one another as their bases.

Thus (12 pentagons) : (20 triangles)
= (12 pyramids on pentagons) : (20 pyramids on triangles),
or (surface of dodecahedron) : (surface of icosahedron)
= (content of dod.) : (content of icos.).

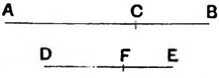
Therefore
(content of dodecahedron) : (content of icosahedron)
= (side of cube) : (side of icosahedron). [Prop. 6]

Lemma.

If two straight lines be cut in extreme and mean ratio, the segments of both are in one and the same ratio.

Let AB be cut in extreme and mean ratio at C, AC being the greater segment;

and let DE be cut in extreme and mean ratio at F, DF being the greater segment.



I say that $AB : AC = DE : DF$.
Since $AB . BC = AC^2$,
and $DE . EF = DF^2$,
$AB . BC : AC^2 = DE . EF : DF^2$,
and $4AB . BC : AC^2 = 4DE . EF : DF^2$.

Componendo,

$$(4AB \,.\, BC + AC^2) : AC^2 = (4DE \,.\, EF + DF^2) : DF^2,$$

or $$(AB + BC)^2 : AC^2 = (DE + EF)^2 : DF^2; \qquad \text{[II. 8]}$$

therefore $$(AB + BC) : AC = (DE + EF) : DF.$$

Componendo,

$$(AB + BC + AC) : AC = (DE + EF + DF) : DF,$$

or $$2AB : AC = 2DE : DF;$$

that is, $$AB : AC = DE : DF.$$

Summary of results.

If AB be *any* straight line divided at C in extreme and mean ratio, AC being the greater segment, and if we have a cube, a dodecahedron and an icosahedron inscribed in one and the same sphere, then:

(1) (side of cube) : (side of icosahedron) $= \sqrt{(AB^2 + AC^2)} : \sqrt{(AB^2 + BC^2)}$;

(2) (surface of dod.) : (surface of icos.)
= (side of cube) : (side of icosahedron);

(3) (content of dod.) : (content of icos.)
= (surface of dod.) : (surface of icos.);

and (4) (content of dodecahedron) : (content of icos.)
$= \sqrt{(AB^2 + AC^2)} : \sqrt{(AB^2 + BC^2)}$.

II. NOTE ON THE SO-CALLED "BOOK XV."

The second of the two Books added to the genuine thirteen is also supplementary to the discussion of the regular solids, but is much inferior to the first, "Book XIV." Its contents are of less interest and the exposition leaves much to be desired, being in some places obscure and in others actually inaccurate. It consists of three portions unequal in length. The first (Heiberg, Vol. v. pp. 40—48) shows how to inscribe certain of the regular solids in certain others, (*a*) a tetrahedron ("pyramid") in a cube, (*b*) an octahedron in a tetrahedron ("pyramid"), (*c*) an octahedron in a cube, (*d*) a cube in an octahedron and (*e*) a dodecahedron in an icosahedron. The second portion (pp. 48—50) explains how to calculate the number of edges and the number of solid angles in the five solids respectively. The third (pp. 50—66) shows how to determine the angle of inclination between faces meeting in an edge of any one of the solids. The method is to construct an isosceles triangle with vertical angle equal to the said angle of inclination; from the middle point of any edge two perpendiculars are drawn to it, one in each of the two faces intersecting in that edge; these perpendiculars (forming an angle which is the inclination of the two faces to one another) are used to determine the two equal sides of an isosceles triangle, and the base of the triangle is easily found from the known properties of the particular solid. The rules for drawing the respective isosceles triangles are first given all together in general terms (pp. 50—52); and the special interest of the passage consists in the fact that the rules are attributed to "Isidorus our great teacher." This Isidorus is no doubt Isidorus of Miletus, the architect of the Church of St Sophia at Constantinople (about 532 A.D.), whose pupil Eutocius also was; he is often referred to by Eutocius (*Comm. on Archimedes*) as ὁ Μιλήσιος μηχανικὸς Ἰσίδωρος ἡμέτερος διδάσκαλος. Thus the third portion of the Book at all events was written by a pupil of Isidorus in the sixth century. Kluge (*De Euclidis elementorum libris qui feruntur XIV et XV*, Leipzig, 1891) has closely examined the language and style of the three portions and conjectures that they may be the work of different authors; the first portion may, he thinks, date from the end of the third century (the time of Pappus), and the second portion too may be older than the third. Hultsch however (art. "Eukleides" in Pauly-Wissowa's *Real-Encyclopädie der classischen Altertumswissenschaft*, 1907) does not think his arguments convincing.

It may be worth while to set out the particulars of Isidorus' rules for constructing isosceles triangles with vertical angles equal respectively to the angles of inclination between faces meeting in an edge of the several regular solids. A certain base is taken, and then with its extremities as centres and a certain other straight line as radius two circles are drawn; their point of intersection determines the vertex of the particular isosceles triangle. In the case of the cube the triangle is of course right-angled; in the other cases the bases and the equal sides are as shown below.

	Base of isosceles triangle	*Equal sides of isosceles triangle*
For the tetrahedron	the side of a triangular face	the perpendicular from the vertex of a triangular face to its base
For the octahedron	the diagonal of the square on one side of a triangular face	ditto
For the icosahedron	the chord joining two non-consecutive angular points of the regular pentagon on an edge (the "pentagon of the icosahedron")	ditto
For the dodecahedron	the chord joining two non-consecutive angular points of a pentagonal face [*BC* in the figure of Eucl. XIII. 17]	the perpendicular from the middle point of the chord joining two non-consecutive angular points of a face to the parallel side of that face [*HX* in the figure of Eucl. XIII. 17]

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